THE J-INVARIANT AND TITS INDICES FOR GROUPS OF INNER TYPE E_6

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Abstract. A connection between the indices of the Tits algebras of a split linear algebraic group G and the degree one parameters of its motivic J-invariant was introduced by Quéguiner-Mathieu, Semenov and Zainoulline through use of the second Chern class map in the Riemann-Roch theorem without denominators. In this paper we extend their result to higher Chern class maps and provide applications to groups of inner type E_6.

Introduction

For a linear algebraic group G, an invariant known as the *Tits algebra* was introduced by J. Tits in [18] and has proven to be an invaluable tool for the computation of the K-theory of twisted flag varieties by Panin [11] and for the index reduction formulas by Merkurjev, Panin and Wadsworth [10]. Furthermore, the Tits algebra has applications to both the classification of linear algebraic groups and the study of the associated homogeneous varieties.

The J-invariant, as defined by Petrov, Semenov and Zainoulline in [13], is an invariant of G which describes the motivic behaviour of the variety of Borel subgroups of G. For a prime p, the J-invariant of G modulo p is given by an r-tuple of integers $J_p(G) = (j_1, \ldots, j_r)$. We consider also a (possibly empty) subtuple $J_p^{(1)}(G) = (j_1, \ldots, j_s)$, $s \leq r$, consisting of the parameters of the J-invariant of degree 1.

Motivated by the work [5], Quéguiner-Mathieu, Semenov and Zainoulline discovered a connection between these degree 1 parameters and the indices of the Tits algebras of G. This connection is developed in [14], through use of the second Chern class map in the Riemann-Roch theorem without denominators. The goal of this paper is to extend their result through use of higher Chern class maps (see Theorem 4.1). We then apply this result to a group G of inner type E_6. We provide an explicit connection between the values of $J_3^{(1)}(G)$ and the index of the Tits algebra of G (see Proposition 5.2).

This paper is organized as follows. In the first section, we review the notion of characteristic classes and introduce the topological and γ-filtrations on the Grothendieck group K_0 of a smooth projective variety. In Section 2, we look more closely at the γ-filtration of the variety of Borel subgoups X, and give a simplified definition through use of the Steinberg basis. We then consider a twisted form ${}_\xi X$ of X by means of a G-torsor $\xi \in H^1(k, G)$ and the γ-filtration of $\mathrm{K}_0({}_\xi X)$. In Section

3, we recall the definition of the Tits map and its relation to our primary objects of concern, the common index i_c and the J-invariant of ξ. Section 4 provides the main result of the paper, a relationship between the common index and the possible values of the indices of the J-invariant of degree 1. In the final section, we give an application of this result to a group of inner type E_6.

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1. Gamma filtration on K_0 and the Chern class map

In the present section we recall several useful properties of the topological filtration and the γ-filtration on Grothendieck's K_0 of a smooth projective variety. The reader is advised to look at [16], [3] and [4, ch.15] for more details.

Let X be a smooth projective variety over a field k. Consider the topological filtration on $\mathrm{K}_0(X)$ (see [4, Ex. 15.1.5]) given by

$$\tau^i\,\mathrm{K}_0(X) = \langle [\mathcal{O}_V] \mid codim\ V \geq i \rangle,$$

where $\mathcal{O}_V$ is the structure sheaf of a closed subvariety V in X. We denote by $\tau^{i/i+1}(X)$, $i \geq 0$ the i-th subsequent quotient $\tau^i\,\mathrm{K}_0(X)/\tau^{i+1}\,\mathrm{K}_0(X)$. There is a surjection

$$p\colon\ \mathrm{CH}^i(X) \twoheadrightarrow \tau^{i/i+1}K_0(X), \quad V \mapsto [\mathcal{O}_V],$$

from the Chow group of codimension i cycles.

For a vector bundle $\mathcal{E}$ on X, the total Chern class $c(\mathcal{E}) = 1+c_1(\mathcal{E})t+c_2(\mathcal{E})t^2+\dots$ is an element of $\mathrm{CH}(X)$, and by the Whitney sum formula, it defines a group homomorphism

$$c : \mathrm{K}_0(X) \to \mathrm{CH}(X)$$

For $\alpha \in \mathrm{K}_0(X)$, $c_i(\alpha)$ is the component of $c(\alpha)$ in $\mathrm{CH}^i(X)$, giving a group homomorphism

$$c_i\colon \tau^i\,\mathrm{K}_0(X) \to \mathrm{CH}^i(X)$$

defined by taking the i-th Chern class.

1.1. **Lemma.** *$c_i(\tau^j\,\mathrm{K}_0(X)) = 0$ for all $0 < i < j$.*

Proof. From [4, Ex 15.3.6] it can be seen that $c_i(\tau^{i+1}\,\mathrm{K}_0(X)) = 0$ for all $i \in \mathbb{Z}_{>0}$. By the definiton of the topological filtration, we have $\tau^j\,\mathrm{K}_0(X) \subseteq \tau^i\,\mathrm{K}_0(X)$ for all $i \leq j$, and so $i \lneq j$ implies that in particular we have $\tau^j \subseteq \tau^{i+1}\,\mathrm{K}_0(X)$. The result follows immediately. □

Thus we have an induced homomorphism

$$c_i\colon \tau^{i/i+1}(X) \to \mathrm{CH}^i(X)$$

such that the composite $c_i \circ p$ is the multiplication by $(-1)^{i-1}(i-1)!$. This result implies that c_i is an isomorphism for $i \leq 2$ (see [4, Ex. 15.3.6]) and, moreover,

1.2. **Lemma.** *The map c_i is an isomorphism over the coefficient ring $\mathbb{Z}[\frac{1}{(i-1)!}]$.*

Let γ_i denote the i-th characteristic class with values in K_0 (see [4, Ex. 3.2.7(b)]. We follow the convention $\gamma_1([\mathcal{L}]) = 1 - [\mathcal{L}^\vee]$ for any line bundle $\mathcal{L}$ over X, where $\mathcal{L}^\vee$ denotes the dual of $\mathcal{L}$.

1.3. **Example.** Using the Whitney sum formula we obtain the following results for $i = 1, 2$ by computing the total Chern classes,

$$\begin{aligned} c(\gamma_1([\mathcal{L}])) &= c(1 - [\mathcal{L}^\vee]) \\ &= \frac{1}{1 - c_1(\mathcal{L})t} \\ &= 1 + c_1(\mathcal{L})t + c_1(\mathcal{L})^2 t^2 + \dots \end{aligned}$$

This gives $c_1(\gamma_1([\mathcal{L}])) = c_1(\mathcal{L})$. Similarly,

$$c(\gamma_1([\mathcal{L}_1])\gamma_1([\mathcal{L}_2])) = c((1 - [\mathcal{L}_1^\vee])(1 - [\mathcal{L}_2^\vee])) = c(1 - [\mathcal{L}_1^\vee] - [\mathcal{L}_2^\vee] + [(\mathcal{L}_1 \otimes \mathcal{L}_2)^\vee])$$

$$= \frac{c(1) \cdot c([(\mathcal{L}_1 \otimes \mathcal{L}_2)^\vee])}{c([\mathcal{L}_1^\vee]) \cdot c([\mathcal{L}_2^\vee])} = \frac{1 - (c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2))t}{(1 - c_1(\mathcal{L}_1)t)(1 - c_1(\mathcal{L}_2)t)}$$

$$= (1-(c_1(\mathcal{L}_1)+c_1(\mathcal{L}_2))t)(1+c_1(\mathcal{L}_1)t+c_1(\mathcal{L}_1)^2t^2+\dots)(1+c_1(\mathcal{L}_2)t+c_1(\mathcal{L}_2)^2t^2+\dots)$$

$$= 1 - c_1(\mathcal{L}_1)c_1(\mathcal{L}_2)t^2 + \dots$$

Hence $c_2(\gamma_1([\mathcal{L}_1])\gamma_1([\mathcal{L}_2])) = -c_1(\mathcal{L}_1)c_1(\mathcal{L}_2)$.

By the definition of these characteristic classes (see [4, Ex 15.3.6]), we have in general

$$c_i\big(\gamma_1([\mathcal{L}_1]) \dots \gamma_1([\mathcal{L}_i])\big) = (-1)^{i-1}(i-1)! \cdot c_1(\mathcal{L}_1) \cdot \ldots \cdot c_1(\mathcal{L}_i). \tag{1}$$

The Grothendieck γ-filtration on $K_0(X)$ is defined by

$$\gamma^i \mathrm{K}_0(X) = \langle \gamma_{i_1}(x_1) \cdot \ldots \cdot \gamma_{i_m}(x_m) \mid i_1 + \ldots + i_m \geq i,\ x_l \in \mathrm{K}_0(X) \rangle,$$

(see [4, Ex.15.3.6], [3, Ch.3 and 5]). Let $\gamma^{i/i+1}(X)$ denote the i-th subsequent quotient $\gamma^i \mathrm{K}_0(X)/\gamma^{i+1} \mathrm{K}_0(X)$.

It is known that $\gamma^i \mathrm{K}_0(X)$ is contained in $\tau^i \mathrm{K}_0(X)$ for every $i \geq 0$, and they coincide for $i \leq 2$ (see [8, Prop.2.14]). Therefore, by Lemma 1.1 the Chern class map c_i restricted to $\gamma^i \mathrm{K}_0(X)$ vanishes on $\gamma^{i+1} \mathrm{K}_0(X)$, and hence induces a map

$$c_i \colon \gamma^{i/i+1} \mathrm{K}_0(X) \to \mathrm{CH}^i(X).$$

1.4. **Example.** For $i = 1$ we have $\gamma^{1/2} \mathrm{K}_0(X) = \tau^{1/2} \mathrm{K}_0(X)$ and $c_1 \circ p = id_{\mathrm{CH}^1(X)}$, giving an isomorphism

$$c_1 \colon \gamma^{1/2} \mathrm{K}_0(X) \to \mathrm{CH}^1(X).$$

In the previous example, we saw that the map c_1 sends $\gamma_1([L])$ to $c_1(L)$.
For $i = 2$ we again have $\gamma^2 \mathrm{K}_0(X) = \tau^2 \mathrm{K}_0(X)$, but this time $\gamma^3 \mathrm{K}_0(X)$ does not necessarily coincide with $\tau^3 \mathrm{K}_0(X)$. We may form an exact sequence,

$$0 \to \tau^3 \mathrm{K}_0(X)/\gamma^3 \mathrm{K}_0(X) \to \gamma^{2/3} \mathrm{K}_0(X) \to \tau^{2/3} \mathrm{K}_0(X) \to 0.$$

Replacing $\tau^{2/3}\,\mathrm{K}_0(X)$ with $\mathrm{CH}^2(X)$, we see that the map $c_2 : \gamma^{2/3}\,\mathrm{K}_0(X) \to \mathrm{CH}^2(X)$ is surjective. In addition, $\ker(c_2) \cong \tau^3\,\mathrm{K}_0(X)/\gamma^3\,\mathrm{K}_0(X)$. Note that for all $i \geq 0$, (see [8, Prop. 2.14])

$$\tau^i\,\mathrm{K}_0(X) \otimes \mathbb{Q} \cong \gamma^i\,\mathrm{K}_0(X) \otimes \mathbb{Q},$$

thus $\ker(c_2)$ is torsion.

1.5. **Proposition.** *The Chern class map* $c_i \colon \gamma^{i/i+1}(X) \to \mathrm{CH}^i(X)$ *is surjective over the coefficient ring* $\mathbb{Z}[\frac{1}{(i-1)!}]$.

Proof. By Lemma 1.2, the map $c_{\tau,i} : \tau^{i/i+1}(X) \to \mathrm{CH}^i(X)$ is surjective over the coefficient ring $\mathbb{Z}[\frac{1}{(i-1)!}]$. Since $\gamma^i\,\mathrm{K}_0(X) \subseteq \tau^i\,\mathrm{K}_0(X)$ for all i, we have an obvious map $\gamma^{i/i+1}\,\mathrm{K}_0(X) \to \tau^{i/i+1}\,\mathrm{K}_0(X)$ defined by sending $x + \gamma^{i+1}\,\mathrm{K}_0(X) \mapsto x + \tau^{i+1}\,\mathrm{K}_0(X)$. By definition, $c_{\gamma,i}$ is the composition of these two maps.

$$\begin{tikzcd} \gamma^{i+1}\,\mathrm{K}_0(X) \arrow[r] \arrow[rr, bend right, "c_{\gamma,i}"'] & \tau^{i/i+1}\,\mathrm{K}_0(X) \arrow[r, two heads, "c_{\tau,i}"] & \mathrm{CH}^i(X) \end{tikzcd}$$

Over $\mathbb{Z}[\frac{1}{(i-1)!}]$ we have $im(c_{\gamma,i}) \subseteq im(c_{\tau,i}) = \mathrm{CH}^i(X)$, and so it remains to show that $\mathrm{CH}^i(X) \subseteq im(c_{\gamma,i})$ for all i.
Consider an arbitrary element $x \in \mathrm{K}_0(X)$. By the splitting principle we may write $x = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n$ where $\mathcal{L}_1, \ldots, \mathcal{L}_n$ are line bundles over X. By the properties of characteristic classes, $\gamma_i(\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n) = 0$ for all $i > n$ and $\gamma_i(\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n) = s_i(\gamma_1(\mathcal{L}_1), \ldots, \gamma_1(\mathcal{L}_n))$ for all $0 < i \leq n$, where $s_i(\gamma_1(\mathcal{L}_1), \ldots, \gamma_1(\mathcal{L}_n))$ is the i-th elementary symmetric polynomial in variables $\gamma_1(\mathcal{L}_1), \ldots, \gamma_1(\mathcal{L}_n)$. Thus, taking the total Chern class, we have by (1) and Lemma 1.1

$$\begin{aligned} c(\gamma_i(\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n)) &= c(s_i(\gamma_1(\mathcal{L}_1), \ldots, \gamma_1(\mathcal{L}_n))) \\ &= \prod_{1 \leq j_1 < \cdots < j_i \leq n} (1 + (-1)^{i-1}(i-1)! c_1(\mathcal{L}_{j_i}) \cdots c_1(\mathcal{L}_{j_i}) t^i + \ldots) \\ &= 1 + (-1)^{i-1}(i-1)! \cdot s_i(c_1(\mathcal{L}_1), \ldots, c_1(\mathcal{L}_n)) t^i + \ldots \end{aligned}$$

Thus, $c_i(\gamma_i(\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n)) = (-1)^{i-1}(i-1)! s_i(c_1(\mathcal{L}_1), \ldots, c_1(\mathcal{L}_n))$. In general, we have

$$c_i(\gamma_i(x)) = (-1)^{i-1}(i-1)! c_i(x) \text{ for all } x \in \mathrm{K}_0(X). \tag{2}$$

Let $y \in \mathrm{CH}^i(X)$. Then by the sujectivity of the Chern class map, $y = c_i(x)$ for some $x \in \tau^{i/i+1}\,\mathrm{K}_0(X)$. By (2), $c_i(x) = \frac{(-1)^{i-1}}{(i-1)!} \cdot c_i(\gamma_i(x))$, where $\gamma_i(x) \in \gamma^{i/i+1}\,\mathrm{K}_0(X)$ by definiton of the γ-filtration. Thus $y \in im(c_{\gamma,i})$ over the coefficient ring $\mathbb{Z}[\frac{1}{(i-1)!}]$, as required. $\square$

2. Gamma filtration on twisted flag varieties

In the present section we discuss the γ-filtration on the variety of Borel subgroups. A reader is encouraged to look at [14] and [5] for further details.

Let G be a split simple linear algebraic group over a field k. We fix a split maximal torus T and a Borel subgroup $B \supset T$ of G. Let W denote the Weyl group of G. Let X denote the variety of Borel subgroups of G.

Let $\{g_w\}_{w\in W}$ be the Steinberg basis of $\mathrm{K}_0(X)$ (cf. [17]). For each $w \in W$, $g_w = [\mathcal{L}_w]$ is the class of a line bundle over X, and together they form a $\mathbb{Z}$-basis of $\mathrm{K}_0(X)$.

2.1. **Lemma.** *The i-th term of the γ-filtration on X is generated by products*

$$\gamma^{i/i+1}\,\mathrm{K}_0(X) = \{\gamma_1(g_{w_1})\cdot\ldots\cdot\gamma_1(g_{w_i}),\ w_1,\ldots,w_i \in W\}.$$

Proof. As a temporary notation, let

$$\Gamma^i = \langle\gamma_1(g_{w_1})\cdot\ldots\cdot\gamma_1(g_{w_i}),\ w_1,\ldots,w_i \in W\rangle.$$

By the definition of $\gamma^i\,\mathrm{K}_0(X)$, it is clear that $\Gamma^i \subseteq \gamma^i\,\mathrm{K}_0(X)$ for all i. It remains to show the other inclusion. Consider first the case $i = 1$. Let $x \in \mathrm{K}_0(X)$ such that $x = \sum_{w\in W} a_w g_w$ for some $a_w \in \mathbb{Z}$. Computing the total characteristic class, we have

$$\gamma(x) = \gamma(\sum_{w\in W} a_w g_w) = \prod_{w\in W}\gamma(g_w)^{a_w} = \prod_{w\in W}(1+\gamma_1(g_w)t)^{a_w}$$

$$= \prod_{w\in W}\Big(\sum_{k=0}^{a_w}\binom{a_w}{k}\gamma_1(g_w)^k t^k\Big) = \prod_{w\in W}\Big(1 + a_w\gamma_1(g_w)t + \binom{a_w}{2}\gamma_1(g_w)^2t^2+\ldots\Big)$$

$$(3)\quad = 1+\Big(\sum_{w\in W} a_w\gamma_1(g_w)\Big)t+\Big(\sum_{w\in W}\binom{a_w}{2}\gamma_1(g_w)^2+\sum_{v\neq w} a_v a_w\gamma_1(g_v)\gamma_1(g_w)\Big)t^2+\ldots$$

Thus we have $\gamma_1(x) = \sum_{w\in W} a_w\gamma_1(g_w) \in \Gamma^1$, as required.

In general, it is clear that Γ defines a graded ring. That is, if $x \in \Gamma^i$ and $y \in \Gamma^j$, then $xy \in \Gamma^{i+j}$. Suppose for induction that $\gamma^j\,\mathrm{K}_0(X) = \Gamma^j$ for all $j < i$ and consider an element $y = \gamma_{i_1}(x_1)\cdots\gamma_{i_m}(x_m) \in \gamma^i\,\mathrm{K}_0(X)$ such that $i_1+\cdots+i_m = i$. In the case that $m = 1$ (i.e. $y = \gamma_i(x)$) the expansion (3) shows that $y \in \Gamma^i$. If instead $m > 1$, then we have $i_l < i$ for all $l = 1,\ldots,m$. By induction, $\gamma_{i_l}(x_l) \in \Gamma^{i_l}$ for each $l = 1,\ldots,m$, and so

$$y = \gamma_{i_1}(x_1)\cdots\gamma_{i_m}(x_m) \in \Gamma^{i_1}\cdots\Gamma^{i_m} \subseteq \Gamma^i.$$

□

Consider the twisted form ${}_\xi X$ of X by means of a G-torsor $\xi \in H^1(k,G)$. In general the group $\mathrm{K}_0({}_\xi X)$ is not generated by classes of line bundes. Hence, in the definition of $\gamma^i\,\mathrm{K}_0({}_\xi X)$ some higher characteristic classes $\gamma_{j>1}(-)$ may appear.

Note that characteristic classes commute with restrictions, i.e. for every field extension k'/k and variety Y over k we have

$$res_{k'/k} \circ \gamma_j = \gamma_j \circ res_{k'/k} \text{ and } c_j \circ res_{k'/k} = res_{k'/k} \circ c_j$$

where $res_{k'/k} \colon \mathrm{K}_0(Y) \to \mathrm{K}_0(Y \times_k k')$. Moreover, for every k'/k the restriction map $res \colon \mathrm{K}_0(X) \to \mathrm{K}_0(X \times_k k')$ is an isomorphism.

We will use the following commutative diagram

$$\begin{tikzcd} \gamma^{i/i+1}({}_\xi X) \arrow[r, "res_\gamma"] \arrow[d, "c_i"'] & \gamma^{i/i+1}(X) \arrow[d, "c_i"] \\ \mathrm{CH}^i({}_\xi X) \arrow[r, "res_{k'/k}"] & \mathrm{CH}^i(X). \end{tikzcd}$$

Let l be a splitting field of ξ. The main result of [11] says that the image of the restriction map

$$res_{l/k} \colon \mathrm{K}_0({}_\xi X) \to \mathrm{K}_0({}_\xi X \times_k l) \simeq \mathrm{K}_0(X \times_k l) \simeq \mathrm{K}_0(X)$$

coincides with the sublattice

$$\langle i_{w,\xi} g_w \rangle_{w \in W},$$

where $i_{w,\xi} \geq 1$ are indices of the respective Tits algebras, which will be introduced in the next section.

2.2. **Proposition.** *Consider the composite*

$$\pi_i \colon \gamma^{i/i+1}({}_\xi X) \xrightarrow{res_\gamma} \gamma^{i/i+1}(X) \xrightarrow{c_i} \mathrm{CH}^i(X).$$

The image of π_1 is generated by $i_{w,\xi} c_1(g_w)$ for all $w \in W$. The image of π_2 is generated by the elements $i_{w_1,\xi} i_{w_2,\xi} c_1(g_{w_1}) c_1(g_{w_2})$ and $\binom{i_{w,\xi}}{2} c_1(g_w)^2$ for all $w_1, w_2, w \in W$. In general, the image of π_i is generated by the elements

$$(i-1)! \binom{i_{w_1}}{i_1} \cdots \binom{i_{w_m}}{i_m} c_1(g_{w_1})^{i_1} \cdots c_1(g_{w_m})^{i_m}$$

where $i_1 + \cdots + i_m = i$ for all $w_1, \ldots, w_m \in W$.

Proof. By the defintions of the restriction map and the γ-filtration on $\mathrm{K}_0(X)$, we can see that the image of $res_\gamma^{(i)}$ is generated by products

$$res_\gamma(\gamma^{i/i+1} \mathrm{K}_0({}_\xi X)) = \{\gamma_{i_1}(i_{w_1} g_{w_1}) \cdots \gamma_{i_m}(i_{w_m} g_{w_m}) | i_1 + \cdots + i_m = i\},$$

where $w_1, \ldots, w_m \in W$.

We also note that since the g_w's are line bundles, we have

$$\begin{aligned} \gamma(i_w g_w) &= \gamma(g_w)^{i_w} \\ &= (1 + \gamma_1(g_w) t)^{i_w} \\ &= \sum_{k=1}^{i_w} \binom{i_w}{k} \gamma_1(g_w)^k t^k \end{aligned}$$

and hence for all $0 \leq j \leq i_w$ we have

$$\gamma_j(i_w g_w) = \binom{i_w}{j} \gamma_1(g_w)^j. \tag{4}$$

This property holds for all characteristic classes, and hence

$$c_j(i_w g_w) = \binom{i_w}{j} c_1(g_w)^j. \tag{5}$$

We consider first the case $i = 1$. Then $im(res_\gamma^{(1)})$ is generated by the elements $\gamma_1(i_w g_w) = i_w \gamma_1(g_w)$ for any $w \in W$. Thus, by (2) and (5), the image of π_1 is generated by elements of the form

$$\begin{aligned} c_1(\gamma_1(i_w g_w)) &= c_1(i_w g_w) \\ &= i_w c_1(g_w) \end{aligned}$$

for all $w \in W$.

In general, consider an element of the form $x = \gamma_{i_1}(i_{w_1} g_{w_1}) \cdots \gamma_{i_m}(i_{w_m} g_{w_m})$ such that $i_1 + \cdots + i_m = i$ for some $w_1, \ldots, w_m \in W$. By (4)

$$\gamma_{i_1}(i_{w_1} g_{w_1}) \cdots \gamma_{i_m}(i_{w_m} g_{w_m}) = \binom{i_{w_1}}{i_1} \cdots \binom{i_{w_m}}{i_m} \gamma_1(g_{w_1})^{i_1} \cdots \gamma_1(g_{w_m})^{i_m}.$$

Taking total Chern classes then gives

$$\begin{aligned} c(x) &= c\Big(\binom{i_{w_1}}{i_1} \cdots \binom{i_{w_m}}{i_m} \gamma_1(g_{w_1})^{i_1} \cdots \gamma_1(g_{w_m})^{i_m}\Big) \\ &= c(\gamma_1(g_{w_1})^{i_1} \cdots \gamma_1(g_{w_m})^{i_m})^{\binom{i_{w_1}}{i_1} \cdots \binom{i_{w_m}}{i_m}} \\ &= (1 + (-1)^{i-1}(i-1)! c_1(g_{w_1})^{i_1} \cdots c_1(g_{w_m})^{i_m} t^i)^{\binom{i_{w_1}}{i_1} \cdots \binom{i_{w_m}}{i_m}}, \end{aligned}$$

and so,

$$c_i(x) = (-1)^{i-1}(i-1)! \binom{i_{w_1}}{i_1} \cdots \binom{i_{w_m}}{i_m} c_1(g_{w_1})^{i_1} \cdots c_1(g_{w_m})^{i_m}.$$

□

3. Tits algebras and the J-invariant

Recall that we have defined G to be a split linear algebraic group of rank n over a field k. Also, we have fixed a split maximal torus $T \subset G$ and a Borel subgrup $B \supset T$. Let T^* be the character group of T, $\{\alpha_1, \ldots, \alpha_n\}$ a set of simple roots with respect to B and $\{\omega_1, \ldots, \omega_n\}$ the respective set of fundamental weights, so that $\alpha_i^\vee(\omega_j) = \delta_{ij}$. We have $\Lambda_r \subset T^* \subset \Lambda$, where Λ_r is the root lattice and Λ is the weight lattice. Consider the simply connected cover $\tilde{G}$ of G with corresponding Borel subgroup $\tilde{B}$ and maximal split torus $\tilde{T}$. Given any $\lambda \in \Lambda = Hom(\tilde{T}, \mathbb{G}_m)$, we can lift $\lambda : \tilde{T} \to \mathbb{G}_m$ uniquely to $\lambda : \tilde{B} \to \mathbb{G}_m$. Letting

$$\tilde{G} \times^{\tilde{B}} V_1 = \tilde{G} \times V_1/(g, v) \sim (g \cdot b, \lambda(b)^{-1} \cdot v),$$

the projection map $\tilde{G} \times^{\tilde{B}} V_1 \to \tilde{G}/\tilde{B}$ defines a line bundle $\mathcal{L}(\lambda)$ over $\tilde{G}/\tilde{B} = G/B$ (see [2, §1.5]), the variety of Borel subgroups of G.

For a fixed $\xi \in H^1(k,G)$, we can associate to each weight λ a central simple k-algebra $A_{\xi,\lambda}$, called a Tits algebra of G (cf. [18]). We define the Tits map

$$\beta_\xi : \Lambda/\Lambda_r \to Br(k)$$

by sending $\bar{\lambda} \mapsto [A_{\xi,\bar{\lambda}}]$, its class in the Brauer group. We note that β_ξ is a group homomorphism for a fixed ξ, with $\bar{\lambda_1} + \bar{\lambda_2} \mapsto [A_{\xi,\bar{\lambda_1}}] \otimes [A_{\xi,\bar{\lambda_2}}]$. Consider all Tits algebras $A_{\xi,\bar{\lambda}}$, $\lambda \in \Lambda$ of ξ and let H be the subgroup in $Br(k)$ generated by the classes of all such $A_{\xi,\bar{\lambda}}$. In other words, H is generated by the non-constant elements in the image of the Tits map β_ξ. Since Λ/Λ_r is a finite abelian group, H is a finite abelian group as well. If H is non-trivial, we define the *common index* i_c of ξ as

$$i_c := gcd\{ind(a) \mid a \in H \setminus \{1\}\}.$$

where the gcd is taken over all indices of non-trivial elements of H.

3.1. **Example.** Let G be a group of inner type E_6. G has precisely one non-trivial Tits algebra, with index 3^d for $d = 0, \dots, 3$. Therefore, by the definition of the common index, we have $i_c = 3^d$ as well.

Let X be the vairety of Borel subgroups of G. The degree 1 characteristic map in the simply connected case

$$\mathfrak{c}_{sc}^{(1)} : \Lambda \tilde{\to} \mathrm{CH}^1(X)$$

defines an isomorphim such that the cycles $h_i = c_1(\mathcal{L}(\omega_i))$, $i = 1, \dots, n$ form a $\mathbb{Z}$-basis of $\mathrm{CH}^1(X)$. The degree 1 characteristic map is the restriction of this isomorphism to the character group T^*

$$\mathfrak{c}^{(1)} : T^* \to \mathrm{CH}^1(X),$$

mapping $\lambda = \sum_{i=1}^n a_i\omega_i \mapsto c_1(\mathcal{L}(\lambda)) = \sum_{i=1}^n a_i h_i$. In general, this defines the characteristic map $\mathfrak{c}\colon S^*(T^*) \to \mathrm{CH}(X)$.

We denote by $\pi : \mathrm{CH}^*(X) \to \mathrm{CH}^*(G)$ the pull-back induced by the natural projection $G \to X$. By [6, Section 4, Rem. 2], π is sujective and its kernel is given by the ideal $I \subset \mathrm{CH}^*(X)$ generated by the non-constant elements in the image of the characteristic map. In particular, we have $I^{(1)} = im(\mathfrak{c}^{(1)})$, and

$$\mathrm{CH}^1(G) \simeq \mathrm{CH}^1(X)/(im(\mathfrak{c}^{(1)})) \simeq \Lambda/T^*.$$

Given a prime p, set $Ch(X) = \mathrm{CH}(X) \otimes \mathbb{F}_p$. Taking $\mathbb{F}_p$-coefficients, we have

$$Ch^1(G) \simeq Ch^1(X)/(im(\mathfrak{c}^{(1)})) \simeq \Lambda/T^* \otimes_{\mathbb{Z}} \mathbb{F}_p.$$

It is known (see [7]) that $Ch(X)/I$ is isomorphic (as an $\mathbb{F}_p$ algebra, as well as a Hopf algebra) to

$$Ch(X)/I \cong \mathbb{F}_p[x_1, \dots, x_r]/(x_1^{p^{k_1}}, \dots, x_r^{p^{k_r}})$$

for some integers r and $k_i \geq 0$ for $i = 1, \dots, r$, which are dependent on the group G. For each i, we let d_i be the degree of the generator x_i. The number of generators of degree 1 is given by the dimension over $\mathbb{F}_p$ of the vector space $\Lambda/T^* \otimes_{\mathbb{Z}} \mathbb{F}_p$.

Let $s = dim_{\mathbb{F}_p}(Ch^1(G))$. Then, since $\omega_1, \dots, \omega_n$ generate Λ we may choose a minimal set $\{i_1, \dots, i_s\} \subset \{1, \dots, n\}$ such that the classes of $\omega_{i_1}, \dots, \omega_{i_s}$ generate

$\Lambda/T^* \otimes \mathbb{F}_p$. Then, $h_{i_l} = c_1(\mathcal{L}(\omega_{i_l}))$, $l = 1,\dots,s$ generate $Ch^1(X)$ and so we may take $x_l = \pi(h_{i_l})$, $l = 1,\dots,s$ to be the generators of $Ch^1(G)$.

In fact, this definition of the generators $x_1,\dots,x_s$ can be simplified using properties of the Steinberg basis. We recall that for all $i = 1,\dots,n$, $g_i = \mathcal{L}(\rho_{s_i})$, where $\rho_{s_i} = \omega_i - \alpha_i$. Thus, we have $\mathcal{L}(\omega_i) = g_i + \mathcal{L}(\alpha_i)$. But, $\Lambda_r \subset T^*$ implies that $c_1(\mathcal{L}(\alpha_i)) \in im(\mathfrak{c}^{(1)})$ and hence $c_1(\mathcal{L}(\alpha_i)) \in ker(\pi)$ for all $i = 1,\dots,n$. So, for each $l = 1,\dots,s$, we have

$$\begin{aligned}\pi(h_{i_l}) &= \pi(c_1(\mathcal{L}(\omega_{i_l})))\\ &= \pi(c_1(g_{i_l}) + c_1(\mathcal{L}(\alpha_{i_l})))\\ &= \pi(c_1(g_{i_l})) + \pi(c_1(\mathcal{L}(\alpha_{i_l})))\\ &= \pi(c_1(g_{i_l})).\end{aligned}$$

Thus, we may take the generators to be $x_l = \pi(c_1(g_{i_l}))$ for $l = 1,\dots,s$.

We impose an well-ordering on the set of generators, that is, on the monomials $x_1^{m_1}\cdots x_r^{m_r}$ known as the *DegLex* order [13]. For ease of notation, we denote the monomial $x_1^{m_1}\cdots x_r^{m_r}$ by x^M, where M is the r-tuple of integers $(m_1,\dots,m_r)$, and set $|M| = \sum_{i=1}^r d_i m_i$. Given two r-tuples $M = (m_1,\dots,m_r)$ and $N = (n_1,\dots,n_r)$, we say that $x^M \le x^N$ (or equivalently $M \le N$) if either $|M| < |N|$, or $|M| = |N|$ and $m_i \le n_i$ for the greatest i such that $m_i \neq n_i$.

Consider the restriction map $res_{l/k}\colon \mathrm{CH}^*({}_\xi X) \to \mathrm{CH}^*(X)$. Let I_ξ denote the ideal generated by non-constant elements of degrees ≥ 1 from the image of $res_{l/k}$. Observe that $I_\xi \supset I$ (see [9]) . Furthermore, it is known that there exists some ξ such that $I_\xi = I$, and such a ξ is called a "generic" torsor.

Now, since $I \subseteq I_\xi$, we have $\mathrm{CH}(X)/I_\xi \subseteq \mathrm{CH}(X)/I$ and hence $Ch(X)/I_\xi \subseteq Ch(X)/I$. Again, there is an isomorphism

$$Ch(X)/I_\xi \cong \mathbb{F}_p[x_1,\dots,x_r]/(x_1^{p^{j_1}},\dots,x_n^{p^{j_r}})$$

where $j_i \le k_i$ for each $1 \le i \le r$. While r, d_i and k_i for $i = 1,\dots,r$ depend only on the group G, the values $j_1,\dots,j_r$ depend also on the choice of ξ. Thus given the *DegLex* ordering defined above, we have a well-defined r-tuple $J_{p,\xi}(G) = (j_1,\dots,j_r)$, called the J-invariant of G. We note that $(0,\dots,0) \le (j_1,\dots,j_r) \le (k_1,\dots,k_r)$ for any choice of ξ.

Let $J_p^{(1)} = \{j_1,\dots,j_s\}$ denote the set of indices of the J-invariant of ξ of degree 1. We say that $J_p^{(1)} > m$ if for every index j_l such that $k_l > m$ we have $j_l > m$.

4. The main result

For a fixed prime p, we have defined a minimal subset $\{\omega_{i_1},\dots,\omega_{i_s}\} \subset \Lambda$ such that the elements $x_l = \pi(c_1(g_{i_l}))$, $l = 1,\dots,s$ generate $Ch^1(G)$. By the general definiton of the common index, we may now state that

$$i_c = gcd\{ind(A_{\omega_{i_1}}^{\otimes b_1} \otimes \cdots \otimes A_{\omega_{i_s}}^{\otimes b_n}),\ \text{where at least one of the } b_i \text{ is coprime to } p\}.$$

Let $I \subset \mathrm{CH}(X)$ be the ideal generated by the non-constant elements in the image of the characteristic map $\mathfrak{c} : S^*(T^*) \to \mathrm{CH}(X)$ and $I_\xi \subset \mathrm{CH}(X)$ be be ideal generated by the non-constant elements in the image of the restriction map $res_{l/k} : \mathrm{CH}({}_\xi X) \to \mathrm{CH}(X)$. For any integer m, we let $I^{(m)} \subset \mathrm{CH}^m(X)$ and $I_\xi^{(m)} \subset \mathrm{CH}^m(X)$ denote the homogeneous parts of these ideals of degree m.

4.1. **Theorem.** *If $v_p(i_c) > 0$, then $I_\xi^{(1)} = I^{(1)}$.*

If $v_p(i_c) > 1$, then $I_\xi^{(m)} = I^{(m)}$ for $m = 2, \dots, p$.

Proof. Since we know already that $I \subset I_\xi$, it suffices to prove that $I_\xi^{(m)} \subset I^{(m)}$ for all $m = 1, \dots, p$ under the relevant hypothesis on i_c. By Proposition 1.5 and the commutative diagram in Section 2, we have that for any $i \geq 0$,

$$im(res_{l/k}^{(i)}) = c_i(im(res_\gamma^{(i)})$$

over the coefficient ring $\mathbb{Z}[\frac{1}{(i-1)!}]$.

We begin first with the case $m = 1$. By the definiton of I_ξ, we have $I_\xi^{(1)} = im(res_{l/k}^{(1)})$. Therefore, to show that $I_\xi^{(1)} \subseteq I^{(1)}$, we must prove that if $v_p(i_c) > 0$, then for any $w \in W$, the element $i_w c_1(g_w)$ belongs (after tensoring with $\mathbb{F}_p$) to $I^{(1)} = im(\mathfrak{c}^{(1)})$. Recall that $g_w = \mathcal{L}(\rho_w)$, and that we may write $\rho_w = \sum_{i=1}^n a_i \omega_i$. Therefore, we have

$$\begin{aligned} g_w &= \mathcal{L}(\rho_w) \\ &= \mathcal{L}(\sum_{i=1}^n a_i \omega_i) \\ &= \mathcal{L}(\omega_1)^{\oplus a_1} \oplus \cdots \oplus \mathcal{L}(\omega_n)^{\oplus a_n}. \end{aligned}$$

Taking total chern classes gives

$$\begin{aligned} c(g_w) &= c(\mathcal{L}(\omega_1)^{\oplus a_1} \oplus \cdots \oplus \mathcal{L}(\omega_n)^{\oplus a_n}) \\ &= c(\mathcal{L}(\omega_1))^{a_1} \cdots c(\mathcal{L}(\omega_n))^{a_n} \\ &= (1 + c_1(\mathcal{L}(\omega_1))t)^{a_1} \cdots (1 + c_1(\mathcal{L}(\omega_n))t)^{a_n} \\ &= (1 + a_1 c_1(\mathcal{L}(\omega_1))t + \dots) \cdots (1 + a_n c_1(\mathcal{L}(\omega_n))t + \dots) \\ &= 1 + \Big(\sum_{i=1}^n a_i c_1(\mathcal{L}(\omega_i))\Big)t + \dots, \end{aligned}$$

and hence $c_1(g_w) = \sum_{i=1}^n a_i c_1(\mathcal{L}(\omega_i))$. For each $i = 1, \dots, n$ we have shown in the previous sections that $c_1(g_i) = c_1(\mathcal{L}(\omega_i)) - c_1(\mathcal{L}(\alpha_i))$, and $c_1(\mathcal{L}(\alpha_i)) \in im(\mathfrak{c}^{(1)})$. Now, since $Ch^1(G)$ is generated by $x_l = \pi(c_1(g_{i_l}))$ for $l = 1, \dots, s$, we may write

$$c_1(g_w) = \sum_{l=1}^s a_{i_l} c_1(g_{i_l}) \bmod im(c^{(1)}).$$

If all $a_{i_l} \in \mathbb{Z}$ are divisible by p, we are done. So we assume at least one a_{i_l} is coprime to p.

Recall that $\bar{\rho}_w = \sum_{l=1}^{s} a_{i_l}\bar{\omega}_{i_l} \mod T^*$ for the same coefficients a_{i_l}. Applying the Tits map β_ξ, we get

$$\begin{aligned}\beta_\xi(\bar{\rho}_w) &= \beta_\xi\big(\sum_{l=1}^{s} a_{i_l}\bar{\omega}_{i_l}\big)\\ &= \bigotimes_{l=1}^{s}\beta_\xi(\bar{\omega}_{i_l})^{\otimes a_{i_l}}\\ &= \bigotimes_{l=1}^{s}[A_{\xi,\bar{\omega}_{i_l}}]^{\otimes a_{i_l}}\end{aligned}$$

By the assumption that at least one of the a_{i_l} is coprime to p, we have $\beta_\xi(\bar{\rho}_w) \in H \setminus \{1\}$. Thus by the hypothesis that $v_p(i_c) > 0$, we have $p | i_w = ind(\beta_\xi(\bar{\rho}_w))$. Therefore $i_w c_1(g_w) = 0$ in $Ch^1(X)$.

For the case $m > 1$ we work under the hypothesis that $v_p(i_c) > 1$, and proceed by induction. We assume that the result $I_\xi^{(m')} \subseteq I^{(m')}$ holds for all $m' < m$. It can be seen that

$$I_\xi^{(m)} = \left(\bigoplus_{j=1}^{m-1} \mathrm{CH}^{m-j}(X)\cdot im(res_{l/k}^{(j)})\right) \oplus im(res_{l/k}^{(m)}).$$

By the inductive hypothesis, $im(res_{l/k}^{(j)}) \subset I_\xi^{(j)} \subseteq I^{(j)}$ for $1 \leq j \leq m-1$, which implies that $\mathrm{CH}^{m-j}(X)\cdot im(res_{l/k}^{(j)}) \subset I^{(m)}$ for $1 \leq j \leq m-1$. It remains to show that $im(res_{l/k}^{(m)}) \subset I^{(m)}$.

By Proposition 2.2 we know that $im(res_{l/k}^{(m)}) = c_m(im(res_\gamma^{(m)}))$, and is generated by elements of the form

$$a = (m-1)!\binom{i_{w_1}}{i_1}\cdots\binom{i_{w_k}}{i_k}c_1(g_{w_1})^{i_1}\cdots c_1(g_{w_k})^{i_k},$$

where $i_1 + \cdots + i_k = m$, and $w_1, \ldots, w_k \in W$.

If $i_l < m$ for all $l = 1, \ldots, k$, then $\binom{i_{w_l}}{i_l}c_1(g_{w_l})^{i_l} \in I^{(i_l)}$ by the inductive hypothesis. Therefore, $a \in I^{(i_1)}\cdots I^{(i_k)} \subseteq I^{(m)}$. If, on the other hand, a is of the form $a = \binom{i_w}{m}c_1(g_w)^m$, then we apply the previous argument. Namely, we have

$$\rho_w = \sum_{i=1}^{n} a_i\omega_i,$$

which implies

$$\begin{aligned}(c_1(g_w))^m &= \Big(\sum_{i=1}^{n} a_i c_1(\mathcal{L}(\omega_i))\Big)^m\\ &= \Big(\sum_{i=1}^{n} a_i(c_1(g_i) + c_1(\mathcal{L}(\alpha_i)))\Big)^m.\end{aligned}$$

Again, $c_1(\mathcal{L}(\alpha_i)) \in im(\mathfrak{c}^{(1)})$ implies $c_1(\mathcal{L}(\alpha_i)) \in I^{(1)}$ for all $i = 1, \ldots, n$, and so all terms in the above expansion that are divisible by some $c_1(\mathcal{L}(\alpha_i))$ are contained in $I^{(m)}$. As in the previous case, we may write

$$(c_1(g_w))^m = \Big(\sum_{l=1}^{s} a_{i_l} c_1(g_{i_l})\Big)^m \bmod I^{(m)}.$$

If a_{i_l} is divisible by p for all $l = 1, \ldots, s$, then we are done, so we assume that at least one a_{i_l} is coprime to p. As before, this ensures that $i_c \mid ind(\beta_\xi(\bar{\rho_w})$, and so $v_p(i_w) \geq v_p(i_c)$. It is clear that for any $b \in \mathbb{Z}_{>0}$ if $v_p(b) > 1$ then $p \mid \binom{b}{l}$ for all $1 \leq l \leq p$. Thus, under the hypothesis that $v_p(i_c) > 1$, we have $p \mid \binom{i_w}{m}$, and so $\binom{i_w}{m}(c_1(g_w))^m = 0$ in $Ch^m(X)$. □

4.2. **Corollary.** *Let p be a prime number. If $v_p(i_c) > 0$, then $J_p^{(1)} > 0$. If $v_p(i_c) > 1$, then $J_p^{(1)} > 1$.*

Proof. Consider the diagram

$$Ch({}_\xi X) \xrightarrow{res_{l/k}} Ch(X) \xrightarrow{\pi} Ch(G).$$

We begin first with the hypothesis that $v_p(i_c) > 0$.

Let $R_\xi = im(\pi \circ res_{l/k})$. Then $a \in R_\xi^{(1)}$ implies that $a \in im(res_{l/k}^{(1)}) = I^{(1)}$ by Theorem 4.1. Thus, $\pi(a) = 0 \in Ch^1(X)/I$ and so $R_\xi^{(1)} = \{0\}$. Let $x_1, \ldots, x_s$ be generators of degree 1 in $Ch^1(G)$. By the definition of the J-invariant and the *DegLex* order, we have that for every $1 \leq i \leq s$, j_i is the smallest non-negative integer such that R_ξ contains an element of the form

$$x_i^{p^{j_i}} + \sum_{x^M < x_i^{p^{j_i}}} a_M x^M, \ a_M \in \mathbb{F}_p. \tag{6}$$

Thus j_1 is the smallest non-negative integer m such that $x_1^{p^m} \in R_\xi$. Since x_1 is non-trivial, we must have $x_1^{p^0} = x_1 \notin R_\xi^{(1)}$ by the above argument. Therefore, $j_1 > 0$.

The same argument applies for the remaining generators. Let $1 < i \leq s$, then $x^M < x_i$ implies that $x^M = x_j$ for some $j < i$. Since $x_i + a_{i-1}x_{i-1} + \cdots + a_1 x_1$ is non-trivial for any $a_1, \ldots, a_{i-1} \in \mathbb{F}_p$, it cannot belong to $R_\xi^{(1)}$, and we have by (6) that $j_i > 0$. Thus, $J_p^{(1)} > 0$.

Under the hypothesis that $v_p(i_c) > 1$, suppose again that $x_1, \ldots, x_s$ are a minimal set of generators of degree 1 in $Ch(G)$. We have the inclusion $im(res_{l/k}^{(p)}) \subset I_\xi^{(p)}$ and by Theorem 4.1, $I_\xi^{(p)} = I^{(p)}$. Therefore, we again have $R_\xi^{(p)} = im(\pi \circ res_{l/k}^{(p)}) = \{0\}$. To show that $J_p^{(1)} > 1$, we begin with the generator x_1. If $k_1 \leq 1$ we are done, so suppose $k_1 > 1$. Then, $x_1^{p^1} = x_1^p \in Ch^p(G)$ is non-trivial, and so $x^p \notin R_\xi^{(p)}$ and we must have $j_1 > 1$. Again, we extend the argument for the remaining generators. We suppose that $k_i > 1$ for somel $1 < i \leq s$. Then, the element

$$x_i^p + \sum_{(x^M < x_i^p) \cap (|M| = p)} a_M x^M$$

is non-trivial for any $a_M \in \mathbb{F}_p$ and hence cannot belong to $R_\xi^{(p)}$, and so $j_i > 1$. Thus $J_p^{(1)} > 1$. □

5. Applications

We now apply the results of the previous section to some E_6 varieties.

Let G be a group of inner type E_6. It has one Tits algebra A of index 3^d for some $d = 0, \ldots, 3$. Consider the J-invariant of G modulo $p = 3$. We note that $Ch(G)$ has precisely two generators, with $d_1 = 1$ and $d_2 = 4$, where the 3-power relations are defined by $k_1 = 2$ and $k_2 = 1$ [7, Table II]. Thus

$$J_3(G) = (j_1, j_2),$$

where $j_1 = 0, 1, 2$, $j_2 = 0, 1$. These values are independent on the characteristic of the base field.

We will use the following result concerning the possible values of the J-invariant.

5.1. **Lemma.** *Let G be a semisimple algebraic group of inner type over k, p a prime integer and $J_p(G) = (j_1, \ldots, j_r)$. Assume $d_i = 1$ for some $i = 1, \ldots, r$. Then $j_i \leq \max_A v_p(indA)$, where A runs through all Tits algebras of G. Conversely, if $j_i > 0$, then there exists a Tits algebra A of G with $v_p(indA) > 0$.*

Proof. See Proposition 4.3 part 3 in [15]. □

Observe that $J_3(G) = (0, 0)$ iff G splits by a field extension of degree coprime to 3 [13, Corollary 6.7]. Suppose $indA = 27$, then for any splitting field l/k of A, $27 \mid [l : k]$. By [13, Prop. 6.6], we must then have $3 \leq j_1 + j_2$. Since $j_1 \leq k_1 = 2$ and $j_2 \leq k_2 = 1$, the only possible value for the J-invariant is $J_3(G) = (2, 1)$.

5.2. **Proposition.** *$indA = 1$ if and only if $j_1 = 0$. $indA = 3$ if and only if $j_1 = 1$.*

Proof. Suppose first that $indA = 1$. Since G has only one Tits algebra, this implies that $v_3(indB) = 0$ for all $[B] \in H$, where H is the subgroup of $Br(k)$ generated by the non-constant elements in the image of the Tits map β_ξ. Thus by Lemma 5.1, $j_1 = 0$. Suppose conversely that $j_1 = 0$. Since there is only one generator of degree 1, we have $i_c = indA$ and so $v_3(i_c) = d$, where $d = 0, \ldots, 3$. By Corollary 4.2, $j_1 = 0$ implies $d = 0$ and hence $indA = 1$.

For the second case, suppose first $j_1 = 1$. By Corollary 4.2 this implies that $indA = 1$ or 3. However, by the first case, $j_1 \neq 0$ implies $indA \neq 1$ and so $indA = 3$. Conversely, suppose $indA = 3$. Then $v_3(indB) \leq 1$ for all Tits algebras

$[B] \in Br(k)$. By Lemma 5.1, $j_1 \leq \max_B(indB) = 1$ and so $j_1 \leq 1$. Again by the first case, $indA \neq 1$ implies $j_1 \neq 0$ and so $j_1 = 1$. □

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