WITT GROUPS OF ALGEBRAIC GROUPS

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Abstract. Let G_k be a split reductive group over a field k of $ch(k) = 0$ corresponding to a simply connected Lie group G. Let T be a maximal torus of G. When k is an algebraically closed, the Balmer's Witt group $W^*(G_k)$ is isomorphic to $KO^{2*-1}(G/T)$ but not to $KO^{2*-1}(G)$.

1. Introduction

Let X be a (quasi projective) smooth variety over a field k with $k \subset \mathbb{C}$. The Witt group $W(X)$ is the quotient of the Grothendieck group of vector bundles with quadratic forms over X, by the subgroup generated by bundles V with quadratic forms which admit Lagrangian subbundles E. The generalized Witt group $W^*(X;L)$ is defined by Balmer [Ba] for $* \in \mathbb{Z}/4$ and for a line bundle L on X so that $W^0(X;O_X) = W^0(X) = W(X)$. (Moreover if $L = L'$ in $Pic(X)/2$, there is an isomorphism $W^*(X;L) \cong W^*(W;L')$.) We can define a natural map [Ya3], [Zi]

$$q^* : W^*(X) \to KO^{2*}(X(\mathbb{C}))/KU^{2*}(X(\mathbb{C})) \stackrel{\times\eta}{\to} KO^{2*-1}(X(\mathbb{C}))$$

where $KO^*(-)$ and $KU^*(-)$ are (toplogical) real and complex K-theories, and $0 \neq \eta \in KO^{-1}(pt.) \cong \mathbb{Z}/2$.

Let G be a compact connected Lie group and T a maximal torus of G and B be the Borel subroup with $T \subset B$. Let us denote by G_k (resp. T_k, B_k) the split reductive group (resp. its maximal torus, the Borel subgroup) over k which corresponds G (resp. T, B). Let $T_k^1 \subset ... \subset T_k^\ell = T_k$ be a sequence of tori of G_k where $T_k^i \cong (\mathbb{A}^1 - \{0\})^{\times i}$. Recently Calmès and Fasel [Ca-Fa] proved that $W^*(G_k/B_k;L) = 0$ whenever $0 \neq L \in Pic(G_k/T_k)/2$. (Note that $W^*(G_k/B_k;L) \cong W^*(G_k/T_k;L)$ for all L.) There is a localization exact sequence in the Witt theory

$$\stackrel{\delta}{\to} W^{*-1}(G_k/T_k^i;t_i) \stackrel{j_*}{\to} W^*(G_k/T_k^i) \to W^*(G_k/T_k^{i-1}) \stackrel{\delta}{\to} ...$$

where $(t_1,...,t_n)$ is a basis of $Pic(G/T)/2$. Here we see $W^{*-1}(G_k/T_k^i;t_i) = 0$ from the result by Calmès and Fasel. By induction on i, we easily prove

Theorem 1.1. *If $rank_2 Pic(G_k/T_k)/2 = rank(T_k) = \ell$ (e.g., G is a simply connected and k is algebraically closed), then $W^*(G_k) \cong W^*(G_k/T_k)$.*

Now let k be algebraically closed with $ch(k) = 0$. Also recently, Zibrowius showed [Zi] that the above map q^* is always an isomorphism for each cellular variety X when $k = \mathbb{C}$, and it can be generalized to any algebraically closed field. Of course, the flag variety G_k/B_k is cellular. Hence we have the isomorphism

$$q^* : W^*(G_k) \cong KO^{2*}(G/T)/KU^{2*}(G/T) \cong KO^{2*-1}(G/T).$$

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Let us consider the Atiyah-Hirzebruch spectral sequence converges to $KO^*(G/T)$. It is known that the differential d_2 is the Steenrod squareing $Sq^2\ mod(2)$. Kishimoto, Kono and Ohsita [Ki-Ko-Oh], [Ki-Oh], [Oh] computed this Atiyah-Hirzebruch spectral sequence and showed that it collapses from the $E_3^{*,*'}$-term for simply connected simple groups except for E_7 and E_8. Moreover the above $KO^{2*-1}(G/T)$ is isomorphic to

$$H^*(G/T; Sq^2) = H^*(H^{2*}(G/T; \mathbb{Z}/2); Sq^2)$$

for these groups.

On the other hand, there are spectral sequences $E(P)_r^{*,*'}$ (by Gille and Pardon [Gi], [Pa]) and $E(BW)_r^{*,*'}$ (by Balmer-Walter [Ba-Wa]) such that $E(P)_r^{*,*'}$ converges to $grE(BW)_2^{*,*'}$ and $E(BW)_r^{*,*'}$ converges to $W^*(G_k)$.

By the Borel theorem, the cohomolgy $H^*(G; \mathbb{Z}/2)$ is isomorphic to a tensor algebra of a polynomial algebra generated by even dimensional elements y_i and an exterior algebra generated by x_j of odd dimensional. We consider the graded algebra $grH^*(G; \mathbb{Z}/2)$ defined by the filtration $w(y_i) = 0$ and $w(x_j) = 1$. Then Sq^2 acts on $grH^*(G; \mathbb{Z}/2)$ as a differential. Let us write its homology by

$$H^*(G; Sq^2) = H^*(grH^*(G; \mathbb{Z}/2); Sq^2).$$

Then we can prove that $E(P)_3^{*,*'} \cong H^*(G; Sq^2)$ with

$$deg(y_i) = (1/2|y_i|, 1/2|y_i|) \quad and \quad deg(x_j) = (1/2(|x_j| - 1), 1/2(|x_j| + 1)).$$

In fact, by Totaro [To], it is known that $d_2 = Sq^2$ on $E(P)_2^{*,*'}$ when $* = *'$. Moreover we will see

Theorem 1.2. *Let k be an algebraically closed field in $\mathbb{C}$. For each simply connected simple Lie group G, we have isomorphisms*

$$W^*(G_k) \cong H^*(G/T; Sq^2) \cong \Lambda(z_1, ..., z_s) \quad deg(z_i) = odd,$$

which is also isomorphic to

$$\begin{cases} E(P)_4^{*,*'} \cong H^*(H^*(G; Sq^2); d_3) & for\ G = E_6, E_7, E_8, \\ E(P)_3^{*,*'} \cong H^*(G; Sq^2) & otherwise. \end{cases}$$

The explicit value of $deg(z_i)$ is given in §7 below.

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2. THE TOTAL WITT GROUP

We can consider the generalized Witt group $W^i(X; L)$ for $i \in \mathbb{Z}/4$ and for a line bundle L on X such that the usual Witt group $W^i(X) = W^i(X; O_X)$ (see [Ba], [Ba-Ca2], [Zi] for example). We also note if $L = L'\ mod(2)$, then we have a (non canonical) isomorphism $W^*(X; L) \cong W^*(X; L')$. Let us write the total Witt group

$$W^{total}(X) = \oplus_{i \in \mathbb{Z}/4,\ L \in Pic(X)/2} W^i(X; L).$$

The Gysin and the boundary maps are defined as the maps for $W^{total}(-)$. (Gysin maps are constructed by Calmès and Hornbostel in [Ca-Ho], also see §4.1 in [Ne].) Let $g : Z \subset X$ be a regular embedding of $codim = c$, and $U = X - Z$. Let ω_g be the relative canonical bundle (for the definition see [Ba-Ca1]). Then we have the natural exact sequence ((11) in [Ba-Ca1])

$$\to W^{*-c}(Z,\omega_g\otimes L|_Z)\stackrel{g_*}{\to} W^*(X,L)\stackrel{f^*}{\to} W^*(U,L|_U)\stackrel{\delta}{\to}\ldots$$

Here we note that the Witt group with the coefficient L is written by using Thom space $Th(L)$ of the bundle L ([Ne], [Zi]), namely, $W^{*+1}(X;L)\cong \tilde{W}^*(Th(L))$ where $\tilde{W}^*(-)$ is the reduced theory. Then we also have for a vector bundle V on X

$$W^{*+c}(X;det(V))\cong \tilde{W}^*(Th(V))\qquad for\ dim(V)=c.$$

Let G_k be a split reductive group over a field k of $ch(k)=0$ corresponding to a compact Lie group G. Let T_k be a maximal (split) torus of G_k. We consider the sequence of tori

$$T_k^1\subset T_k^2\subset\ldots\subset T_k^\ell=T_k\quad where\ T_k^i\cong(\mathbb{A}^1-\{0\})^{\times i}\cong(\mathbb{G}_m)^{\times i}.$$

Here we assume that $rank_{\mathbb{Z}/2}(Pic(G_k/T_k)/2)=rank(T)=\ell$ and

$$\mathbb{Z}/2\{t_1,\ldots,t_\ell\}\cong Pic(G_k/T_k)/2.$$

We consider the $\mathbb{G}_m$-bundle

$$\mathbb{G}_m\to G/T_k^{i-1}\to G/T_k^i$$

where we take the base t_i corresponding the above $\mathbb{G}_m$-bundle. Let $E(G_k/T_k^i)$ be the line bundle corresponding to the above $\mathbb{G}_m$-bundle so that G/T_k^{i-1} is an open subset of $E(G_k/T_k^i)$. Then we have maps

$$Z=G_k/T_k^i\stackrel{g}{\subset} X=E(G_k/T_k^i)\stackrel{f}{\supset} U=X-Z=G_k/T_k^{i-1}.$$

Thus we have its localization exact sequence

$$\to W^{*-1}(G_k/T_k^i;t_i+L)\stackrel{g_*}{\to} W^*(G_k/T_k^i;L)\stackrel{f^*}{\to} W^*(G_k/T_k^{i-1},L)\stackrel{\delta}{\to}\ldots$$

Let B_k be the Borel subgroup of G_k. Then we have the fibering

$$U_k\to G_k/T_k\to G_k/B_k$$

for the unipotent group U_k. Hence we have an isomorphism $W^*(G_k/T_k;L)\cong W^*(G_k/B_k;L)$. We recall the result Calmès and Fasel

Lemma 2.1. *([Ca-Fa]) If $L\neq 0\in Pic(G_k/T_k)/2$, then*

$$W^*(G_k/T_k;L)\cong W^*(G_k/B_k;L)=0.$$

For an algebraically closed field k, we will give a topological proof of this fact in §4 below.

Lemma 2.2. *Let $L_i=e_1t_1+\ldots+e_it_i\neq 0\in Pic(G_k/T_k)/2$, $e_i=0$ or 1. Then we have $W^*(G_k/T_k^i;L_i)=0$ for $1\leq i\leq\ell$.*

Proof. Consider the localization exact sequence

$$\to W^{*-1}(G_k/T_k^\ell;t_\ell+L_{\ell-1})\stackrel{g_*}{\to} W^*(G_k/T_k^\ell;L_{\ell-1})\stackrel{f^*}{\to} W^*(G_k/T_k^{\ell-1},L_{\ell-1})\stackrel{\delta}{\to}\ldots$$

Since the first and the second term are zero from the result by Calmès and Fasel, the third term $W^*(G_k/T_k^{\ell-1},L_{\ell-1})=0$.

Next consider the localization exact sequence

$$\to W^{*-1}(G_k/T_k^i;t_i+L_{i-1})\stackrel{g_*}{\to} W^*(G_k/T_k^i;L_{i-1})\stackrel{f^*}{\to} W^*(G_k/T_k^{i-1},L_{i-1})\stackrel{\delta}{\to}\ldots$$

By induction, we assume the first and the second term are zero. Then we have $W^*(G_k/T_k^{i-1}, L_{i-1}) = 0$. □

Theorem 2.3. *If $rank_2 Pic(G_k/T_k)/2 = rank(T) = \ell$ (e.g., G is a simply connected and k is algebraically closed), then $W^*(G_k) \cong W^*(G_k/T_k)$.*

Proof. We consider the localization exact sequence

$$\to W^{*-1}(G_k/T_k^i; t_i) \xrightarrow{g_*} W^*(G_k/T_k^i) \xrightarrow{f^*} W^*(G_k/T_k^{i-1}) \xrightarrow{\delta} ...$$

Since the first term is zero from the preceding lemma, we have an isomorphism

$$W^*(G_k/T_k^i) \cong W^*(G_k/T_k^{i-1}).$$

When $rank_2(Pic(G/T)/2) = \ell$, then the above isomorphism holds for $i = 1$ also. □

Next consider the case $rank_2(Pic(G_k/T_k)/2) + 1 = \ell$ and $\mathbb{Z}/2\{t_2, ..., t_\ell\} \cong Pic(G_k/T_k)/2$. Letting $L_i = e_2t_2 + ... + e_it_i$, the arguments above also work and we have

Corollary 2.4. *If $rank_2(Pic(G_k/T_k)/2) + 1 = \ell$, then we have an isomorphism*

$$W^*(G_k) \cong W^*(G_k/T_k) \otimes \Lambda(x_0), \quad |x_0| = 0.$$

Proof. We consider the localization exact sequence

$$\to W^{*-1}(G_k/T_k^1; t_1) \xrightarrow{g_*} W^*(G_k/T_k^1) \xrightarrow{f^*} W^*(G_k) \xrightarrow{\delta} ...$$

where $t_1 \in L_\ell$. But $Pic(G_k/T_k^1) = 0$ (compare the results in §3 e.g. Theorem 3.4). So the normal bundle for $G_k/T_k^1 \subset E(G_k/T_k^1)$ is trivial. Hence the Gysin map $g_* = 0$. So we have

$$W^*(G_k/T_k^1; t_1) \cong W^*(G_k/T_k^1) \cong W^*(G_k/T_k).$$

Hence we get the desired result. □

3. Cohomology theories of compact Lie group G

Let G be a compact connected Lie group. By the Borel theorem, we have the ring isomorphism

$$grH^*(G; \mathbb{Z}/2) \cong P(y) \otimes \Lambda(x_1, ..., x_l) \quad with \quad P(2) = \otimes_i \mathbb{Z}/2[y_i]/(y_i^{2^{r_i}})$$

where $|y_i| = even$ and $|x_j| = odd$. Moreover for each y_i, there is x_j with $x_j^2 = y_i$. (In fact, $H^*(G; \mathbb{Z}/2)$ is multiplicatively generated by odd dimensional elements, e.g., see [Ka].)

Let T be a maximal torus of G and BT the classifying space of T. We consider the fibering $G \xrightarrow{\pi} G/T \xrightarrow{i} BT$ and the induced spectral sequence

$$E_2^{*,*'} = H^*(BT; H^{*'}(G; \mathbb{Z}/2)) \Longrightarrow H^*(G/T; \mathbb{Z}/2).$$

The cohomology of the classifying space of the torus is given by

$$H^*(BT; \mathbb{Z}/2) \cong S(t) = \mathbb{Z}/2[t_1, ..., t_\ell] \quad with \quad |t_i| = 2,$$

where ℓ is also the number of the odd degree generators x_i in $H^*(G; \mathbb{Z}/2)$. It is known that y_i are permanent cycles and that there is a regular sequence ([Tod],[Mi-Ni]) $(b_1, ..., b_\ell)$ in $H^*(BT; \mathbb{Z}/2)$ such that $d_{|x_i|+1}(x_i) = b_i$. Thus we get

$$E_\infty^{*,*'} \cong grH^*(G/T; \mathbb{Z}/2) \cong P(y) \otimes S(t)/(b_1, ..., b_\ell).$$

Since $H^*(G/T;\mathbb{Z})$ is no torsion, we get the isomorphism

$$H^*(G/T;\mathbb{Z}) \cong \mathbb{Z}[y_1,...,y_k,t_1,...,t_\ell]/(f_1,...,f_k,\bar{b}_1,...,\bar{b}_\ell)$$

where $f_i = y_i^{2^{r_i}}\ mod(Ideal(t_1,...,t_\ell))$ and $\bar{b}_j = b_j\ mod(2)$.

In particular, if $|y|>2$ and $|b_1|>2$, then $Pic(G_k/T_k) \cong \mathbb{Z}/2\{t_1,...,t_\ell\}$ and

$$dim_2(Pic(G_k/T_k)/2) = \ell = dim(T).$$

Let $T^1 \subset ... \subset T^\ell = T$ be a sequence of tori of G where $T^i \cong (S^1)^{\times i}$. The fibering $S^1 \to G/T^{i-1} \to G/T^i$ induces the Gysin exact sequence

$$\xrightarrow{\delta} H^{*-2}(G/T^i;\mathbb{Z}/2) \xrightarrow{j_*=\times t_i} H^*(G/T^i;\mathbb{Z}/2) \to H^*(G/T^{i-1};\mathbb{Z}/2) \xrightarrow{\delta} ...$$

From this arguments, we can compute $H^*(G/T^i;\mathbb{Z}/2)$ from $H^*(G/T;\mathbb{Z}/2)$.

Lemma 3.1. *We can take tori T^i such that $b_1,...,b_i$ is regular in $S(t)/(t_{i+1},...,t_\ell)$, and that $b_i = t_ig_i$ in $S(t)/(b_1,...,b_{i-1},t_{i+1},...,t_\ell)$ for some $g_i \in S(t)$.*

Proof. In §4 in [Ya2], we see that the above assumption holds for each simply connected simple Lie group G (when $p=2$ in Lemma 2.1 in [Ya2]) except for $G = Sp(n), E_8$. (Note $H^*(Spin(n)/T;\mathbb{Z}/2) \cong H^*(SO(n)/T;\mathbb{Z}/2)$ see §3 in [Tod].) The case $G = Sp(n)$ is almost immediate since

$$H^*(G/T;\mathbb{Z}/2) \cong \mathbb{Z}/2[t_1,...,t_n]/(c_1^2,...,c_n^2).$$

where c_i is the i-th elementary symmetric function on $t_1,...,t_n$. In fact $c_1^2,...,c_n^2$ is regular and $c_i^2 = g_it_i\ mod(t_{i+1},...,t_n)$.

For the case $G = E_8$, we use the result by Kono-Ishitoya, Ohsita [Ko-Is], [Oh]. It is known that $grH^*(G/T;\mathbb{Z}/2)/(P(y))$ is isomorphic to

$$S(t)/(b_1,...,b_{\ell=8}) \cong S(t)/(c_2,c_3,c_5',c_9',I_8,I_{12},I_{14},I_{15})$$

in the notation in [Ko-Is] (see [Ko-Is] for details). Here $c_9' = c_1(c_8+c_7c_1+c_0c_1^2)$ and we think $b_\ell = c_9'$ and $t_\ell = c_1$. Next we see $I_{15} = c_8c_7\ mod(c_1)$, and think $t_{\ell-1} = t_8$. We see that

$$I_{14} = c_7^2,\ I_{12} = c_6^2,\ c_5' = c_5,\ I_8 = c_4^2 \quad mod(c_1,c_8).$$

This shows the lemma for $G = E_8$. □

Using this lemma, we can prove the following Corollary 3.2, Theorem 3.4 and Corollary 3.5 (see also [Ya2]).

Corollary 3.2. *(Lemma 2.1 in[Ya2]) We have an isomorphism*

$$grH^*(G/T^i;\mathbb{Z}/2) \cong H^*(G/T)/(t_{i+1},...,t_\ell) \otimes \Lambda(x_{i+1},...,x_\ell).$$

Examples. When $G = U(n)$, we still know that $P(y) \cong \mathbb{Z}/2$ and

$$H^*(G;\mathbb{Z}/2) \cong \Lambda(x_1,x_3,x_5,...,x_{2n-1}) \quad with\ |x_j| = j,$$

$$H^*(G/T;\mathbb{Z}/2) \cong \mathbb{Z}/2[t_1,...,t_n]/(c_1,c_2,...,c_n).$$

Of course $b_i = c_i$ is regular and satisfies (by $g_i = t_1...t_{i-1}$) the lemma above

$$H^*(G/T^i;\mathbb{Z}/2) \cong \mathbb{Z}/2[t_1,...,t_i]/(c_1,...,c_i) \otimes \Lambda(x_{2i-1},...,x_{2n-1}).$$

Let X be an algebraic variety over k. Let $H^{*,*'}(X;\mathbb{Z}/2)$ be the $mod(2)$ motivic cohomology constructed by Suslin and Voevodsky [Vo1]. For nonzero element $x \in H^{m,n}(X;\mathbb{Z}/2)$, we define the weight degree and the different degree by

$$w(x) = 2n-m, \quad d(x) = m-n.$$

When X is smooth, it is known that $w(x) \geq 0,\ d(x) \leq dim(X)$ for $0 \neq x$. (For example see Corollary 2.3 in [Vo1].) Moreover from the affirmative answer by Voevodsky to the Milnor conjecture (and hence the Beilinson-Lichtenbaum conjecture) implies

$$H^{*,*'}(pt.;\mathbb{Z}/2) \cong \mathbb{Z}/2[\tau] \otimes K_*^M(k)$$

where $0 \neq \tau \in H^{0,1}(pt.;\mathbb{Z}/2) \cong \mathbb{Z}/2$ and the mod (2) Milnor K-theory is $K_*^M(k)/2 \cong H^{*,*}(pt.;\mathbb{Z}/2)$.

Let us denote by G_k the split reductive group over k corresponding to the compact Lie group G and T_k^i the spit torus. Since G_k/B_k is cellular and $H^{*,*'}(G_k/B_k;\mathbb{Z}/2) \cong H^{*,*'}(G_k/T_k;\mathbb{Z}/2)$, we have the isomorphism

$$H^{*,*'}(G_k/T_k;\mathbb{Z}/2) \cong K_*^M(k)/2 \otimes H^{*,*'}(G_{\mathbb{C}}/T_{\mathbb{C}};\mathbb{Z}/2)$$

$$\cong H^{*,*'}(pt.;\mathbb{Z}/2) \otimes H^*(G/T;\mathbb{Z}/2) \quad with\ w(H^*(G/T;\mathbb{Z}/2)) = 0.$$

In particular, we note that the realization map $t_{\mathbb{C}} : G_k/T_k \to G_{\mathbb{C}}/T_{\mathbb{C}}$ induces an isomorphism $Pic(G_k/T_k)/2 \cong Pic(G_{\mathbb{C}}/T_{\mathbb{C}})/2$.

For the motivic theory, we have also the Thom isomorphism and hence the Gysin exact sequence

$$\xrightarrow{\delta} H^{*-2,*'-1}(G_k/T_k^i;\mathbb{Z}/2) \xrightarrow{\times t_i} H^{*,*'}(G_k/T_k^i;\mathbb{Z}/2) \to H^{*,*'}(G_k/T_k^{i-1};\mathbb{Z}/2) \xrightarrow{\delta} ...$$

Since $H^{2*+1,*}(G_k/T_k^i;\mathbb{Z}/2) = 0$ (the weight degree< 0), we have

$$H^{2*-2,*-1}(G_k/T_k^i;\mathbb{Z}/2) \xrightarrow{\times t_i} H^{2*,*}(G_k/T_k^i;\mathbb{Z}/2) \to H^{2*,*}(G_k/T_k^{i-1};\mathbb{Z}/2) \xrightarrow{\delta} 0.$$

By descending induction on i, we easily show

$$H^{2*,*}(G_k/T^i;\mathbb{Z}/2) \cong H^{2*}(G/T;\mathbb{Z}/2)/(t_{i+1},...,t_\ell)$$

and there is $x_i \in H^{2*-1,*}(G_k/T^{i-1};\mathbb{Z}/2)$ with $\delta(x_i) = b_i$. Moreover, we have the following theorems

Theorem 3.3. *(Theorem 3.1 in [Ya2]) There is an $H^*(pt.;\mathbb{Z}/2)$-module isomorphism*

$$H^{*,*'}(G_k/T_k^i;\mathbb{Z}/2) \cong H^*(G/T^i;\mathbb{Z}/2) \otimes H^{*,*'}(pt.;\mathbb{Z}/2)$$

where the bidegree in $H^*(G/T^i;\mathbb{Z}/2)$ is given for nonzero element $u \in H^*(G/T;\mathbb{Z}/2)$ by $w(u) = 0$ and $w(x_i) = 1$.

Corollary 3.4. *(Corollary 3.2 in [Ya2]) Define a filtration F_i' to be the $H^*(pt.;\mathbb{Z}/2)$-module generated by elements x with $w(x) \leq i$. Then we have*

$$grH^{*,*'}(G_k;\mathbb{Z}/2) = \oplus F_i'/F_{i-1}' \cong H^{*,*'}(pt.;\mathbb{Z}/2) \otimes P(y) \otimes \Lambda(x_1,...,x_\ell)$$

where $w(P(y)) = 0$ and $w(x_i) = 1$.

Proof. Since $t_{\mathbb{C}} : H^{2*,*}(G_k;\mathbb{Z}/2) \to H^{2*}(G;\mathbb{Z}/2)$ is an isomorphism, we have $y_i^{2^{r_i}} = 0 \in H^{2*,*}(G_k;\mathbb{Z}/2)$. Let $x_i^2 = y_j$. Since $w(x_i^2) = 2$ and $t_{\mathbb{C}}(\tau) = 1$ and $K_+^M(k) \subset Ker(t_{\mathbb{C}})$, we see

$$x_i^2 = \tau y_j\ mod(Ideal(K_+^M(k)/2)).$$

□

Theorem 3.5. *(Theorem 3.3 in [Ya2]) Suppose that $H^{*,*'}(X;\mathbb{Z}/2)$ is $\mathbb{Z}/2[\tau]$-free. Then*

$$H^{*,*'}(X \times G_k/T_k^i;\mathbb{Z}/2) \cong H^{*,*'}(X;\mathbb{Z}/2) \otimes_{H^{*,*'}(pt.;\mathbb{Z}/2)} H^{*,*'}(G_k/T_k^i;\mathbb{Z}/2).$$

Corollary 3.6. *(Corollary 3.4 in [Ya2]) By the assumption in the preceding theorem, the Künneth formula holds for $H^{*,*'}(G_k;\mathbb{Z}/2)$. Hence it is a Hopf algebra.*

4. KO-theory

We explain the KO-theory of flag manifolds G/T according to Hara, Kishimoto, Kono and Ohsita [Ha], [Ki-Ko-Oh]. Recall that the coefficient rings of the (topological) KO^*-theory and KU^*-theory are (e.g., see §1 in [Ha])

$$KO^*\cong \mathbb{Z}[\mu,\mu^{-1},\eta,w]/(2\eta,\eta^3,w^2-4\mu,\eta w),$$
$$KU^*\cong \mathbb{Z}[\beta,\beta^{-1}]$$

with $|\mu|=-8,|w|=-4,|\eta|=-1$ and $|\beta|=-2$. To compute $KO^*(G/T)$, we consider the Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*'}\cong H^*(G/T;KO^{*'})\Longrightarrow KO^{*+*'}(G/T).$$

It is well known that the first differential is ((3.1) in [Ha], [Fuj])

$$d_2(\lambda\otimes x)=\lambda\eta\otimes Sq^2(\bar{x}),\quad \lambda\in KO^*$$

where $\bar{x}\in H^*(G/T;\mathbb{Z}/2)$ is the mod 2 reduction of x.

Note $Sq^2Sq^2=Sq^3Sq^1$ from the Adem relation. So $Sq^2Sq^2(x)=0$ in $H^*(G/T;\mathbb{Z}/2)$ (in fact $H^{odd}(G/T;\mathbb{Z}/2)=0$). Let us write

$$H^*G/T;Sq^2)=H(H^{2*}(G/T;\mathbb{Z}/2);Sq^2)$$

the homology with the differential Sq^2. To compatible with $W^*(-)$, we define degree of element $x\in H^*(G/T;Sq^2)$ by $half$ of that in $H^*(G/T;\mathbb{Z}/2)$, i.e., $deg(x)=1/2|x|$. Hence we have

$$E_3^{2*,odd}\cong E_3^{2*,8*'-1}\cong \mathbb{Z}/2\{\eta\}[\mu,\mu^{-1}]\otimes H^*(G/T;Sq^2).$$

Hence $E_\infty^{2*,odd}$ is isomorphic to a subquotient of $H^*(G/T;Sq^2)$.

Kishimoto, Kono and Ohsita [Ki-Ko-Oh], [Ki-Oh] get this homology for $G=U(n),Sp(n),O(n),G_2,F_4,E_6$. For example

$$H^*(U(2m+1)/T;Sq^2)\cong \Lambda(z_3,z_7,...,z_{4m-1})$$

where $z_{4s-1}=\sum_{i_1<...<i_s}t_{i_1}t_{i_2}^2...t_{i_s}^2$ in $H^*(G/T;\mathbb{Z}/2)$, (in fact $Sq^2(z_{4s-1})=0$).

Since $deg(d_r)=(r+1,-r)$, we see the Atiyah-Hirzebruch spectral sequence collapses form the $E_3^{*,*'}$-term for the case $G=U(2m+1)$. The same fact happens for other groups, e.g., Kishimoto, Kono and Ohsita proved that the following assumption is satisfied for all above groups G.

Assumption 4.1. *The Atiyah-Hirzebruch spectral sequence for $KO^*(G/T)$ collapses from the $E_3^{*,*'}$-term.*

Let Y be a topological space (e.g., a finite dimensional CW-complex). We have the following well known (Bott) exact sequence ((1.1) in [Ha], (3.4) in [At])

$$(1)\quad \to KO^{*+1}(Y)\overset{\times\eta}{\to} KO^*(Y)\overset{c}{\to} KU^*(Y)\overset{r\cdot\beta^{-1}}{\to} KO^{*+2}(Y)\to ...$$

where c is the complexification map and r is the real restriction map. The map $\beta^{-1}:KU^*(Y)\to KU^{*+2}(Y)$ is an isomorphism. Let us write

$$KO^*(Y)/KU^*(Y)=KO^*(Y)/(rKU^*(Y))=KO^*(Y)/(r\beta^{-1}KU^*(Y)).$$

Then we have

$$KO^*(Y)/KU^*(Y) \cong KO^*(Y)/(Ker\eta) \cong Im(\eta)(KO^*(Y))$$

(which is 2-torsion since so is η), e.g., $KO^*/KU^* \cong \mathbb{Z}/2\{1,\eta\}[\mu,\mu^{-1}]$.

Hereafter, in this paper, let us write

$$KO^{2*}(Y)/KU^{2*}(Y) = \oplus_{r\in\mathbb{Z}/4} KO^{2r}(Y)/KU^{2r}(Y),$$

(while $KO^*(Y)$ means usually $\oplus_{r\in\mathbb{Z}} KO^r(Y)$). Taking the degree modulo 4 in $H^*(G/T; Sq^2)$, we have a ring isomorphism ;

Corollary 4.2. *The graded ring $KO^{2*}(G/T)/KU^{2*}(G/T)$ is isomorphic to a subquotient of $H^*(G/T; Sq^2)$. Moreover if Assumption 4.1 is satisfied, then*

$$KO^{2*}(G/T)/KU^{2*}(G/T) \cong H^*(G/T; Sq^2).$$

By Künneth theorem, we have

$$H^*(X \times Y; Sq^2) \cong H^*(X; Sq^2) \otimes H^*(Y; Sq^2).$$

Hereafter this section, we assume that k is algebraically closed. When X is cellular variety over $\mathbb{C}$, Zibrouwius shows that $W^*(X) \cong KO^{2*}(X(\mathbb{C}))/KU^{2*}(X(\mathbb{C}))$. (For the another proof, see §5 below.) This fact is easily extended to any algebraically closed field $k \in \mathbb{C}$. Hence we get

Corollary 4.3. *If Assumption 4.1 holds, then $KO^*(G/T)/KU^*(G/T)$ has the Künneth formula and (from Lemma 2.5)*

$$W^*(G_k) \cong H^*(G/T; Sq^2)$$

is a Hopf algebra (for an algebraically closed field k).

Example. Let $G = SU(3)$. Then $H^*(G;\mathbb{Z}/2) \cong \Lambda(x_3, x_5)$ and

$$H^*(G/T;\mathbb{Z}/2) \cong \mathbb{Z}/2[t_1,t_2,t_3]/(c_1', c_2', c_3') \cong \mathbb{Z}/2[t_1,t_2]/(c_1^2 + c_2, c_1c_2)$$

where c_i' (resp. c_i) is the i-th elementary symmetric function of 3-variables (resp. 2-variables). Hence we see

$$W^*(G_k/T_k) \cong KO^{2*}(G/T)/KU^{2*}(G/T) \cong H^*(G/T; Sq^2) \cong \Lambda(z_3)$$

where $z_3 = t_1t_2^2$. ($SQ^2(z_3) = t_1^2t_2^2 = c_2^2 = 0$.) Then we have $W^*(G_k) \cong W^*(G_k/T_k)$ and it is a primitive Hopf algebra $\Lambda(z_3)$. On the other hand, we consider the Atiyah-Hirzebruch spectral sequence converging to $KO^*(G)$. Since $Sq^2(x_3) = x_5$, we see

$$E_3^{*',-1} \cong E_3^{2*,-1} \cong H^*(H^*(G;\mathbb{Z}/2); Sq^2) \cong \Lambda(v_4)$$

where $v_4 = x_3x_5$. We easily see $E_3^{*,*'} \cong E_\infty^{*,*'}$ from dimensional reason. Thus the map

$$q^* : W^*(G_k) \cong \Lambda(z_3) \to KO^{2*}(G)/KU^{2*}(G) \cong \Lambda(v_4)$$

is not injective nor surjective since $deg(v_4) = 4$ but $deg(z_3) = 3$.

We will prove the Camlès and Fasel result for an algebraically closed field k. Let L be a line bundle over X and $Th(L)$ be its Thom class. Recall $W^{*-1}(X;L) \cong \tilde{W}^*(Th(L))$.

Lemma 4.4. *(Zibrowius (2.4) in [Zi])*

$$H^*(\tilde{H}^*(Th(L);\mathbb{Z}/2); Sq^2) \cong H^*(H^*(X;\mathbb{Z}/2); Sq^2 + c_1(L)).$$

Proof. We recall the Steenrod algebra A_2-strucure of the Thom space

$$\tilde{H}^*(Th(L);\mathbb{Z}/2) \cong H^*(X;\mathbb{Z}/2)\{c_1\} \subset H^*(X;\mathbb{Z}/2)\{1,c_1\} \cong H^*(P(L);\mathbb{Z}/2)$$

where $P(L)$ is the associated projective bundle and c_1 is the first Chern class so that $c_1^2 = c_1(L)c_1$. Hence for $x \in H^*(X);\mathbb{Z}/2)$, we have (by Cartan formula)

$$Sq^2(xc_1) = Sq^2(x)c_1 + xc_1(L)c_1 = (Sq^2 + c_1(L))xc_1.$$

Each element in $\tilde{H}^*(Th(L);\mathbb{Z}/2)$ is expressed as xc_1, and we have the result. $\square$

Recall that $grH^*(G/T;\mathbb{Z}/2) \cong P(y) \otimes S(t)/(b_1,...,b_\ell)$ and $S(t) = \mathbb{Z}/2[t_1,...,t_\ell]$.

Lemma 4.5. *For each* $0 \neq t \in \mathbb{Z}/2\{t_1,...,t_\ell\}$, *we get*

$$H^*(H^*(Th(t);\mathbb{Z}/2);Sq^2) \cong H^*(H^*(G/T;\mathbb{Z}/2);Sq^2 + t) = 0.$$

Proof. Let $t = t_1$ and $d = Sq^2 + t$. Give a weight by $w(t_1) = 1$ and $w(t_i) = 0$ for $i \geq 2$ and consider the graded ring $grS(t)$. Then $d = Sq^2 + t$ acts on $\mathbb{Z}/2[t_i]$ as $Sq^2 + 0$ for $i \geq 2$. Decompose $grS(t) = \mathbb{Z}/2[t_1] \otimes B$ as a d-module with $B = \otimes_{i\geq 2}\mathbb{Z}/2[t_i]$. Then

$$H^*(grS(t);d) \cong H^*(B;d) \otimes H^*(\mathbb{Z}/2[t];d) \cong \mathbb{Z}/2 \otimes 0 = 0$$

since $H^*(\mathbb{Z}/2[t],d) = 0$, $(d = Sq^2 + t : t^{even} \mapsto t^{even+1})$ while $H^*(\mathbb{Z}/2[t];Sq^2) \cong \mathbb{Z}/2$, $(Sq^2 : t^{odd} \mapsto t^{odd+1})$. Hence $H^*(S(t);d)$ itself is also zero.

We use the induction on i with $|x_i| \leq |x_{i+1}|$. The Sq^2-action on $H^*(G;\mathbb{Z}/2)$ is a derivative $(mod(Sq^1))$. Hence Sq^2 acts on the ring generators to ring generators or zero, i.e., we can take generators x_i so that $Sq^2(x_i) = x_{i+1}$ or $Sq^2(x_i) = 0$. (Moreover if $|x_i| + 2 = |x_j|$, then $Sq^2x_i = x_j$ for simply connected simple Lie groups.) Let $Sq^2(x_1) = x_2$. Write $d_r(x_1) = b_1$ and $d_{r+2}(x_2) = b_2$. Then the Cartan-Serre transgression theorem implies that

$$Sq^2(b_1) = Sq^2(d_r(x_1)) = d_{r+2}(Sq^2(x_1)) = d_{r+2}(x_2) = b_2.$$

Similarly if $Sq^2(x_1) = 0$, then $Sq^2(b_1) = 0$.

Suppose that $Sq^2(b_1) = 0$ (i.e., $Sq^2(x_1) = 0$ by the arguments above). Then $S(t)/(b_1)$ is a d-module because

$$(Sq^2 + t)(xb_1) = Sq^2(x)b_1 + txb_1 \in Ideal(b_1)$$

by the Cartan formula. We consider the short exact sequence

$$0 \to S(t) \xrightarrow{b_1} S(t) \to S(t)/(b_1) \to 0$$

of d-modules, and consider the induced long exact sequence

$$\to H^*(S(t);d) \to H^*(S(t);d) \to H^*(S(t)/(b_1);d) \to ...$$

The first and the second terms in the above sequence are zero, and so is the third, namely $H^*(S(t)/(b_1);d) = 0$.

Suppose that $Sq^2(b_1) \neq 0$. Then take b_2 with $Sq^2(b_1) = b_2$ (i.e., $Sq^2(x_1) = x_2$). Then $S(t)/(b_2)$ is a d-module and $H^*(S(t)/(b_2);d) = 0$ by the arguments above. Next consider a short exact sequence of d-modules

$$0 \to S(t)/(b_2) \xrightarrow{b_1} S(t)/(b_2) \to S(t)/(b_1,b_2) \to 0.$$

Considering the induced long exact sequence of d-homology, we get $H^*(S(t)/(b_1,b_2);d) = 0$. Similarly, we have $H^*(S(t)/(b_1,...,b_\ell);d) = 0$.

Thus we see

$$H^*(grH^*(G/T;\mathbb{Z}/2);d) \cong H^*(P(y);d) \otimes H^*(S(t)/(b_1,...,b_\ell);d) \cong 0.$$

$\square$

Theorem 4.6. *(Calmès and Fasel theorem for algebraically closed field) Let k be an algebraically closed and $0 \neq L \in Pic(G_k/T_k)/2$. Then $W^*(G_k/T_k;L) = 0$.*

Proof. We have

$$W^{*-1}(G_k/T_k;L) \cong \tilde{W}^*(Th(L)) \cong \tilde{K}O^{2*}(Th(L))/\tilde{K}O^{2*}(Th(L)).$$

The last term is isomorphic to a subquotient of $H^*(\tilde{H}^*(Th(L);\mathbb{Z}/2);Sq^2)$, which is isomorphic to $H^*(H^*(G/T;\mathbb{Z}/2);Sq+c_1(L)) = 0$ from the preceding lemma. $\square$

5. HERMITIAN K-THEORY

The Balmer Witt group can be extended as a generalized cohomology theory for the stable $\mathbb{A}^1$-homotopy category as follows. By Hornbostel, Schlichting, Panin and Walter [Ho], [Sch], [Pa-Wa], there is a spectrum KO in the stable $\mathbb{A}^1$-category such that the hermitian K-theory is written as

$$KO^{*,*'}(X) \cong Hom_{\mathbb{A}^1}(X, S^{*,*'} \wedge KO)$$

where $S^{*,*'}$ is the sphere of $deg = (*,*')$, e.g., $S^{2,1} \cong \mathbb{P}^1$. Moreover the Witt group is written as

$$W^i(X) \cong KO^{i+*,*}(X) \quad for\ i - * > 0$$

(namely, $W^{d(*,*')}(X) \cong KO^{*,*'}(X)$ for $w(*,*') < 0$).

Since $KO^{*,*'}(-)$ theory has a good product [Pa-Wa], so does $W^*(X)$, namely, $W^*(X)$ is a graded ring.

Let $t_{\mathbb{C}} : KO^{*,*'}(X) \to KO^*(X(\mathbb{C}))$ be the realization map (§3.4 in [Vo1]). So we have a natural map

$$W^*(X) \cong KO^{2*-1,*-1}(X) \to KO^{2*-1}(X).$$

Zibrowius proves (Theorem 2.5 in [Zi]) that the above map is an isomorphism when X is cellular and $k = \mathbb{C}$.

We give its (a bit different) proof here for an algebraically closed k. Let X be cellular (of $dimX = n$). Then by the definition, X has a filtration by closed subvarieties

$$\emptyset = Z_{-1} \subset Z_0 \subset ... \subset Z_n = X$$

such that the open complement of Z_{k-1} in Z_k is isomorphic to $\coprod \mathbb{A}^k$. In general, Z_k are not smooth. Let us write $X_i = X - Z_i$. So we have $Z_n - Z_{n-1} = X_{n-1} \subset X_2 \subset ... \subset X_{-1} = X$. Since $Z_i - Z_{i-1} \subset X_{i-1}$ is smooth, we have the long exact sequences

$$\to W^*(Th(Z_{n-1} - Z_{n-2})) \xrightarrow{i_*} W^*(X_{n-2}) \xrightarrow{j^*} W^*(X_{n-1} = Z_n - Z_{n-1}) \xrightarrow{\delta} ...,$$

$$\to W^*(Th(Z_i - Z_{i-1})) \xrightarrow{i_*} W^*(X_{i-1}) \xrightarrow{j^*} W^*(X_i) \xrightarrow{\delta} ...,$$

$$\to W^*(Th(Z_0 - Z_{-1})) \xrightarrow{i_*} W^*(X_{-1} = X) \xrightarrow{j^*} W^*(X_0) \xrightarrow{\delta} ...$$

By induction, we easily show the Zibrowius theorem from the following lemma.

Lemma 5.1. *Let k be an algebraically closed field. Let $U \to X \to T$ be a cofiber sequence in the stable $\mathbb{A}^1$-homotopy category. For both $Y = T, U$, if the following (1),(2) are satisfied*

$$(1) \quad KU^{odd}(t_{\mathbb{C}}(Y)) = 0,$$

$$(2) \quad t_{\mathbb{C}} : W^*(Y) \cong KO^{2*}(t_{\mathbb{C}}(Y))/KU^{2*}(t_{\mathbb{C}}(Y)) \cong KO^{2*-1}(t_{\mathbb{C}}(Y)),$$

then so are for $Y = X$.

Proof. Let us write $KO^*(t_{\mathbb{C}}(-))$ by $KO^*(-)$ simply. We consider the following diagram for long exact sequences

$$\begin{tikzcd}
W^{*-1}(U) \arrow[r, "\delta"] \arrow[d, "t_1"] & W^*(T) \arrow[r, "g^*"] \arrow[d, "\cong", "t_2"'] & W^*(X) \arrow[r, "f^*"] \arrow[d, "t_3"] & W^*(U) \arrow[r, "\delta"] \arrow[d, "\cong", "t_4"'] & W^{*+1}(T) \arrow[d, "t_5"] \\
KO^{2*-2}(U) \arrow[r, "\delta"] & KO^{2*-1}(T) \arrow[r, "g^*"] & KO^{2*-1}(X) \arrow[r, "f^*"] & KO^{2*-1}(U) \arrow[r, "\delta"] & KO^{2*}(T).
\end{tikzcd}$$

First we see that the map t_5 is injective from the following diagram

$$\begin{tikzcd}
W^{*+1}(T) \cong KO^{2*+1,*}(T) \arrow[r, "\eta", "\cong"'] \arrow[d, "\cong"] & W^{*+1}(T) \cong KO^{2*,*-1}(T) \arrow[d, "t_5"] \\
KO^{2*+1}(T) \arrow[r, "\eta"] & KO^{2*}(T).
\end{tikzcd}$$

Indeed, $\eta : KO^{odd}(T) \to KO^{even}(T)$ is injective from the Bott exact sequence since $KU^{odd}(T) = 0$ (note that $\eta : KO^{even}(T) \to K^{odd}(T)$ is surjective.)

Next we study the map t_1. Since $\delta | KU^{2*-2}(U) = 0$ (from $K^{2*-1}(T) = 0$), the map δ factors as

$$\delta : KO^{2*-2}(U) \to KO^{2*-2}(U)/KU^{2*-2}(U) \to KO^{2*-1}(T).$$

Moreover, we have the diagram

$$\begin{tikzcd}
KO^{8N+2*-2,*-1}(U) \arrow[r, "\cong"] \arrow[d, "t_1"] & W^{*-1}(U) \arrow[r] \arrow[d, "\cong"] & W^*(T) \arrow[d, "t_2"] \\
KO^{2*-2}(U) \arrow[r] & KO^{2*-2}(U)/KO^{2*-2}(U) \arrow[r] & KO^{2*-1}(T).
\end{tikzcd}$$

Then we can prove the lemma from the five lemma. If $t_3(x) = 0$, then $f^*(x) = 0$, so there is $x' \in W^*(T)$ with $g^*(x') = x$. Since $g^*(t_2(x')) = 0$, there is $\tilde{x} \in KO^{2*-2}(U)$ with $\delta(\tilde{x}) = .t_2(x')$. From the above diagram, we can take $x'' \in W^{*-1}(U)$ such that $t_1(x'') - \tilde{x} = 0 \in KO^{2*-2}(U)/KU^{2*-2}(U)$. Since $t_2(\delta(x'')) = \delta(\tilde{x}) = t_2(x')$, we see that $\delta x'' = x'$ and $x = 0$. Hence t_3 is injective.

For $y \in KO^{2*-1}(X)$, we can take $y' \in W^*(U)$ with $t_4(y') = f^*(y)$. Then $\delta(y') = 0$ since t_5 is injective. So there is $y'' \in W^*(X)$ such that $f^*(t_3(y'') - y) = 0$. Hence there is $z \in KO^{2*-1}(X)$ with $g^*(z) = t_3(y'') - y$. Then we see $t_3(y'' - g^* t_2^{-1}(z)) = y$. Therefore we also have the surjectivity of t_3. □

We will study the case which is not cellular but (1), (2) in the above lemma are satisfied.

Lemma 5.2. *Let k be an algebraically closed field. Let $U \to X \to T$ be a cofiber sequence in the stable $\mathbb{A}^1$-homotopy category. For both $Y = X, T$ (resp. $Y = X, U$), if (1),(2) in Lemma 5.1 are satisfied, and moreover, $g^* : KU^*(t_{\mathbb{C}}(T)) \to$*

$KU^*(t_{\mathbb{C}}(X))$ is injective (resp. $f^* : KU^*(t_{\mathbb{C}}(X)) \to KU^*(t_{\mathbb{C}}(U))$ is surjective), then (1), (2) are satisfied for $Y = U$ (resp. $Y = T$).

Proof. Consider the diagram for exact sequences

$$\begin{array}{ccccccccc}
W^*(T) & \xrightarrow{g^*} & W^*(X) & \xrightarrow{f^*} & W^*(U) & \xrightarrow{\delta} & W^{*+1}(T) & \xrightarrow{g_1^*} & W^{*+1}(X) \\
\cong\downarrow t_2 & & t_3\downarrow\cong & & \downarrow t_4 & & t_5\downarrow & & t_6\downarrow \\
KO^{2*-1}(T) & \xrightarrow{g^*} & KO^{2*-1}(X) & \xrightarrow{f^*} & KO^{2*-1}(U) & \xrightarrow{\delta} & KO^{2*}(T) & \xrightarrow{g_2^*} & KO^{2*}(X).
\end{array}$$

We want to see that t_4 is an isomorphism. Consider the Bott exact sequence

$$\begin{array}{ccccc}
0 \to W^{*+1}(T) \cong KO^{2*+1}(T) & \xrightarrow{\eta} & KO^{2*}(T) & \longrightarrow & KU^{2*}(T) \\
g_1^*\downarrow & & g_2^*\downarrow & & g_3^*\downarrow injective \\
0 \to W^{*+1}(X) \cong KO^{2*+1}(X) & \xrightarrow{\eta} & KO^{2*}(X) & \longrightarrow & KU^{2*}(X)
\end{array}$$

Here note that $\times\eta|KO^{odd}(-)$ is injective, and g_3^* are also injective by the assumption. If $x \in Ker(g_2^*)$, then $x \in Im(\eta)$ from the injectivity of g_3^*. Hence we easily see (from the injectivity of $\times\eta$) that $Ker(g_2^*) \cong Ker(g_1^*)$, and t_4 is isomorphic from the five lemma.

The second case is similarly proved by using the following diagram (to see $Im(f_2^*) \cong Im(f_3^*)$).

$$\begin{array}{ccccc}
KU^{2*-2}(X) & \longrightarrow & KO^{2*-2}(X) & \longrightarrow & KU^{2*-2}(X)/KU^{2*-2}(X) \cong W^{*-1}(X) \\
f_1^*\downarrow surjective & & f_2^*\downarrow & & f_3^*\downarrow \\
KU^{2*-2}(U) & \longrightarrow & KO^{2*-2}(U) & \longrightarrow & KU^{2*-2}(U)/KU^{2*-2}(U) \cong W^{*-1}(U)
\end{array}$$

□

Remark. When $X = G/T, T = Th(t_\ell)$ and $U = G/T^{\ell-1}$, of course, $t_\ell : KU^*(T) \to KU^*(X)$ is not injective. In fact, $KU^*(G/T) \cong KU^* \otimes H^*(G/T)$ and g_ℓ (given in Lemma 3.1) is in $Ker(t_\ell)$.

We consider the classifying space BG for a finite group G. First consider the case $G = \mathbb{Z}/2^r$. In the stable $\mathbb{A}^1$-homotopy category, we have the cofibering (see (6.4) in [Vo3] for details)

$$B\mathbb{Z}/2^r \to E \stackrel{q_r}{\to} Th(E)$$

where $E = O(-2^r)$ is the -2^r-th twisted line bundle (of the canonical one) of $\mathbb{P}^\infty$ and $Th(E)$ is its Thom space and q_r is the composition map with 2^r-times twist on $\mathbb{P}^\infty$ and the quotient map.

Of course $\mathbb{P}^n$ is cellular, and (see §7 in [Ya1])

$$W^*(\mathbb{P}^n) \cong \begin{cases} \mathbb{Z}/2\{1, y_n\} & for\ deg(y_n) = n \quad if\ n : odd \\ \mathbb{Z}/2\{1\}, & otherwise. \end{cases}$$

Hence we have $W^*(\mathbb{P}^\infty) \cong \mathbb{Z}/2$. (For Witt groups of infinite spaces, see the remark after Lemma 6.1 below.) . We also see $W^*((\mathbb{P}^\infty)^{\times n}) \cong \mathbb{Z}/2$. Moreover $E = O(-2^r)$ represents zero in $Pic(\mathbb{P}^\infty)/2$, and so we have the Thom isomorphism

$$\tilde{W}^{*+1}(Th(E)) \cong W^*(E) \cong W^*(\mathbb{P}^\infty) \cong \mathbb{Z}/2.$$

Therefore we have the exact sequence

$$\to W^{*-1}(\mathbb{P}^\infty) \stackrel{\times 2^r=0}{\to} W^*(\mathbb{P}^\infty) \to W^*(B\mathbb{Z}/2^r) \to ...$$

Hence we obtain $W^*(B\mathbb{Z}/2^r) \cong \Lambda(x)$ with $deg(x) = 0$. (In fact $x^2 = 0$ since so in $E(P)_2^{*,*'}$ and $x^2 \neq 1$ by the naturality.)

For each space X, we have the cofibering

$$B\mathbb{Z}/2^r \times X \to E \times X \stackrel{q_r \times id}{\to} Th(E) \times X.$$

By induction on i, starting $(\mathbb{P}^\infty)^{\times n}$, we can prove with $deg(x_j) = 0$

$$W^*((\times_{s=1}^i B\mathbb{Z}/(2^{r_s})) \times (\mathbb{P}^\infty)^{\times(n-i)}) \cong \Lambda(x_1, ..., x_i).$$

Theorem 5.3. *Let k be algebraically closed. Let G be a 2-group of rank $= n$ i.e., $G \cong \oplus_{s=1}^n \mathbb{Z}/(2^{r_s})$. Then there are isomorphisms*

$$q^* : W^*(BG) \cong KO^{2*}(BG)/KU^{2*}(BG) \cong \Lambda(x_1, ..., x_n), \quad deg(x_i) = 0.$$

Proof. We only need to see the first isomorphism. It is well known that $KU^*(BG)$ is torsion free for each compact group G. In particular

$$\times 2^{r_i} | KU^*((\times_{s=1}^{i-1} B\mathbb{Z}/(2^{r_s})) \times (\mathbb{P}^\infty)^{\times(n-i+1)})$$

is injective. Hence this satisfies the assumptions in the preceding lemma. Hence (2) in the lemma also satisfies for $U = (\times_{s=1}^i B\mathbb{Z}/(2^{r_s})) \times (\mathbb{P}^\infty)^{\times(n-i)}$. □

The isomorphisms are still given in Theorem 7.4, 7.7 in [Ya1] for $G = (\mathbb{Z}/2)^{\oplus n}$. (However it was not proved that the map q^* induces these isomorphisms.)

Recall $grH^*(B\mathbb{Z}/2^r) \cong \mathbb{Z}/2[y] \otimes \Lambda(x)$, and $Pic(B\mathbb{Z}/2^r)/2 \cong \mathbb{Z}/2\{y\}$. We easily see $H^*(H^*(\mathbb{Z}/2[y] \otimes \Lambda(x); Sq^2 + y) = 0$ as the proof of Lemma 4.5. The analogue of the theorem by Calmès and Fasel also holds.

Corollary 5.4. *Let k be algebraically closed and let $G \cong \oplus_{s=1}^n \mathbb{Z}/(2^{r_s})$. Then*

$$W^*(BG; L) = 0 \quad for\ 0 \neq L \in Pic(BG)/2.$$

Let S be a 2-Sylow subgroup of a finite group G. Since we can identify the induced map $g : BS \to BG$ as a finite covering, we have the Gysin map g_* in also Witt theory (so that $g_* g^* = [G; S]$).

Corollary 5.5. *Let k be algebraically closed. Let G be a finite group having an abelian 2-Sylow subgroup. Then*

$$q^* : W^*(BG) \cong KO^{2*}(BG)/KU^{2*}(BG).$$

6. Gille-Pardon spectral sequence.

Balmer and Walter ((1) in [Ba-Wa]) define the Gersten-Witt complex

$$0 \to W(k(X)) \to \oplus_{x \in X^{(1)}} W(k(x)) \to ... \to \oplus_{x \in X^{(n)}} W(k(x)) \to 0.$$

Let $H^*(W(X))$ denote the cohomology group of the above cochain complex, with $W(k(X))$ in degree 0. Then Balmer-Walter constructed the spectral sequence (Theorem in [Ba-Wa])

$$E(BW)_2^{r,t} \cong \begin{cases} H^r(W(X))\ (t = 0\ mod(4)) & \Longrightarrow W^{r+t}(X). \\ 0 \quad (t \neq 0 (mod(4)) \end{cases}$$

By the affirmative answer to the Milnor conjecture of quadratic forms by Orlov-Vishik-Voevodsky (Theorem 4.1 in [Or-Vi-Vo]), we have an isomorphism of graded rings $H^*_{et}(k(x);\mathbb{Z}/2)\cong grW(k(x))$. Using this fact, we can reinterpret the Pardon-Gille spectral sequence (Corollary 0.13 in [Pa] and [Gi]) as a spectral sequence

$$E(P)_2^{r,s}\cong H^r_{Zar}(X;H^s_{\mathbb{Z}/2})\Longrightarrow H^r(W(X))\cong E(BW)_2^{r,4t}$$

so that the differential d_r has degree $(1,r-1)$ for $r\geq 2$. Here $H^s_{\mathbb{Z}/2}$ is the Zariski sheaf associated to the presheaf $V\mapsto H^s_{et}(V;\mathbb{Z}/2)$ for any open subscheme V of X.

The above sheaf cohomology $H^r_{Zar}(X;H^s_{\mathbb{Z}/2})$ is related to the motivic cohomology $H^{*,*'}(X;\mathbb{Z}/2)=\oplus_{r,s\in\mathbb{Z}}H^{r,s}(X;\mathbb{Z}/2)$ (for details, see [Vo1-3]) as follows.

Recall that $\tau\in H^{0,1}(Spec(k);\mathbb{Z}/2)\cong\mathbb{Z}/2$ be a generator. (For example, if k is algebraically closed, then $H^{*,*'}(Spec(k);\mathbb{Z}/2)\cong\mathbb{Z}/2[\tau]$.) Then we get the long exact sequence from the solution of the Beilinson-Lichtenbaum conjecture (Theorem 1.3 in [To], Lemma 2.4 (5) in [Or-Vi-Vo]),

$$\to H^{m,n-1}(X;\mathbb{Z}/2)\stackrel{\times\tau}{\to}H^{m,n}(X;\mathbb{Z}/2)$$

$$\to H^{m-n}_{Zar}(X;H^n_{\mathbb{Z}/2})\to H^{m+1,n-1}(X;\mathbb{Z}/2)\stackrel{\times\tau}{\to}...$$

All elements in the above cohomology groups are 2-torsion, and we have the following additive isomorphisms

Lemma 6.1. $E(P)_2^{m-n,n}\cong H^{m-n}_{Zar}(X;H^n_{\mathbb{Z}/2})\cong$

$$H^{m,n}(X;\mathbb{Z}/2)/(Im(\tau))\oplus Ker(\tau)|_{H^{m+1,n-1}(X;\mathbb{Z}/2)}.$$

Remark. Let X_n be smooth and $colim_nX_n=X$. Suppose $\oplus_sH^*(X_n;H^s_{\mathbb{Z}/2})$ is a finite group. (Note $H^*(X_n;H^{*'}_{\mathbb{Z}/2})=0$ for $*>dim(X_n)$.) Then $\oplus_{m,s}E(P)_r^{m,s}(X_n)$ is a finite group, and so is $\oplus_mE(BW)_\infty^{m,0}$. Therefore $W^*(X_n)$ is a finite group for each $*\in\mathbb{Z}/4$. So $lim^1=0$, and we have

$$W^*(X)\cong lim_nW^*(X_n).$$

It is well known that $CH^m(X)\cong H^{2m,m}(X;\mathbb{Z})$ (Corollary 2 in [Vo2]). Since $H^{2m+1,m}(X;\mathbb{Z})=0$ (Corollary 2.3 in [Vo1]), from the Bockstein exact sequence, we have an isomorphism $H^{2m,m}(X;\mathbb{Z}/2)\cong H^{2m,m}(X;\mathbb{Z})/2$. In particular,

$$E(P)_2^{m,m}\cong H^m_{Zar}(X;H^m_{\mathbb{Z}/2})\cong H^{2m,m}(X;\mathbb{Z}/2)\cong CH^m(X)/2,$$

$$E(P)_2^{m,m+1}\cong H^m_{Zar}(X;H^{m+1}_{\mathbb{Z}/2})\cong H^{2m,m+1}(X;\mathbb{Z}/2).$$

In $H^{*,*'}(X;\mathbb{Z}/2)$, we can define the cohomology operation Sq^i from Voevodsky (§9 in [Vo3], §3.3 in [Vo1]) (or Brosnan [Br] for $CH^*(X)/2$). This Sq^i is compatible with that in the usual (topological) mod 2 cohomology via the realization map $t_{\mathbb{C}}$ when $k\subset\mathbb{C}$ (§3.4 in [Vo1]). Moreover $Sq^2Sq^2=\tau Sq^3Sq^1$ (Theorem 10.2 in [Vo3]). Then Totaro proved

Lemma 6.2. *(Theorem 1.1 in [To]) If $x\in E(P)_2^{m,m}\cong CH^m(X)/2$, then $d_2(x)=Sq^2(x)$.*

We assume the following assumption throughout this section

Assumption 6.3. *If $x\in E(P)_2^{m,m+1}\cong H^{2m+1,m+1}(X;\mathbb{Z}/2)$, then $d_2(x)=Sq^2(x)$.*

This assumption should be proved if the Gersten-Witt complex can be defined for each object X in the $\mathbb{A}^1$-homotopy category (see Proposition 7.8 in [Ya1]).

Hereafter in this paper, we always assume that k is an algebraically closed field (of $ch(k)=0$). Hence $H^{*,*'}(pt.;\mathbb{Z}/2)\cong\mathbb{Z}/2[\tau]$. Since $W^{*''}(X)=BO^{*,*'}(X)$ has the ring structure from the results by Schlichting and Panin-Walter, $W^*(-)$ is a multiplicative cohomology theory. So $W^*(X)$ is a $W^*(pt.)\cong\mathbb{Z}/2$-algebra, i.e., we have the cup product

$$W^*(X)\otimes_{\mathbb{Z}/2}W^{*'}(X)\to W^{*+*'}(X\times X)\xrightarrow{\Delta}W^{*+*'}(X).$$

For a map $X\to Y$ in $\mathbb{A}^1$-homotopy category, we have the long exact sequence

$$\to\tilde{W}^*(Y/X)\to W^*(Y)\to W^*(X)\xrightarrow{\delta}\tilde{W}^{*+1}(Y/X)\to ...$$

The coboundary map δ is a derivation (over $\mathbb{Z}/2$) in the following sense. There is a commutative diagram

$$\begin{tikzcd} W^*(X)\otimes W^{*'}(X') \arrow[r, "\delta'"] \arrow[d] & W^{*+1}(Y,X)\otimes W^*(X')\oplus W^*(X)\otimes W^{*'+1}(Y',X') \arrow[d] \\ W^{*+*'}(X\times X') \arrow[r, "\delta"] & W^{*+*'+1}(X\times Y'\cup Y\times X', X\times X') \end{tikzcd}$$

where $\delta'(a\otimes b)=\delta(a)\otimes b+a\otimes\delta(b)$ from the standard arguments (as topological cases).

Let us consider

$$Z^n\subset Z^{n-1}\subset ...\subset Z^1\subset Z$$

series of smooth embedding with $codim_Z(Z^i)=i$. Then we have the diagram

$$\begin{tikzcd} W^*(X;L^0) & W^*(Z^1;L^1) \arrow[l, "g_*"'] & W^*(Z^2;L^2) \arrow[l, "g_*"'] & ... \arrow[l, "g_*"'] \\ & W^*(X-Z^1;L^0) \arrow[u, "\delta"] \arrow[r, "f^*\delta"] & W^*(Z^1-Z^2;L^1) \arrow[u, "\delta"] \arrow[r, "f^*\delta"] & ... \end{tikzcd}$$

$\searrow f^*$ $\qquad$ $\searrow f^*$

Roughly speaking, the Gersten-Witt complex is constructed by taking limit of the above diagram [Bl-Og], [Ka]. Indeed, we have the commutative diagram

$$\begin{tikzcd} W^*(X-Z^1;L^0) \arrow[r, "f^*\delta"] \arrow[d] & W^*(Z^1-Z^2;L^1) \arrow[r, "f^*\delta"] \arrow[d] & W^*(Z^2-Z^3;L^2) \arrow[r] \arrow[d] & ... \\ W^*(k(X)) \arrow[r, "d"] & \oplus_{x\in X^{(1)}}W^*(k(x)) \arrow[r, "d"] & \oplus_{x\in X^{(2)}}W^*(k(x)) \arrow[r, "d"] & ... \end{tikzcd}$$

Moreover we have a natural map

$$\oplus_{x\in X^{(i)},x'\in X'^{(j)}}W^*(k(x))\otimes W^{*'}(k(x'))\to\oplus_{x\in(X\times X')^{(i+j)}}W^{*+*'}(k(x)).$$

Then we can prove that the differential d of the Gersten-Witt complex is a derivation by using that the coboundary map δ^* is a derivation in the sense above.

Hence spectral sequences $E(P)_r^{*,*'}$ and $E(BW)_r^{*,*'}$ are that of graded rings, e.g., the differential is a derivation.

For $X=G_k, Y=G_k'$, we still know from Corollary 3.7,

$$H^{*,*'}(X\times Y;\mathbb{Z}/2)\cong H^{*,*'}(X;\mathbb{Z}/2)\otimes_{\mathbb{Z}/2[\tau]}H^{*,*'}(Y;\mathbb{Z}/2).$$

Hence the spectral sequences has the same property

$$E(P)_r^{*,*'}(X\times Y)\cong E(P)_r^{*,*'}(X)\otimes E(P)_r^{*,*'}(Y)$$
$$E(BW)_r^{*,*'}(X\times Y)\cong E(BW)_r^{*,*'}(X)\otimes E(BW)_r^{*,*'}(Y)$$

Therefore $W^*(X\times Y)\cong W^*(X)\otimes W^*(Y)$. In particular the spectral sequences for X are differential Hopf algebras. It is known (e.g., Araki [Ar],Toda [Tod]) that the image of differential of a ring generator (indecomposable element) is also a ring generator or zero. (We can identify $grE_2^{*,*'}$ to be primitively generated, and use the fact that the image of the differential of a primitive element is primitive.)

Lemma 6.4. *The Witt ring $W^*(G_k)$ is a Hopf algebra. So spectral sequences $E(P)_r^{*,*'}$ and $E(BW)_r^{*,*'}$ are differential Hopf algebras, that implies differential d_r works as ring generators maps to ring generators or zero.*

From Corollary 3.5, we have

$$grH^{*,*'}(G_k;\mathbb{Z}/2)\cong \mathbb{Z}/2[\tau]\otimes P(y)\otimes \Lambda(x_1,...,x_\ell)$$

with $w(P(y))=0$ and $w(x_i)=1$. So we get

$$H^{*,*'}(G_k;\mathbb{Z}/2)/(Im(\tau))\cong P(y)\otimes\Lambda(x_1,...,x_\ell),$$
$$and\quad Ker(\tau)|_{H^{*,*'}(G_k;\mathbb{Z}/2)}=0.$$

Thus we get a bigraded ring isomorphism

$$E(P)_2^{*,*'}\cong P(y)\otimes\Lambda(x_1,...,x_\ell)$$

with $deg(z)=(1/2|z|,1/2|z|)$ for $z\in P(y)$ and $deg(x_i)=(1/2(|x_i|-1),1/2(|x_i|+1))$.

On the other hand, we give a (coniveau) filtration to topological $H^*(G;\mathbb{Z}/2)$ by $F_i=\{x|w(x)\le i\}$ and denote by $grH^*(G;\mathbb{Z}/2)=\oplus F_i/F_{i-1}$ the associated algebra. Then we have the ring isomorphism

$$grH^*(G;\mathbb{Z}/2)\cong P(y)\otimes\Lambda(x_1,...,x_\ell).$$

Here $Sq^2Sq^2=Sq^3Sq^1=0$ and Sq^2 define a differential of $grH^*(G;\mathbb{Z}/2)$. Let us write its homology by $H^*(G;Sq^2)$ with $deg(y_i)=1/2|y_i|$ and $deg(x_i)=1/2(|x_i|-1)$.

Then we have the Sq^2-bidegree ring isomorphism

$$E(P)_2^{*,*'}\cong grH^*(G;\mathbb{Z}/2)$$

identifying $w(y_i)=0$ and $w(x_i)=1$. From the result of Totaro and Assumption 6.3, we have

$$E(P)_3^{*,*'}\cong H^*(grH^{2*}(G;\mathbb{Z}/2);Sq^2)=H^*(G;Sq^2).$$

Recall the degree of differential $deg(d_r)=(1,r-1)$. Here we consider the assumption

Assumption 6.5. *There is an algebra isomorphism*

$$E(P)_3^{*,*'}\cong H^*(G;Sq^2)\cong\Lambda(z_1,...,z_s)\quad with\ first.deg(z_i)=odd.$$
$$or\quad E(P)_4^{*,*'}\cong H^*(H^*G;Sq^2);d_3)\cong\Lambda(z_1,...,z_s),$$

Lemma 6.6. *There is a spectral sequence*

$$E_2\cong H^*(G;Sq^2)\Rightarrow H^*(G/T;Sq^2).$$

Proof. (Recall the proof of Lemma 4.5 and see also §2 in [Ki-Oh].) Let us write $d = Sq^2$ simply. Recall $S(t) \cong \otimes_{i=1}^{\ell}\mathbb{Z}/2[t_i]$. We give $deg(t_i) = 1$ here to compatible with degree of $W^*(-)$. We get

$$H^*(S(t); d) \cong \otimes_i^{\ell} H^*(\mathbb{Z}/2[t_i]; d) \cong \mathbb{Z}/2$$

since $H^*(\mathbb{Z}/2[t]; Sq^2) \cong \mathbb{Z}/2$, $(Sq^2 : t^{odd} \mapsto t^{odd+1})$.

Suppose that $Sq^2(b_1) = 0$ (i.e., $Sq^2(x_1) = 0$). Then $S(t)/(b_1)$ is a d-module. We consider the short exact sequence

$$0 \to S(t) \stackrel{b_1}{\to} S(t) \to S(t)/(b_1) \to 0$$

of d-modules, and consider the induced long exact sequence

$$\to H^*(S(t); d) \stackrel{\times b_1}{\to} H^*(S(t); d) \to H^*(S(t)/(b_1); d) \to ...$$

The first and the second terms in the above sequence are isomorphic to $\mathbb{Z}/2$ and $\times b_1 = 0$. Hence we have

$$H^*(S(t)/(b_1); d) \cong \mathbb{Z}/2\{1, \tilde{b}_1\} \cong \Lambda(\tilde{b}_1).$$

Here $d(\tilde{b}_1) = b_1$, which is a cycle in $S(t)/(b_1)$ (but not in $S(t)$), and $\tilde{b}_1^2 \in Im(d) \subset S(t)$).

Suppose that $Sq^2(b_1) \neq 0$. Then can take b_2 with $Sq^2(b_1) = b_2$ (i.e., $Sq^2(x_1) = x_2$). Then $S(t)/(b_2)$ is a d-module and $H^*(S(t)/(b_2); d) \cong \Lambda(\tilde{b}_2)$ here note $\tilde{b}_2 = b_1$, by the arguments above. Next consider a short exact sequence

$$0 \to S(t)/(b_2) \stackrel{b_1}{\to} S(t)/(b_2) \to S(t)/(b_1, b_2) \to 0.$$

and induced long exact sequence of d-homology,

$$\to \Lambda(b_1) \stackrel{b_1}{\to} \Lambda(b_1) \to H^*(S(t)/(b_1, b_2); d) \to ...$$

Hence we have

$$H^*(S(t)/(b_1, b_2); d) \cong \Lambda(\tilde{b}_{1,2}) \quad d(\tilde{b}_{1,2}) = b_1^2.$$

Similarly, we can compute $H^*(S(t)/(b_1, ..., b_\ell); d) \cong \Lambda(\bar{b}_1, ..., \bar{b}_{\ell'})$.

On the other hand, we consider the homology of $\Lambda(x_1, ..., x_\ell)$. If $Sq^2(x_1) = 0$. then $H^*(\Lambda(x_1); d_1) \cong \Lambda(\tilde{x}_1)$. If $Sq^2(x_1) = x_2$, then

$$H^*(\Lambda(x_1, x_2), d) \cong \Lambda(x_{1,2}) \quad identifying\ x_{1,2} = x_1x_2.$$

Similarly we have an isomorphism $H^*(\Lambda(x_1, ..., x_\ell); d) \cong \Lambda(\bar{x}_1, ..., \bar{x}_{\ell'})$. Thus we can construct an isomorphism

$$H^*(\Lambda(x_1, ..., x_\ell); d) \cong H^*(S(t)/(b_1, ..., b_\ell); d).$$

Therefore we have

$$H^*(G; Sq^2) \cong H^*(P(y) \otimes \Lambda(x_1, ..., x_\ell); d)$$

$$\cong H^*(P(y) \otimes S(t)/(b_1, ..., b_\ell); d) \cong H^*(grH^*(G/T; \mathbb{Z}/2); d).$$

Hence we have a spectral sequence

$$H^*(G; Sq^2) \cong H^*(grH^*(G/T; \mathbb{Z}/2); d) \Rightarrow H^*(H^*(G/T; \mathbb{Z}/2); d).$$

□

Theorem 6.7. *Let k be an algebraically closed field in $\mathbb{C}$. For each simply connected simple Lie group G, Assumption 4.1, 6.3, 6.5 are satisfied and*

$$W^*(G_k) \cong H^*(G/T; Sq^2) \cong \Lambda(z_1, ..., z_s) \quad deg(z_i) = odd.$$

which is also isomorphic to

$$\begin{cases} E(P)_4^{*,*'} \cong H^*(H^*(G; Sq^2); d_3) & for\ G = E_6, E_7, E_8 \\ E(P)_3^{*,*'} \cong H^*(G; Sq^2) & otherwise. \end{cases}$$

Proof. We assume here 6.3 and 6.5 which are shown for each simple group G in the next section. From Assumption 6.5, Gill-Pardon and Balmer-Walter spectral sequences collapse because, from Lemma 6.4, the d_r-image of z_i must be some z_j $(mod(decomposable\ el.))$. However $deg(z_i) - deg(z_j) = even$ but $deg(d_r) = odd$, and this means $d_r = 0$. Hence $W^*(G_k) \cong E(P)_4^{*,*'} \cong H^*(H^*(G; Sq^2); d_3)$.

Recall that $W^*(G_k)$ is a subquotient of $H^*(G/T; Sq^2)$ from Corollary 4.2 (and is isomorphic to it if and only if Assumption 4.1 is satisfied).

On the other hand, by the preceding lemma, we know $H^*(G/T; Sq^2)$ is a subquotient of $H^*(G; Sq^2)$. Here we also assume that

$(*)$ $H^*(G/T; Sq^2)$ is a subquotient of $E(P)_4^{*,*'} \cong H^*(G; Sq^2); d_3)$.

(Of course $(*)$ holds when $d_3 = 0$. For the cases $G = E_6, E_7, E_8$, it is proved in the next section.) Then we get

$$W^*(G_k) \cong E(P)_4^{*,*'} \cong H^*(G/T; Sq^2).$$

□

The homology $H^*(G; Sq^2)$ is easier computed than $H^*(G/T; Sq^2)$. However, unfortunately, in this paper, we use the ring structure of $H^*(G/T; \mathbb{Z}/2)$ (for each simple group) to see Lemma 3.1 (for Corollary 3.5 and Theorem 3.6) and to see Assumption 6.3. We are hoping to obtain alternative proofs that do not rely on the detailed ring structure of $H^*(G/T; \mathbb{Z}/2)$.

Here we note the map of spectral sequences induced from $G_k \to G_k/T_k$. Since G_k/B_k is cellular, we have isomorphisms

$$E(P)_\infty^{*,*'}(G/T) \cong E(P)_3^{*,*'}(G/T) \cong E(P)_3^{*,*}(G/T) \cong H^*(G/T; Sq^2).$$

From the above theorem, we also see that the Balmer-Walter spectral sequence also collapses.

$$E(P)_\infty^{*,*}(G/T) \cong grE(BW)_\infty^{*,4*'}(G/T) \cong grW^*(G_k/T_k).$$

The projection $G_k \to G_k/T_k$ induces the isomorphism of the Balmer-Walter spectral sequence $E(BW)_r^{*,*'}(G/T) \cong E(BW)_r^{*,*'}(G)$, but not for the Gille-Pardon spectral sequence, indeed

$$E(P)_4^{*,*}(G/T) \cong E(P)_4^{*,*'}(G/T) \to E(P)_4^{*,*'}(G)$$

is not injective nor surjective (while $grE(BW)_2^{*,*''} \cong E(P)_\infty^{*,0}$).

7. Simple Lie groups

(I) classical groups.

For classical groups, we will see Assumption 6.3 and 6.5. First, we consider the case $G = U(2m+1)$. Its cohomology is

$$H^*(G;\mathbb{Z}/2) \cong \Lambda(x_1, x_3, ..., x_{4m+1})$$

$$\cong \Lambda(x_1) \otimes \otimes_s^m \Lambda(x_{4s-1}, x_{4s+1}) \quad with\ Sq^2 x_{4s-1} = x_{4s+1}.$$

(Throughout this section, subscripts indicate degree, e.g, $|x_i| = i$ for convenience.) Therefore we have the Sq^2 homology

$$H^*(G; Sq^2) \cong \Lambda(x_1) \otimes \otimes_{s=1}^m \Lambda(x_{4s-1}x_{4s+1}),$$

which is (changing degree so that $deg(x_i) = 1/2(i-1)$) isomorphic to

$$\Lambda(u_1) \otimes H^*(G/T; Sq^2) \cong \Lambda(z_0) \otimes \otimes \Lambda(z_{4s-1}) \quad deg(z_i) = i.$$

This result coincides with that given by Kishimoto, Kono and Ohsita. When $G = 2m+2$, $H^*(G; Sq^2) \cong H^*(U(2m+1); Sq^2) \otimes \Lambda(x'_{2m+1})$. This element x'_{2m+1} corresponds the element z in the notation of [Ki-Ko-Oh]. We also get $H^*(SU(n); Sq^2) \cong H^*(U(n); Sq^2)/(z_0)$.

The case $G = Sp(n)$ is easy. Infact, $H^*(G;\mathbb{Z}/2) \cong \Lambda(x_3, x_7, ..., x_{4n-1})$ and so $Sq^2 = 0$. Hence $H^*(G; Sq^2) \cong \Lambda(x'_1, ..., x'_{2n-1})$ where x'_{2i+1} corresponds x_{4i+3} in $H^*(G;\mathbb{Z}/2)$.

Next we consider the case $G = SO(4m+2)$. Then the mod 2-cohomology is written as (see for example [Ni])

$$grH^*(SO(4m+1);\mathbb{Z}/2) \cong \Lambda(x_1, x_2, ..., x_{4m+1})$$

where the multiplications are given by $x_s^2 = x_{2s}$. We write $y_{2(odd)} = x_{odd}^2$. Let us write

$$grH^*(G;\mathbb{Z}/2) \cong \Lambda(y_2, y_4, ..., y_{4m}) \otimes \Lambda(x_1, x_3, ..., x_{4m+1}).$$

Here $grP(y) \cong \Lambda(y_2, y_4, ..., y_{4m})$ and $Sq^2(y_{4s+2}) = y_{4s+4}$. Hence

$$H^*(P(y); Sq^2) \cong \otimes_s^m \Lambda(y_{4s-2}y_{4s}).$$

This result also coincides with the result of Kishimoto-Kono-Ohsita

$$H^*(SO(4m+1)/T; Sq^2) \cong \otimes_{s=1}^m \Lambda(z_{4s-1}, w_{4s-1})$$

where z_{4s-1} (resp. w_{4s-1}) corrsponds $x_{4s-1}x_{4s+1}$ (resp. $y_{4s-2}y_{4s}$). The other n cases are similar (see also [Ki-Ko-Oh]).

Here we give some note for $G = Spin(n)$ case. The cohomology is

$$H^*(Spin(n);\mathbb{Z}/2) \cong H^*(SO(m);\mathbb{Z}/2)/(x_1, y_1) \otimes \Lambda(a)$$

$$\cong \Lambda(y_2, y_4, ..., y_{4m})/(y_{2^s}|s \geq 1) \otimes \Lambda(x_3, ..., x_{4m+1}) \otimes \Lambda(a)$$

where $|a| = 2^t - 1$ for $2^{t-1} < m \leq 2^t$ ([Mi-Ni], [Tod]). Then we can prove that $H^*(G; Sq^2)$ is isomorphic to

$$H^*(SO(n); Sq^2)/(z_0, w_{2^s-1}|s \geq 1) \otimes (\otimes_{s=1}^{t-2} \Lambda(y'_{2^s-1})) \otimes \Lambda(a'_{2^{t-1}-1})$$

where w_{2^s-1} (resp. $y'_{2^{s-1}-1}$, $a'_{2^{t-1}-1}$) corresponds $y_{2^s-2}y_{2^s}$ (resp. y_{2^s-2}, a). Identifying $y'_{2^s-1} = w_{2^s-1}$ and $a'_{2^{t-1}-1} = w_{2^{i-1}-1}$, we have an isomorphism $H^*(Spin(n); Sq^2) \cong H^*(SO(n); Sq^2)/(z_0)$. Indeed, we still know $H^*(Sin(n)/T;\mathbb{Z}/2) \cong H^*(SO(n)/T;\mathbb{Z}/2)$ (see §3 in [Tod]).

Proposition 7.1. *Let G be a simple classical group. Then Assumption 6.3 and 6.5 are satisfied.*

Proof. Assumption 6.5 is satisfied from above arguments. We will prove Assumption 6.3 for $G = SO(2m+1)$. Then the other cases are shown by the naturality of maps e.g., $SU(n) \to SO(2n)$.

Let $G' = SO(2m-3)$ and suppose the assumption for G'. The cohomology is written

$$H^*(G;\mathbb{Z}/2) \cong H^*(G';\mathbb{Z}/2) \otimes \Lambda(y_{4m-2}, y_{4m}, x_{4m-1}, x_{4m+1})$$
$$with \quad Sq^2(y_{4m-2}) = y_{4m}, \quad Sq^2(x_{4m-1}) = x_{4m+1}.$$

In the Gille-Pardon spectral sequence if $d_2(x_{4m-1}) \neq x_{4m+1}$, then $d_2(x_{4m-1}) = 0$ by Lemma 6.4. By th dimensional reason, x_{4m-1} is a permanent cycle. However this element is zero in $H^*(G/T;Sq^2)$ (there is no ring generator in $dim = 4m-1$). This is a contradiction. So $d_2(x_{4m-1}) = x_{4m+1}$. Of course we know $d_2(y_{4m-2}) = y_{4m}$ from Lemma 5.3. □

(II) exceptional groups G_2 and F_4.

The cohomology is given

$$grH^*(G_2;\mathbb{Z}/2) \cong \Lambda(y_6, x_3, x_5) \quad with\ Sq^2x_3 = x_5.$$

We have the natural inclusion $SU(3) \subset G_2$. Hence in the Gille-Pardon spectral sequence $d_2(x_3) = x_5$ and hence

$$H^*(G_2;Sq^2) \cong \Lambda(z_3, y_3')$$

where z_3 (resp. y_3') corresponds x_3x_5 (resp. y_6). The cohomology

$$H^*(F_4;\mathbb{Z}/2) \cong H^*(G_2;\mathbb{Z}/2) \otimes \Lambda(x_{15}, x_{23}).$$

Hence $H^*(F_4;Sq^2) \cong H^*(G_2;Sq^2) \otimes \Lambda(x_7', x_{11}')$ where x_7' (resp. x_{11}') corresponds x_{15} (resp. x_{23}). (See also [Ki-Oh].) Thus Assuption 6.3, 6.5 are satisfied for these cases.

(III) exceptional Lie groups E_6, E_7, E_8.

First consider the case $G = E_6$. Its cohomology is

$$grH^*(E_6;\mathbb{Z}/2) \cong \Lambda(y_6, x_3, x_5, x_9, x_{15}, x_{17}, x_{23}).$$

where $Sq^2x_i = x_{i+2}$. We easily see

$$H^*(G;Sq^2) \cong \Lambda(y_3', z_3, x_4', z_{15}, x_{11}')$$

by the notation similar to (I),(II) (e.g., $x_4' = x_9, z_{15} = x_{15}x_{17}, x_{11}' = x_{23}$).

The spectral sequence in Lemma 6.5 does not collapse by the following reason. In $H^*(G/T;\mathbb{Z}/2)$, we know $Sq^2y_6 = c_4$ by (§4 in [Ki-Oh], Theorem 5.9 in [Ko-Is], y_6 is written by γ_3 in there). So $d_3(y_3') = x_4'$, and we have

$$H^*(G/T;\mathbb{Z}/2) \cong \Lambda(z_3, v_7, z_{15}, x_{11}')$$

where v_7 corresponds y_6x_9. Indeed, this result is still given in [Ki-Oh].

Now return to the Gille-Pardon spectral sequence

$$E(P)_3^{*,*} \cong \begin{cases} \Lambda(y_3', z_3, x_4', z_{15}, x_{11}') & if\ d_2(x_{15}) = y_{17} \\ \Lambda(y_3', z_3, x_4', x_7', x_8', x_{11}') & otherwise. \end{cases}$$

For the second case above if $d_3(y_6) \neq x_9$, then $x_9 = x_4'$ is a permanent cycle in the Gille-Pardon and Balmer-Walter spectral sequences by Lemma 6.4 (in fact

$deg(x'_i) - deg(x'_4) \neq 1\ mod(4)$). This is a contradiction to the fact that $W^*(G)$ is a subquotient of $H^*(G/T; Sq^2)$, where there are no generator of $dim = 4$. For the first case, if $d_3(y_6) \neq x_9$, then $y_6 = y'_3$ is a permanent cycle, ($w(z_3 = x_3x_5) = 2$ but $w(x'_4 = x_9) = 1$, and hence there is no differential $d(z_3) = x'_4$) and this is also contradicts to $H^*(G/T; Sq^2)$.

Thus we see $d_3(y_6) = x_9$. Then we easily prove $d_2(x_{15}) = x_{17}$ by the same reason. Therefore we have Theorem 6.7 for $G = E_6$. (In fact we have showed $(*)$ in the proof of Theorem 6.7.)

Next we consider the case $G = E_7$. The cohomology is

$$grH^*(E_7; \mathbb{Z}/2) \cong grH^*(E_6; \mathbb{Z}/2) \otimes \Lambda(y_{10}, y_{18}, x_{27}).$$

Hence we easily see

$$H^*(G/T; Sq^2) \cong H^*(E_6/T; \mathbb{Z}/2) \otimes \Lambda(y'_5, y'_9, x'_{13})$$
$$\cong \Lambda(v_7, y'_5, y'_9, z_3, z_{15}, x'_{11}, x'_{13})$$

where $y'_5 = y_{10}, y'_9 = y_{18}, x'_{13} = x_{27}$.

At last we consider the case $G = E_8$. The chomology is

$$grH^*(E_8; \mathbb{Z}/2) \cong grH^*(E_7; \mathbb{Z}/2) \otimes \Lambda(y_{12}, y_{24}, y_{20}, y_{30}, x_{29})$$
$$\cong \Lambda(y_i | i = 6, 12, 24, 10, 20, 18, 30) \otimes \Lambda(x_j | j = 3, 5, 9, 15, 17, 23, 27, 29).$$

The Steenrod operation acts as $Sq^2(y_i) = y_{i+2}$, $Sq^2(x_i) = x_{i+2}$ and $Sq^2(y_{12}y_{10}) = y_{24}$ (in fact, $y_{24} = y_{12}^2$). Hence we can compute

$$H^*(G/T; Sq^2) \cong \Lambda(v_7, w_{23}, w_{19}, y'_{15}, z_3, z_{15}, x'_{11}, z_{27})$$

where $w_{23} = y_{10}y_{12}y_{24}, y'_{15} = y_{30}, z_{27} = x_{27}x_{29}$. Ohsita first computed this homology by using the Sq^2-algebra structure of $H^*(G/T; \mathbb{Z}/2)$, and of course, our result coincides with him. We can easily show $d_2(x_{27}) = x_{29}$ in the Gille-Pardon spectral sequence and see Theorem 6.7.

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