VARIATIONS ON A THEME OF RATIONALITY OF CYCLES

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Abstract. We prove certain weak versions of some celebrated results due to Alexander Vishik comparing rationality of algebraic cycles over the function field of a quadric and over the base field. The original proofs use Vishik's symmetric operations in the algebraic cobordism theory and work only in characteristic 0. Our proofs use the modulo 2 Steenrod operations in the Chow theory and work in any characteristic $\neq 2$. Our weak versions are still sufficient for all existing applications. In particular, Vishik's construction of fields of u-invariant $2^r + 1$ (for $r \geq 3$) is extended to arbitrary characteristic $\neq 2$.

The main results of this note are Theorem 1.1 (the basic result) with its enhancement 2.1, Proposition 3.1 with its enhancement 4.1 implying Theorems 3.2 and 3.3 (which go a little bit beyond the basic result), and (a quite special) Proposition 5.3 (going in a special situation even more beyond the basic result). The main application is Theorem 5.1.

In characteristic 0, all of this has been proved several years ago by Alexander Vishik in [3] and [5] (exact references are given right before each statement) with a help of the algebraic cobordism theory and especially *symmetric operations* of [4]. In fact, the original versions of the most results are stronger. They do not involve the assumption that the group $\mathrm{CH}(\bar{Y})$ (notation introduced in the beginning of Section 1) is 2-torsion-free (and therefore has no 2-primary torsion), made here in Theorem 1.1 and Proposition 3.1. Our versions with the 2-torsion-free $\mathrm{CH}(\bar{Y})$ (let us call them *very weak*) are even weaker than the weak versions of [3, Remark on Page 370] where, roughly speaking, for Y with arbitrary $\mathrm{CH}(\bar{Y})$ the results are obtained up to an element of exponent 2 in $\mathrm{CH}(\bar{Y})$ and which (the weak versions) can be obtained (still in characteristic 0 only) with a help of the Landweber-Novikov operations (still in the algebraic cobordism theory) replacing the symmetric operations.

Although the very weak versions are already sufficient for all existing applications, we prove the weak versions as well (see Theorem 2.1 and Proposition 4.1). The proofs here are only a bit more complicated (than in the very weak case) and have an advantage: they avoid induction by dimension of the quadric[1] and therefore can be adopted to serve for the proof of Proposition 5.3 of the last section, where dimension of the quadric is specific. Proposition 5.3 is the final step in extending construction of fields of u-invariant $2^r + 1$ for any $r \geq 3$ given in [5] for characteristic 0 (the case of $r = 3$ has been done earlier and

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[1]The inductive proofs of the very weak versions do not work for the weak versions because the Steenrod operations under use do not map the subgroup of the modulo 2 classes of exponent 2 elements to itself in general.

for arbitrary characteristic $\neq 2$ by Oleg Izhboldin, [2]) to arbitrary characteristic $\neq 2$, see Theorem 5.1.

In our proofs (for both weak and very weak versions), the base field is allowed to be of any characteristic different from 2 because the Landweber-Novikov operations are replaced here by the Steenrod operations on the modulo 2 Chow groups.

Although the proofs given here are inspired by the original ones, they are not completely parallel. In particular, our proofs employ essentially less computations.

We refer to [3] for an introduction into the subject. Notation is introduced in the beginning of Section 1.

Contents

1. Basic result: very weak version

Let F be a field of characteristic $\neq 2$, Q a smooth projective quadric over F of dimension $n \geq 0$, Y a smooth quasi-projective F-variety (a *variety* is a separated scheme of finite type over a field).

The function field $F(Q)$ is defined if $n \geq 1$ or if Q is anisotropic. In the case of $n = 0$ and isotropic Q we have $Q = \operatorname{Spec} F \coprod \operatorname{Spec} F$ and we set $F(Q) := F$.

We write $\operatorname{CH}(Y)$ for the integral Chow group of Y (see [1, Chapter X]) and we write $\operatorname{Ch}(Y)$ for $\operatorname{CH}(Y)$ modulo 2. We write $\operatorname{CH}(\bar{Y})$ for the colimit of $\operatorname{CH}(Y_L)$ and we write $\operatorname{Ch}(\bar{Y})$ for the colimit of $\operatorname{Ch}(Y_L)$, where L runs over all field extensions of F. An element of $\operatorname{Ch}(\bar{Y})$ (or of $\operatorname{CH}(\bar{Y})$) is *$L$-rational*, if it is in the image of $\operatorname{Ch}(Y_L) \to \operatorname{Ch}(\bar{Y})$ (resp., $\operatorname{CH}(Y_L) \to \operatorname{CH}(\bar{Y})$) ($F$-rational elements are sometimes simply called *rational*).

A stronger version of the following result has been proved in characteristic 0 in [3, Corollary 3.5(1)]:

Theorem 1.1. *Assume that the group $\operatorname{CH}(\bar{Y})$ is 2-torsion-free. Then for any integer $m < n/2$, any $F(Q)$-rational element of $\operatorname{Ch}^m(\bar{Y})$ is F-rational.*

Proof. We induct on n and m. The statement being trivial for negative m, we may assume that $m \geq 0$. In particular, $n \geq 1$. Let y be an element of $\operatorname{Ch}^m(Y_{F(Q)})$. We are going to show that the image $\bar{y} \in \operatorname{Ch}^m(\bar{Y})$ of y is rational.

Let us fix an element $x \in \operatorname{Ch}^m(Q \times Y)$ mapped to y under the surjection

$$\operatorname{Ch}^m(Q \times Y) \twoheadrightarrow \operatorname{Ch}^m(Y_{F(Q)})$$

given by the pull-back with respect to the generic point of Q times the identity of Y. Since over some field extension of F the variety Q becomes cellular (with the Chow classes of the cells in codimensions $\leq m$ given by the powers of the hyperplane section class, see e.g. [1, §68]), the image $\bar{x} \in \mathrm{Ch}^m(\bar{Q} \times \bar{Y})$ of x decomposes as

$$\bar{x} = h^0 \times \bar{y} + h^1 \times y^{m-1} + \cdots + h^m \times y^0$$

with some $y^i \in \mathrm{Ch}^i(\bar{Y})$, where $h^i \in \mathrm{Ch}^i(\bar{Q})$ is the ith power of the hyperplane section class. By induction, all the elements y^i are rational. Indeed, the element y^{m-1} is $F(Q)(Q')$-rational, the element y^{m-2} is $F(Q)(Q')(Q'')$-rational, and so on, where Q' is a projective quadric over $F(Q)$ of dimension $n-2$ Witt-equivalent to $Q_{F(Q)}$. (The element y^{m-1} is $F(Q)(Q')$-rational, because $y^{m-1} = pr_*(l_1 \cdot \bar{x})$, where pr is the projection $Q \times Y \to Y$ and $l_1 \in \mathrm{Ch}_1(\bar{Q})$ is the class of a line which is $F(Q)(Q')$-rational. Similarly, the element y^{m-2} is $F(Q)(Q')(Q'')$-rational, because $y^{m-2} = pr_*(l_2 \cdot \bar{x})$, where $l_2 \in \mathrm{Ch}_2(\bar{Q})$ is the class of a plane which is $F(Q)(Q')(Q'')$-rational. And so on.)

Since moreover all the elements h^i are rational, it follows that the element

$$h^0 \times \bar{y} = [\bar{Q}] \times \bar{y} \in \mathrm{Ch}^m(\bar{Q} \times \bar{Y})$$

is rational. Changing notation, let now $x \in \mathrm{Ch}^m(Q \times Y)$ be a representative of $h^0 \times \bar{y}$. For every $i = 0, 1, \ldots, m$, let s^i be the image in $\mathrm{CH}^{m+i}(\bar{Q} \times \bar{Y})$ of an *integral* class in $\mathrm{CH}^{m+i}(Q \times Y)$ representing the modulo 2 class $S^i(x) \in \mathrm{Ch}^{m+i}(Q \times Y)$, where S^i is the ith cohomological Steenrod operation [1, Definition 61.7]. (This choice of s^i is important for Lemma 1.3; in Lemma 1.2 s^i can be any representative of $S^i(\bar{x})$.) We also set $s^i := 0$ for $i > m$ as well as for $i < 0$. Therefore, for any integer i, s^i is the image in $\mathrm{CH}^{m+i}(\bar{Q} \times \bar{Y})$ of an integral representative (in $\mathrm{CH}^{m+i}(Q \times Y)$) of $S^i(x)$.

From now on we are mostly working with the *integral* Chow groups and we use the notation h^i for the ith power of the *integral* hyperplane section class in $\mathrm{CH}^i(\bar{Q})$ as well. As before, pr stands for the projection $Q \times Y \to Y$ and pr_* for the corresponding push-forward homomorphism of Chow groups.

Lemma 1.2. *For any i with $0 \leq i \leq n-1$, $pr_*(h^i s^{n-i}) \equiv 0 \pmod 4$ in $\mathrm{CH}^m(\bar{Y})$.*

Proof. Since $s^{n-i} = 0$ for $n - i > m$, we may assume that $i \geq n - m$ in which case $h^i \equiv 0 \pmod 2$ in $\mathrm{CH}^i(\bar{Q})$ (namely, $h^i = 2l_{n-i}$). Since $s^{n-i} \pmod 2 = S^{n-i}([\bar{Q}] \times \bar{y}) = [\bar{Q}] \times S^{n-i}(\bar{y})$, we are done. □

Let d be any integer satisfying $m < d \leq n$. Let P be a smooth subquadric of Q of dimension d; we write *in* for the imbedding

$$(P \hookrightarrow Q) \times \mathrm{id}_Y : P \times Y \hookrightarrow Q \times Y.$$

Lemma 1.3. *For any integer r, the element*

$$pr_* \sum_{i=0}^{r} c_i(-T_P) \cdot in^*\, s^{r-i} \in \mathrm{CH}^{r+m-d}(\bar{Y})$$

(where T_P is the tangent bundle of P, c_i are the Chern classes, and pr is the projection $P \times Y \to Y$) is twice a rational element.

Proof. We induct on r. For $r \leq -1$ the statement is trivial because the sum is empty. Thus we may assume that $r \geq 0$.

Since $pr_* \, in^* \, x \in \mathrm{Ch}^{m-d}(Y) = 0$ (because $m - d < 0$) and the rth *homological* Steenrod operation S_r commutes with pr_*, the modulo 2 Chow class $pr_* \, S_r \, in^* \, x$ is 0. Therefore the integral Chow class

$$pr_* \sum_{i,j=0}^{r} c_i(-T_P) \cdot c_j(-T_Y) \cdot in^* \, s^{r-i-j}$$

is equal to a rational element multiplied by 2. By the induction hypothesis, for any fixed $j > 0$, the partial sum

$$pr_* \sum_{i=0}^{r-j} c_i(-T_P) \cdot c_j(-T_Y) \cdot in^* \, s^{r-i-j} = c_j(-T_Y) \cdot pr_* \sum_{i=0}^{r-j} c_i(-T_P) \cdot in^* \, s^{r-i-j}$$

is also a rational element multiplied by 2. The remaining part of the sum is the sum of the statement. $\square$

We apply Lemma 1.3 taking as d the maximal integer $\leq n$ of the shape a power of 2 minus 1 (note that $d \geq n/2 > m$) and with $r = d$. For any $i \neq d$, the ith summand of the sum of the statement of Lemma 1.3 is a multiple of

$$pr_*(h^i \cdot in^* \, s^{d-i}) = pr_*(h^{n-d+i} s^{d-i})$$

(the first pr here is the projection $P \times Y \to Y$ while the second pr is $Q \times Y \to Y$; the first h is the hyperplane section class of P, the second – of Q), which is 0 modulo 4 by Lemma 1.2. Therefore the remaining (dth) summand

$$pr_* \left(c_d(-T_P) \cdot in^* \, s^0 \right)$$

is congruent modulo 4 to twice a rational element $a \in \mathrm{CH}^m(\bar{Y})$. By [1, Lemma 78.1] we have $c_d(-T_P) = \binom{-d-2}{d} h^d$. The binomial coefficient $\binom{-d-2}{d} = \binom{2d+1}{d}$ is odd (because d is a power of 2 minus 1, cf. [1, Lemma 78.6]). Since $h^d \in \mathrm{CH}^d(\bar{P})$ modulo 2 is 0 and $in^* \, s^0 \in \mathrm{CH}^m(\bar{P} \times \bar{Y})$ is congruent modulo 2 to $[\bar{P}] \times \mathbf{y}$, where $\mathbf{y} \in \mathrm{CH}^m(\bar{Y})$ is an integral representative of $\bar{y} \in \mathrm{Ch}^m(\bar{Y})$, the product $c_d(-T_P) \cdot in^* \, s^0$ is congruent modulo 4 to $h^d \times \mathbf{y}$. Finally,

$$pr_* \left(h^d \times \mathbf{y} \right) = 2\mathbf{y},$$

and we get the congruence $2\mathbf{y} \equiv 2a$ modulo 4 in $\mathrm{CH}^m(\bar{Y})$. Since the group $\mathrm{CH}^m(\bar{Y})$ is 2-torsion-free, it follows dividing by 2 that the element $\bar{y} = \mathbf{y} \pmod 2 \in \mathrm{Ch}^m(\bar{Y})$ is the class modulo 2 of the rational element $a \in \mathrm{CH}^m(\bar{Y})$. Thus Theorem 1.1 is proved. $\square$

2. Basic result: weak version

In this section we continue to use notation introduced in the beginning of Section 1. We are going to prove a stronger version of Theorem 1.1 (which is still weaker than the result proved in characteristic 0 in [3, Corollary 3.5(1)] and is precisely the *weak version* mentioned in [3, Remark on Page 370]):

Theorem 2.1. *For any integer $m < n/2$, any $F(Q)$-rational element of $\mathrm{CH}^m(\bar{Y})$ is congruent modulo 2 and 2-torsion to an F-rational element.*

Proof. We assume that $m \geq 0$ in the proof. Let y be an element of $\mathrm{CH}^m(Y_{F(Q)})$. We are going to show that the image $\bar{y} \in \mathrm{CH}^m(\bar{Y})$ of y is congruent modulo 2 to the sum of a rational element and an element of exponent 2.

Let us fix an element $x \in \mathrm{Ch}^m(Q \times Y)$ mapped to $y \mod 2$ under the surjection

$$\mathrm{Ch}^m(Q \times Y) \twoheadrightarrow \mathrm{Ch}^m(Y_{F(Q)}).$$

The image $\bar{x} \in \mathrm{Ch}^m(\bar{Q} \times \bar{Y})$ of x decomposes as

$$\bar{x} = h^0 \times y^m + h^1 \times y^{m-1} + \cdots + h^m \times y^0$$

with some $y^i \in \mathrm{Ch}^i(\bar{Y})$, where $y^m = \bar{y} \mod 2$.

For every $i = 0, 1, \ldots, m$, let s^i be the image in $\mathrm{CH}^{m+i}(\bar{Q} \times \bar{Y})$ of an element in $\mathrm{CH}^{m+i}(Q \times Y)$ representing $S^i(x) \in \mathrm{Ch}^{m+i}(Q \times Y)$. We also set $s^i := 0$ for $i > m$ as well as for $i < 0$.

We still have Lemma 1.3 for the elements s^i (with the same proof). In particular, for the maximal integer $d \leq n$ of the shape a power of 2 minus 1 and a smooth subquadric $P \subset Q$ of dimension d, the element

$$pr_* \sum_{i=0}^{d} c_i(-T_P) \cdot in^* \, s^{d-i} \in \mathrm{CH}^m(\bar{Y})$$

(where in is the imbedding $(P \hookrightarrow Q) \times \mathrm{id}_Y : P \times Y \hookrightarrow Q \times Y$, T_P is the tangent bundle of P, c_i are the Chern classes, and pr is the projection $P \times Y \to Y$) is twice a rational element. Since $c_i(-T_P) = \binom{-d-2}{i} \cdot h^i$ and the binomial coefficient $\binom{-d-2}{i} = \binom{d+i+1}{i}$ is odd for any $i = 0, 1, \ldots, d$, we get that the element

$$pr_* \sum_{i=0}^{d} h^i \cdot in^* \, s^{d-i} \in \mathrm{CH}^m(\bar{Y})$$

is twice a rational element. Finally, since $pr_*(h^i \cdot in^* \, s^{d-i}) = pr_*(h^{n-d+i} \cdot s^{d-i})$, where pr on the right hand side is the projection $Q \times Y \to Y$, we get that the sum

$$\sum_{i=0}^{d} pr_*(h^{n-d+i} \cdot s^{d-i}) \in \mathrm{CH}^m(\bar{Y})$$

is twice a rational element.

We would like to compute the sum obtained modulo 4. Since $s^{d-i} = 0$ if $d - i > m$, the ith summand is 0 for any $i < d - m$. Otherwise – if $i \geq d - m$ – the factor h^{n-d+i} is divisible by 2 (because $n - d + i \geq n - m > n/2$) and in order to compute the ith summand modulo 4 it suffices to compute s^{d-i} modulo 2, that is, to compute $S^{d-i}(\bar{x})$.

We recall that

$$\bar{x} = h^0 \times y^m + h^1 \times y^{m-1} + \cdots + h^m \times y^0.$$

Therefore $S^{d-i}(\bar{x})$ is represented by

$$\sum_{k=0}^{m} \sum_{l=0}^{d-i} \binom{k}{d-i-l} (h^{d+k-i-l} \times \varepsilon_{k,l}),$$

where $\varepsilon_{k,l} \in \mathrm{CH}^{m-k+l}(\bar{Y})$ is an integral representative of $S^l(y^{m-k})$ which in the case of $l > m - k$ we choose to be 0. Besides, we choose $\varepsilon_{0,0} = \bar{y}$.

It follows that for any $i \geq d-m$, the summand $pr_*(h^{n-d+i} \cdot s^{d-i})$ is congruent modulo 4 to

$$2\sum_{k=0}^{m}\binom{k}{d-i-k}\varepsilon_k,$$

where $\varepsilon_k := \varepsilon_{k,k}$. Note that $\varepsilon_k = 0$ for $k > m-k$, that is for $k > m/2$. We get that the sum

$$2\sum_{i=d-m}^{d}\sum_{k=0}^{[m/2]}\binom{k}{d-i-k}\varepsilon_k$$

is congruent modulo 4 to twice a rational element $a \in \mathrm{CH}^m(\bar{Y})$.

For every $k = 0, 1, \dots, [m/2]$, the total coefficient near ε_k is twice the sum of all binomial coefficients $\binom{k}{\cdot}$ which (the sum) is equal to 2^k and for $k \geq 1$ is even. It follows that $2\varepsilon_0 \equiv 2a \pmod 4$. Dividing by 2, we get that ε_0 is congruent modulo 2 to the rational element a plus an element of exponent 2. Since $\varepsilon_0 = \bar{y}$, we are done with the proof of Theorem 2.1. $\square$

3. Beyond basic result: very weak version

In this section we continue to use notation introduced in the beginning of Section 1 and we are assuming that the variety Y is geometrically irreducible. The main result of this section is the following proposition (a stronger version of it has been proved in characteristic 0 in [3, Proposition 3.3(2)]):

Proposition 3.1. *Assume that $n = 2m$ or $n = 2m-1$ for some integer $m \geq 1$. Assume that the group $\mathrm{CH}(\bar{Y})$ is 2-torsion-free. Let x be an element of $\mathrm{Ch}^m(Q \times Y)$. If the image of x under the composition*

$$\mathrm{Ch}^m(Q \times Y) \to \mathrm{Ch}^m(Q_{F(Y)}) \to \mathrm{Ch}^m(\bar{Q})$$

is rational, then the image of x under the composition

$$\mathrm{Ch}^m(Q \times Y) \to \mathrm{Ch}^m(Y_{F(Q)}) \to \mathrm{Ch}^m(\bar{Y})$$

is also rational.

The following two theorems are consequences of Proposition 3.1. A stronger version of the first one has been proved in characteristic 0 in [3, Corollary 3.5(2)]:

Theorem 3.2. *Assume that $n = 2m$ or $n = 2m-1$ for some integer $m \geq 1$. Assume that the group $\mathrm{CH}(\bar{Y})$ is 2-torsion-free. Assume that the quadric $Q_{F(Y)}$ is not completely split. Then any $F(Q)$-rational element of $\mathrm{Ch}^m(\bar{Y})$ is F-rational.*

Proof. Let y be an arbitrary element of $\mathrm{Ch}^m(Y_{F(Q)})$. Let x be an element of $\mathrm{Ch}^m(Q \times Y)$ mapped to y under the surjection

$$\mathrm{Ch}^m(Q \times Y) \to \mathrm{Ch}^m(Y_{F(Q)}).$$

Since $Q_{F(Y)}$ is not completely split, the group of $F(Y)$-rational elements in $\mathrm{Ch}^m(\bar{Q})$ is generated by h^m (where the modulo 2 Chow class h^m is trivial if $n = 2m-1$). In particular, any $F(Y)$-rational element of $\mathrm{Ch}^m(\bar{Q})$ is F-rational. Therefore, by Proposition 3.1, the image of x under the composition

$$\mathrm{Ch}^m(Q \times Y) \to \mathrm{Ch}^m(Y_{F(Q)}) \to \mathrm{Ch}^m(\bar{Y})$$

(which coincides with the image of $y \in \mathrm{Ch}^m(Y_{F(Q)})$ in $\mathrm{Ch}^m(\bar{Y})$) is rational. □

To formulate the second theorem (whose stronger version has been proved in characteristic 0 in [3, Statement 3.8]), we need an additional notation. Let G be the maximal orthogonal grassmannian associated to Q as in [1, §85]. Let $z \in \mathrm{Ch}(\bar{G})$ be the class of the subvariety in G of the maximal linear subspaces in $Q_{F(Q)}$ passing through a fixed rational point of $Q_{F(Q)}$ (this z is one of the generators of the ring $\mathrm{Ch}(\bar{G})$ given in [1, §86], namely the generator of maximal codimension).

Theorem 3.3. *Assume that $n = 2m$ or $n = 2m - 1$ for some integer $m \geq 1$. Assume that the group $\mathrm{CH}(\bar{Y})$ is 2-torsion-free. Finally, assume that the element z is rational. Then any $F(Q)$-rational element of $\mathrm{Ch}^m(\bar{Y})$ is F-rational.*

Proof. According to [3, Statement 3.10], the rationality of z ensures that for any element $x \in \mathrm{Ch}^m(Q \times Y)$ there exists an element $x' \in \mathrm{Ch}^m(Q \times Y)$ such that the image of x' under the composition

$$\mathrm{Ch}^m(Q \times Y) \to \mathrm{Ch}^m(Q_{F(Y)}) \to \mathrm{Ch}^m(\bar{Q})$$

is rational and the image of x' under the composition

$$\mathrm{Ch}^m(Q \times Y) \to \mathrm{Ch}^m(Y_{F(Q)}) \to \mathrm{Ch}^m(\bar{Y})$$

coincides with the image of x. The proof of [3, Statement 3.10] does not use the theory of algebraic cobordism and is valid over fields of any characteristic (even including 2). Theorem 3.3 follows by Proposition 3.1. □

Proof of Proposition 3.1. We induct on m. We may assume that Q is anisotropic. In this case, the condition on x ensures that

$$\bar{x} = h^0 \times y + h^1 \times y^{m-1} + \cdots + h^m \times y^0$$

for some $y^i \in \mathrm{Ch}^i(\bar{Y})$, $i = 0, \ldots, m-1$, and some $y \in \mathrm{Ch}^m(\bar{Y})$. (Note that $h^m = 0$ in the case of $n = 2m - 1$.) The image of x under the composition

$$\mathrm{Ch}^m(Q \times Y) \to \mathrm{Ch}^m(Y_{F(Q)}) \to \mathrm{Ch}^m(\bar{Y})$$

is equal to y, and we will show y is rational.

By induction, all the elements y^i are rational. Indeed, applying the incidence correspondence of [1, Lemma 72.3] to the element $x_{F(Q)} \in \mathrm{Ch}^m(Q \times Y)_{F(Q)}$, we get an element $x' \in \mathrm{Ch}^{m-1}(Q' \times Y_{F(Q)})$ (where Q' is a projective quadric over $F(Q)$ of dimension $n-2$ Witt-equivalent to $Q_{F(Q)}$) such that $\bar{x'} = h^0 \times y^{m-1} + h^1 \times y^{m-2} + \cdots + h^{m-1} \times y^0$. It follows by induction that the elements $y^{m-1}, \ldots, y^0$ are $F(Q)$-rational. Therefore, by Theorem 1.1, they are F-rational.

Since moreover all the elements h^i are rational, it follows that the element

$$h^0 \times y = [\bar{Q}] \times y \in \mathrm{Ch}^m(\bar{Q} \times \bar{Y})$$

is rational. Changing notation, let now $x \in \mathrm{Ch}^m(Q \times Y)$ be a representative of $h^0 \times y$. For every $i = 0, 1, \ldots, m-1$, let s^i be the image in $\mathrm{CH}^{m+i}(\bar{Q} \times \bar{Y})$ of an element in $\mathrm{CH}^{m+i}(Q \times Y)$ representing the modulo 2 class $S^i(x) \in \mathrm{Ch}^{m+i}(Q \times Y)$. We also set $s^i := 0$ for $i > m$ as well as for $i < 0$. Finally, we set $s^m := (s^0)^2$. Therefore, for any integer i, s^i is the image in $\mathrm{CH}^{m+i}(\bar{Q}\times\bar{Y})$ of an integral representative (in $\mathrm{CH}^{m+i}(Q\times Y)$) of $S^i(x)$.

Lemma 3.4. *For any i with $0\leq i\leq n-1$, $pr_*(h^i s^{n-i})\equiv 0 \pmod 4$ in $\mathrm{CH}^m(\bar{Y})$.*

Proof. Since $s^{n-i}=0$ for $n-i>m$, we may assume that $i\geq n-m$. If $i>n-m$, then $h^i\equiv 0 \pmod 2$ in $\mathrm{CH}^i(\bar{Q})$. Since $s^{n-i} \pmod 2 = S^{n-i}([\bar{Q}]\times y)=[\bar{Q}]\times S^{n-i}(y)$, we are done in the case of $i>n-m$.

To finish the proof, let us consider the case of $i=n-m$. Since the element s^0 is congruent modulo 2 to $\bar{Q}\times\mathbf{y}$, where $\mathbf{y}\in\mathrm{CH}^m(\bar{Y})$ is an integral representative of $y\in\mathrm{Ch}^m(\bar{Y})$, the element $s^m=(s^0)^2$ is congruent modulo 4 to $\bar{Q}\times\mathbf{y}^2$. Therefore $pr_*(h^{n-m}s^m)$ modulo 4 is 0. ☐

Let d be any integer satisfying $m\leq d\leq n$. Let P be a smooth subquadric of Q of dimension d; we write in for the imbedding

$$(P\hookrightarrow Q)\times\mathrm{id}_Y : P\times Y\hookrightarrow Q\times Y.$$

Lemma 3.5. *For any integer r, the element*

$$pr_*\sum_{i=0}^{r} c_i(-T_P)\cdot in^*\, s^{r-i}\in\mathrm{CH}^{r+m-d}(\bar{Y})$$

(where T_P is the tangent bundle of P, c_i are the Chern classes, and pr is the projection $P\times Y\to Y$) is twice a rational element.

Proof. We almost repeat the proof of Lemma 1.3, but the case of $d=m$ here is new.

We induct on r. For $r\leq -1$ the statement is trivial because the sum is empty. Thus we may assume that $r\geq 0$.

Note that the element $pr_*\, in^*\, x\in\mathrm{Ch}^{m-d}(Y)$ is 0. Indeed, if $m<d$, then the whole group $\mathrm{Ch}^{m-d}(Y)$ is 0. Otherwise we have $m=d$. Since the group $\mathrm{Ch}^0(Y)$ imbeds into $\mathrm{Ch}^0(\bar{Y})$, triviality of $pr_*\, in^*\, x$ follows from triviality of $pr_*\, in^*\, \bar{x}$.

Since the *homological* Steenrod operation S_r commutes with pr_*, the modulo 2 Chow class $pr_*\, S_r\, in^*\, x$ is 0. Therefore the integral Chow class

$$pr_*\sum_{i,j=0}^{r} c_i(-T_P)\cdot c_j(-T_Y)\cdot in^*\, s^{r-i-j}$$

is equal to a rational element multiplied by 2. By the induction hypothesis, for any fixed $j>0$, the partial sum

$$pr_*\sum_{i=0}^{r-j} c_i(-T_P)\cdot c_j(-T_Y)\cdot in^*\, s^{r-i-j} = c_j(-T_Y)\cdot pr_*\sum_{i=0}^{r-j} c_i(-T_P)\cdot in^*\, s^{r-i-j}$$

is also a rational element multiplied by 2. The remaining part of the sum is the sum of the statement. ☐

We apply Lemma 3.5 taking as d the maximal integer $\leq n$ of the shape a power of 2 minus 1 (note that $d\geq m$) and with $r=d$. For any $i\neq d$, the ith summand of the sum of the statement of Lemma 3.5 is a multiple of

$$pr_*(h^i\cdot in^*\, s^{d-i}) = pr_*(h^{n-d+i}s^{d-i})$$

which is 0 modulo 4 by Lemma 3.4. Therefore the remaining (dth) summand

$$pr_*\left(c_d(-T_P)\cdot in^*\,s^0\right)$$

is congruent modulo 4 to twice a rational element $a\in \mathrm{CH}^m(\bar{Y})$. We have $c_d(-T_P)=\binom{-d-2}{d}h^d$. The binomial coefficient $\binom{-d-2}{d}=\binom{2d+1}{d}$ is odd. Since $h^d\in\mathrm{CH}^d(\bar{P})$ modulo 2 is 0 and $in^*\,s^0\in\mathrm{CH}^m(\bar{P}\times\bar{Y})$ is congruent modulo 2 to $[\bar{P}]\times\mathbf{y}$, where $\mathbf{y}\in\mathrm{CH}^m(\bar{Y})$ is an integral representative of $y\in\mathrm{Ch}^m(\bar{Y})$, the product $c_d(-T_P)\cdot in^*\,s^0$ is congruent modulo 4 to $h^d\times\mathbf{y}$. Finally,

$$pr_*\left(h^d\times\mathbf{y}\right)=2\mathbf{y},$$

and we get the congruence $2\mathbf{y}\equiv 2a$ modulo 4 in $\mathrm{CH}^m(\bar{Y})$. Since the group $\mathrm{CH}^m(\bar{Y})$ is 2-torsion-free, it follows dividing by 2 that the element $y=\mathbf{y}\pmod 2\in\mathrm{Ch}^m(\bar{Y})$ is the class modulo 2 of the rational element $a\in\mathrm{CH}^m(\bar{Y})$. Thus Proposition 3.1 is proved. □

4. Beyond basic result: weak version

In this section we continue to use notation introduced in the beginning of Section 1 and we are assuming that the variety Y is geometrically irreducible. The main result of this section is the following stronger version of Proposition 3.1 (which is still weaker than the result proved in characteristic 0 in [3, Proposition 3.3(2)] and is precisely the *weak version* mentioned in [3, Remark on Page 370]):

Proposition 4.1. *Assume that $n=2m$ or $n=2m-1$ for some integer $m\geq 1$. Let x be an element of $\mathrm{Ch}^m(Q\times Y)$. If the image of x under the composition*

$$\mathrm{Ch}^m(Q\times Y)\to\mathrm{Ch}^m(Q_{F(Y)})\to\mathrm{Ch}^m(\bar{Q})$$

is rational, then the image of x under the composition

$$\mathrm{Ch}^m(Q\times Y)\to\mathrm{Ch}^m(Y_{F(Q)})\to\mathrm{Ch}^m(\bar{Y})$$

differs from a rational element by the modulo 2 class of an exponent 2 element of $\mathrm{CH}^m(\bar{Y})$.

As a consequence of Proposition 4.1, we get the corresponding stronger versions of Theorems 3.2 and 3.3.

Proof of Proposition 4.1. We may assume that Q is anisotropic. In this case, the condition on x ensures that

$$\bar{x}=h^0\times y^m+h^1\times y^{m-1}+\cdots+h^m\times y^0$$

for some $y^i\in\mathrm{Ch}^i(\bar{Y})$, $i=0,1,\dots,m$ (where $h^m=0$ in the case of $n=2m-1$). The image of x under the composition

$$\mathrm{Ch}^m(Q\times Y)\to\mathrm{Ch}^m(Y_{F(Q)})\to\mathrm{Ch}^m(\bar{Y})$$

is equal to y^m, and we will show for an integral representative $\mathbf{y}\in\mathrm{CH}^m(\bar{Y})$ of y^m that $\mathbf{y}$ modulo 2 and 2-torsion is rational.

The elements $y^0,\dots,y^{m-1}$ are the modulo 2 classes of some elements

$$\mathbf{y}^0\in\mathrm{CH}^0(\bar{Y}),\dots,\mathbf{y}^{m-1}\in\mathrm{CH}^{m-1}(\bar{Y}).$$

For every $i = 1, \dots, m-1$, let s^i be the image in $\mathrm{CH}^{m+i}(\bar{Q} \times \bar{Y})$ of an element of $\mathrm{CH}^{m+i}(Q \times Y)$ representing the element $S^i(x) \in \mathrm{Ch}^{m+i}(Q \times Y)$. We also set $s^i := 0$ for $i > m$ as well as for $i < 0$. Finally, we set

$$s^0 := h^0 \times \mathbf{y} + h^1 \times \mathbf{y}^{m-1} + \dots + h^m \times \mathbf{y}^0 \in \mathrm{CH}^m(\bar{Y})$$

and we set $s^m := (s^0)^2$. Therefore, for any integer i, s^i is the image in $\mathrm{CH}^{m+i}(\bar{Q} \times \bar{Y})$ of an integral representative of $S^i(x)$.

Let d be the maximal integer $\leq n$ of the shape a power of 2 minus 1. Similarly as in Lemma 3.5 and in the proof of Theorem 2.1, we get that the sum

$$\sum_{i=d-m}^{d} pr_*(h^{n-d+i} \cdot s^{d-i}) \in \mathrm{CH}^m(\bar{Y})$$

is twice a rational element. We are going to compute this sum modulo 4.

For any $i > d-m$, the factor h^{n-d+i} present in the ith summand is divisible by 2. The other factor modulo 2 is $S^{d-i}(\bar{x})$ and is represented by

$$\sum_{k=0}^{m} \sum_{l=0}^{d-i} \binom{k}{d-i-l} (h^{d+k-i-l} \times \varepsilon_{k,l}),$$

where $\varepsilon_{k,l} \in \mathrm{CH}^{m-k+l}(\bar{Y})$ is an integral representative of $S^l(y^{m-k})$ which in the case of $l > m-k$ we choose to be 0. Besides, we choose $\varepsilon_{0,0} = \mathbf{y}$. Finally, in the case of even m, we choose $\varepsilon_{m/2,m/2} = (\mathbf{y}^{m/2})^2$.

It follows that for any $i > d-m$, we have the congruence

$$pr_*(h^{n-d+i} \cdot s^{d-i}) \equiv 2 \sum_{k=0}^{[m/2]} \binom{k}{d-i-k} \varepsilon_k \pmod 4, \tag{4.2}$$

where $\varepsilon_k := \varepsilon_{k,k}$.

For $i = d-m$ we have

$$pr_*(h^{n-m} \cdot s^m) = pr_*\Big(h^{n-m} \cdot (h^0 \times \mathbf{y} + h^1 \times \mathbf{y}^{m-1} + \dots + h^m \times \mathbf{y}^0)^2\Big)$$

which is 0 modulo 4 in the case of odd m. In the case of even m, this is congruent modulo 4 to $2(\mathbf{y}^{m/2})^2 = 2\varepsilon_{m/2}$. Therefore the congruence (4.2) holds for $i = d-m$ as well.

We get that the sum

$$2 \sum_{i=d-m}^{d} \sum_{k=0}^{[m/2]} \binom{k}{d-i-k} \varepsilon_k$$

is congruent modulo 4 to twice a rational element $a \in \mathrm{CH}^m(\bar{Y})$ and we finish as in the proof of Theorem 2.1: for every $k = 0, 1, \dots, [m/2]$, the total coefficient near ε_k is 2^{k+1}; it follows that $2\varepsilon_0 \equiv 2a \pmod 4$ and therefore ε_0 is congruent modulo 2 to the rational element a plus an element of exponent 2. Since $\varepsilon_0 = \mathbf{y}$, we are done with the proof of Proposition 4.1. $\square$

5. More beyond basic result: u-invariant

The aim of this section is the following result, proved for characteristic 0 in [5, Corollary 5.2]:

Theorem 5.1. *For any integer $r \geq 3$, any field F of characteristic $\neq 2$ is a subfield of a field of u-invariant $2^r + 1$.*

As explained in [5], Theorem 5.1 is a consequence of the following result (proved for characteristic 0 in [5, Theorem 5.1]):

Theorem 5.2. *Let P be a smooth projective quadric over F of dimension $2^r - 1$ (for some $r \geq 3$). Let G be the maximal and let G' be the "previous" (the "almost maximal") orthogonal grassmannians associated to P. For $i = 1, 2, \dots, 2^{r-1}$, let $e_i \in \mathrm{Ch}^i(\bar{G})$ be the standard generators of the ring $\mathrm{Ch}(\bar{G})$ as defined in [1, §86]. Let $e' \in \mathrm{Ch}^{2^{r-1}+1}(\bar{G}')$ be the class of the subvariety in G' of the linear subspaces in $P_{F(P)}$ passing through a fixed rational point. Let Q be a smooth projective quadric over F of dimension $2^r = \dim P + 1$. If the elements $e_1, e_2, \dots, e_{2^{r-1}-1}, e'$ are F-irrational, then they are also $F(Q)$-irrational.*

The statement on $e_1, \dots, e_{2^{r-1}-1}$ being given by Theorem 1.1 (note that the groups $\mathrm{CH}(\bar{G})$ and $\mathrm{CH}(\bar{G}')$ are torsion-free), we only need to prove irrationality of e'. The codimension of e' is $2^{r-1} + 1 = (\dim Q)/2 + 1$ so that even the results of Section 3 or 4 (where the codimension is $(\dim Q)/2$ or $(\dim Q + 1)/2$) are not appropriate. In order to deal with e', we prove the following result which constitutes the main content of this section and replaces [5, Proposition 3.5 and Corollary 3.6] in the proof of Theorem 5.2:

Proposition 5.3. *Let Q be a smooth projective quadric over F of dimension $n = 2^r$ for some $r \geq 2$. For $m := 2^{r-1} + 1$, let x be an element of $\mathrm{Ch}^m(Q \times Y)$ such that in the decomposition*

$$\bar{x} = h^0 \times y^m + h^1 \times y^{m-1} + \dots + h^{m-1} \times y^1 + l_{m-1} \times y' + l_{m-2} \times y^0 \in \mathrm{Ch}^m(\bar{Q} \times \bar{Y})$$

with $y^i \in \mathrm{Ch}^i(\bar{Y})$, $i = 0, 1, \dots, m$, and $y' \in \mathrm{Ch}^1(\bar{Y})$, the element y^0 is trivial. Then the element

$$y^m + S^1(y^{m-1}) + y^{m-1} \cdot y' \in \mathrm{Ch}^m(\bar{Y})$$

is rational modulo the classes modulo 2 of integral elements of exponent 2.

Remark 5.4. The condition on x of Proposition 5.3 is automatically fulfilled if the element $l_{m-2} \in \mathrm{Ch}(\bar{Q})$ is $F(Y)$-irrational.

Proof of Proposition 5.3. For every $i = 0, 1, \dots, m-1$, let s^i be the image in $\mathrm{CH}^{m+i}(\bar{Q} \times \bar{Y})$ of an element of $\mathrm{CH}^{m+i}(Q \times Y)$ representing the element $S^i(x) \in \mathrm{Ch}^{m+i}(Q \times Y)$. We also set $s^i := 0$ for $i > m$ as well as for $i < 0$. Finally, we set $s^m := (s^0)^2$.

Note that we have

$$s^0 := h^0 \times \mathbf{y}^m + h^1 \times \mathbf{y}^{m-1} + \dots + h^{m-1} \times \mathbf{y}^1 + l_{m-1} \times \mathbf{y}' + l_{m-2} \times \mathbf{y}^0 \in \mathrm{CH}^m(\bar{Y})$$

with some $\mathbf{y}^i \in \mathrm{CH}^i(\bar{Y})$, $i = 0, 1, \dots, m$, $\mathbf{y}' \in \mathrm{CH}^1(\bar{Y})$ (such a decomposition exists for every element of $\mathrm{CH}^m(\bar{Q} \times \bar{Y})$). Since $s^0 \mod 2 = \bar{x}$, $\mathbf{y}^0$ is divisible by 2. Since the element $2l_{m-2} = h^m \in \mathrm{CH}^m(\bar{Y})$ is rational, we may assume that the last summand in the above decomposition of s^0 is absent.

Let d be any integer with $m \le d \le n$. Similarly as in Lemma 3.5 and in the proof of Theorem 2.1, we get that the sum

$$\sum_{i=d-m}^{d} \binom{-d-2}{i} \cdot pr_*(h^{n-d+i} \cdot s^{d-i}) \in \mathrm{CH}^m(\bar{Y}) \tag{5.5}$$

is twice a rational element. We are going to use this statement for various values of d (actually, for 2 values). Note that the sum is a linear combination of always the same elements $pr_*(h^{n-m}s^m) = pr_*(h^{m-2}s^m)$, $pr_*(h^{n-m+1}s^{m-1}) = pr_*(h^{m-1}s^{m-1})$, ..., $pr_*(h^n s^0)$, only the coefficients vary with d.

Let us compute the ith element $pr_*(h^{n-d+i} \cdot s^{d-i})$ modulo 4. For any $i \ge d-m+2$, the factor h^{n-d+i} is divisible by 2. The other factor modulo 2 is $S^{d-i}(\bar{x})$ and it follows that

$$\text{for any } i \ge d-m+2,\quad pr_*(h^{n-d+i} \cdot s^{d-i}) \equiv 2\sum_{k=0}^{[m/2]} \binom{k}{d-i-k} \varepsilon_k \pmod 4,$$

where ε_k is an integral representative of $S^k(y^{m-k})$ which in the case of $k > m-k$ we choose to be 0. Besides, we choose $\varepsilon_{0,0} = \mathbf{y}^m$.

For $i = d-m$, the ith summand is

$$\begin{aligned} pr_*(h^{m-2} \cdot s^m) &= pr_*\Big(h^{m-2} \cdot (h^0 \times \mathbf{y}^m + h^1 \times \mathbf{y}^{m-1} + \cdots + h^{m-1} \times \mathbf{y}^1 + l_{m-1} \times \mathbf{y}')^2\Big) \\ &\equiv 2\, pr_*\Big(h^{m-2} \cdot (h^1 \times \mathbf{y}^{m-1}) \cdot (l_{m-1} \times \mathbf{y}')\Big) = 2 \cdot \mathbf{y}^{m-1} \cdot \mathbf{y}' \end{aligned}$$

(where the congruence is modulo 4).

We do not compute by now the remaining summand $pr_*(h^{m-1}s^{m-1})$ (corresponding to $i = d-m+1$).

We are going to consider the sum (5.5) for two following values of d: $d = 2^r - 1 = n-1$ (this is the biggest integer of the shape a power of 2 minus 1 non-exceeding n, the choice we always use) and $d = 2^r = n$. For the first choice of d, since the binomial coefficient $\binom{-d-2}{i}$ is odd for every $i = 0, 1, \ldots, d$, we get that the sum

$$2\mathbf{y}^{m-1}\mathbf{y}' + pr_*(h^{m-1}s^{m-1}) + 2\sum_{i=d-m+2}^{d} \sum_{k=0}^{[m/2]} \binom{k}{d-i-k} \varepsilon_k$$

is congruent modulo 4 to twice a rational element $a \in \mathrm{CH}^m(\bar{Y})$.[2] For every k with $0 < k < (m-1)/2$, the coefficient near ε_k is twice the sum of all binomial coefficients $\binom{k}{\cdot}$ and therefore is divisible by 4. The coefficient near ε_0 is 2 and the coefficient near $\varepsilon_{(m-1)/2}$ is also 2 (in the sum of the binomial coefficients $\binom{(m-1)/2}{\cdot}$ occurring near $\varepsilon_{(m-1)/2}$ the coefficient $\binom{(m-1)/2}{(m-1)/2} = 1$ is missing). Therefore the congruence we get with the first choice of d is

$$2\mathbf{y}^{m-1}\mathbf{y}' + pr_*(h^{m-1}s^{m-1}) + 2\varepsilon_0 + 2\varepsilon_{(m-1)/2} \equiv 2a \pmod 4. \tag{5.6}$$

[2] A priori, we should put an integer coefficient representing $\binom{-d-2}{d-m+1}$ modulo 4 near the summand with pr_*, that is, $+1$ or -1. (This looks like the first case where we have to compute a binomial coefficient modulo 4, not just modulo 2.) But a fortiori, the summand is 0 modulo 2 so that we can put any of ± 1. Another argument is rationality of the summand so that changing the sign we do not change the statement.

For the second choice of d, the binomial coefficient $\binom{-d-2}{i}$ with $i = 0, 1, \ldots, d$ is odd for even $i < d$ and is even otherwise (that is, for odd i as well as for $i = d$). Since the integer $d - m = 2^{r-1} - 1$ is odd, we get that the sum

$$-pr_*(h^{m-1}s^{m-1}) + 2\sum_{\substack{i=d-m+3\\ \text{even } i}}^{d-2} \sum_{k=0}^{[m/2]} \binom{k}{d-i-k}\varepsilon_k$$

is congruent modulo 4 to twice a rational element $b \in \mathrm{CH}^m(\bar{Y})$.[2] Note that the coefficient near ε_0 is 0 here. Since for any $k \geq 1$, the sum of all binomial coefficients $\binom{k}{\cdot}$ with $\cdot$ of a fixed parity is equal to 2^{k-1}, only the coefficients near ε_1 and $\varepsilon_{(m-1)/2}$ survive modulo 4, where the coefficient near $\varepsilon_{(m-1)/2}$ survives because the binomial coefficient $\binom{(m-1)/2}{(m-1)/2}$ is missing. Therefore the congruence we get with the second choice of d is

$$-pr_*(h^{m-1}s^{m-1}) + 2\varepsilon_1 + 2\varepsilon_{(m-1)/2} \equiv 2b \pmod 4. \tag{5.7}$$

Adding together the congruences (5.6) and (5.7), we get that

$$2\mathbf{y}^{m-1}\mathbf{y}' + 2\varepsilon_0 + 2\varepsilon_1 \equiv 2(a+b) \pmod 4.$$

Dividing by 2, we get Proposition 5.3 because $\mathbf{y}^{m-1} \mod 2 = y^{m-1}$, $\mathbf{y}' \mod 2 = y'$, $\varepsilon_0 \mod 2 = \mathbf{y}^m \mod 2 = y^m$, and $\varepsilon_1 \mod 2 = S^1(y^{m-1})$. □

Proof of Theorem 5.2. All parts of the proof of Theorem 5.2 given in [5, Theorem 5.1] are free from the algebraic cobordism theory and work in any characteristic $\neq 2$ except for [5, Theorem 3.1] (replaced here by Theorem 1.1 or its stronger version Theorem 2.1), [5, Proposition 3.11] (replaced here by Theorem 3.3), and [5, Corollary 3.6] (replaced here by Proposition 5.3 with Remark 5.4).

□

References

[1] Elman, R., Karpenko, N., and Merkurjev, A. *The algebraic and geometric theory of quadratic forms*, vol. 56 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2008.
[2] Izhboldin, O. T. Fields of u-invariant 9. *Ann. of Math. (2) 154*, 3 (2001), 529–587.
[3] Vishik, A. Generic points of quadrics and Chow groups. *Manuscripta Math. 122*, 3 (2007), 365–374.
[4] Vishik, A. Symmetric operations in algebraic cobordism. *Adv. Math. 213*, 2 (2007), 489–552.
[5] Vishik, A. Fields of u-invariant $2^r + 1$. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II*, vol. 270 of *Progr. Math.* Birkhäuser Boston Inc., Boston, MA, 2009, pp. 661–685.

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