COHOMOLOGICAL INVARIANTS OF CENTRAL SIMPLE ALGEBRAS OF DEGREE 4

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Abstract. In this paper, we prove a result of Rost, which describes the cohomological invariants of central simple algebras of degree 4 with values in μ_2 when the base field contains a square root of -1.

Introduction

In [5], Rost, Serre and Tignol defined a cohomological invariant of central simple algebras of degree 4 with values in μ_2 when the base field contains a square root of -1. On the other hand, taking the Brauer class of the tensor square of a central simple algebra of degree 4 yields a cohomological invariant of degree 2 with values in μ_2. In this paper, we prove a result of Rost (unpublished), which asserts that these two invariants are essentially the only ones (see Section 4 for a more precise statement).

This paper is organized as follows. After proving some preliminary results in Section 1, we determine the cohomological invariants of cyclic and bicyclic algebras in Section 2. Section 3 is devoted to the construction of a generic central simple algebra of degree 4. Finally, in Section 4, we prove that a cohomological invariant which vanishes on cyclic algebras and biquaternion algebras is identically zero (which another result due to Rost). As a corollary, we obtain a complete description of cohomological invariants of central simple algebras of degree 4.

The proofs of all the results of this paper rely heavily on the use of valuations and residue maps. We let the reader refer to [2] for the basic definitions and results on these topics.

We are grateful to Markus Rost for permitting us to publish his results and to Alexander Merkurjev for providing us his private lecture notes on Rost's theorem, and for pointing us a mistake in an earlier version of this paper.

1. Preliminaries

Let F be a field of characteristic different from 2. We will denote by **Sets** the category of sets, by **Rings** the category of commutative rings, by $\mathbf{Fields}_F$ the category of fields extensions of F and by $\mathbf{Alg}_F$ the category of commutative F-algebras.

For any field extension K/F, we will denote by $H^*(K)$ the cohomology ring of K with coefficients in μ_2. We then get a functor $H^* : \mathbf{Fields}_F \longrightarrow \mathbf{Rings}$.

If K/F is a field extension, then $H^*(K)$ carries a natural structure of a $H^*(F)$-module, given by the external law

$$\begin{aligned} H^*(F) \times H^*(K) &\longrightarrow H^*(K) \\ (a,\xi) &\longmapsto a\cdot\xi = \mathrm{Res}_{K/F}(a) \cup \xi. \end{aligned}$$

We start with some results on the cohomology of rational extensions.

Let $n \geq 1$ be an integer, let $t_1, \ldots, t_n$ be algebraically independent indeterminates over k, and let $K_n = k(t_1, \ldots, t_n)$. Let us denote by $\mathcal{P}_n$ the set of subsets of $\{1, \ldots, n\}$. If $I = \{i_1, \ldots, i_r\} \in \mathcal{P}_n$, we set

$$(\mathbf{t}_I) = (t_{i_1}) \cup \cdots \cup (t_{i_r}).$$

Notice that this definition does not depend on the numbering of the elements of I, since the cup-product in $H^*(K)$ is commutative.

If $I = \emptyset$, then $(\mathbf{t}_I) = 1$ (and thus $a\cdot(\mathbf{t}_I) = \mathrm{Res}_{K/k}(a)$ in this case).

Lemma 1.1. *The cohomology classes $(\mathbf{t}_I), I \in \mathcal{P}_n$ are linearly independent over $H^*(F)$.*

Proof. We proceed by induction on n. Assume that $n = 1$, and let $a_0, a_1 \in H^*(F)$ such that $a_0\cdot 1 + a_1\cdot(t_1) = 0$. Taking residues with respect to the valuation v_{t_1} shows that $a_1 = 0$. Then multiplying by $(-t_1)$ and taking residues shows that $a_0 = 0$.

Assume now that the result is proved for $n \geq 1$, and let us prove it for $n+1$ indeterminates. Let $a_I \in H^*(F), I \in \mathcal{P}_{n+1}$ such that $\sum_I a_I\cdot(\mathbf{t}_I) = 0$. Then we have

$$\sum_J a_J\cdot(\mathbf{t}_J) + \sum_J a_{J\cup\{n+1\}}\cdot(\mathbf{t}_J) \cup (t_{n+1}) = 0,$$

where J describes $\mathcal{P}_n$. Reasoning as in the case $n = 1$, we get

$$\sum_J a_J\cdot(\mathbf{t}_J) = \sum_J a_{J\cup\{n+1\}}\cdot(\mathbf{t}_J) = 0,$$

and by induction we have $a_J = a_{J\cup\{n+1\}} = 0$ for all $J \in \mathcal{P}_n$, that is $a_I = 0$ for all $I \in \mathcal{P}_{n+1}$. This concludes the proof. □

Proposition 1.2. *Let $\xi \in H^*(K_n)$ be a cohomology class which is unramified at every discrete k-valuation $v \neq v_{t_i}, i = 1, \ldots, n$. Then there exist unique cohomology classes $a_I \in H^*(F)$ such that $\xi = \sum_I a_I\cdot(\mathbf{t}_I)$.*

Proof. We prove it by induction on n. Assume first that $n = 1$, and let $a_1 \in H^*(F)$ be the residue of ξ with respect to the t_1-adic valuation. Then $\xi - \mathrm{Res}_{K/F}(a_1) \cup (t_1)$ is unramified at every discrete F-valuation $v \neq v_{t_1}$ by assumption on ξ and because a_1 is constant. By choice of a_1, it is also unramified at v_{t_1}, so $\xi - \mathrm{Res}_{K_1/F}(a_1) \cup (t_1)$ is a constant class, and we may finally write

$$\xi = \mathrm{Res}_{K_1/F}(a_0) + \mathrm{Res}_{K_1/F}(a_1) \cup (t_1) = a_0\cdot 1 + a_1\cdot(t_1).$$

Now assume that the result is proved for $n \geq 1$, and let $\xi \in H^*(K_{n+1})$ be a cohomology class which is unramified at every discrete F-valuation $v \neq v_{t_i}, i = 1, \ldots, n+1$. In particular, it is unramified at every discrete K_n-valuation $v \neq$

$v_{t_{n+1}}$. By the case $n = 1$, there exist $b_0, b_1 \in H^*(K_n)$ such that $\xi = b_0 \cdot 1 + b_1 \cdot (t_{n+1})$. Let v be a discrete F-valuation on $K_n, v \neq v_{t_i}, i = 1, \ldots, n$, and let w the unique F-valuation on K_{n+1} extending v such that $w(t_{n+1}) = 0$. Notice that the corresponding ramification index is equal to 1. Since ξ is unramified at w, taking residues shows that $\mathrm{Res}_{\kappa(w)/\kappa(v)}(r_v(b_1)) \cup (t_{n+1}) = 0$. Now $\kappa(w) = \kappa(v)(t_{n+1})$. The previous equality rewrites $r_v(b_1) \cdot (t_{n+1}) = 0$, and by the case $n = 1$ (applied to the base field $\kappa(v)$), we get $r_v(b_1) = 0$. Moreover, $\xi \cup (-t_{n+1}) = b_0 \cdot (-1) + b_0 \cdot (t_{n+1})$ is also unramified at w, and reasoning as before shows that $r_v(b_0) = 0$. Hence b_0 and b_1 are unramified at any discrete F-valuation v on $K_n, v \neq v_{t_i}, i = 1, \ldots, n$. Now use the induction hypothesis to conclude that $\xi = \sum_I a_I \cdot (\mathbf{t}_I)$ for some $a_I \in H^*(F)$.

The uniqueness of the classes a_I comes from the previous lemma. This completes the proof. $\square$

Definition 1.3. Let $\mathbf{F} : \mathbf{Fields}_F \longrightarrow \mathbf{Sets}$ be a covariant functor. A *cohomological invariant of* $\mathbf{F}$ over F is a natural transformation $\mathbf{F} \longrightarrow H^*$ of functors $\mathbf{Fields}_F$ from $\mathbf{Sets}$.

Cohomological invariants of $\mathbf{F}$ form an $H^*(F)$-module, that we will denote by $\mathrm{Inv}(\mathbf{F}, H^*)$.

We will need in the sequel the notion of a classifying pair for a functor $\mathbf{F}$.

Definition 1.4. Let $\mathbf{F} : \mathbf{Alg}_F \longrightarrow \mathbf{Sets}$ be a covariant functor. Let R be an F-algebra, and let $a \in \mathbf{F}(R)$. We say that the pair (R, a) is classifying for $\mathbf{F}$, if the following conditions hold:

(1) The ring R is a noetherian domain;
(2) For every field extension L/F, and every $a' \in \mathbf{F}(L)$, there exists a maximal ideal $\mathfrak{m}$ of R and a morphism of F-algebras $f : R \longrightarrow L$ such that $\ker(f) \supset \mathfrak{m}$ and $\mathbf{F}(f)$ maps a onto a'.

The Specialization Theorem (cf. [2, Theorem 12.2]) then immediately gives:

Lemma 1.5. *Let* $\mathbf{F} : \mathbf{Alg}_F \longrightarrow \mathbf{Sets}$ *be a subfunctor of* $H^1(\_, G)$*, where* G *is an algebraic group over* F*. Assume that* (R, a) *is classifying for* $\mathbf{F}$*, and let* $\alpha, \beta \in \mathrm{Inv}(\mathbf{F}, H^*)$*. Let* K *be the quotient field of* R*. If* $\alpha(a_K) = \beta(a_K)$*, then* $\alpha = \beta$*.*

2. Invariants of cyclic and bicyclic algebras

For $n \geq 1$, we will denote by $\mathbf{CSA}_n : \mathbf{Alg}_F \longrightarrow \mathbf{Sets}$ the functor of isomorphism classes of Azumaya algebras of rank n.

Let R be a commutative ring. Assume that $n \nmid \mathrm{char}(R)$ and that R contains a primitive n-th root of 1, that we will denote by ζ_n. For all $u, v \in R^\times$, the R-algebra $\{u, v\}_{n,R}$ generated by two elements e and f subject to the relations

$$e^n = u, f^n = v, fe = \zeta_n ef$$

is an Azumaya R-algebra of rank n, called a symbol algebra.

For $r \geq 1$, we will denote by $\mathbf{MS}_{n,r}$ the subfunctor of $\mathbf{CSA}_{n^r}$ of isomorphism classes of tensor products of r symbols algebras of degree n.

We will denote by 1 the constant cohomological invariant of $\mathbf{CSA}_n$ (where $\mathbf{CSA}_n$ is now viewed as a functor from $\mathbf{Fields}_F$ to $\mathbf{Sets}$). If $n = 2m$, then for every field extension K/F and every central simple K-algebra A of exponent dividing n, the class $m[A]$ is killed by 2, and therefore defines a cohomology class of $H^2(K)$ via the usual isomorphism. This defines a cohomological invariant of $\mathbf{CSA}_n$, as well as a cohomological invariant of $\mathbf{MS}_{n,r}$ for all $r \geq 1$, that we will denote by f_m in both cases. In particular, we have

$$f_m(\{a_1, b_1\}_{n,K} \otimes_F \cdots \otimes_F \{a_r, b_r\}_{n,K}) = (a_1) \cup (b_1) + \ldots + (a_r) \cup (b_r)$$

for every field extension K/F and all $a_i, b_i \in K^\times$.

If A is a central simple F-algebra, the trace form of A is the quadratic form

$$q_A \colon \begin{array}{l} A \longrightarrow F \\ a \longmapsto \mathrm{Trd}_A(a^2). \end{array}$$

Lemma 2.1. *Assume that $n \nmid \mathrm{char}(F)$ and that $\mu_n \in F$. Assume also that $n = 2m$. For all $u, v \in F^\times$, the trace form of $\{u, v\}_{n,F}$ is Witt-equivalent to*

$$\langle n, nu, nv, (-1)^m nuv \rangle.$$

Proof. One may check that $\mathrm{Trd}_A(e^i f^j) = 0$ if $(i, j) \neq (0, 0)$. It easily follows that the subspaces

$$F{\cdot}1, F{\cdot}e^m, F{\cdot}f^m, F{\cdot}e^m f^m \text{ and } F{\cdot}e^i f^j \oplus F{\cdot}e^{n-i} f^{m-j},$$

where $0 \leq i \leq j \leq m, (i, j) \neq (0, 0), (0, m), (m, 0), (m, m)$ are mutually orthogonal. Moreover, the $2m^2 - 2$ planes above are hyperbolic since $e^i f^j$ is isotropic. The result then follows from the fact that the reduced traces of $1, e^m, f^m$ and $e^m f^m$ are respectively n, nu, nv and $(-1)^m nuv$. This concludes the proof. □

Assume that $n \nmid \mathrm{char}(F)$ and that $\mu_n \subset F$. Moreover, assume that $-1 \in F^{\times 2}$ (this condition is automatically satisfied if $n \equiv 0[4]$).

If $A = \{u_1, v_1\}_{n,F} \otimes_F \cdots \otimes_F \{u_r, v_r\}_{n,F}$, it follows from the previous lemma that q_A is Witt-equivalent to $\langle n^r \rangle \langle\langle u_1, v_1, \ldots, u_r, v_r \rangle\rangle$. Therefore, $q_A \in I^{2r} F$, and we have

$$e_{2r}(q_A) = (u_1) \cup (v_1) \cup \cdots \cup (u_r) \cup (v_r).$$

This defines an element of $\mathrm{Inv}(\mathbf{MS}_{n,r}, H^*)$ that we denote by e_{2r}.

Proposition 2.2. *Let F be a field of characteristic different from 2, and let $n = 2m$.*

1. *Assume that $n \nmid \mathrm{char}(F)$, and that $\zeta_n \in F$, where ζ_n is a primitive n^{th}-root of 1. Then $\mathrm{Inv}(\mathbf{MS}_{n,1}, H^*)$ is a free $H^*(F)$-module with basis $1, f_m$.*
2. *Assume moreover that $-1 \in F^{\times 2}$. Then $\mathrm{Inv}(\mathbf{MS}_{n,2}, H^*)$ is a free $H^*(F)$-module with basis $1, f_m, e_4$.*

Proof. We first prove (1). Let $\alpha \in \mathrm{Inv}(\mathbf{MS}_{n,1}, H^*)$. Let t_1, t_2 be two indeterminates over F, and let $K = F(t_1, t_2)$. Set $A = \{t_1, t_2\}_{n,K}$, and let $v \neq v_{t_i}, i = 1, 2$ be a discrete F-valuation. By assumption on $v, t_1, t_2 \in \mathcal{O}_v^\times$, and we have $A \simeq \{t_1, t_2\}_{n,\mathcal{O}_v} \otimes_{\mathcal{O}_v} K$. It follows from [2, Theorem 11.7] that the class $\alpha_K(\{t_1, t_2\}_{n,K})$ is unramified at v. By Proposition 1.2, there exist $a_0, a_1, a_2, a_3 \in H^*(F)$ such that

$$\alpha_K(\{t_1,t_2\}_{n,K}) = a_0\cdot 1 + a_1\cdot(t_1) + a_2\cdot(t_2) + a_3\cdot(t_1)\cup(t_2).$$

Now let v be the (t_1-1)-adic valuation on $F(t_2)$. Applying [2, Theorem 11.7] shows by specialization that

$$\alpha_K(\{1,t_2\}_{n,F(t_2)}) = a_0\cdot 1 + a_2\cdot(t_2),$$

that is

$$\alpha_K(\mathrm{M}_n(F(t_2))) = a_0\cdot 1 + a_2\cdot(t_2).$$

Now $\mathrm{M}_n(F(t_2))$ is unramified at v_{t_2}, and therefore, so is $\alpha_K(\mathrm{M}_n(F(t_2)))$. Taking residues then yields $a_2 = 0$. Similar arguments show that $a_1 = 0$, and thus

$$\alpha_K(\{t_1,t_2\}_{n,K}) = a_0\cdot 1 + a_3\cdot(t_1)\cup(t_2).$$

Let $R = F[t_1, t_1^{-1}, t_2, t_2^{-1}]$. It is clear that the pair $(R, \{t_1,t_2\}_{n,R})$ is classifying for $\mathbf{MS}_{n,1}$. The previous equality shows that the invariants α and $a_0\cdot 1+a_3\cdot f_m$ coincide on a classifying pair, hence they are equal by Lemma 1.5. Moreover, if $a_0\cdot 1+a_3\cdot f_m = 0$, then applying this equality to $\{t_1,t_2\}_{n,K}$ shows that $a_0\cdot 1 + a_3\cdot(t_1)\cup(t_2) = 0$. Lemma 1.1 then yields $a_0 = a_3 = 0$. This proves (1).

Assume now that $-1 \in F^{\times 2}$. The fact that $1, f_m, e_4$ are linearly independent over $H^*(F)$ easily comes from Lemma 1.1. Let $\alpha \in \mathrm{Inv}(\mathbf{MS}_{n,2}, H^*)$, let K/F be a field extension, and let A be a symbol algebra over K. Any field extension L/K yields a field extension L/F, and the maps

$$\beta_L\colon \begin{array}{rcl} \mathbf{MS}_{n,r}(L) & \longrightarrow & H^*(L) \\ B & \longmapsto & \alpha_L(B\otimes_L A_L) \end{array}$$

fit together into a cohomological invariant $\beta \in \mathrm{Inv}(\mathbf{MS}_{n,1}, H^*)$. By (1), there exists $\alpha_{i,K}(A) \in H^*(K)$ such that

$$\beta = \alpha_{0,K}(A)\cdot 1 + \alpha_{1,K}(A)\cdot f_m.$$

By linear independence of $1, f_m$ (over the base field K), these classes are unique. It easily follows that $\alpha_{i,K}(A)$ only depends on the isomorphism class of A. Moreover, it follows from the uniqueness of $\alpha_{i,K}(A)$ and the fact that α is a cohomological invariant that the maps $\alpha_{i,K}$ fit together into a cohomological invariant $\alpha_i \in \mathrm{Inv}(\mathbf{MS}_{n,1}, H^*)$. By (1), we may write $\alpha_i = a_i\cdot 1 + b_i\cdot f_m$, for some $a_i, b_i \in H^*(F)$.

Taking $K = F(u_1,v_1,u_2,v_2)$, where u_i, v_i are indeterminates, $B = \{u_1,v_1\}_{n,K}$ and $A = \{u_2,v_2\}_{n,K}$, we get

$$\begin{array}{rcl} \alpha_K(\{u_1,v_1\}_{n,K}\otimes_K\{u_2,v_2\}_{n,K}) & = & a_0\cdot 1 + b_0\cdot(u_2)\cup(v_2) \\ & & +(a_1\cdot 1 + b_1\cdot(u_2)\cup(v_2))\cup(u_1)\cup(v_1). \end{array}$$

The F-automorphism of K which exchanges u_1 and u_2, and exchanges v_1 and v_2, induces maps $\mathbf{MS}_{n,2}(K) \longrightarrow \mathbf{MS}_{n,2}(K)$ and $H^*(K) \longrightarrow H^*(K)$. Notice that the first one maps the isomorphism class of $\{u_1,v_1\}_{n,K}\otimes_K\{u_2,v_2\}_{n,K}$ onto itself. Since α commutes with induced maps, we then get

$$\begin{array}{rcl} \alpha_K(\{u_1,v_1\}_{n,K}\otimes_K\{u_2,v_2\}_{n,K}) & = & a_0\cdot 1 + b_0\cdot(u_1)\cup(v_1) \\ & & +(a_1\cdot 1 + b_1\cdot(u_1)\cup(v_1))\cup(u_2)\cup(v_2). \end{array}$$

Comparing with the previous equality and using Lemma 1.1, we get that $b_0 = a_1$. Therefore, $\{u_1,v_1\}_{n,K}\otimes_K\{u_2,v_2\}_{n,K}$ has same image by α and by $a_0\cdot 1 + a_1\cdot f_m +$

$b_1 \cdot e_4$. Now let $R = F[u_i, u_i^{-1}, v_i, v_i^{-1}, i = 1, 2]$. Since the pair $(R, \{u_1, v_1\}_{n,R} \otimes_R \{u_2, v_2\}_{n,R})$ is classifying for $\mathbf{MS}_{n,2}$, it follows from the previous considerations that the invariants α and $a_0 \cdot 1 + a_1 \cdot f_m + b_2 \cdot e_4$ are equal. This concludes the proof. □

3. Central simple algebras of degree 4

We start this section with a parametrization of central simple algebras of degree 4.

Let R be a commutative ring such that $2 \in R^\times$. If $a \in R^\times$, we will denote by $R[\sqrt{a}]$ the étale quadratic R-algebra $R[X]/(X^2 - a)$.

Let L be a biquadratic étale R-algebra, that is an R-algebra L generated by two elements α, β subject to the relations

$$\alpha^2 = d, \beta^2 = d', \alpha\beta = \beta\alpha,$$

for some $d, d' \in R^\times$. Such an algebra will be denoted by $F[\sqrt{d}, \sqrt{d'}]$.

The group G of automorphisms of the R-algebra $L = F[\sqrt{d}, \sqrt{d'}]$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, generated by the automorphisms σ and τ which are uniquely determined by the following relations:

$$\sigma(\alpha) = \alpha, \sigma(\beta) = -\beta, \tau(\alpha) = -\alpha, \tau(\beta) = \beta.$$

Let $a \in R[\sqrt{d}]^\times, b \in R[\sqrt{d'}]^\times, c \in R[\sqrt{dd'}]^\times$, and set

$$n_a = N_{R[\sqrt{d}]/R}(a), n_b = N_{R[\sqrt{d'}]/R}(b), n_c = N_{R[\sqrt{dd'}]/R}(c).$$

Notice that $n_a, n_b, n_c \in R^\times$ and that we have

$$n_a = a\tau(a), n_b = b\sigma(b), n_c = c\sigma(c) = c\tau(c).$$

Assume that $n_c = n_a n_b$, and let $1, e_\sigma, e_\tau, e_{\sigma\tau}$ be the canonical basis of the L-vector space $\mathrm{Map}(G, L)$. We define a product law on $\mathrm{Map}(G, L)$ by imposing the relations

$$e_\sigma^2 = a, e_\tau^2 = b, e_{\sigma\tau}^2 = c, e_\sigma e_\tau = e_{\sigma\tau},$$

$$e_\sigma \lambda = \sigma(\lambda) e_\sigma, e_\tau \lambda = \tau(\lambda) e_\tau, \text{ for all } \lambda \in L,$$

and extending by distributivity.

By definition, the elements $e_\rho, \rho \in G$ are invertible with respect to this product law. Notice that the missing products $e_\rho e_{\rho'}, \rho, \rho' \in G$ may be obtained using the relations above, so that the product law on $\mathrm{Map}(G, L)$ is completely determined. For example, we have

$$e_\tau e_{\sigma\tau} = e_\sigma^{-1} e_\sigma e_\tau e_{\sigma\tau} = e_\sigma^{-1}(e_{\sigma\tau})^2 = a^{-1} e_\sigma c = a^{-1}\sigma(c) e_\sigma.$$

One can check that we obtain an Azumaya R-algebra of rank 4, that we will denote by $(a, b, c, L/R)$. Notice that if we set $e = e_\sigma$ and $f = e_\tau$, then we have $(a, b, c, L/R) = L \oplus Le \oplus Lf \oplus Lef$, and

$$e^2 = a, f^2 = b, (ef)^2 = c, e\lambda = \sigma(\lambda)e, f\lambda = \tau(\lambda)f, \text{ for all } \lambda \in L.$$

Notice also for later use that we have $fe = a^{-1}\sigma(cb^{-1})ef$, since

$$fe = e^{-1}(ef)^2 f^{-1} = a^{-1}ecb^{-1}f = a^{-1}\sigma(cb^{-1})ef.$$

Lemma 3.1. *Let F be a field of characteristic different from 2. Then every central simple F-algebra of degree 4 is isomorphic to some $(a, b, c, L/F)$. Moreover, one may assume that $a, b, c \notin F^\times$.*

Proof. Assume first that A is a division F-algebra. By a theorem of Albert (see [1, Theorem 11.9] for example), A contains a biquadratic étale F-algebra L. By Skolem-Noether's theorem, σ and τ extend to inner automorphisms $\mathrm{Int}(e)$ and $\mathrm{Int}(f)$ of A, for some $e, f \in A^\times$. This rewrites

$$e\lambda = \sigma(\lambda)e, f\lambda = \tau(\lambda)f, \text{ for all } \lambda \in L.$$

Since $\mathrm{Int}(e)_{|_L} = \sigma$, we have

$$\mathrm{Int}(e^2)_{|_L} = (\mathrm{Int}(e)_{|_L})^2 = \sigma^2 = \mathrm{Id}_L,$$

so $a = e^2 \in C_A(L) = L$ (since L is a maximal subfield of A) . Since $e \in A^\times$, we have $a \in L^\times$. Similarly, $b = f^2 \in L^\times$ and $c = (ef)^2 \in L^\times$.

Associativity of the product law of A implies that

$$\sigma(a) = a, \tau(b) = b, \sigma\tau(c) = c, n_c = n_a n_b.$$

Indeed, we have

$$e^3 = ee^2 = ea = \sigma(a)e = e^2e = ae,$$

and since e is invertible, we get $\sigma(a) = a$. Similar arguments show that $\tau(b) = b$ and $\sigma\tau(c) = c$. We also have

$$\begin{array}{rcl}
be & = & f^2e \\
& = & f(fe) \\
& = & fa^{-1}\sigma(cb^{-1})ef \\
& = & \tau(a^{-1}\sigma(cb^{-1}))f(ef) \\
& = & c\tau(a)^{-1}\sigma(b)^{-1}(fe)f \\
& = & c\tau(a)^{-1}\sigma(b)^{-1}a^{-1}\sigma(cb^{-1})(ef)f \\
& = & c\tau(a)^{-1}\sigma(b)^{-1}a^{-1}\sigma(c)\sigma(b)^{-1}eb \\
& = & c\tau(a)^{-1}\sigma(b)^{-1}\sigma(c)a^{-1}\sigma(b)^{-1}\sigma(b)e \\
& = & n_c n_a^{-1}\sigma(b)^{-1}e.
\end{array}$$

Since e is invertible, we get $b = n_c n_a^{-1}\sigma(b)^{-1}$, that is $n_c = n_a n_b$. The proof of the linear independence of $1, e, f, ef$ over L is straightforward and left to the reader. Hence A contains a subalgebra isomorphic to $(a, b, c, L/F)$, and therefore $A \simeq (a, b, c, L/F)$ since they have same dimension over F.

Assume now that A is not a division algebra. Then $A \simeq \mathrm{M}_2(Q)$, where $Q = (d, d')$ is a (not necessarily division) quaternion F-algebra.

Let $L = F[\sqrt{d}, \sqrt{d'}]$ be a biquadratic extension with generators α and β, and let us consider the F-algebra $(d, 1, d, L/F)$. We then have

$$e^2 = d, f^2 = 1, (ef)^2 = d, e\alpha = \alpha e, e\beta = -\beta e, f\alpha = -\alpha f, f\beta = \beta f.$$

Notice that we also have $fe = d\sigma(d^{-1})ef = ef$.

Set $i = e, j = \beta f, i' = f, j' = \alpha e$. It is straightforward to check that the F-algebra B generated by i and j is isomorphic to Q, and that the F-algebra B' generated by i' and j' is isomorphic to $(1, d^2) \simeq \mathrm{M}_2(F)$. Moreover one may check that B' is the centralizer of B. Since B' is a central simple F-algebra, we get

$$(d, 1, d, L/F) \simeq B \otimes_F B' \simeq Q \otimes_F \mathrm{M}_2(F) \simeq \mathrm{M}_2(Q) \simeq A.$$

We now prove the last part. Write $A \simeq (a, b, c, L/F)$, for some biquadratic étale F-algebra $L = F[\sqrt{d}, \sqrt{d'}]$. Replacing f by $(1+\alpha+\beta)f$, one may assume that $b \notin F^\times$. Indeed, if $b \in F^\times$, we then have

$$[(1+\alpha+\beta)f]^2 = (1+\alpha+\beta)(1-\alpha+\beta)f^2 = b(1+d'-d+2\beta) \notin F^\times.$$

Write $a = a_0 + a_1\alpha$. Notice that, for all $s \in F^\times$, we have

$$\begin{aligned} [(s+\alpha+\beta)e]^2 &= (s^2+d-d'+2s\alpha)(a_0+a_1\alpha) \\ &= (a_0(s^2+d'-d)+2sa_1d) + (a_1(s^2+d-d')+2a_0s)\alpha, \end{aligned}$$

and

$$[(s+\alpha+\beta)ef]^2 = (s+\alpha+\beta)(s-\alpha-\beta)c = c(s^2-d-d'-2\alpha\beta).$$

Thus, after replacing e by $(s+\alpha+\beta)e$ for a suitable $s \in F^\times$, one may also assume that $a, c \notin F^\times$. This concludes the proof. □

Remark 3.2. For any biquadratic F-algebra L, we have $\mathrm{M}_4(F) \simeq (1, 1, 1, L/F)$.

Indeed, the subalgebra B generated by e and β is isomorphic to $(1, d') \simeq \mathrm{M}_2(F)$, and its centralizer C is the subalgebra generated by f and αe, which is isomorphic to $(1, d^2) \simeq \mathrm{M}_2(F)$. Since B is a central simple F-algebra, we have

$$(1, 1, 1, L/F) \simeq B \otimes_F C \simeq \mathrm{M}_2(F) \otimes_F \mathrm{M}_2(F) \simeq \mathrm{M}_4(F).$$

Proposition 3.3. *Let F be a field of characteristic different from 2, and let $A = (a, b, c, L/F)$. Then*

$$2[A] = [(d, n_c)] \in \mathrm{Br}(F).$$

Proof. Recall from [4] that for every central simple F-algebra of degree n, we have

$$\det(q_A) = \det(q_{\mathrm{M}_n(F)}), w_2(q_A) = w_2(q_{\mathrm{M}_n(F)}) + \frac{n(n-1)}{2}[A].$$

Since q_A and $q_{\mathrm{M}_n(F)}$ have same dimension and determinant, the quadratic form $q_A \perp -q_{\mathrm{M}_n(F)}$ lies in $I^2(F)$, and the second equality may be rewritten as

$$c(q_A \perp -q_{\mathrm{M}_n(F)}) = \frac{n(n-1)}{2}[A],$$

where c denotes the Clifford invariant. In particular, if $n = 4$, we have

$$c(q_A \perp -q_{\mathrm{M}_n(F)}) = 6[A] = 2[A].$$

Let us introduce some notation. Let E be an F-algebra. For $u \in E^\times$, we denote by $q_{E,u}$ the quadratic form

$$q_{E,u} \colon \begin{aligned} E &\longrightarrow F \\ x &\longmapsto \mathrm{Tr}_{E/F}(ux^2). \end{aligned}$$

If $u = 1$, we denote it by q_E.

Let ρ be an F-automorphism of L satisfying $\rho^2 = \mathrm{Id}_E$. If $u \in E^\times, \rho(u) = u$, we denote by $q_{E,\rho,u}$ the quadratic form

$$q_{E,\rho,u} \colon \begin{array}{l} E \longrightarrow F \\ x \longmapsto \mathrm{Tr}_{E/F}(ux\rho(x)) \end{array}$$

If $u = 1$, we denote it by $q_{E,\rho}$.

Now assume that $A = (a, b, u, L/F)$. It is easy to check that we have

$$\mathrm{Trd}_A(x + ye + zf + tef) = \mathrm{Tr}_{L/F}(x), \text{for all } x, y, z, t, \in L.$$

This implies that the subspaces L, Le, Lf and Lef are mutually orthogonal. It follows from the previous observation that we have

$$q_A \simeq q_L \perp q_{L,\sigma,a} \perp q_{L,\tau,b} \perp q_{L,\sigma\tau,c}.$$

Let ρ be an F-automorphism of L, and assume that $\rho \neq \mathrm{Id}_L$, so we may write $L^{\langle\rho\rangle} = F[\sqrt{\Delta}]$, and $L = L^{\langle\rho\rangle}[\sqrt{\Delta'}]$. Notice that $ux\rho(x) \in L^{\langle\rho\rangle}$, so that we have $\mathrm{Tr}_{L/F}(ux\rho(x)) = 2\mathrm{Tr}_{L^{\langle\rho\rangle}/F}(ux\rho(x))$.

Now if $x = x_0 + x_1\alpha'$ where $\alpha'^2 = \Delta'$ and $x_i \in L^{\langle\rho\rangle}$, we have $x\rho(x) = x_0^2 - \Delta' x_1^2$, and therefore

$$q_{L,\rho,u} \simeq \langle 2, -2\Delta'\rangle \otimes q_{L^{\langle\rho\rangle},u}.$$

Write $u = u_0 + u_1\sqrt{\Delta}, u_i \in F$, and set $n_u = u_0^2 - \Delta u_1^2$. The representative matrix of the previous quadratic form in the basis $1, \sqrt{\Delta}$ is

$$\begin{pmatrix} 2u_0 & 2u_1\Delta \\ 2u_1\Delta & 2u_0\Delta \end{pmatrix}.$$

If $u_0 = 0$, this 2-dimensional quadratic form is isotropic, hence hyperbolic. If $u_0 \neq 0$, this form then represents $2u_0$ and has determinant $4n_u\Delta$, so it is isomorphic to $\langle 2u_0\rangle \otimes \langle 1, n_u\Delta\rangle$. We then get

$$q_{L,\rho,u} \simeq \langle u_0\rangle \otimes \langle\langle -n_u\Delta, \Delta'\rangle\rangle,$$

where this form has to be understood as the hyperbolic form of dimension 4 if $u_0 = 0$. Taking $u = 1$, we get

$$q_{L,\rho} \simeq \langle\langle -\Delta, \Delta'\rangle\rangle,$$

and thus

$$q_{L,\rho,u} \perp -q_{L,\rho} \simeq \langle 1, -\Delta'\rangle \otimes \langle u_0, u_0 n_u\Delta, 1, \Delta\rangle.$$

Since $c(\varphi \otimes \varphi') = [(\mathrm{disc}(\varphi), \mathrm{disc}(\varphi'))]$ for every even-dimensional forms φ, φ', we get

$$c(q_{L,\rho,u} \perp -q_{L,\rho}) = [(\Delta', n_u)].$$

Remark 3.2 implies that $q_A \perp -q_{\mathrm{M}_4(F)}$ is Witt-equivalent to

$$(q_{L,\sigma,a} \perp -q_{L,\sigma}) \perp (q_{L,\tau,b} \perp -q_{L,\tau}) \perp (q_{L,\sigma\tau,c} \perp -q_{L,\sigma\tau}).$$

We then finally get

$$c(q_A \perp -q_{\mathrm{M}_4(F)}) = [(d', n_a)] + [(d, n_b)] + [(d, n_c)].$$

Since the quaternion algebras $(d, n_a), (d', n_b)$ and (dd', n_c) are split, we get

$$\begin{array}{rcl} c(q_A \perp -q_{\mathrm{M}_4(F)}) & = & [(dd', n_a)] + [(dd', n_b)] + [(d, n_c)] \\ & = & [(dd', n_a n_b)] + [(d, n_c)] \\ & = & [(dd', n_c)] + [(d, n_c)] \\ & = & [(d, n_c)]. \end{array}$$

This concludes the proof. □

We now define a classifying pair for $\mathbf{CSA}_4$. Let us consider the affine variety $\mathcal{V} \subset \mathbb{A}^6$ defined by the equation

$$x^2 - uy^2 - vz^2 + uvt^2 + uv = 0.$$

This is a rational variety, with coordinate ring

$$F[\mathcal{V}] = F[X, Y, Z, T, U, V]/(X^2 - UY^2 - VZ^2 + UVT^2 + UV).$$

Let us denote by $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t}, \mathbf{u}, \mathbf{v}$ the images of X, Y, Z, T, U, V in $F[\mathcal{V}]$ respectively. Notice that $\mathbf{u}$ and $\mathbf{v}$ are algebraically independent over F. Indeed, we have a surjective F-algebra morphism

$$\begin{array}{rcl} F[\mathcal{V}] & \longrightarrow & F[V, Z] \\ P(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t}, \mathbf{u}, \mathbf{v}) & \longmapsto & P(0, i, Z, i, VZ^2, V) \end{array}$$

which maps $\mathbf{u}$ and $\mathbf{v}$ onto VZ^2 and V respectively. Since VZ^2 and V are algebraically independent over F, so are $\mathbf{u}$ and $\mathbf{v}$.

Now let us consider the open subset $\mathcal{U}$ of $\mathcal{V}$ defined by the equations

$$u \neq 0, v \neq 0, t \neq 0, y^2 - vt^2 \neq 0, z^2 - u \neq 0.$$

Then $\mathcal{U}$ is also a rational variety, whose coordinate ring $F[\mathcal{U}]$ is the localization of $F[\mathcal{V}]$ at $\mathbf{u}, \mathbf{v}, \mathbf{t}, \mathbf{y}^2 - \mathbf{v}\mathbf{t}^2$ and $\mathbf{z}^2 - \mathbf{u}$. Set $R_0 = F[\mathcal{U}]$. Then R_0 is a noetherian ring, whose quotient field F_0 is a rational extension of F. Moreover, $\mathbf{u}, \mathbf{v} \in R_0$ are algebraically independent over F. In particular, there exists a transcendence basis of F_0/F containing $\mathbf{u}$ and $\mathbf{v}$.

Set $\mathbf{w} = \mathbf{u}(\mathbf{y}^2 - \mathbf{v}\mathbf{t}^2) \in R_0^\times$ and let $L_0 = R_0[\sqrt{\mathbf{u}}, \sqrt{\mathbf{u}\mathbf{w}}]$. We now define three elements $\mathbf{a}_0, \mathbf{b}_0, \mathbf{c}_0 \in L_0^\times$ by

$$\mathbf{a}_0 = \mathbf{z} + \sqrt{\mathbf{u}} \in R_0[\sqrt{\mathbf{u}}]^\times, \mathbf{b}_0 = \frac{\mathbf{y}}{\mathbf{t}} + \frac{\sqrt{\mathbf{u}\mathbf{w}}}{\mathbf{t}\mathbf{u}} \in R_0[\sqrt{\mathbf{u}\mathbf{w}}]^\times, \mathbf{c}_0 = \mathbf{x} + \sqrt{\mathbf{w}} \in R_0[\sqrt{\mathbf{w}}]^\times.$$

Let us check that $n_{\mathbf{a}_0} n_{\mathbf{b}_0} = n_{\mathbf{c}_0}$. We have

$$n_{\mathbf{a}_0} n_{\mathbf{b}_0} = \frac{(\mathbf{z}^2 - \mathbf{u})(\mathbf{y}^2\mathbf{u}^2 - \mathbf{u}\mathbf{w})}{\mathbf{t}^2\mathbf{u}^2} = \frac{(\mathbf{z}^2 - \mathbf{u})(\mathbf{y}^2\mathbf{u} - \mathbf{w})}{\mathbf{t}^2\mathbf{u}}.$$

Since $\mathbf{y}^2\mathbf{u} - \mathbf{w} = \mathbf{u}\mathbf{v}\mathbf{t}^2$, we get $n_{\mathbf{a}_0} n_{\mathbf{b}_0} = (\mathbf{z}^2 - \mathbf{u})\mathbf{v}$. Now $\mathbf{z}^2\mathbf{v} - \mathbf{u}\mathbf{v} = \mathbf{x}^2 - \mathbf{u}\mathbf{y}^2 + \mathbf{u}\mathbf{v}\mathbf{t}^2 = \mathbf{x}^2 - \mathbf{w} = n_{\mathbf{c}_0}$, and we are done. Hence we may consider the Azumaya algebra

$$\mathcal{A}_0 = (\mathbf{a}_0, \mathbf{b}_0, \mathbf{c}_0, L_0/R_0).$$

Lemma 3.4. *The pair $(R_0, \mathcal{A}_0)$ is classifying for $\mathbf{CSA}_4$.*

Proof. Let E/F be a field extension, and let A be a central simple E-algebra of degree 4. By Lemma 3.1, we have $A \simeq (a,b,c,L/E)$, with $a,b,c \notin E^\times$. Write $L = E[\sqrt{d},\sqrt{d'}]$, with generators α,β. Write

$$a = \lambda_0 + \lambda_1\alpha, b = \mu_0 + \mu_1\beta, c = \gamma_0 + \gamma_1\alpha\beta.$$

We set

$$\overline{x} = \gamma_0, \overline{y} = \frac{\mu_0\gamma_1}{\lambda_1\mu_1}, \overline{z} = \lambda_0, \overline{t} = \frac{\gamma_1}{\lambda_1\mu_1}, \overline{u} = \lambda_1^2 d, \overline{v} = \mu_0^2 - \mu_1^2 d'.$$

Using the equality

$$(\lambda_0^2 - \lambda_1^2 d)(\mu_0^2 - \mu_1^2 d) = (\gamma_0^2 - \gamma_1^2 dd'),$$

and the fact that $\lambda_1,\mu_1,\gamma_1 \in F^\times$, one may check that $\mathbf{p} = (\overline{x},\overline{y},\overline{z},\overline{t},\overline{u},\overline{v}) \in \mathcal{U}(E)$. Then evaluation at $\mathbf{p}$ yields an F-algebra morphism $f : R_0 \longrightarrow E$ such that

$$\mathcal{A}_0 \otimes_{R_0} E \simeq A.$$

Now it suffices to notice that $\ker(f) = (\mathbf{x}-\overline{x},\mathbf{y}-\overline{y},\mathbf{z}-\overline{z},\mathbf{t}-\overline{t},\mathbf{u}-\overline{u},\mathbf{v}-\overline{v})$ is a maximal ideal to conclude. □

4. Cohomological invariants of $\mathbf{CSA}_4$

We now prove the following theorem, due to Rost (unpublished):

Theorem 4.1 (Rost)**.** *Let F be a field of characteristic different from 2. Assume that $-1 \in F^{\times 2}$. Then the map*

$$\mathrm{Inv}(\mathbf{CSA}_4, H^*) \longrightarrow \mathrm{Inv}(\mathbf{MS}_{2,2}, H^*) \times \mathrm{Inv}(\mathbf{MS}_{4,1}, H^*)$$

is injective. In other words, a cohomological invariant which is zero on biquaternion algebras and cyclic algebras is identically zero.

Proof. Let $\alpha \in \mathrm{Inv}(\mathbf{CSA}_4, H^*)$. Let $(R_0, \mathcal{A}_0)$ be the classifying pair of $\mathbf{CSA}_4$ defined in the previous section, and let F_0 be the quotient field of R_0. Finally, set

$$A_0 = \mathcal{A}_0 \otimes_{R_0} F_0 = (\mathbf{z}+\sqrt{\mathbf{u}}, \frac{\mathbf{y}}{\mathbf{t}} + \frac{\sqrt{\mathbf{uw}}}{\mathbf{tu}}, \mathbf{x}+\sqrt{\mathbf{w}}, F_0[\sqrt{\mathbf{u}},\sqrt{\mathbf{uw}}]/F_0).$$

Notice that we have $2[A_0] = [(\mathbf{u},\mathbf{v})]$ in $\mathrm{Br}(F_0)$. Indeed, by Proposition 3.3, we have

$$\begin{array}{rcl} 2[A_0] & = & [(\mathbf{u}, \mathbf{x}^2 - \mathbf{w})] \\ & = & [(\mathbf{u}, \mathbf{x}^2 - \mathbf{u}\mathbf{y}^2 + \mathbf{u}\mathbf{v}\mathbf{t}^2)] \\ & = & [(\mathbf{u}, \mathbf{v}\mathbf{z}^2)] \\ & = & [(\mathbf{u},\mathbf{v})]. \end{array}$$

Let υ be a discrete F-valuation on F_0. If $\upsilon \neq \upsilon_\pi$, with $\pi = \mathbf{u},\mathbf{v},\mathbf{t},\mathbf{y}^2-\mathbf{v}\mathbf{t}^2,\mathbf{z}^2-\mathbf{u}$, then $\mathbf{u},\mathbf{v},\mathbf{w},\mathbf{t} \in \mathcal{O}_\upsilon^\times$ and $\mathbf{z}+\sqrt{\mathbf{u}}, \frac{\mathbf{y}}{\mathbf{t}} + \frac{\sqrt{\mathbf{uw}}}{\mathbf{tu}}, \mathbf{x}+\sqrt{\mathbf{w}} \in \mathcal{O}_\upsilon[\sqrt{\mathbf{u}},\sqrt{\mathbf{uw}}]^\times$. We then get that

$$A_0 \simeq (\mathbf{z}+\sqrt{\mathbf{u}}, \frac{\mathbf{y}}{\mathbf{t}} + \frac{\sqrt{\mathbf{uw}}}{\mathbf{tu}}, \mathbf{x}+\sqrt{\mathbf{w}}, \mathcal{O}_\upsilon[\sqrt{\mathbf{u}},\sqrt{\mathbf{uw}}]/\mathcal{O}_\upsilon) \otimes_{\mathcal{O}_\upsilon} F_0,$$

and therefore $\alpha_{F_0}(A_0)$ is unramified at υ by [2, Theorem 11.7] .

Assume now that $\upsilon = \upsilon_\pi$, with $\pi = \mathbf{t}, \mathbf{z}^2 - \mathbf{u}$ or $\mathbf{y}^2 - \mathbf{t}\mathbf{v}^2$. Then $\mathbf{u},\mathbf{v} \in \mathcal{O}_\upsilon^\times$, and thus we have

$$(\mathbf{u}) \cup (\mathbf{v}) = j_\upsilon((\overline{\mathbf{u}}) \cup (\overline{\mathbf{v}})),$$

where $j_\upsilon : H^2(\kappa(\upsilon)) \hookrightarrow H^2((F_0)_\upsilon)$ is the canonical injection. Now in all three cases, the quadratic form $\langle 1, -\overline{\mathbf{u}}, -\overline{\mathbf{v}}, \overline{\mathbf{uv}}\rangle$ is isotropic over $\kappa(\upsilon)$. It is clear if $\pi = \mathbf{z}^2 - \mathbf{u}$ or $\mathbf{y}^2 - \mathbf{vt}^2$, and it follows from the equalities

$$0 = \mathbf{x}^2 - \mathbf{uy}^2 - \mathbf{vz}^2 + \mathbf{uvt}^2 + \mathbf{uv} = \mathbf{x}^2 - \mathbf{uy}^2 - \mathbf{vz}^2 + \mathbf{uv} \mod \mathbf{t},$$

if $\pi = \mathbf{t}$. Thus $(\overline{\mathbf{u}}) \cup (\overline{\mathbf{v}}) = 0$, and therefore $(\mathbf{u}) \cup (\mathbf{v}) = 0$ as well. Hence the quaternion algebra $(\mathbf{u}, \mathbf{v})$ splits over $(F_0)_\upsilon$, and thus $2[A_0]$ splits over $(F_0)_\upsilon$. By a theorem of Albert, $A_0 \otimes_{F_0} (F_0)_\upsilon$ is a biquaternion algebra. By assumption, we get

$$\mathrm{Res}_{(F_0)_\upsilon/F_0}(\alpha_{F_0}(A_0)) = \alpha_{(F_0)_\upsilon}(A_0 \otimes_{F_0} (F_0)_\upsilon) = 0.$$

Consequently, $r_\upsilon(\alpha_{F_0}(A_0)) = 0$ and $\alpha_{F_0}(A_0)$ is once again unramified at υ.

Finally, $\alpha_{F_0}(A_0)$ is unramified at every discrete F-valuation $\upsilon \neq \upsilon_{\mathbf{u}}, \upsilon_{\mathbf{v}}$. In particular, it is unramified at every F_1-valuation, where $F_1 = F(\mathbf{u}, \mathbf{v})$. Since there exists a transcendence basis of F_0/F containing $\mathbf{u}$ and $\mathbf{v}$, the field extension F_0/F_1 is rational, and we deduce that $\alpha_{F_0}(A_0) = \mathrm{Res}_{F_0/F_1}(\beta)$, for some $\beta \in H^*(F_1)$. Let υ_1 be a discrete F-valuation on F_0, which different from the $\mathbf{u}$-adic and the $\mathbf{v}$-adic valuations. Let us show that β is unramified at υ_1.

Let $\mathbf{u}, \mathbf{v}, t_1, t_2, t_3$ be a transcendence basis of F_0/F, and let us extend υ_1 to a discrete F-valuation υ on F_0 by setting

$$\upsilon_{F_1} = \upsilon_1, \upsilon(t_i) = 0, i = 1, 2, 3.$$

Then the corresponding ramification index is 1 and $\kappa(\upsilon_1) = \kappa(\upsilon)(t_1, t_2, t_3)$, so that $\kappa(\upsilon)/\kappa(\upsilon_1)$ is a rational extension. Since $\alpha_{F_0}(A_0)$ is unramified at υ, we get

$$0 = r_\upsilon(\alpha_{F_0}(A_0)) = \mathrm{Res}_{\kappa(\upsilon)/\kappa(\upsilon_1)}(r_{\upsilon_1}(\beta)).$$

Since $\kappa(\upsilon)/\kappa(\upsilon_1)$ is rational, the map $\mathrm{Res}_{\kappa(\upsilon)/\kappa(\upsilon_1)}$ is injective, and we get that β is unramified at υ_1. By Proposition 1.2, we have

$$\beta = a_0{\cdot}1 + a_1{\cdot}(\mathbf{u}) + a_2{\cdot}(\mathbf{v}) + a_3{\cdot}(\mathbf{u}) \cup (\mathbf{v}), a_i \in H^*(F).$$

Hence we get

$$\alpha_{F_0}(A_0) = a_0{\cdot}1 + a_1{\cdot}\mathrm{Res}_{F_0/F_1}((\mathbf{u})) + a_2{\cdot}\mathrm{Res}_{F_0/F_1}((\mathbf{v})) + a_3{\cdot}\mathrm{Res}_{F_0/F_1}((\mathbf{u}) \cup (\mathbf{v})).$$

Let us consider the $(\mathbf{v} - 1)$-adic valuation on F_0, and let F_2 its residue field. Since A_0 is unramified at this valuation, one may specialize A_0 to a central simple F_2-algebra B. Then $2[A_0]$ specializes to $2[B]$. But $2[A_0] = [(\mathbf{u}, \mathbf{v})]$ specializes to 0, and therefore B is a biquaternion algebra. Then $\alpha_{F_2}(B) = 0$ by assumption, and therefore

$$a_0{\cdot}1 + a_1{\cdot}\mathrm{Res}_{F_2/F_1}((\mathbf{u})) = 0.$$

In other words, $\mathrm{Res}_{F_2/F_1}(a_0{\cdot}1 + a_1{\cdot}(\mathbf{u})) = 0$. Since $F_1 \subset F_2 \subset F_0$, and F_0/F_1 is rational, the map Res_{F_2/F_1} is injective and thus

$$a_0{\cdot}1 + a_1{\cdot}(\mathbf{u}) = 0 \text{ in } H^*(F_1).$$

By Lemma 1.1, we get $a_0 = a_1 = 0$. Considering the $(\mathbf{u} - 1)$-adic valuation yields that $a_2 = 0$ in a similar way, so that

$$\alpha_{F_0}(A_0) = a_3{\cdot}\mathrm{Res}_{F_0/F_1}((\mathbf{u}) \cup (\mathbf{v})).$$

Since $\mathrm{Res}_{F_0/F_1}((\mathbf{u}) \cup (\mathbf{v}))$ corresponds to $[(\mathbf{u}, \mathbf{v})] = 2[A_0]$ in $\mathrm{Br}_2(F_0)$ via the usual isomorphism, the previous equality rewrites

$$\alpha_{F_0}(A_0) = a_3{\cdot}f_{2,F_0}(A_0).$$

By Lemma 1.5, we get $\alpha = a_3 \cdot f_2$. By assumption on α, we then have

$$0 = \alpha_{F_1}(\{\mathbf{u}, \mathbf{v}\}_{4,F_1}) = a_3 \cdot (\mathbf{u}) \cup (\mathbf{v}) \in H^*(F_1).$$

By Lemma 1.1, we get $a_3 = 0$, and thus $\alpha = 0$. This concludes the proof. $\square$

We now describe the cohomological invariants of $\mathbf{CSA}_4$. Let F be a field of characteristic different from 2. Assume that $-1 \in F^{\times 2}$, and let K/F be a field extension. By [5], for every central simple K-algebra A of degree 4, there exists a unique 2-fold Pfister form q_2 and a unique 4-fold Pfister form q_4 such that

$$q_A \sim q_2 + q_4 \in W(K).$$

Taking the cohomology class $e_4(q_4) \in H^4(K)$ yields a cohomological invariant of $\mathrm{Inv}(\mathbf{CSA}_4, H^*)$. This invariant restricts to zero on cyclic algebras, and its restriction to $\mathbf{MS}_{2,2}$ obviously coincide with the invariant e_4 defined in Section 2. Therefore, we still denote this invariant by e_4.

Corollary 4.2. *Let F be a field of characteristic different from* 2*. Assume that $-1 \in F^{\times 2}$. Then $\mathrm{Inv}(\mathbf{CSA}_4, H^*)$ is a free $H^*(F)$-module with basis $1, f_2, e_4$.*

Proof. We first prove that $1, f_2, e_4$ are linearly independent. Assume that we have

$$a_0 \cdot 1 + a_1 \cdot f_2 + a_2 \cdot e_4 = 0 \text{ for some } a_i \in H^*(F).$$

Since the restriction of e_4 to $\mathbf{MS}_{4,1}$ is zero, it follows that $a_0 + a_1 \cdot f_2 = 0 \in \mathrm{Inv}(\mathbf{MS}_{4,1}, H^*)$. By Proposition 2.2 (1), we get $a_0 = a_1 = 0$. Thus, we get $a_2 \cdot e_4 = 0 \in \mathrm{Inv}(\mathbf{CSA}_4, H^*)$. In particular, $a_2 \cdot e_4 = 0 \in \mathrm{Inv}(\mathbf{MS}_{2,2}, H^*)$. By Proposition 2.2 (2), we get $a_2 = 0$.

We now prove that $1, f_2, e_4$ span $\mathrm{Inv}(\mathbf{CSA}_4, H^*)$ as an $H^*(F)$-module. Let $\alpha \in \mathrm{Inv}(\mathbf{CSA}_4, H^*)$. By Proposition 2.2, we have

$$\alpha_{|\mathbf{MS}_{2,2}} = a_0 \cdot 1 + a_1 \cdot f_2 + a_2 \cdot e_4,$$

for some $a_i \in H^*(F)$, and also

$$\alpha_{|\mathbf{MS}_{4,1}} = b_0 \cdot 1 + b_1 \cdot f_2,$$

for some $b_i \in H^*(F)$. Since $\mathrm{M}_4(F) \simeq \{1,1\}_{4,F} \simeq (1,1) \otimes_F (1,1)$, we may apply $\mathrm{M}_4(F)$ to both equalities to get $a_0 = b_0$.

Let u, v be two independent indeterminates over F. We have

$$\mathrm{M}_2((u,v)) \simeq (1,1) \otimes_{F(u,v)} (u,v) \simeq \{u^2, v^2\}_{4,F(u,v)},$$

and applying $\mathrm{M}_2((u,v))$ to both equalities yields

$$a_1 \cdot (u) \cup (v) = b_1 \cdot (u) \cup (v) \in H^*(F(u,v)).$$

It follows from Lemma 1.1 that $a_1 = b_1$. Since e_4 is zero on $\mathbf{MS}_{4,1}$, we conclude that α and $a_0 \cdot 1 + a_1 \cdot f_2 + a_2 \cdot e_4$ coincide on $\mathbf{MS}_{4,1}$ and $\mathbf{MS}_{2,2}$. By the previous theorem, we get $\alpha = a_0 \cdot 1 + a_1 \cdot f_2 + a_2 \cdot e_4$, and this concludes the proof. $\square$

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