EQUIVARIANT PRETHEORIES AND INVARIANTS OF TORSORS

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Abstract. In the present paper we introduce and study the notion of an equivariant pretheory: basic examples include equivariant Chow groups, equivariant K-theory and equivariant algebraic cobordism. To extend this set of examples we define an equivariant (co)homology theory with coefficients in a Rost cycle module and provide a version of Merkurjev's (equivariant K-theory) spectral sequence. As an application we generalize the theorem of Karpenko-Merkurjev on G-torsors and rational cycles; to every G-torsor E and a G-equivariant pretheory we associate a graded ring which serves as an invariant of E. In the case of Chow groups this ring encodes the information concerning the J-invariant of E and in the case of Grothendieck's K_0 – indexes of the respective Tits algebras.

1. Introduction

In the present paper we introduce and study the notion of a (graded) equivariant pretheory. Roughly speaking, it is defined to be a contravariant functor from the category of G-varieties, where G is an algebraic group, to (graded) abelian groups which satisfies localization and homotopy invariance properties. All known examples of equivariant oriented cohomology theories (equivariant Chow groups, K-theory, algebraic cobordism, etc.) are pretheories in our sense.

We generalize the equivariant Chow groups of Edidin-Graham by introducing equivariant (co)homology theory with coefficients in a Rost cycle module. We also prove a version of Merkurjev's equivariant K-theory spectral sequence for equivariant cycle homology. This provides many new examples of equivariant pretheories.

One of the key results of Karpenko-Merkurjev [16, Thm. 6.4] tells us that the characteristic subring of the Chow ring of a variety of Borel subgroups of a split linear algebraic group G is contained in the image of the restriction map, i.e. always consists of rational cycles. This fact plays a fundamental role in computations of canonical/essential dimensions, discrete motivic invariants of G and in the study of splitting properties of G-torsors.

In the present paper we generalize this result to an arbitrary equivariant pretheory (see Theorem 4.4). In particular, we obtain versions of [16, Thm. 6.4] for Grothendieck's K_0 and algebraic cobordism Ω of Levine-Morel.

As an application we define for any equivariant pretheory $\mathtt{h}$ and G-torsor E a commutative ring $\hat{\mathtt{h}}_B(E)$ (see Def. 4.5). If E is generic and $\mathtt{h}$ is either the Chow ring CH^* or Grothendieck's K_0 or algebraic cobordism Ω, this ring coincides with the cohomology ring $\mathtt{h}(G)$ of G. In general, it is always a quotient of $\mathtt{h}(G)$ which

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in the case of the Chow ring is related to the motivic J-invariant of E and in the case of K_0 – to the indexes of the Tits algebras of E. This provides a fascinating link between these two discrete invariants, totally unrelated at the first sight: one observes that the p-exceptional degrees of V. Kac for Chow groups [15] play the same role as the maximal Tits indexes for K_0 [22].

The paper is organized as follows: In the first two sections we introduce the notion of an equivariant pretheory and provide several examples including equivariant cycle (co)homology. In Section 4 we generalize the result of Karpenko-Merkurjev to an arbitrary equivariant pretheory. In the last section we provide applications to equivariant oriented cohomology theories (Chow groups, Grothendieck's K_0 and algebraic cobordism of Levine-Morel). Appendix is devoted to the construction of a spectral sequence for cycle homology which generalizes the long exact localization sequence.

1.1 (**Notations).** Unless otherwise indicated, all schemes/varieties are defined over the base field k. By a scheme over a field k (k-scheme) we mean a reduced separated Noetherian scheme over k. By a variety over a field k (k-variety) we mean a quasi-projective scheme over k (note that it has to be of finite type over k). If l/k is a field extension and X is a k-scheme, we define $X_l = X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, l$ to be the respective base change. By pt we denote $\mathrm{Spec}\, k$.

By an algebraic group we mean an affine smooth group scheme over k. By a subgroup we always understand a closed algebraic subgroup. By an action of an algebraic group G on a scheme X we mean a morphism $G \times_{\mathrm{Spec}\, k} X \to X$ of schemes over k (all group actions are assumed to be on the left), subject to the usual axioms, see [23, Def. 0.3]. By a G-scheme we mean a scheme X endowed with an action of an algebraic group G.

We denote by G-Sm_k the category of smooth G-varieties over k with equivariant G-morphisms. A localization of a smooth variety over k is called essentially smooth. We denote by G-Ess_k the category of essentially smooth G-schemes over k with G-equivariant flat morphisms. We denote by $\mathfrak{Ab}$ the category of abelian groups.

2. Equivariant pretheories.

In the present section we introduce the notion of a (graded) equivariant pretheory and provide several examples.

Let G be an algebraic group over a field k. Consider a contravariant functor from the category of smooth G-varieties over k to the category of abelian groups

$$\mathtt{h}_G\colon\ G\text{-}\mathrm{Sm}_k \longrightarrow \mathfrak{Ab}, \quad X \mapsto \mathtt{h}_G(X).$$

Given X, $Y \in G$-Ess_k and a G-equivariant map $f\colon X \to Y$ the induced functorial map $\mathtt{h}_G(Y) \to \mathtt{h}_G(X)$ is called a *pull-back* and is denoted by f^*_G.

2.1. **Definition.** The functor $\mathtt{h}_G\colon\ G$-$\mathrm{Sm}_k \to \mathfrak{Ab}$ is called a *G-equivariant pretheory* over k if it satisfies the following two axioms:

H. (homotopy invariance) For a G-equivariant map $p\colon \mathbb{A}^n_k \to \mathrm{pt}$ (where G acts trivially on pt) the induced pull-back

$$p^*_G\colon \mathtt{h}_G(\mathrm{pt}) \longrightarrow \mathtt{h}_G(\mathbb{A}^n_k)$$

is an isomorphism.

L. (localization) For a smooth G-variety X and a G-equivariant open embedding $\iota\colon U \hookrightarrow X$ the induced pull-back

$$\iota_G^*\colon \mathtt{h}_G(X) \longrightarrow \mathtt{h}_G(U)$$

is surjective.

Let $\mathcal{U}$ be a G-scheme over k such that $\mathcal{U}$ is the localization of a smooth irreducible G-variety X with respect to G-equivariant open embeddings $f_{ij}\colon U_j \to U_i$, $U_i \subset X$, i.e. $\mathcal{U} = \varprojlim_{f_{ij}} U_i$. Observe that $\mathcal{U}$ is essentially smooth over k.

Let $\bar{\mathtt{h}}_G(\mathcal{U})$ denote the induced colimit $\varinjlim_{(f_{ij})_G^*} \mathtt{h}_G(U_i)$. Note that the canonical maps $\mathtt{h}_G(U_i) \to \bar{\mathtt{h}}_G(\mathcal{U})$ are surjective by the localization property (L).

2.2. **Definition.** We call $\mathtt{h}_G$ an *essential* G-equivariant pretheory if $\mathtt{h}_G$ can be extended to the category $G\text{-Ess}_k$ of essentially smooth G-schemes over k with G-equivariant flat morphisms, i.e.

$$\mathtt{h}_G\colon\ G\text{-Ess}_k \longrightarrow \mathfrak{Ab},$$

such that the following additional axiom holds:

C. Given $\mathcal{U}$ as above, the map induced by flat pull-backs $\mathtt{h}_G(U_i) \to \mathtt{h}_G(\mathcal{U})$

$$\bar{\mathtt{h}}_G(\mathcal{U}) \to \mathtt{h}_G(\mathcal{U})$$

is surjectve.

Note that (C) holds if and only if the induced pull-back $\mathtt{h}_G(U_i) \to \mathtt{h}_G(\mathcal{U})$ is surjective for some i.

2.3. **Example** (Equivariant K-theory)**.** We recall definitions and basic properties of equivariant K-groups as defined by Thomason [27], see also the survey article [21] of Merkurjev.

Let G be an algebraic group over k and let X be a smooth G-variety. Then the category $\mathcal{P}(G,X)$ of locally free G-modules on X (in the sense of Mumford [23, I, §3]) is an exact category. Following Thomason [27] one defines the i-th G-equivariant K-group $\mathrm{K}_i(G,X)$ as Quillen's i-th K-group of the exact category $\mathcal{P}(G,X)$.

Let $\mathtt{h}_G(X) = \mathrm{K}_0(G,X)$. Then according to [21, Thm. 2,7 and Lem. 4.1] it satisfies localization and homotopy invariance, and by [9, 52.F] it satisfies (C). Hence, it provides an example of an essential G-equivariant pretheory.

2.4. **Example** (Equivariant cobordism)**.** This theory has been recently defined by Heller and Malagón-López [14].

Assume that $char(k) = 0$. Consider the ring $\Omega_*(X)$ of algebraic cobordism of a smooth k-variety X as defined by Levine and Morel [19]. Since $\Omega_i(X)$ does not vanish for i big enough (as Chow groups do) one can not copy word by word the definition of equivariant Chow groups given by Edidin and Graham [8], see also Section 3.

Instead Heller and Malagón-López consider [14] (what they call) *good systems of representations.* These are families of pairs $(V_i, U_i)_{i\in\mathbb{N}}$ of vector spaces with $U_i \subseteq V_i$ endowed with an action of an algebraic group G such that

(i) G acts freely on U_i and $U_i \to U_i/G$ is a G-torsor,
(ii) $V_{i+1} = V_i \oplus W_i$ for some k-subspace W_i, such that $U_i \oplus W_i \subseteq U_{i+1}$,

(iii) $\sup\dim V_i = \infty$, and
(iv) $\operatorname{codim}_{V_i}(V_i \setminus U_i) < \operatorname{codim}_{V_{i+1}}(V_{i+1} \setminus U_{i+1})$, where we consider V_i as an affine space over k.

Observe that assumption (i) ensures that the quotient $X \times^G U := (X \times_k U)/G$ is a quasi-projective variety over k (see [8, Prop. 23]). Moreover, it is smooth over k by the descent, since $X \times_k U \to X \times^G U$ is faithfully flat.

Let G be connected. Then the n-th *equivariant cobordism group* of a smooth G-variety X is defined by

$$\Omega_n^G(X) := \varprojlim_i \Omega_{n-\dim G+\dim U_i}(X \times^G U_i)\,.$$

This is well defined, see [14, Cor. 3.4], and the functor

$$\mathtt{h}_G\colon X \longmapsto \bigoplus_{n\in\mathbb{Z}} \Omega_n^G(X)$$

satisfies the localization and homotopy invariance axioms by [*loc.cit.* Thm. 4.2 and Cor. 4.6]. Hence, it provides an example of a G-equivariant pretheory.

A further example is the equivariant Chow-theory of Edidin and Graham [8]. We consider this later (see Example 3.16) when we take a closer look at equivariant cycle (co)homology.

There is also a graded version of a G-equivariant pretheory

2.5. **Definition.** A pair of varieties (X, U) is called a G-pair if $X \in G\text{-}\mathrm{Sm}_k$ and $U \subseteq X$ is a G-equivariant open subvariety. Consider the category of G-pairs over k with G-equivarant morphisms of pairs.

A contravariant functor

$$(X, U) \longmapsto \mathtt{h}_G^*(X, U)$$

from the category of G-pairs to graded abelian groups is called a *graded G-equivariant pretheory* if it satisfies (H) homotopy invariance, and for any G-pair (X, U) there is a long exact localization sequence

$$\cdots \longrightarrow \mathtt{h}_G^i(X) \xrightarrow{\iota_G^*} \mathtt{h}_G^i(U) \xrightarrow{\partial} \mathtt{h}_G^{i+1}(X, U) \longrightarrow \cdots\,,$$

where $\iota\colon U \hookrightarrow X$ is the corresponding G-equivariant open embedding, and we have set $\mathtt{h}_G^*(Y) := \mathtt{h}_G^*(Y, Y)$. It is called a *graded essential G-equivariant pretheory* if given an inverse limit $(\mathcal{X}, \mathcal{U}) = \varprojlim_i (X_i, U_i)$ of G-equivariant open embeddings of pairs, there is the induced surjection

$$\varinjlim_i \mathtt{h}_G^*(X_i, U_i) \longrightarrow \mathtt{h}_G^*(\mathcal{X}, \mathcal{U}).$$

3. Equivariant cycle (co)homology

In this section we generalize the equivariant Chow groups of Edidin and Graham [8]. This theory has been considered for the cycle module Galois cohomology by Guillot [13]. We will use freely Rost's [26] theory of cycle modules for which we refer also to the book [9] of Elman, Karpenko and Merkurjev, as well as to the article of Déglise [5] where several important properties of the generalized "intersection" product in cycle cohomology are proven (defined in [26, Sect. 14]).

Since Rost's theory for algebraic spaces is not yet developed we have to restrict ourseives to quasi-projective schemes, i.e. to varieties. This assumption guarantees that certain quotients by groups actions which we consider here do exist.

3.1 (**Equivariant cycle homology).** To fix notations we recall briefly the definition of cycle homology. A *cycle module* over the field k is a (covariant) functor M_* from the category of field extensions of k to the category of graded abelian groups subject to several axioms, see [26, Sects. 1,2]. The prototype of such a functor is *Milnor K-theory* K^M_*, and by the very definition $\mathrm{M}_*(E) = \bigoplus_{i\in\mathbb{Z}} \mathrm{M}_i(E)$ is a graded $\mathrm{K}^M_*(E)$-module for all field extensions $E \supseteq k$.

Given a k-variety X (not necessarily smooth) and a cycle module M_* over k Rost [26] has defined a complex, the so called *cycle complex* (generalizing a construction of Kato [17] for Milnor K-theory):

$$\cdots \longrightarrow \bigoplus_{x\in X_{(2)}} \mathrm{M}_{n+2}(k(x)) \xrightarrow{d_2} \bigoplus_{x\in X_{(1)}} \mathrm{M}_{n+1}(k(x)) \xrightarrow{d_1} \bigoplus_{x\in X_{(0)}} \mathrm{M}_n(k(x)),$$

where $X_{(i)} \subseteq X$ denotes the set of points of dimension i in X. We denote this complex $\mathrm{C}_\bullet(X,\mathrm{M}_n)$ and consider it as a homological complex with the direct sum $\bigoplus_{x\in X_{(i)}} \mathrm{M}_{n+i}(k(x))$ in degree i.

The i-th cycle homology group $\mathrm{H}_i(X,\mathrm{M}_n)$ of M_n over X is then defined as $\mathrm{H}_i(\mathrm{C}_\bullet(X,\mathrm{M}_n))$. Note that there is a natural isomorphism $\mathrm{H}_i(X,\mathrm{K}^M_{-i}) \simeq \mathrm{CH}_i(X)$ for all $i \geq 0$, where we have set $\mathrm{K}^M_{-i} \equiv 0$ for $i < 0$.

To introduce the equivariant cycle homology we adapt the definition of equivariant Chow groups due to Edidin and Graham [8], see also Guillot [13] and Totaro [28].

Let G be an algebraic group over k of dimension s and X a G-variety. To define the i-th cycle homology group with coefficients in the cycle module M_* we chose a linear representation V of G, such that there is an open subscheme $U \hookrightarrow V$ with $\mathrm{codim}_V(V \setminus U) \geq c = \dim X$ on which G acts freely. By shrinking U we can moreover assume that $U \longrightarrow U/G$ is a principal bundle. The later assumption assures that $X \times^G U := (X \times_k U)/G$ exists in the category of k-varieties, see [8, Prop. 23] (recall that we assume that X is quasi-projective, see 1.1). We call the pair (U,V) an (X,G)-*admissible pair* for the G-variety X. Note that for a finite number of G-varieties there always exist a pair (U,V) which is admissible for all of them.

3.2. **Definition.** Let G be an algebraic group over k. Then the i-th *G-equivariant cycle homology group* with values in the cycle module M_* over k is defined as

$$\mathrm{H}^G_i(X,\mathrm{M}_*) := \mathrm{H}_{i+l-s}(X \times^G U, \mathrm{M}_{*-(l-s)}),$$

where $s = \dim G$ and (U,V) is a (X,G)-admissible pair with $\dim V = l$ and $\mathrm{codim}_V V \setminus U \geq \dim X$.

It remains to check that this definition does not depend on the choice of the (X,G)-admissible pair (U,V). Since $\mathrm{H}_j(Y,\mathrm{M}_*) = 0$ for any k-variety Y if $j > \dim Y$ this can be proven as for equivariant Chow groups using (the so called) Bogomolov's double filtration argument, see [8] or [28]. We recall briefly the details:

Let $U_1 \subset V_1$ be another (X, G)-admissible pair with $l_1 = \dim V_1$. Then there exists an open subvariety W of $V_1 \oplus V$, which contains $U_1 \oplus V$ and $V_1 \oplus U$ as open subvarieties, and such that G acts on W with principal bundle quotient W/G. Then the quotient $X \times^G W$ exists in the category of varieties.

We use the following fact from [8, Prop. 2 and Lem. 1], which is not hard to verify if $U \longrightarrow U/G$ is a trivial G-torsor and follows in general by descent from the "trivial" case.

3.3. **Lemma.** *Let G, U, V and X be as above and $f\colon X \to Y$ be a G-morphism. Denote $f \times^G \mathrm{id}_U \colon X \times^G U \to Y \times^G U$ the induced morphism.*

Consider the following properties $\mathbf{P}$ *of f: vector bundle, flat, smooth, proper, regular immersion, open or closed immersion. Then if f as property* $\mathbf{P}$ *implies that also $f \times^G \mathrm{id}_U$ has property* $\mathbf{P}$.

3.4**.** We also need the following fact. Let

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be a cartesian square of G-varieties and (U, V) a pair which is admissible for all varieties in the square. Then there is a natural and unique morphism

$$X' \times^G U \longrightarrow X \times^G U \times_{Y \times^G U} Y' \times^G U\,,$$

which is an isomorphism if $U \longrightarrow U/G$ is a trivial G-torsor. Hence by descent, see *e.g.* [10, Thm. 2.55 and Lem. 4.44], it is an isomorphism in general.

The morphism

$$\mathrm{id}_X \times^G p_U \colon X \times^G (V_1 \oplus U) \longrightarrow X \times^G U\,,$$

where $p_U \colon V_1 \oplus U \longrightarrow U$ is the projection, and the inclusion

$$\mathrm{id}_X \times^G \iota_U \colon X \times^G (V_1 \oplus U) \longrightarrow X \times^G W\,,$$

where $\iota_U \colon W \hookrightarrow V_1 \oplus U$, induce homomorphisms

$$\begin{tikzcd}[column sep=small] \mathrm{H}_{i+l_1+l-s}(X \times^G W, \mathrm{M}_{n-l_1-l+s}) \arrow[dr, "(\mathrm{id}_X \times^G \iota_U)^*"] & \\ & \mathrm{H}_{i+l_1+l-s}(X \times^G (V_1 \oplus U), \mathrm{M}_{n-l_1-l+s})\ . \\ \mathrm{H}_{i+l-s}(X \times^G U, \mathrm{M}_{n-l+s}) \arrow[ur, "(\mathrm{id}_X \times^G p_U)^*"'] & \end{tikzcd}$$

Both maps are isomorphisms. The first since the dimension of the closed complement of $X \times^G (V_1 \oplus U)$ in $X \times^G W$ is smaller than $i + l_1 + l - s$, and the second by homotopy invariance (recall that $\mathrm{id}_X \times^G p_U$ is a vector bundle by Lemma 3.3). Similarly we have an isomorphism

$$\mathrm{H}_{i+l_1+l-s}(X \times^G W, \mathrm{M}_{n-l_1-l+s}) \xrightarrow{\simeq} \mathrm{H}_{i+l_1-s}(X \times^G U_1, \mathrm{M}_{n-l_1+s})\,,$$

and hence both pairs define natural isomorphic groups.

3.5 (Pull-backs and push-forwards). Let now $f\colon X \longrightarrow Y$ be a morphism of G-varieties, and (U,V) and (U_1,V_1) two (X,G)- and (Y,G)-admissible pairs as above. We set $d = \dim X - \dim Y$. Then we have by the functorial properties of push-forward and pull-back maps in cycle homology (using Lemma 3.3 to see that the maps in question are defined) two commutative diagrams:

(i) If f is flat of constant relative dimension or Y is a smooth k-variety then

$$\begin{tikzcd} \mathrm{H}_{*+l-s}(Y\times^G U,\mathrm{M}_{n-l+s}) \arrow[r, "(f\times^G \mathrm{id}_U)^*"] \arrow[d, dash, "\simeq"] & \mathrm{H}_{*+d+l-s}(X\times^G U,\mathrm{M}_{n-l-d+s}) \arrow[d, dash, "\simeq"'] \\ \mathrm{H}_{*+l_1-s}(Y\times^G U_1,\mathrm{M}_{n-l+s}) \arrow[r, "(f\times^G \mathrm{id}_{U_1})^*"] & \mathrm{H}_{*+d+l_{1-s}}(X\times^G U_1,\mathrm{M}_{n-l_1-d+s}) \end{tikzcd}$$

is a commutative diagram whose column arrows are natural isomorphisms, and

(ii) If f is proper there is another commutative diagram

$$\begin{tikzcd} \mathrm{H}_{*+l-s}(X\times^G U,\mathrm{M}_{n-l+s}) \arrow[r, "(f\times^G \mathrm{id}_U)_*"] \arrow[d, dash, "\simeq"] & \mathrm{H}_{*+l-s}(X\times^G U,\mathrm{M}_{n-l+s}) \arrow[d, dash, "\simeq"'] \\ \mathrm{H}_{*+l_1-s}(X\times^G U_1,\mathrm{M}_{n-l_1+s}) \arrow[r, "(f\times^G \mathrm{id}_{U_1})_*"] & \mathrm{H}_{*+l_1-s}(X\times^G U_1,\mathrm{M}_{n-l_1+s}) \end{tikzcd} ,$$

whose column arrows are again natural isomorphisms.

Let now $f\colon X \to Y$ be a morphism of G-varieties and (U,V) a (G,X)- and (G,Y)-admissible pair.

(i) If either f is flat of constant relative dimension or Y is smooth, we define the pull-back morphism

$$f_G^*\colon\ \mathrm{H}_i^G(Y,\mathrm{M}_n) \longrightarrow \mathrm{H}_{i+d}^G(X,\mathrm{M}_{n+d})$$

as

$$(f\times^G \mathrm{id}_U)^*\colon\ \mathrm{H}_{i+l-s}(Y\times^G U,\mathrm{M}_{n-l+s}) \longrightarrow \mathrm{H}_{i+l-s+d}(X,\mathrm{M}_{n+l-s-d})\,,$$

where $d = \dim X - \dim Y$.

(ii) If f is proper, we define the push-forward morphism

$$f_{G*}\colon\ \mathrm{H}_i^G(X\,\mathrm{M}_n) \longrightarrow \mathrm{H}_i^G(Y,\mathrm{M}_n)$$

as

$$(f\times^G \mathrm{id}_U)_*\colon\ \mathrm{H}_i(X,\mathrm{M}_n) \longrightarrow \mathrm{H}_i(Y,\mathrm{M}_n)\,.$$

With these definitions G-equivariant cycle homology is a contravariant functor on the category of smooth and G-varieties and also covariant for proper morphisms. Moreover, if

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

is a cartesian square with f proper and g flat of constant relative dimension or all varieties in the diagram are smooth over k then we have

$$f'_{G*}\circ g_G^* \;=\; g_G'^*\circ f_{G*}\,.$$

3.6 (Axioms)**.** There is the localization sequence

$$\cdots \xrightarrow{\iota_{G*}} \mathrm{H}^G_i(X, \mathrm{M}_n) \xrightarrow{j^*_G} \mathrm{H}^G_i(X \setminus Z, \mathrm{M}_n) \xrightarrow{\partial} \mathrm{H}^G_{i-1}(Z, \mathrm{M}_n) \xrightarrow{\iota_{G*}} \cdots$$

for a closed G-embedding $\iota\colon Z \hookrightarrow X$ with open G-equivariant complement $j\colon X \setminus Z \hookrightarrow X$ which follows from the localization sequence in ordinary cycle homology. And if $\pi\colon E \longrightarrow X$ is a G-vector bundle of rank r, *i.e.* a G-linear bundle, then

$$\pi \times^G \mathrm{id}_U\colon E \times^G U \longrightarrow X \times^G U$$

is a vector bundle, see [8, Lem. 1], and therefore the pull-back

$$\pi^*\colon \mathrm{H}^G_i(X, \mathrm{M}_n) \longrightarrow \mathrm{H}^G_{i+r}(E, \mathrm{M}_{n-r})$$

is an isomorphism by homotopy invariance of cycle homology. Finally, $\mathrm{H}^G_i(X, \mathrm{M}_n)$ can be extended to G-Ess_k and satisfies (C) by [9, 52.F] or [26, p.320].

3.7. **Example.** If $f\colon X \to Y = X/G$ is a G-torsor then we have a natural isomorphism

$$\mathrm{H}_{i-s}(Y, \mathrm{M}_{n+s}) \simeq \mathrm{H}^G_i(X, \mathrm{M}_n)$$

where $s = \dim X - \dim Y = \dim G$, for all $i \in \mathbb{N}$ and $n \in \mathbb{Z}$. This can be seen as follows, *cf.* [13, Expl. 2.3.2]:

We choose a (X, G)-admissible pair (U, V). Let $l = \dim V$. Then the closed complement of $X \times^G U = (X \times_k U)/G$ in $(X \times_k V)/G$ has dimension less than $i + l - s$ and therefore we have by the localization sequence an isomorphism

$$\mathrm{H}_{i+l-s}((X \times_k V)/G, \mathrm{M}_{n-(l-s)}) \xrightarrow{j^*} \mathrm{H}_{i+l-s}(X \times^G U, \mathrm{M}_{n-(l-s)}) = \mathrm{H}^G_i(X, \mathrm{M}_n),$$

where $j\colon (X \times_k U)/G \hookrightarrow (X \times_k V)/G$ is the corresponding open immersion (note that the target of j exists in the category of varieties since by assumption $X \to X/G$ is a G-torsor). By [8, Lem. 1] we know that $(X \times_k V)/G \to X/G$ is a vector bundle of rank l and so by homotopy invariance we have

$$\mathrm{H}_{i+l-s}((X \times_k V)/G, \mathrm{M}_{n-(l-s)}) \simeq \mathrm{H}_{i-s}(X/G, \mathrm{M}_{n+s}).$$

3.8 (Restriction map)**.** If $G_1 \subseteq G$ is a closed subgroup and X a G-variety over k, we can choose a (X, G)- and (X, G_1)-admissible pair (U, V). Then we have a morphism of k-varieties $(X \times_k U)/G_1 \to (X \times_k U)/G$ which induces (via pull-back) a homomorphism of equivariant cycle homology groups

$$\mathrm{res}^G_{G_1}\colon \mathrm{H}^G_i(X, \mathrm{M}_n) \longrightarrow \mathrm{H}^{G_1}_i(X, \mathrm{M}_n)$$

for any cycle module M_* called *restriction homomorphism.* In particular if G_1 is the trivial group we have a (forgetful) morphism $\mathrm{H}^G_i(X, \mathrm{M}_n) \longrightarrow \mathrm{H}_i(X, \mathrm{M}_n)$ from G-equivariant cycle homology to ordinary cycle homology.

3.9 (The first Chern class)**.** Let $\pi\colon L \to X$ be a G-equivariant line bundle with zero section $\sigma\colon X \to L$. Then σ is a G-equivariant closed embedding and so induces a morphism $\sigma \times^G \mathrm{id}_U\colon X \times^G U \to L \times^G U$ which is the zero section of the line bundle (see Lemma 3.3)

$$\pi \times^G \mathrm{id}_U\colon L \times^G U \longrightarrow X \times^G U.$$

The *first Chern class* (or also called *Euler class*) of L is then defined as the operator

$$\mathrm{c}_1(L)^G := (\pi^*_G)^{-1} \circ \sigma_{G*}\colon \mathrm{H}^G_*(X, \mathrm{M}_*) \longrightarrow \mathrm{H}^G_{*-1}(X, \mathrm{M}_{*+1}).$$

This map commutes with push-forwards and pull-backs. More precisely, assume we have a cartesian square of G-equivariant morphisms, where L and L' are G-equivariant line bundles over X and X', respectively:

$$\begin{tikzcd} L' \arrow[r, "f'"] \arrow[d, "\pi'"'] & L \arrow[d, "\pi"] \\ X' \arrow[r, "f"] & X \end{tikzcd} .$$

Denote by σ and σ' the zero sections of L and L', respectively. These are also G-equivariant morphisms and so we get a cartesian square

$$\begin{tikzcd} L'_G \arrow[r, "f'_G"] \arrow[d, bend left, "\pi'_G"] & L_G \arrow[d, bend left, "\pi_G"] \\ X'_G \arrow[r, "f_G"] \arrow[u, bend left, "\sigma'_G"] & X_G \arrow[u, bend left, "\sigma_G"] \end{tikzcd} ,$$

where we have set $Y_G := Y \times^G U$ for any G-variety Y, $g_G := g \times^G \mathrm{id}_U$ for all G-equivariant morphisms $g\colon Y' \longrightarrow Y$, and assumed that the pair (U,V) is admissible for all varieties in question. A straightforward computation using this diagram, see [9, Prop. 53.3] for the analogous result in ordinary cycle homology, shows

$$\mathrm{c}_1^G(L') \circ f_G^* = f_G^* \circ \mathrm{c}_1^G(L) \quad \text{if } f \text{ is flat, and}$$
$$\mathrm{c}_1^G(L) \circ f_{G*} = f_{G*} \circ \mathrm{c}_1^G(L') \quad \text{if } f \text{ is proper.}$$

3.10. **Remark.** (i) The Chern class homomorphism

$$\mathrm{c}_1^G(L)\colon \mathrm{H}_i^G(X, \mathrm{K}^M_{-i}) = \mathrm{CH}_i^G(X) \longrightarrow \mathrm{CH}_{i-1}^G(X) = \mathrm{H}_{i-1}^G(X, \mathrm{K}^M_{-i+1})$$

coincides with the first Chern class defined in Edidin and Graham [8, Sect. 2.4].

(ii) As for ordinary cycle homology in [9] one can use the Euler class of a vector bundle to prove the projective bundle theorem and then use this to define the higher Chern classes in G-equivariant cycle homology. We leave this to the interested reader.

3.11 (An equivariant spectral sequence)**.** We provide now a version of Merkurjev's [21] equivariant K-theory spectral sequence for equivariant cycle homology.

Let T be a k-split torus of rank m, and $\chi\colon T \to \mathbb{G}_m$ a character. The algebraic group T acts via χ on the affine line $\mathbb{A}^1_k$ defining a T-equivariant line bundle $\mathbb{A}^1_k \to \mathrm{pt}$ which we denote by $L(\chi)$ (trivial action of T on the base point pt). If $p\colon X \to \mathrm{pt}$ is a T-scheme we denote the pull-back $p^*L(\chi)$ by $L_X(\chi)$. This is also a T-equivariant vector bundle.

Let X be a G-variety, where G is an algebraic group over k, and $T \subseteq G$ a split torus of rank m. Let $\chi_1, \dots, \chi_m$ be a basis of the character group $T^* = \mathrm{Hom}(T, \mathbb{G}_m)$, and let T act on the affine space $\mathbb{A}^m_k = \mathrm{Spec}\, k[x_1, \dots, x_m]$ by

$$t \cdot (a_1, \dots, a_m) \longmapsto (\chi_1(t) \cdot a_1, \dots, \chi_m(t) \cdot a_m)\,, \tag{1}$$

and on $\mathbb{A}^m_X = X \times_k \mathbb{A}^m_k$ diagonally. Let $Z_i \subset \mathbb{A}^m_k$ be the hyperplan defined by $x_i = 0$ for $i = 1, \dots, m$. Then $X \times_k Z_i$ are T-subvarieties of $\mathbb{A}^m_X$ and therefore, see Lemma 3.3, we have closed subschemes

$$(X \times_k Z_1) \times^G U\,,\ \dots\ ,\ (X \times_k Z_m) \times^G U$$

of $\mathbb{A}_X \times^G U$, where (U,V) is a $(\mathbb{A}^m_X, T)$-admissible pair. Since $U \to U/T$ is a T-torsor (by assumption) we have

$$\bigcap_{j\notin I}(X\times_k Z_j)\times^G U \;=\; (X\times_k Z_I)\times^G U$$

for all $I \in \{1,\dots,m\}$, where $Z_I = \bigcap\limits_{j\notin I} Z_j$. From Example A.7 we get then a convergent spectral sequence

$$\tilde{E}_1^{p,q} \;=\; \bigoplus_{|I|=p} \mathrm{H}^T_{-q-m}(X\times_k Z_I, \mathrm{M}_n) \;\Longrightarrow\; \mathrm{H}^T_{-p-q}(X\times_k T, \mathrm{M}_n) \tag{2}$$

for all cycle modules M_*.

By Example 3.7 we have $\mathrm{H}^T_{-p-q}(X\times_k T, \mathrm{M}_n) \simeq \mathrm{H}_{-p-q-m}(X, \mathrm{M}_{n+m})$ and since

$$Z_I \;=\; \mathbb{A}^{|I|}_k = \operatorname{Spec} k[x_i, i\in I] \hookrightarrow \mathbb{A}^m_k$$

is a T-equivariant vector bundle over the base the pull-back

$$\pi^*_{I\,T}\colon\; \mathrm{H}^T_{-q-m-|I|}(X, \mathrm{M}_{n+|I|}) \;\longrightarrow\; \mathrm{H}^T_{-q-m}(X\times_k Z_I, \mathrm{M}_n)$$

is an isomorphism, where $\pi_I\colon X\times_k Z_I \to X$ is the projection. Replacing q by $q+m$ the spectral sequence takes therefore the following form

$$E_1^{p,q} \;=\; \bigoplus_{|I|=p} \mathrm{H}^T_{-q-p}(X, \mathrm{M}_{n+p}) \;\Longrightarrow\; \mathrm{H}_{-p-q}(X, \mathrm{M}_{n+m})\,. \tag{3}$$

We compute the differential

$$d_1^{p,q}\colon \bigoplus_{|I|=p} \mathrm{H}^T_{-q-p}(X, \mathrm{M}_{n+p}) \;\longrightarrow\; \bigoplus_{|J|=p+1} \mathrm{H}^T_{-q-p-1}(X, \mathrm{M}_{n+p+1})\,.$$

If $J \not\supset I$ the the IJ-component of $d_1^{p,q}$ is zero. If $J\supset I$ let $J=\{i_1,\dots,i_{p+1}\}$ and $I = J\setminus\{i_r\}$ for some $1\le r\le p+1$. Then by Example A.7 the IJ-component of the differential $\tilde{d}_1^{p,q}$ of the spectral sequence (2) is equal $(-1)^{r-1}$-times the push-forward along the closed embedding (see Lemma 3.3)

$$\iota_{IJ}\times^T \mathrm{id}_U\colon (X\times_k Z_I)\times^T U \;\hookrightarrow\; (X\times_k Z_J)\times^T U\,,$$

where ι_{IJ} is the closed immersion $X\times_k Z_I \hookrightarrow X\times_k Z_J$. Using the above identification we have then a commutative diagram

$$\begin{array}{ccccc}
E_1^{p,q} \;= & \mathrm{H}^T_{-q-p}(X, \mathrm{M}_{n+p}) & \xrightarrow[\simeq]{\pi^*_{I\,T}} & \mathrm{H}^T_{-q}(X\times_k Z_I, \mathrm{M}_n) \\
 & \Big\downarrow{\scriptstyle d_1^{p,q}} & & \Big\downarrow{\scriptstyle (-1)^{r-1}\iota_{IJ,T\,*}} \\
E_1^{p+1,q} \;= & \mathrm{H}^T_{-q-p-1}(X, \mathrm{M}_{n+p+1}) & \xrightarrow[\simeq]{\pi^*_{J\,T}} & \mathrm{H}^T_{-q}(X\times_k Z_J, \mathrm{M}_n) & ,
\end{array}$$

and therefore

$$\begin{aligned}
(-1)^{r-1}d_1^{p,q}(x) &= (\pi^*_{J\,T})^{-1}\big[\iota_{IJ,T\,*}\big(\pi^*_{I\,T}(x)\big)\big] \\
&= (\pi^*_{J\,T})^{-1}\Big[\iota_{IJ,T\,*}\Big(\mathbb{I}^T_{X\times Z_I}\cap\big((\pi_J\times^T \mathrm{id}_U)\circ(\iota_{IJ}\times^T\mathrm{id}_U)^*(x)\big)\Big)\Big] \\
&= \big[(\pi^*_{J\,T})^{-1}(\iota_{IJ,T\,*}(\mathbb{I}^T_{X\times Z_I}))\big]\cap x \\
&= \mathrm{c}_1(L_X(\chi_r))\cap x
\end{aligned}$$

for all $x\in \mathrm{H}^T_{-q-p}(X, \mathrm{M}_{n+p})$. The last equation since ι_{IJ} is the zero section of the pull-back of $L_X(\chi)$ along the projection $X\times_k Z_I \to X$.

3.12 (**Equivariant cycle cohomology).** From now on all varieties are assumed to be smooth. Given a smooth equidimensional variety X we define its i-th *cycle cohomology group* as

$$\mathrm{H}^i(X,\mathrm{M}_n) \;:=\; \mathrm{H}_{\dim X - i}(X,\mathrm{M}_{n-\dim X})\,.$$

A pairing

$$\mathrm{K}^M_i(E)\,\times\,\mathrm{M}_j(E)\;\longrightarrow\;\mathrm{M}_{i+j}(E)$$

induces the so called *intersection product* of cycle cohomology groups

$$\mathrm{H}^i(X,\mathrm{K}^M_m)\,\times\,\mathrm{H}^j(X,\mathrm{M}_n)\;\;\longrightarrow\;\mathrm{H}^{i+j}(X,\mathrm{M}_{m+n})\,,\;\;(\alpha,x)\;\longmapsto\;\alpha\cap x\,,$$

see [26, Sect. 14]. This generalizes the usual intersection product of Chow groups as defined for instance in the book [11] of Fulton.

3.13. **Definition.** It follows by descent (since $U \longrightarrow U/G$ is a principal bundle) that $X\times^G U$ is an equidimensional smooth variety, too. Hence we can define

$$\mathrm{H}^i_G(X,\mathrm{M}_n) \;:=\mathrm{H}^G_{\dim X-i}(X,\mathrm{M}_{n-\dim X})\;=\;\mathrm{H}^i(X\times^G U,\mathrm{M}_n)$$

and call it the i-th G-*equivariant cycle cohomology group* of X with values in M_*.

The pairing of $\mathrm{H}^*(X\times^G U,\mathrm{K}^M_*)$ with $\mathrm{H}^*(X\times^G U,\mathrm{M}_*)$ induces a pairing

$$\mathrm{H}^i_G(X,\mathrm{K}^M_m)\,\times\,\mathrm{H}^j_G(X,\mathrm{M}_n)\;\longrightarrow\;\mathrm{H}^{i+j}_G(X,\mathrm{M}_{m+n})\,,\;\;(\alpha,x)\;\longmapsto\;\alpha\cap x\,.$$

Since the pull-back of ordinary cycle cohomology groups respects the product, we can use Bogomolov's double filtration argument to check that this definition does not depend (up to natural isomorphism) on the choice of an (X,G)-admissible pair.

It follows from the properties of the ordinary "intersection" product in cycle cohomology that $\mathrm{H}^*_G(X,\mathrm{K}^M_*)$ becomes a skew-commutative ring with this product, *i.e.*

$$\alpha\cap\beta\;=\;(-1)^{(i+m)\cdot(j+n)}\cdot\beta\cap\alpha$$

for all $\alpha\in\mathrm{H}^i_G(X,\mathrm{K}^M_m)$ and $\beta\in\mathrm{H}^j_G(X,\mathrm{K}^M_n)$, *cf.* [9, Prop. 56.4]. We denote the neutral element of this multiplication by $\mathbb{I}^G_X$. This is the rational equivalence class of $X\times^G U$ in $\mathrm{H}^0_G(X,\mathrm{K}^M_0)=\mathrm{H}^0(X\times^G U,\mathrm{K}^M_0)$.

3.14 (Projection formulas)**.** From the properties of the ordinary intersection product shown in [5, Prop. 5.9] we obtain the following:

Let $f\colon X\longrightarrow Y$ be a morphism of equidimensional smooth G-varieties. Then

(i) $f^*_G(\beta\cup y)\;=\;f^*_G(\beta)\cap f^*_G(y)$, and

(ii) if f is proper the equations (projection formulas)

$$f_{G*}(\alpha\cap f^*_G(y))\;=\;f_{G*}(\alpha)\cap y\quad\text{and}\quad f_{G*}(f^*_G(\beta)\cap x)\;=\;\beta\cap f_{G*}(x)$$

for all $\alpha\in\mathrm{H}^i_G(X,\mathrm{K}^M_m)$, $\beta\in\mathrm{H}^i_G(Y,\mathrm{K}^M_m)$, $x\in\mathrm{H}^j_G(X,\mathrm{M}_n)$, and $y\in\mathrm{H}^j_G(Y,\mathrm{M}_n)$.

We denote by $\mathrm{c}_1(L)$ also the element $\mathrm{c}_1(L)(\mathbb{I}^G_X)\in\mathrm{CH}^1_G(X)$. With this notation we have by the projection formula

$$\mathrm{c}_1(L)(x)\;=\;\mathrm{c}_1(L)\cap x$$

for all $x\in\mathrm{H}^i_G(X,\mathrm{M}_n)$ and all $i\in\mathbb{N}$ and $n\in\mathbb{Z}$ if the variety X is smooth k.

This section can be summarized by the following (see 3.13 and 3.6)

3.15. **Theorem.** *The functor* $X\mapsto\mathrm{H}^*_G(X,\mathrm{M}_*)$ *provides an example of a graded essential* G*-equivariant pretheory.*

3.16. **Example** (Equivariant Chow groups)**.** We have

$$\mathrm{H}_i^G(X, \mathrm{K}_{-i}^M) \;=\; \mathrm{CH}_i^G(X)\,,$$

where CH_*^G denotes the G-equivariant Chow-theory of Edidin and Graham [8].

Since $\mathrm{H}_j(Y, \mathrm{K}_{-i}^M) = 0$ for $j \leq i-1$ we know by the localization sequence that for a G-equivariant open embedding $\iota\colon W \hookrightarrow X$ the pull-back

$$\iota_G^*\colon\; \mathrm{CH}_i^G(X) \;\longrightarrow\; \mathrm{CH}_i^G(W)$$

is surjective. Identifying $\mathrm{CH}^i_G(X)$ with $\mathrm{CH}^G_{\dim X - i}(X)$ we obtain that the functor

$$\mathtt{h}_G\colon X \;\longmapsto\; \mathrm{CH}^*_G(X) \;=\; \bigoplus_{i\in\mathbb{Z}} \mathrm{CH}^i_G(X)$$

provides an example of an essential G-equivariant pretheory.

4. Torsors and equivariant maps

In the present section we generalize the result of Karpenko and Merkurjev [16, Thm. 6.4] to an arbitrary equivariant pretheory. Our arguments follow closely the exposition of [16, §6].

Let $S = \mathrm{GL}(V)$ be the group of automorphisms of a finite dimensional k-vector space V. Let H be an algebraic subgroup of S. Consider S as a (left) H-variety.

Let $\mathtt{h}_H$ be a H-equivariant pretheory over k. Following the proof of [16, Prop. 6.2] we embed S into the affine space $\mathrm{End}_k(V)$ as a S-equivariant (and, hence, H-equivariant) open subset.

Let $\phi\colon S \to \mathrm{pt}$ denote the structure map. The induced pull-back ϕ_H^* factors as the composite of pull-backs

$$\mathtt{h}_H(\mathrm{pt}) \xrightarrow{\simeq} \mathtt{h}_H(\mathrm{End}(V)) \twoheadrightarrow \mathtt{h}_H(S),$$

where the first map is an isomorphism by homotopy invariance and the second map is surjective by the localization property. This proves that

4.1. **Lemma.** *The induced pull-back ϕ_H^* is surjective.*

Let $\mu_s\colon S \to S$ denote the right multiplication by $s \in S(k)$. Since $\phi \circ \mu_s = \varphi$ as morphisms over k and μ_s is H-equivariant, we have $(\mu_s)_H^* \circ \phi_H^* = \phi_H^*$. Since ϕ_H^* is surjective by Lemma 4.1, this proves that

4.2. **Lemma.** *The induced pull-back $(\mu_s)_H^*\colon \mathtt{h}_H(S) \to \mathtt{h}_H(S)$ is the identity.*

Let G be an algebraic subgroup of S such that $H \subseteq G \subseteq S$ so that G is considered as a (left) H-variety. Let E be a (left) G-variety over k and let $\eta_E\colon \operatorname{Spec} K \to E$ denote its generic point, where $K = k(E)$.

Consider the G-equivariant (and, hence, H-equivariant) map

$$\psi_E\colon G_K \;=\; G \times_{\operatorname{Spec} k} \operatorname{Spec} K \xrightarrow{(\mathrm{id}, \eta_E)} G \times_{\operatorname{Spec} k} E \;\longrightarrow\; E$$

which takes the identity of G to the generic point of E. Suppose that there is a G-equivariant map $\rho\colon E \to S$ over k. Then there is a commutative diagram of

H-equivariant maps

$$\begin{tikzcd} G_K \arrow[r, "\psi_E"] \arrow[d, "i"] & E \arrow[r, "\rho"] & S \\ S_K \arrow[rr, "\mu_{\rho(\eta_E)}"] & & S_K \arrow[u, "p"] \end{tikzcd}$$

where the map i is the embedding, p is the projection $S_K = S \times_{\operatorname{Spec} k} \operatorname{Spec} K \to S$ to the first factor and the bottom horizontal map is the multiplication by $\rho(\eta_E)$.

By the diagram the pull-back of the composite $(\psi_E)^*_H \circ \rho^*_H = (\rho \circ \psi_E)^*_H$ coincides with the pull back $(p \circ \mu_{\rho(\eta_E)} \circ i)^*_H$. Here the map $(\psi_E)^*_H \colon \mathtt{h}_H(E) \to \bar{\mathtt{h}}_H(G_K)$ is the canonical map to the colimit. By Lemma 4.2 the latter coincides with the pull-back $i^*_H \circ p^*_H$, hence, proving the following

4.3. **Lemma.** *Let E be a G-variety together with a G-equivariant map $\rho\colon E \to S$. Then we have*

$$(\psi_E)^*_H \circ \rho^*_H = i^*_H \circ p^*_H \colon \mathtt{h}_H(S) \to \bar{\mathtt{h}}_H(G_K).$$

We are now in position to prove the main result of this section

4.4. **Theorem.** *Let $H \subset G$ be algebraic groups and let $\mathtt{h}_H(-)$ be a H-equivariant pretheory. Then for any G-torsor E with $K = k(E)$ we have*

$$\operatorname{Im}(\varphi^*_H) \subseteq \operatorname{Im}((\psi_E)^*_H) \text{ in } \bar{\mathtt{h}}_H(G_K),$$

where $\varphi\colon G_K \to \mathrm{pt}$ is the structure map.

Proof. By Lemma 4.1 we have

$$\operatorname{Im}(\varphi^*_H) = \operatorname{Im}(i^*_H \circ p^*_H \circ \phi^*_H) = \operatorname{Im}(i^*_H \circ p^*_H).$$

Theorem then follows from Lemma 4.3 and the fact that there exists a finite dimensional k-vector space V and a G-equivariant map $E \longrightarrow S = \mathrm{GL}(V)$, see [16, Prop. 6.4]. □

4.5. **Definition.** Let $H \subset G$ be algebraic groups over k and $\mathtt{h}_H(-)$ be an equivariant pretheory with values in the category of commutative rings. To each G-torsor E we associate a commutative ring

$$E \mapsto \bar{\mathtt{h}}_H(G_K) \otimes_{\mathtt{h}_H(E)} \bar{\mathtt{h}}_H(H_K),$$

where $\bar{\mathtt{h}}_H(G_K)$ is the $\mathtt{h}_H(E)$-module via $(\psi_E)^*_H$ and $\bar{\mathtt{h}}_H(H_K)$ is the $\mathtt{h}_H(E)$-module via the composite $\mathtt{h}_H(E) \overset{(\psi_E)^*_H}{\to} \bar{\mathtt{h}}_H(G_K) \to \bar{\mathtt{h}}_H(H_K)$ with the last map induced by the embedding $H \subset G$.

This ring will be denoted by $\widehat{\mathtt{h}}_H(E)$ and will play the central role in the last section of this paper. In particular, it will be shown that for known examples of equivariant pretheories it is always a quotient of the cohomology ring $\mathtt{h}(G)$ of G.

4.6. **Corollary.** *Let $H \subset G$ be algebraic groups over k and let $\mathtt{h}_H(-)$ be an essential H-equivariant pretheory. Then there exists a field extension l/k and a G-torsor E over l with $L = l(E)$ such that*

$$\operatorname{Im}(\varphi^*_{H_l}) = \operatorname{Im}((\psi_E)^*_{H_l}) \text{ in } \bar{\mathtt{h}}_{H_l}(G_L)$$

Proof. We fix an embedding $G \longrightarrow S = \mathrm{GL}(V)$ for some finite dimensional k-vector space V. The quotient $S \longrightarrow G\backslash S$ (for the right action of G on S) is a (left) G-torsor. Let l be its function field and consider the cartesian square

$$\begin{tikzcd} E \arrow[r, "\rho"] \arrow[d] & S \arrow[d] \\ \operatorname{Spec} l \arrow[r] & G\backslash S\,. \end{tikzcd}$$

Since $S \longrightarrow G\backslash S$ is a G-torsor the map $E \longrightarrow \operatorname{Spec} l$ is a G-torsor, too. The l-scheme E is a localization of S and, therefore, by (C) and localization property (L) the pull-back $\rho_H^*\colon \mathtt{h}_H(S) \to \bar{\mathtt{h}}_H(E) \to \mathtt{h}_H(E)$ is surjective. This implies that the pull-back $\rho_{H_l}^*\colon \mathtt{h}_{H_l}(S_l) \to \mathtt{h}_{H_l}(E)$ is surjective. It remains to apply the proof of Theorem 4.4 over l and to observe that $\mathrm{Im}(\varphi_{H_l}^*) = \mathrm{Im}((\psi_E)_{H_l}^* \circ \rho_{H_l}^*) = \mathrm{Im}((\psi_E)_{H_l}^*)$. □

4.7. **Definition.** A G-torsor E over k which satisfies the equality of Corollary 4.6 will be called a *generic torsor* with respect to the pretheory $\mathtt{h}_H(-)$. Note that in the proof we provided an example of a G-torsor which is generic over some field extension for all equivariant pretheories.

4.8. **Example.** Observe that for a generic G-torsor E over k we have

$$\widehat{\mathtt{h}}_H(E) = \bar{\mathtt{h}}_H(G_K) \otimes_{\mathtt{h}_H(\mathrm{pt})} \bar{\mathtt{h}}_H(H_K),$$

where $\bar{\mathtt{h}}_H(G_K)$ is an $\mathtt{h}_H(\mathrm{pt})$-module via φ_H^*, and for the trivial G-torsor G we have

$$\widehat{\mathtt{h}}_H(E) = \bar{\mathtt{h}}_H(G_K) \otimes_{\mathtt{h}_H(G)} \bar{\mathtt{h}}_H(H_K).$$

5. Equivariant oriented cohomology

In the present section we apply Theorem 4.4 to the case of a B-equivariant oriented cohomology, where B is a Borel subgroup of a split semisimple linear algebraic group.

Let G be a split semisimple linear algebraic group of rank n over a field k and let T be a split maximal torus of G. Similarly to 3.11 consider the action (1) of T on the affine space $\mathbb{A}_k^n$ with weights $\chi_1, \ldots, \chi_n$ together with an action of T on G by left multiplication. Then T embeds into $\mathbb{A}_k^n = \operatorname{Spec} k[x_1, \ldots, x_n]$ as the complement of the coordinates hyperplanes Z_i, $i = 1, \ldots, n$. Let $V = \mathbb{A}_k^n \times^T G$ be the associated vector bundle over G/T (see [3, p.22]). By definition $V = L_{G/T}(\chi_1) \oplus \ldots \oplus L_{G/T}(\chi_n)$ and $G = T \times^T G$ embeds into V as the complement of the union of zero-sections

$$V_j \;=\; \bigoplus_{j\neq i} L_{G/T}(\chi_i) = Z_j \times^T G\,, \quad i = 1, \ldots, n\,.$$

Note that $e_j\colon V_j \hookrightarrow V$ is a smooth subvariety for every j.

Let now $\mathtt{h}(-)$ be an oriented cohomology theory in the sense of [19], i.e. a contravariant functor from the category of smooth varieties over k to the category of graded commutative rings satisfying certain axioms. In particular, if X is a k-variety with an open subvariety $\iota\colon U \hookrightarrow X$ there is an exacts sequence

$$\mathtt{h}(Z) \xrightarrow{j_*} \mathtt{h}(X) \xrightarrow{\iota^*} \mathtt{h}(U) \longrightarrow 0$$

where $j\colon Z = X \setminus U \hookrightarrow X$ is the closed complement of U, and there is also a first Chern class which we denote by $c_1^{\mathtt{h}}$.

Having such a theory $\mathtt{h}(-)$ we get from this localization sequence (by induction) an exact sequence

$$\bigoplus_{j=1}^{n} \mathtt{h}(V_j) \xrightarrow{\oplus_j (e_j)_*} \mathtt{h}(V) \to \mathtt{h}(G) \to 0$$

By the properties of the first Chern class we have

$$(e_j)_*(1_{V_j}) = c_1^{\mathtt{h}}(L_V(\chi_j))$$

which implies that the image of $\oplus_j (e_j)_*$ is an ideal generated by the first Chern classes $c_1^{\mathtt{h}}(L_V(\chi_j))$, $j = 1, \ldots, n$.

Let B be a Borel subgroup of G containing T and let G/B be the variety of Borel subgroups. By [1, Thm.10.6 of Ch.III] the composite of projections $V \to G/T \to G/B$ is a chain of affine bundles. Therefore, by the homotopy invariance there is an isomorphism $\mathtt{h}(G/B) \xrightarrow{\simeq} \mathtt{h}(V)$ compatible with the Chern classes and we obtain the following

5.1. **Proposition.** *There is an isomorphism of rings*

$$\mathtt{h}(G) \simeq \mathtt{h}(G/B)/\big(c_1^{\mathtt{h}}(L_{G/B}(\chi_1)), \ldots, c_1^{\mathtt{h}}(L_{G/B}(\chi_n))\big)\,,$$

where $\chi_1, \ldots, \chi_n$ is a basis of the character group T^*.

Let $\mathtt{h}_B(-)$ be an B-equivariant pretheory to the category of commutative rings such that

(i) $\mathtt{h}_B(E) = \mathtt{h}(E/B)$ for every G-torsor E, where $\mathtt{h}(-)$ is an oriented cohomology in the sense of [19].

(ii) $\bar{\mathtt{h}}_B(B_K) = \mathtt{h}(\mathrm{pt})$ and $\bar{\mathtt{h}}_B(G_K) \simeq \mathtt{h}(G/B)$.

Then the ring $\hat{\mathtt{h}}_B(E) = \bar{\mathtt{h}}_B(G_K) \otimes_{\mathtt{h}_B(E)} \bar{\mathtt{h}}_B(B_K)$ introduced in Definition 4.5 can be identified with a quotient of $\bar{\mathtt{h}}_B(G_K) \simeq \mathtt{h}(G/B)$ modulo the ideal generated by non-constant elements from the image of the restriction $(\psi_E)_B^*\colon \mathtt{h}(E/B) \to \mathtt{h}(G/B)$.

Consider now the map $\varphi_B^*\colon \mathtt{h}_B(\mathrm{pt}) \longrightarrow \bar{\mathtt{h}}_B(G_K) \simeq \mathtt{h}(G/B)$. By Theorem 4.4 $\mathrm{Im}(\varphi_B^*) \subseteq \mathrm{Im}((\psi_E)_B^*)$, hence, $\hat{\mathtt{h}}_B(E)$ can be identified with a quotient of the factor ring $\mathtt{h}(G/B)/I$, where I denotes the ideal generated by elements from the image of φ_B^* which are in the kernel of the augmentation.

Then by Proposition 5.1 and Corollary 4.6 we obtain the following

5.2. **Corollary.** *Assume that the image of φ_B^* is generated by the Chern classes $c_1^{\mathtt{h}}(L_{G/B}(\chi_i))$ of line bundles associated to the characters $\chi_i \in T^*$ $(i = 1 \ldots n)$. Then $\hat{\mathtt{h}}_B(E)$ is a quotient of $\mathtt{h}(G/B)/I \simeq \mathtt{h}(G)$. Moreover, if $\mathtt{h}_B(-)$ is essential and E is generic, then*

$$\hat{\mathtt{h}}_B(E) \simeq \mathtt{h}(G).$$

5.3. **Example** (Chow groups and the J-invariant)**.** Consider the equivariant Chow groups $\mathtt{h}_B(-) = \mathrm{CH}^B(-)$. Let E be a G-torsor. The ring $\mathtt{h}_B(\mathrm{pt})$ can be identified with the symmetric algebra $S(T^*)$ and the map

$$\varphi_B^*\colon S(T^*) = \mathtt{h}_B(\mathrm{pt}) \longrightarrow \bar{\mathtt{h}}_B(G_K) = \mathrm{CH}(G/B)$$

coincides with the characteristic map for Chow groups studied in [6]. So its image is generated by the first Chern classes $c_1(L_{G/B}(\chi_i))$ of the respective line bundles.

The map $(\psi_E)^*_B$ coincides with the restriction map

$$\mathrm{res}\colon \mathrm{CH}(E/B) \longrightarrow \mathrm{CH}(G/B),$$

where E/B is the twisted form of G/B by means of E and the map

$$S(T^*) = \mathtt{h}_B(\mathrm{pt}) \longrightarrow \mathtt{h}_B(B_K) = \mathrm{CH}(\mathrm{pt}) = \mathbb{Z}$$

is the augmentation map. If E is generic, then we have

$$\widehat{\mathtt{h}}_B(E) \simeq \mathrm{CH}(G/B) \otimes_{S(T^*)} \mathbb{Z} \simeq \mathrm{CH}(G).$$

where the last isomorphism follows by Corollary 5.2. For an arbitrary G-torsor E the ring

$$\widehat{\mathtt{h}}_B(E) = \mathrm{CH}(G/B) \otimes_{\mathrm{Im(res)}} \mathbb{Z}$$

is a quotient ring of $\mathrm{CH}(G/B)$ modulo the ideal generated by non-constant elements from the image of the restriction $\mathrm{CH}(E/B) \to \mathrm{CH}(G/B)$.

Observe that the characteristic map φ^*_B is not surjective in general. However, its image is a subgroup of finite index in $\mathrm{CH}(G/B)$ measured by the torsion index τ of G. This implies that for a G-torsor E we have

$$\widehat{\mathtt{h}}_B(E) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}.$$

If $p \mid \tau$, then there is an isomorphism

$$\widehat{\mathtt{h}}_B(E) \otimes_{\mathbb{Z}} \mathbb{Z}/p \simeq \frac{\mathbb{Z}/p\,[x_1, \ldots, x_r]}{(x_1^{p^{j_1}}, \ldots, x_r^{p^{j_r}})},$$

where $(j_1, \ldots, j_r)$ is the *J-invariant* of G twisted by E as defined in [25]. Observe that $j_i \leq k_i$, $i = 1 \ldots r$, where k_i are defined via the *p-exceptional degrees* introduced by Kac [15], and for a generic torsor E we have equalities $j_i = k_i$ for each i.

5.4. **Example** (Grothendieck's K_0 and indexes of the Tits algebras)**.** Consider the equivariant K_0-groups $\mathtt{h}_B(-) = \mathrm{K}_0(B, -)$. Let E be a G-torsor. The ring $\mathtt{h}_B(\mathrm{pt})$ can be identified with the integral group ring $\mathbb{Z}[T^*]$ and with the representation ring $\mathrm{Rep}\, T$ of T, i.e.

$$\mathtt{h}_B(\mathrm{pt}) = \mathbb{Z}[T^*] = \mathrm{Rep}\, T.$$

The map

$$\varphi^*_B\colon \mathbb{Z}[T^*] = \mathtt{h}_B(\mathrm{pt}) \longrightarrow \bar{\mathtt{h}}_B(G_K) \simeq \mathrm{K}_0(G/B)$$

coincides with the characteristic map $\mathfrak{c}$ for K_0 studied in [6] and again its image is generated by the first Chern classes.

As before the map $(\psi_E)^*_B$ coincides with the restriction map

$$\mathrm{res}\colon \mathrm{K}_0(E/B) \longrightarrow \mathrm{K}_0(G/B),$$

and applying 4.4 we obtain the following K_0-analogue of the Karpenko-Merkurjev result:

5.5. **Corollary.** *Let E be a G-torsor over k and let E/B be a twisted form of G/B by E. Then*

(i) $\mathfrak{c}(\mathbb{Z}[T^*]) \subseteq \mathrm{res}(\mathrm{K}_0(E/B))$*;*

(ii) *there exists a G-torsor E over some field extension of k such that*

$$\mathfrak{c}(\mathbb{Z}[T^*]) = \mathrm{res}(\mathrm{K}_0(E/B)).$$

According to a result of Panin [24] the image of the restriction map is given by the sublattice

$$\{i_{w,E} \cdot g_w\}_{w \in W},$$

where W is the Weyl group of G, $\{g_w\}_{w\in W}$ is the Steinberg basis of $\mathrm{K}_0(G/B)$ and $\{i_{w,E}\}$ are indexes of the respective Tits algebras.

Corollary 5.5 implies that there exists a maximal set of indexes $\{m_w\}_{w\in W}$ such that

- $i_{w,E} \le m_w$ for every $w \in W$ and every torsor E;
- there exists E such that $i_{w,E} = m_w$ for every $w \in W$;
- the image of the characteristic map $\varphi_B^*(\mathbb{Z}[T^*])$ coincides with the sublattice $\{m_w \cdot g_w\}_{w\in W}$, hence, providing a way to compute m_w.

The indexes m_w are called the *maximal Tits indexes*. They have been extensively studied by Merkurjev [20] and Merkurjev, Panin and Wadsworth [22]. They are closely related to the dimensions of irreducible representations of G. Comparing with the case of Chow groups one observes that the maximal Tits indexes in K_0 play the same role as the p-exceptional degrees in Chow groups.

Since the map $\mathbb{Z}[T^*] = \mathtt{h}_B(\mathrm{pt}) \longrightarrow \bar{\mathtt{h}}_B(B_K) = \mathrm{K}_0(\mathrm{pt}) = \mathbb{Z}$ is the augmentation map, for a generic torsor E we have

$$\widehat{\mathtt{h}}_B(E) = \mathrm{K}_0(G/B) \otimes_{\mathbb{Z}[T^*]} \mathbb{Z} \simeq \mathrm{K}_0(G),$$

where the last isomorphism follows by Corollary 5.2. Hence, for an arbitrary G-torsor E

$$\widehat{\mathtt{h}}_B(E) = \mathrm{K}_0(G/B) \otimes_{\mathrm{Im(res)}} \mathbb{Z}$$

is the quotient ring of $\mathrm{K}_0(G/B)$ modulo the ideal generated by elements from the image of the restriction $\mathrm{K}_0(E/B) \to \mathrm{K}_0(G/B)$ which are in the kernel of the augmentation.

5.6. **Example** (Equivariant algebraic cobordism)**.** Consider the equivariant algebraic cobordism $\mathtt{h}_B(-) = \Omega^B(-)$. Let E be a G-torsor. The completion $\mathtt{h}_B(\mathrm{pt})^\wedge$ of $\mathtt{h}_B(\mathrm{pt})$ at the augmentation ideal, (the kernel of $\mathtt{h}_B(pt) \to \mathtt{h}_B(B)$) can be identified with the formal group ring $\mathbb{L}[[T^*]]_U$ introduced in [2, Def. 2.4 and 2.7], where $\mathbb{L}$ is the Lazard ring and U denotes the universal formal group law.

The map

$$\varphi_B^*\colon \mathbb{L}[[T^*]]_U = \mathtt{h}_B(\mathrm{pt})^\wedge \longrightarrow \bar{\mathtt{h}}_B(G_K) = \Omega(G/B)$$

coincides with the characteristic map of [2, Def. 10.2] and its image is generated by the first Chern classes.

The map $(\psi_E)_B^*$ coincides with the restriction map

$$\mathrm{res}\colon \Omega(E/B) \longrightarrow \Omega(G/B),$$

where E/B is the twisted form of G/B by means of E and the map $\mathbb{L}[[T^*]]_U = \mathtt{h}_B(\mathrm{pt})^\wedge \longrightarrow \mathtt{h}_B(B_K) = \Omega(\mathrm{pt}) = \mathbb{L}$ is the augmentation map. By Corollary 5.2 for an arbitrary G-torsor E we have

$$\widehat{\mathtt{h}}_B(E) = \Omega(G/B) \otimes_{\mathrm{Im(res)}} \mathbb{L}.$$

is a quotient of the ring $\Omega(G/B)$ modulo the image of the restriction $\Omega(E/B) \to \Omega(G/B)$ from the kernel of the augmentation.

Appendix A. A spectral sequence

Let X be a k-scheme, $Z_1,\dots,Z_m$ closed subschemes of X (with reduced structure), and M_* a cycle module over k. We construct in this section a spectral sequence which converges to the cycle homology of M_* over the open complement of $\bigcup\limits_{i=1}^{m} Z_i$ in X generalizing the long exact localization sequence (the case $m=1$). A reader familiar with Levine [18, Sect. 1] will notice an analogy with Levine's construction of a similar spectral sequence for Quillen K-theory which should have an explanation in the theory of model categories.

Let m be a positive integer and $\underline{m}$ the set of subsets of $\{1,2,\dots,m\}\subset\mathbb{N}$. We set further $\underline{0}=\emptyset$. We consider $\underline{m}$ as a category with $\mathrm{Mor}_{\underline{m}}(I,J)=\{\emptyset\}$ (one element set) if $I\subseteq J$ and $\emptyset$ otherwise.

A.1. **Definition.** A *m-cube of complexes* is a functor

$$K_*\colon \underline{m} \longrightarrow \mathrm{K}^b(\mathfrak{Ab}) \qquad I \longmapsto K_I$$

from $\underline{m}$ into the category of bounded (homological) complexes of abelian groups. If $I\subseteq J$ we denote r^K_{IJ} the induced morphism $K_I\longrightarrow K_J$, and set $r^K_{IJ}=0$ if $I\not\subseteq J$. We observe that $r^K_{JL}\cdot r^K_{IJ}=r^K_{IL}$ if $I\subseteq J\subseteq K$ since $I\mapsto K_I$ is a functor.

A *morphism* of m-cubes is a natural transformation.

For brevity of notation we define if $l=|J|=|I|+1$ a morphism ϵ^K_{IJ} as follows: If $I\not\subset J$ we set $\epsilon^K_{IJ}=0$ and if $J=\{i_1<i_2<\ldots<i_l\}$ and $I=J\setminus\{i_d\}$ for some $1\le d\le l$ we set $\epsilon^K_{IJ}=(-1)^{d-1}\cdot r^K_{IJ}$. The signs are chosen, such that the matrix product

$$\big(\epsilon_{JL}\big)_{|J|=p+1,|L|=p+2}\cdot\big(\epsilon_{IJ}\big)_{|I|=p,|J|=p+1} \tag{4}$$

is zero.

With a m-cube of complexes K_* we can associate two $(m-1)$-cubes of complexes (as long as $m\ge 2$). First we have the $(m-1)$-cubes of complexes $\tilde{K}_*$ which is the restriction of K_* to $\underline{m-1}$, and second we have K'_*. The latter is defined as the composition of $\underline{m-1}\longrightarrow\underline{m}$, $J\longmapsto J\cup\{m\}$, with K_*, *i.e.*

$$K'_*\colon \underline{m-1}\longrightarrow \mathrm{D}^b(\mathfrak{Ab}) \qquad J\longmapsto K_{J\cup\{m\}}\,.$$

A.2. **Example.** Let X be a k-scheme with closed subschemes $Z_1,\dots,Z_m$, and M_* a cycle module. We set $Z_I:=\bigcap\limits_{j\notin I} Z_j$ for all I in $\underline{m}$. We have than an m-cube of complexes

$$K_*\colon\ \mathrm{C}_\bullet(X,Z_1,\dots,Z_m,\mathrm{M}_n,*)\colon I\longmapsto \mathrm{C}_\bullet(Z_I,\mathrm{M}_n)\,,$$

where for $I\subseteq J$ the morphism r^{C}_{IJ} is the push-forward along the inclusion of closed subschemes $\bigcap\limits_{j\notin I} Z_j\hookrightarrow\bigcap\limits_{j\notin J} Z_j$. Then we have

$$K'_* = \mathrm{C}_\bullet(X,Z_1,\dots,Z_{m-1},\mathrm{M}_n,*)$$

and

$$\tilde{K}_* = \mathrm{C}_\bullet(Z_m,Z_1\cap Z_m\ \dots,Z_{m-1}\cap Z_m,\mathrm{M}_n,*)\,.$$

Note also that $K_\emptyset=\mathrm{C}_\bullet(\bigcap\limits_{i=1}^{m} Z_i,\mathrm{M}_n)$ and $K_{\{1,\dots,m\}}=\mathrm{C}_\bullet(X,\mathrm{M}_n)$.

A.3 (**The functors $\mathfrak{cf}_{\geq i}$**). Recall first the cone of a morphism of complexes $f\colon K_\bullet \longrightarrow L_\bullet$. This is the complex cone f which is given in degree $i+1$ and i by:

$$\cdots \longrightarrow K_i \oplus L_{i+1} \xrightarrow{\begin{pmatrix} d_i^K & f_i \\ 0 & -d_{i+1}^L \end{pmatrix}} K_{i-1} \oplus L_i \longrightarrow \cdots .$$

We have then a (so called) exact triangle $K_\bullet \xrightarrow{f} L_\bullet \longrightarrow \operatorname{cone} f \longrightarrow K_\bullet[1]$ (with obvious morphisms on the right).

We define inductively a functor $\mathfrak{cf}_{\geq i}$ form the category of m-cubes of complexes to $\mathrm{K}^b(\mathfrak{Ab})$ for all $i \in \mathbb{Z}$. Let K_* be such a m-cube. Then we set $\mathfrak{cf}_{\geq i} K_* = 0$ if $i \geq m+1$ and

$$\mathfrak{cf}_{\geq m} K_* \;:=\; K_{\{1,\ldots,m\}}\,.$$

We have then the morphism of complexes

$$\Theta_m^K \;:=\; \sum_{i=1}^{m} \epsilon_{I\{1,\ldots,m\}}\colon \bigoplus_{|I|=m-1} K_I \longrightarrow K_{\{1,\ldots,m\}} \;=\; \mathfrak{cf}_{\geq m} K_*$$

and define $\mathfrak{cf}_{\geq m-1} K_*$ to be the cone of this morphism.

The composition

$$\bigoplus_{|I|=m-2} K_I \xrightarrow{\left(\epsilon_{IJ}\right)_{I,J}} \bigoplus_{|J|=m-1} K_J \xrightarrow{\Theta_m^K} K_{\{1,\ldots,m\}}$$

is the zero morphism, see (4), and therefore induces a morphism of complexes

$$\Theta_{m-1}^K\colon \bigoplus_{|I|=m-2} K_I[1] \longrightarrow \mathfrak{cf}_{\geq m-1} K_* \;=\; \operatorname{cone} \Theta_m^K\,.$$

We set then $\mathfrak{cf}_{\geq m-2} := \operatorname{cone} \Theta_{m-1}^K$. Let now $p \leq m-3$. Then by (descending) induction we have a morphism of complexes

$$\Theta_{p+2}^K\colon \bigoplus_{|L|=p+1} K_L[m-p-2] \longrightarrow \mathfrak{cf}_{\geq p+2} K_*$$

such that we have $\Theta_{p+2}^K \cdot \big(\epsilon_{JL}[m-p-2]\big)_{|J|=p,|L|=p+1} = 0$. Therefore there exists

$$\Theta_{p+1}^K\colon \bigoplus_{|J|=p} K_J[m-p-1] \longrightarrow \mathfrak{cf}_{\geq p+1} K_* \;=\; \operatorname{cone} \Theta_{p+2}^K\,,$$

such that the following diagram commutes:

$$\begin{tikzcd} & \bigoplus_{|J|=p} K_J[m-p-2] \arrow[dl, "\Theta_{p+1}^K[-1]"'] \arrow[d, "(\epsilon_{JL}[m-p-2])_{J,L}"] & \\ \mathfrak{cf}_{\geq p+1}[-1] \arrow[r] & \bigoplus_{|L|=p+1} K_L[m-p-2] \arrow[r, "\Theta_{p+2}^K"] & \mathfrak{cf}_{\geq p+2} K_* \end{tikzcd}\,.$$

More precisely, the morphism of complexes $\Theta^K_{p+1}[p+1-m]$ is in degree t given by

$$\bigoplus_{|J|=p} K_{J\,t} \xrightarrow{\begin{pmatrix} \big((\epsilon_{JL})_{JL}\big)_t \\ 0 \end{pmatrix}} \bigoplus_{|L|=p+1} K_{L\,t} \oplus \mathfrak{cf}_{\geq p+2}[p+1-m]_t \;=\; \mathfrak{cf}_{\geq p+1}[p+1-m]_t\,.$$

Therefore we have by (4) that $\Theta^K_{p+1}\cdot(\epsilon_{IJ}[m-p-1])_{I,J} = 0$ which finishes the induction step and the definition of $\mathfrak{cf}_{\geq p} K_*$ for $p \geq 0$.

If $p \leq 0$ we set $\mathfrak{cf}_{\geq p} K_* := \mathfrak{cf}_{\geq 0} K_*$.

A.4 (**An exact triangle).** By construction we have then for all $0 \leq p \leq m$ a commutative diagram

$$\begin{array}{ccccc}
\bigoplus\limits_{\substack{|I|=p-2\\ m\notin I}} K_{I\cup\{m\}}[m-p] & \longrightarrow & \bigoplus\limits_{|J|=p-1} K_J[m-p] & \longrightarrow & \bigoplus\limits_{\substack{|L|=p-1\\ m\notin I}} K_L[m-p] \\
\Big\downarrow{\scriptstyle \Theta^{K'}_{p-1}} & & \Big\downarrow{\scriptstyle \Theta^{K}_{p}} & & \Big\downarrow{\scriptstyle \Theta^{\tilde K}_{p}} \\
\mathfrak{cf}_{\geq p-1} K'_* & \longrightarrow & \mathfrak{cf}_{\geq p} K_* & \longrightarrow & (\mathfrak{cf}_{\geq p} \tilde K_*)[1] \;,
\end{array}$$

whose lower row is an exact triangle and whose upper row is a short split exact sequence of complexes (with obvious morphisms) for all $p \geq -1$.

Shifting the diagram of the lemma for $p = m-1$ to the left we get a commutative diagram in the bounded derived category of complexes of abelian groups:

$$\begin{array}{ccccc}
\bigoplus\limits_{\substack{|I|=m-2\\ m\notin I}} K_I & \xrightarrow{0} & \bigoplus\limits_{|J|=m-2} K_J[1] & \longrightarrow & \bigoplus\limits_{\substack{|L|=m-3\\ m\notin I}} K_{L\cup\{m\}}[1] \\
\Big\downarrow{\scriptstyle \Theta^{\tilde K}_{m-1}} & & \Big\downarrow{\scriptstyle \Theta^{K}_{m-1}} & & \Big\downarrow{\scriptstyle \Theta^{K'}_{m-2}} \\
K_{\{1,\ldots,m-1\}} & \xrightarrow{\Theta} & \mathfrak{cf}_{\geq m-2} K'_* & \longrightarrow & \mathfrak{cf}_{\geq m-1} K_* \;,
\end{array}$$

where the arrow

$$\Theta\colon K_{\{1,\ldots,m-1\}} \;=\; \mathfrak{cf}_{\geq m-1} \tilde K_* \longrightarrow$$

$$\mathfrak{cf}_{\geq m-2} K'_* \;=\; \mathrm{cone}\Big(\bigoplus_{\substack{|J|=m-2\\ m\notin J}} K_J \xrightarrow{\Theta^{K'}_{m-1}} K_{\{1,\ldots,m\}}\Big)$$

is induced by $\epsilon_{\{1,\ldots m-1\}\{1,\ldots m\}}\colon K_{\{1,\ldots m-1\}} \longrightarrow K_{\{1,\ldots m\}}$.

A.5. **Definition.** The *cofiber* of the m-cube of complexes K_* is the complex

$$\mathfrak{cf}\, K_* \;:=\; \mathfrak{cf}_{\geq 0} K_*\,.$$

The assignment $K_* \mapsto \mathfrak{cf}\, K_*$ is a covariant functor from the category of m-cubes of complexes to $\mathrm{K}^b(\mathfrak{Ab})$.

A.6 (**The spectral sequence).** By the very definition of the complexes $\mathfrak{cf}_{\geq p} K_*$ we have exact triangles

$$\mathfrak{cf}_{\geq p+1} K_* \longrightarrow \mathfrak{cf}_{\geq p} K_* \longrightarrow \bigoplus_{|I|=p} K_I[m-p]$$

for any m-cube of complexes K_* and all $p \geq 0$. The associated long exact homology sequences constitute an exact couple and so we get a convergent spectral sequence of cohomological type

$$E_1^{p,q}(K_*) := \mathrm{H}_{-p-q}\big(\bigoplus_{|I|=p} K_I[m-p]\big) \implies \mathrm{H}_{-p-q}(\mathfrak{cf}\, K_*)\,.$$

(Note that the complexes K_I and so also $\mathfrak{cf}_{\geq p} K_*$ are all bounded.) By construction the IJ-component $\mathrm{H}_{-p-q}(K_I[m-p]) \longrightarrow \mathrm{H}_{-p-q}(K_J[m-p])$ of the differential $d_1^{p,q}\colon E_1^{p,q}(K_*) \longrightarrow E_1^{p+1,q}(K_*)$ is equal $\mathrm{H}_{-p-q}(\epsilon_{IJ}[m-p])$.

A.7. **Example.** Let $X, Z_1, \ldots, Z_m,\ K_*$, and M_* be as in example A.2. We set $W_l := \bigcup_{j=1}^{l} Z_j$ for $1 \leq l \leq m$ and $\mathrm{C}_\bullet(Y) := \mathrm{C}_\bullet(Y, \mathrm{M}_n)$ for any finite type k-scheme Y.

The pull-back along the open immersion $X \setminus W_m \hookrightarrow X$ induces a morphism of complexes $\mathfrak{cf}_{\geq m} K_* = \mathrm{C}_\bullet(X) \longrightarrow \mathrm{C}_\bullet(X \setminus W_m)$. The composition of the morphism with Θ_m^K is zero given a morphism of complexes $\mathfrak{cf}_{\geq m-1} K_* \longrightarrow \mathrm{C}_\bullet(X \setminus W_m)$. Composing this morphism with Θ_{m-1}^K is again zero and hence induce a morphism from $\mathfrak{cf}_{\geq m-2} K_*$ to $\mathrm{C}_\bullet(X \setminus W_m)$. Proceeding further we finally get a morphism of complexes

$$\gamma\colon\ \mathfrak{cf}\, K_* \longrightarrow \mathrm{C}_\bullet(X \setminus W_m)\,.$$

Similarly we have morphisms of complexes $\tilde{\gamma} : \mathfrak{cf}\, \tilde{K}_* \longrightarrow \mathrm{C}_\bullet(Z_m \setminus W_{m-1})$ and $\gamma'\colon\ \mathfrak{cf}\, K'_* \longrightarrow \mathrm{C}_\bullet(X \setminus W_{m-1})$. We claim that this is a quasi-isomorphism. This is obvious for $m = 1$. Let $m \geq 2$. Then by A.4 we have a commutative diagram whose rows are exact triangles

$$\begin{tikzcd}
\mathfrak{cf}\, \tilde{K}_* \arrow[r] \arrow[d, "\tilde{\gamma}"] & \mathfrak{cf}\, K'_* \arrow[r] \arrow[d, "\gamma'"] & \mathfrak{cf}\, K_* \arrow[d, "\gamma"'] \\
\mathrm{C}_\bullet(Z_m \setminus W_{m-1}) \arrow[r, "\iota_*"] & \mathrm{C}_\bullet(X \setminus W_{m-1}) \arrow[r, "j^*"] & \mathrm{C}_\bullet(X \setminus W_m)
\end{tikzcd}\ ,$$

where $\iota\colon Z_m \setminus W_{m-1} \hookrightarrow X \setminus W_{m-1}$ and $j\colon X \setminus W_{m-1} \hookrightarrow X \setminus W_m$ are to the respective subschemes corresponding open respectively closed immersions. The claim follows from this diagram by induction.

Hence we have a convergent spectral sequence

$$E_1^{p,q}(X, Z_1, \ldots, Z_m, \mathrm{M}_n) := \bigoplus_{|I|=p} \mathrm{H}_{-q-m}(Z_I, \mathrm{M}_n) \quad\implies \mathrm{H}_{-p-q}(U, \mathrm{M}_n)\,,$$

where $U = X \setminus \bigcup_{i=1}^{m} Z_i$. The IJ-component of the differential is equal 0 if $I \not\subset J$ and equal $(-1)^{d-1}$ times the push-forward along the closed immersion $Z_I \hookrightarrow Z_J$ if $J = \{i_1 < \ldots < i_l\}$ and $I = J \setminus \{i_d\}$.

A.8. **Remark.** This spectral sequence applies also to Voevodsky's [29] motivic cohomology of a homotopy invariant Nisnevich sheaf with transfers $\mathcal{F}$. By Déglise's Thèse [4] one can associate to such a sheaf a cycle module $\hat{\mathcal{F}}_*$, such that there is a natural isomorphism $\mathrm{H}^i_{\mathrm{Nis}}(X,\mathcal{F}) \simeq \mathrm{H}^i(X,\hat{\mathcal{F}}_0)$ for all $i \in \mathbb{N}$. (Vice versa, if M_* is a cycle module then $X \mapsto \mathrm{H}^i(X,\mathrm{M}_n)$ is a homotopy invariant Nisnevich sheaf with transfers.)

A.9. **Example.** Let X be a topological space with closed subspaces $Z_1,\dots,Z_m$. As in the book [7] of Dold we denote by SX and $S(X,A)$ the singular and relative singular complex of X and the pair (X,A) with $A \subseteq X$ a closed subset, respectively. Let $\mathrm{H}_*(X)$ and $\mathrm{H}_*(X,A)$ be the homology groups of these complexes, *i.e.* the *(relative) singular homology* of the space X and the pair (X,A), respectively.

We set (as above) $Z_I := \bigcap\limits_{j\notin I} Z_j$ for subsets $I \subseteq \{1,\dots,m\}$, and denote by ι_{IJ} the embedding $Z_I \hookrightarrow Z_J$ if $I \subseteq J$.

The map $I \mapsto SZ_I$ is then a m-cube of complexes and we get by the same reasoning as in Example A.7 a convergent spectral sequence of cohomological type

$$E_1^{p,q}(X,Z_1,\dots,Z_m) := \bigoplus_{|I|=p} \mathrm{H}_{-q-m}(Z_I) \implies \mathrm{H}_{-p-q}(X,\bigcup_{i=1}^m Z_i)\,,$$

where the IJ-component of the differential $d_1^{p,q}\colon E_1^{p,q} \longrightarrow E_1^{p+1,q}$ is zero if $I \not\subset J$ and equal $(-1)^{r-1}\cdot \mathrm{H}_{-q-m}(\iota_{IJ})$ if $J=\{i_1,\dots,i_p,i_{p+1}\}$ and $I = J\setminus\{i_r\}$ for some $1 \le r \le p+1$.

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