Torsors, Reductive Group Schemes and Extended Affine Lie Algebras

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Abstract

We give a detailed description of the torsors that correspond to multiloop algebras. These algebras are twisted forms of simple Lie algebras extended over Laurent polynomial rings. They play a crucial role in the construction of Extended Affine Lie Algebras (which are higher nullity analogues of the affine Kac-Moody Lie algebras). The torsor approach that we take draws heavily for the theory of reductive group schemes developed by M. Demazure and A. Grothendieck. It also allows us to find a bridge between multiloop algebras and the work of J. Tits on reductive groups over complete local fields.

Keywords: Reductive group scheme, torsor, multiloop algebra. Extended Affine Lie Algebras.

MSC 2000 17B67, 11E72, 14L30, 14E20.

Contents

1 Introduction

To our good friend Benedictus Margaux

Many interesting infinite dimensional Lie algebras can be thought as being "finite dimensional" when viewed, not as algebras over the given base field, but rather as algebras over their centroids. From this point of view, the algebras in question look like "twisted forms" of simpler objects with which one is familiar. The quintessential example of this type of behaviour is given by the affine Kac-Moody Lie algebras. Indeed the algebras that we are most interested in, Extended Affine Lie Algebras (or EALAs for short), can roughly be thought of as higher nullity analogues of the affine Kac-Moody Lie algebras. Once the twisted form point of view is taken the theory of reductive group schemes developed by Demazure and Grothendieck [SGA3] arises naturally.

Two key concepts which are common to [GP2] and the present work are those of a *twisted form of an algebra,* and of a *multiloop algebra*. At this point we briefly recall what these objects are, not only for future reference, but also to help us redact a more comprehensive Introduction.

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Unless specific mention to the contrary throughout this paper k will denote a field of characteristic 0, and $\overline{k}$ a fixed algebraic closure of k. We denote k–alg the category of associative unital commutative k–algebras, and R and object of k–alg. Let $n \geq 0$ and $m > 0$ be integers that we assume are fixed in our discussion. Consider the Laurent polynomial rings $R = R_n = k[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ and $R' = R_{n,m} = k[t_1^{\pm \frac{1}{m}}, \ldots, t_n^{\pm \frac{1}{m}}]$. For convenience we also consider the direct limit $R'_\infty = \varinjlim R_{n,m}$ taken over m which in practice will allow us to "see" all the $R_{n,m}$ at the same time. The natural map $R \to R'$ is not only faithfully flat but also étale. If k is algebraically closed this extension is Galois and plays a crucial role in the study of multiloop algebras. The explicit description of $\mathrm{Gal}(R'/R)$ is given below.

Let A be a k–algebra. We are in general interested in understanding forms (for the $fppf$-topology) of the algebra $A \otimes_k R$, namely algebras $\mathcal{L}$ over R such that

$$\mathcal{L} \otimes_R S \simeq A \otimes_k S \simeq (A \otimes_k R) \otimes_R S \tag{1.1}$$

for some faithfully flat and finitely presented extension S/R. The case which is of most interest to us is when S can be taken to be a Galois extension R' of R of Laurent polynomial algebras described above.[1]

Given a form $\mathcal{L}$ as above for which (1.1) holds, we say that $\mathcal{L}$ is *trivialized* by S. The R–isomorphism classes of such algebras can be computed by means of cocycles, just as one does in Galois cohomology:

$$\textit{Isomorphism classes of } S/R\textit{–forms of } A \otimes_k R \longleftrightarrow H^1_{fppf}\big(S/R, \mathbf{Aut}(A)\big). \tag{1.2}$$

The right hand side is the part "trivialized by S" of the pointed set of non-abelian cohomology on the flat site of $\mathrm{Spec}(R)$ with coefficients in the sheaf of groups $\mathbf{Aut}(A)$. In the case when S is Galois over R we can indeed identify $H^1_{fppf}\big(S/R, \mathbf{Aut}(A)\big)$ with the "usual" Galois cohomology set $H^1\big(\mathrm{Gal}(S/R), \mathbf{Aut}(A)(S)\big)$ as in [Se].

Assume now that k is algebraically closed and fix a compatible set of primitive m–th roots of unity ξ_m, namely such that $\xi_{me}^e = \xi_m$ for all $e > 0$. We can then identify $\mathrm{Gal}(R'/R)$ with $(\mathbb{Z}/m\mathbb{Z})^n$ where for each $\mathbf{e} = (e_1, \ldots, e_n) \in \mathbb{Z}^n$ the corresponding element $\overline{\mathbf{e}} = (\overline{e}_1, \cdots, \overline{e}_n) \in \mathrm{Gal}(R'/R)$ acts on R' via ${}^{\overline{\mathbf{e}}}t_i^{\frac{1}{m}} = \xi_m^{e_i} t_i^{\frac{1}{m}}$.

The primary example of forms $\mathcal{L}$ of $A \otimes_k R$ which are trivialized by a Galois extension R'/R as above are the multiloop algebras based on A. These are defined as follows. Consider an n–tuple $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ of commuting elements of $\mathrm{Aut}_k(A)$ satisfying $\sigma_i^m = 1$. For each n–tuple $(i_1, \ldots, i_n) \in \mathbb{Z}^n$ we consider the simultaneous eigenspace $A_{i_1 \ldots i_n} = \{x \in A : \sigma_j(x) = \xi_m^{i_j} x \text{ for all } 1 \leq j \leq n\}$. Then $A = \sum A_{i_1 \ldots i_n}$,

[1] The Isotriviality Theorem of [GP1] and [GP3] shows that this assumption is superflous if $\mathbf{Aut}(A)$ is a an algebraic k–group whose connected component is reductive, for example if A is a finite dimensional simple Lie algebra.

and $A = \bigoplus A_{i_1\ldots i_n}$ if we restrict the sum to those n–tuples $(i_1, \ldots, i_n)$ for which $0 \le i_j < m_j$.

The multiloop algebra corresponding to $\boldsymbol{\sigma}$, commonly denoted by $L(A, \boldsymbol{\sigma})$, is defined by

$$L(A, \boldsymbol{\sigma}) = \underset{(i_1,\ldots,i_n)\in\mathbb{Z}^n}{\oplus} A_{i_1\ldots i_n} \otimes t^{\frac{i_1}{m}} \ldots t_n^{\frac{i_n}{m}} \subset A \otimes_k R' \subset A \otimes_k R'_\infty$$

Note that $L(A, \boldsymbol{\sigma})$, which does not depend on the choice of common period m, is not only a k–algebra (in general infinite dimensional), but also naturally an R–algebra. It is when $L(A, \boldsymbol{\sigma})$ is viewed as an R–algebra that Galois cohomology and the theory of torsors enter into the picture. Indeed a rather simple calculation shows that

$$L(A, \boldsymbol{\sigma}) \otimes_R R' \simeq A \otimes_k R' \simeq (A \otimes_k R) \otimes_R R'.$$

Thus $L(A, \boldsymbol{\sigma})$ corresponds to a torsor over $\mathrm{Spec}(R)$ under $\mathbf{Aut}(A)$.

When $n = 1$ multiloop algebras are called simply loop algebras. To illustrate our methods, let us look at the case of (twisted) loop algebras as they appear in the theory of affine Kac-Moody Lie algebras. Here $n = 1$, $k = \mathbb{C}$ and $A = \mathfrak{g}$ is a finite-dimensional simple Lie algebra. Any such $\mathcal{L}$ is naturally a Lie algebra over $R := \mathbb{C}[t^{\pm 1}]$ and $\mathcal{L} \otimes_R S \simeq \mathfrak{g} \otimes_{\mathbb{C}} S \simeq (\mathfrak{g} \otimes_{\mathbb{C}} R) \otimes_R S$ for some (unique) $\mathfrak{g}$, and some finite étale extension S/R. In particular, $\mathcal{L}$ is an S/R–form of the R–algebra $\mathfrak{g} \otimes_{\mathbb{C}} R$, with respect to the étale topology of $\mathrm{Spec}(R)$. Thus $\mathcal{L}$ corresponds to a torsor over $\mathrm{Spec}(R)$ under $\mathbf{Aut}(\mathfrak{g})$ whose isomorphism class is an element of the pointed set $H^1_{\text{ét}}\big(R, \mathbf{Aut}(\mathfrak{g})\big)$. We may in fact take S to be $R' = \mathbb{C}[t^{\pm\frac{1}{m}}]$.

Assume that A is a finite-dimensional. The crucial point in the classification of forms of $A \otimes_k R$ by cohomological methods is the exact sequence of pointed sets

$$H^1_{\acute{e}t}\big(R, \mathbf{Aut}^0(A)\big) \to H^1_{\acute{e}t}\big(R, \mathbf{Aut}(A)\big) \xrightarrow{\psi} H^1_{\acute{e}t}\big(R, \mathbf{Out}(A)\big), \tag{1.3}$$

where $\mathbf{Out}(A)$ is the (finite constant) group of connected components of the algebraic k–group $\mathbf{Aut}(A)$.[2]

Grothendieck's theory of the algebraic fundamental group allows us to identify $H^1_{\acute{e}t}\big(R, \mathbf{Out}(A)\big)$ with the set of conjugacy classes of n–tuples of commuting elements of the corresponding finite (abstract) group $\mathrm{Out}(A)$ (again under the assumption that k is algebraically closed). This is an important cohomological invariant attached to any twisted form of $A \otimes_k R$. We point out that the cohomological information is always about the twisted forms viewed as algebras over R (and*not* k). In practice, as the

[2]Strictly speaking we should be using the affine R–group scheme $\mathbf{Aut}(A \otimes_k R)$ instead of the algebraic k–group $\mathbf{Aut}(A)$. This harmless and useful abuse of notation will be used throughout the paper.

affine Kac-Moody case illustrates, one is interested in understanding these algebras as objects over k (and *not* R). A technical tool (the centroid trick) developed and used in [ABP2] and [GP2] allows us to compare k vs R information.

We begin by looking at the nullity $n = 1$ case. The map ψ of (1.3) is injective [P1]. This fundamental fact follows from a general result about the vanishing of H^1 for reductive group schemes over certain Dedekind rings which includes $k[t^{\pm 1}]$. This result can be thought of as an analogue of "Serre Conjecture I" for some very special rings of dimension 1. It follows from what has been said that we can attach a conjugacy class of the finite group $\mathrm{Out}(A)$ that characterizes $\mathcal{L}$ up to R–isomorphism. In particular, if $\mathbf{Aut}(A)$ is connected, then all forms (and consequently, all twisted loop algebras) of A are *trivial*, i.e. isomorphic to $A \otimes_k R$ as R–algebras. This yields the classification of the affine Kac-Moody Lie algebras by purely cohomological methods. One can in fact *define* the affine algebras by such methods (which is a completely different approach than the classical definition by generators and relations).

Surprisingly enough the analogue of "Serre Conjecture II" for $k[t_1^{\pm 1}, t_2^{\pm 1}]$ fails, as explained in [GP2]. The single family of counterexamples known are the the so-called Margaux algebras. The classification of forms in nullity 2 case is in fact quite interesting and challenging. Unlike the nullity one case there are forms which are not multiloop algebras (the Margaux algebra is one such example). The classification in nullity 2 by cohomological methods, both over R and over k, will be given in §9 as an application of one of our main results (the Acyclicity Theorem). This classification (over k but not over R) can also be attained entirely by EALA methods [ABP3]. The two approaches complement each other and are the culmination of a project started a decade ago. We also provide classification results for loop Azumaya algebras in §13.

Questions related to the classification and characterization of EALAs in arbitrary nullity are at the heart of our work. In this situation $A = \mathfrak{g}$ is a finite dimensional simple Lie algebra over k. The twisted forms relevant to EALA theory are always multiloop algebras based on $\mathfrak{g}$ [ABFP]. It is therefore desirable to try to characterize and understand the part of $H^1_{ét}\big(R, \mathbf{Aut}(\mathfrak{g})\big)$ corresponding to multiloop algebras. We address this problem by introducing the concept of *loop* and *toral* torsors (with k not necessarily algebraically closed). These concepts are key ideas within our work. It is easy to show using a theorem of Borel and Mostow that a multiloop algebra based on $\mathfrak{g}$, viewed as a Lie algebra over R_n, always admits a Cartan subalgebra (in the sense of [SGA3]). We establish that the converse also holds.

Central to our work is the study of the canonical map

$$H^1_{ét}\big(R_n, \mathbf{Aut}(\mathfrak{g})\big) \to H^1_{ét}\big(F_n, \mathbf{Aut}(\mathfrak{g})\big) \tag{1.4}$$

where F_n stands for the iterated Laurent series field $k((t_1))\dots((t_n))$. The Acyclicity Theorem proved in §8 shows that the restriction of the canonical map (1.4) to the subset $H^1_{loopl}\big(R_n, \mathbf{Aut}(\mathfrak{g})\big) \subset H^1_{\text{ét}}\big(F_n, \mathbf{Aut}(\mathfrak{g})\big)$ of classes of loop torsors is bijective. This has strong applications to the classification of EALAs. Indeed $H^1_{\acute{e}t}\big(F_n, \mathbf{Aut}(\mathfrak{g})\big)$ can be studied using Tits' methods for algebraic groups over complete local fields. In particular EALAs can be naturally attached Tits indices and diagrams, combinatorial root data and relative and absolute types. These are important invariants which are extremely useful for classification purposes. Setting any applications aside, and perhaps more importantly, we believe that the theory and methods that we are putting forward display an intrinsic beauty, and show just how powerful the methods developed in [SGA3] really are.

Acknowledgement The authors would like to thank M. Brion and V. Chernousov for their valuable comments.

2 Generalities on the algebraic fundamental group, torsors, and reductive group schemes

Throughout this section $\mathfrak{X}$ will denote a scheme, and $\mathfrak{G}$ a group scheme over $\mathfrak{X}$.

2.1 The fundamental group

Assume that $\mathfrak{X}$ is connected and locally noetherian. Fix a geometric point a of $\mathfrak{X}$ i.e. a morphism $a : \operatorname{Spec}(\Omega) \to \mathfrak{X}$ where Ω is an algebraically closed field.

Let $\mathfrak{X}_{f\acute{e}t}$ be the category of finite étale covers of $\mathfrak{X}$, and F the covariant functor from $\mathfrak{X}_{f\acute{e}t}$ to the category of finite sets given by

$$F(\mathfrak{X}') = \{\text{geometric points of } \mathfrak{X}' \text{ above } a).$$

That is, $F(\mathfrak{X}')$ consists of all morphisms $a' : \operatorname{Spec}(\Omega) \to \mathfrak{X}'$ for which the diagram

$$\begin{tikzcd} & \mathfrak{X}' \arrow[d] \\ \operatorname{Spec}(\Omega) \arrow[ur, "a'"] \arrow[r, "a"'] & \mathfrak{X} \end{tikzcd}$$

commutes. The group of automorphism of the functor F is called the *algebraic fundamental group of* $\mathfrak{X}$ *at* a, and is denoted by $\pi_1(\mathfrak{X}, a)$. If $\mathfrak{X} = \operatorname{Spec}(R)$ is affine, then a corresponds to a ring homomorphism $R \to \Omega$ and we will denote the fundamental group by $\pi_1(R, a)$.

The functor F is pro-representable: There exists a directed set I, objects $(\mathfrak{X}_i)_{i\in I}$ of $\mathfrak{X}_{f\acute{e}t}$, surjective morphisms $\varphi_{ij} \in \operatorname{Hom}_{\mathfrak{X}}(\mathfrak{X}_j, \mathfrak{X}_i)$ for $i \le j$ and geometric points

$a_i \in F(\mathfrak{X}_i)$ such that

$$a_i = \varphi_{ij} \circ a_j \tag{2.1}$$

$$\textit{The canonical map } f : \varinjlim \mathrm{Hom}_{\mathfrak{X}}(\mathfrak{X}_i, \mathfrak{X}') \to F(\mathfrak{X}') \textit{ is bijective,} \tag{2.2}$$

where the map f of (2.2) is as follows: Given $\varphi : \mathfrak{X}_i \to \mathfrak{X}'$ then $f(\varphi) = F(\varphi)(a_i)$. The elements of $\varinjlim \mathrm{Hom}_{\mathfrak{X}}(\mathfrak{X}_i, \mathfrak{X}')$ appearing in (2.2) are by definition the morphisms in the category of pro-objects over $\mathfrak{X}$ (see [EGA IV] §8.13 for details). It is in this sense that $\varinjlim \mathrm{Hom}(\mathfrak{X}_i, -)$ pro-represents F.

Since the $\mathfrak{X}_i$ are finite and étale over $\mathfrak{X}$ the morphisms φ_{ij} are affine. Thus the inverse limit

$$\mathfrak{X}^{sc} = \varprojlim \mathfrak{X}_i$$

exist in the category of schemes over $\mathfrak{X}$ [EGA IV] §8.2. For any scheme $\mathfrak{X}'$ over $\mathfrak{X}$ we thus have a canonical map

$$\mathrm{Hom}_{Pro-\mathfrak{X}}(\mathfrak{X}^{sc}, \mathfrak{X}') \overset{\mathrm{def}}{=} \varinjlim \mathrm{Hom}_{\mathfrak{X}}(\mathfrak{X}_i, \mathfrak{X}') \simeq F(\mathfrak{X}') \to \mathrm{Hom}_{\mathfrak{X}}(\mathfrak{X}^{sc}, \mathfrak{X}') \tag{2.3}$$

obtained by considering the canonical morphisms $\varphi_i : \mathfrak{X}^{\mathrm{sc}} \to \mathfrak{X}_i$.

Proposition 2.1. *Assume $\mathfrak{X}$ is noetherian. Then F is represented by $\mathfrak{X}^{sc}$; that is, there exists a bijection*

$$F(\mathfrak{X}') \simeq \mathrm{Hom}_{\mathfrak{X}}(\mathfrak{X}^{sc}, \mathfrak{X}')$$

which is functorial on the objects $\mathfrak{X}'$ of $\mathfrak{X}_{f\acute{e}t}$.

Proof. Because the $\mathfrak{X}_i$ are affine over $\mathfrak{X}$ and $\mathfrak{X}$ is noetherian, each $\mathfrak{X}_i$ is noetherian; in particular, quasicompact and quasiseparated. Thus, for $\mathfrak{X}'/\mathfrak{X}$ locally of finite presentation, in particular for $\mathfrak{X}'$ in $\mathfrak{X}_{f\acute{e}t}$, the map (2.3) is bijective [EGA IV, prop 8.13.1]. The Proposition now follows from (2.2). □

Remark 2.2. The bijection of Proposition 2.1 could be thought along the same lines as those of (2.2) by considering the "geometric point" $a^{sc} \in \varprojlim F(\mathfrak{X}_i)$ satisfying $a^{sc} \mapsto a_i$ for all $i \in I$.

In computing $\mathfrak{X}^{sc} = \varprojlim \mathfrak{X}_i$ we may replace $(\mathfrak{X}_i)_{i \in I}$ by any cofinal family. This allows us to assume that the $\mathfrak{X}_i$ are (connected) Galois, i.e. the $\mathfrak{X}_i$ are connected and the (left) action of $\mathrm{Aut}_{\mathfrak{X}}(\mathfrak{X}_i)$ on $F(\mathfrak{X}_i)$ is transitive. We then have

$$F(\mathfrak{X}_i) \simeq \mathrm{Hom}_{Pro-\mathfrak{X}}(\mathfrak{X}^{sc}, \mathfrak{X}_i) \simeq \mathrm{Hom}_{\mathfrak{X}}(\mathfrak{X}_i, \mathfrak{X}_i) = \mathrm{Aut}_{\mathfrak{X}}(\mathfrak{X}_i).$$

Thus $\pi_1(\mathfrak{X}, a)$ can be identified with the group $\varprojlim \mathrm{Aut}_{\mathfrak{X}}(\mathfrak{X}_i)^{opp}$. Each $\mathrm{Aut}_{\mathfrak{X}}(\mathfrak{X}_i)$ is finite, and this endows $\pi_1(\mathfrak{X}, a)$ with the structure of a profinite topological group.

The group $\pi_1(\mathfrak{X}, a)$ acts on the right on $\mathfrak{X}^{sc}$ as the inverse limit of the finite groups $\mathrm{Aut}_{\mathfrak{X}}(\mathfrak{X}_i)$. Thus, the group $\pi_1(\mathfrak{X}, a)$ acts on the left on each set $F(\mathfrak{X}') = \mathrm{Hom}_{Pro-\mathfrak{X}}(\mathfrak{X}^{sc}, \mathfrak{X}')$ for all $\mathfrak{X}' \in \mathfrak{X}_{f\acute{e}t}$. This action is continuous since the structure morphism $\mathfrak{X}' \to \mathfrak{X}$ "factors at the finite level", i.e there exists a morphism $\mathfrak{X}_i \to \mathfrak{X}'$ of $\mathfrak{X}$–schemes for some $i \in I$. If $u : \mathfrak{X}' \to \mathfrak{X}''$ is a morphism of $\mathfrak{X}_{f\acute{e}t}$, then the map $F(u) : F(\mathfrak{X}') \to F(\mathfrak{X}'')$ clearly commutes with the action of $\pi_1(\mathfrak{X}, a)$. This construction provides an equivalence between $\mathfrak{X}_{f\acute{e}t}$ and the category of finite sets equipped with a continuous $\pi_1(\mathfrak{X}, a)$–action.

The right action of $\pi_1(\mathfrak{X}, a)$ on $\mathfrak{X}^{sc}$ induces an action of $\pi_1(\mathfrak{X}, a)$ on $\mathfrak{G}(\mathfrak{X}^{sc}) = \mathrm{Mor}_{\mathfrak{X}}(\mathfrak{X}^{sc}, \mathfrak{G})$, namely

$$^{\gamma}f(z) = f(z^{\gamma}) \quad \forall \gamma \in \pi_1(\mathfrak{X}, a), \quad f \in \mathfrak{G}(\mathfrak{X}^{sc}), \quad z \in \mathfrak{X}^{sc}.$$

Proposition 2.3. *Assume $\mathfrak{X}$ is noetherian and that $\mathfrak{G}$ is locally of finite presentation over $\mathfrak{X}$. Then $\mathfrak{G}(\mathfrak{X}^{sc})$ is a discrete $\pi_1(\mathfrak{X}, a)$–module and the canonical map*

$$\varinjlim H^1\big(\mathrm{Aut}_{\mathfrak{X}}(\mathfrak{X}_i), \mathfrak{G}(\mathfrak{X}_i)\big) \to H^1\big(\pi_1(\mathfrak{X}, a), \mathfrak{G}(\mathfrak{X}^{sc})\big)$$

is bijective.

Remark 2.4. Here and elsewhere when a profinite group A acts discretely on a module M the corresponding cohomology $H^1(A, M)$ is the *continuous* cohomology as defined in [Se]. Similarly, if a group H acts in both A and M, then $\mathrm{Hom}_H(A, M)$ stands for the continuos group homomorphism of A into M that commute with the action of H.

Proof. To show that $\mathfrak{G}(\mathfrak{X}^{sc})$ is discrete amounts to showing that the stabilizer in $\pi_1(\mathfrak{X}, a)$ of a point of $f \in \mathfrak{G}(\mathfrak{X}^{sc})$ is open. But if $\mathfrak{G}$ is locally of finite presentation then $\mathfrak{G}(\mathfrak{X}^{sc}) = \mathfrak{G}(\varprojlim \mathfrak{X}_i) = \varinjlim \mathfrak{G}(\mathfrak{X}_i)$ ([EGA IV] prop. 8.13.1), so we may assume that $f \in \mathfrak{G}(\mathfrak{X}_i)$ for some i. The result is then clear, for the stabilizer we are after is the inverse image under the continuous map $\pi_1(\mathfrak{X}, a) \to \mathrm{Aut}_{\mathfrak{X}}(\mathfrak{X}_i)$ of the stabilizer of f in $\mathrm{Aut}_{\mathfrak{X}}(\mathfrak{X}_i)$ (which is then open since $\mathrm{Aut}_{\mathfrak{X}}(\mathfrak{X}_i)$ is given the discrete topology).

By definition

$$H^1\big(\pi_1(\mathfrak{X}, a), \mathfrak{G}(\mathfrak{X}^{sc})\big) = \varinjlim \big(\pi_1(\mathfrak{X}, a)/U, \mathfrak{G}(\mathfrak{X}^{sc})^U\big)$$

where the limit is taken over all open normal subgroups U of $\pi_1(\mathfrak{X}, a)$. But for each such U we can find $U_i \subset U$ so that $U_i = \ker\big(\pi_1(\mathfrak{X}, a) \to \mathrm{Aut}_{\mathfrak{X}}(\mathfrak{X}_i)\big)$. Thus

$$H^1\big(\pi_1(\mathfrak{X}, a), \mathfrak{G}(\mathfrak{X}^{sc}\big) = \varinjlim H^1\big(\mathrm{Aut}_{\mathfrak{X}}(\mathfrak{X}_i), \mathfrak{G}(\mathfrak{X}_i)\big)$$

as desired. $\square$

Suppose now that our $\mathfrak{X}$ is a geometrically connected k–scheme, where k is of arbitrary characteristic. We will denote $\mathfrak{X} \times_k \overline{k}$ by $\overline{\mathfrak{X}}$. Fix a geometric point $\overline{a} : \mathrm{Spec}(\overline{k}) \to \overline{\mathfrak{X}}$. Let a (resp. b) be the geometric points of $\mathfrak{X}$ [resp. $\mathrm{Spec}(k)$] given by the composite maps $a : \mathrm{Spec}(\overline{k}) \xrightarrow{\overline{a}} \overline{\mathfrak{X}} \to \mathfrak{X}$ [resp. $b : \mathrm{Spec}(\overline{k}) \xrightarrow{\overline{a}} \mathfrak{X} \to \mathrm{Spec}(k)$]. Then by [SGA1, théorème IX.6.1] $\pi_1\big(\mathrm{Spec}(k), b\big) \simeq \mathrm{Gal}(k) = \mathrm{Gal}(k_s/k)$ where k_s is the separable closure of k in $\overline{k}$, and the sequence

$$1 \to \pi_1(\overline{\mathfrak{X}}, \overline{a}) \to \pi_1(\mathfrak{X}, a) \xrightarrow{p} \mathrm{Gal}(k) \to 1 \tag{2.4}$$

is exact.

2.2 Torsors

Recall that a (right) *torsor over* $\mathfrak{X}$ *under* $\mathfrak{G}$ (or simply a $\mathfrak{G}$–torsor if $\mathfrak{X}$ is understood) is a scheme $\mathfrak{E}$ over $\mathfrak{X}$ equipped with a right action of $\mathfrak{G}$ for which there exists a faithfully flat morphism $\mathfrak{Y} \to \mathfrak{X}$, locally of finite presentation, such that $\mathfrak{E} \times_{\mathfrak{X}} \mathfrak{Y} \simeq \mathfrak{G} \times_{\mathfrak{X}} \mathfrak{Y} = \mathfrak{G}_{\mathfrak{Y}}$, where $\mathfrak{G}_{\mathfrak{Y}}$ acts on itself by right translation.

A $\mathfrak{G}$–torsor $\mathfrak{E}$ is *locally trivial* (resp. *étale locally trivial*) if it admits a trivialization by an open Zariski (resp. étale) covering of $\mathfrak{X}$. If $\mathfrak{G}$ is affine, flat and locally of finite presentation over $\mathfrak{X}$, then $\mathfrak{G}$–torsors over $\mathfrak{X}$ are classified by the pointed set of cohomology $H^1_{fppf}(\mathfrak{X}, \mathfrak{G})$ defined by means of cocycles à la Čech. If $\mathfrak{G}$ is smooth, any $\mathfrak{G}$–torsor is étale locally trivial (cf. [SGA3], Exp. XXIV), and their classes are then measured by $H^1_{ét}(\mathfrak{X}, \mathfrak{G})$. In what follows the $fppf$-topology will be our default choice, and we will for convenience denote H^1_{fppf} simply by H^1. Given a base change $\mathfrak{Y} \to \mathfrak{X}$, we denote by $H^1(\mathfrak{Y}/\mathfrak{X}, \mathfrak{G})$ the kernel of the base change map $H^1(\mathfrak{X}, \mathfrak{G}) \to H^1(\mathfrak{Y}, \mathfrak{G}_{\mathfrak{Y}})$. As it is customary, and when no confusion is possible, we will denote in what follows $H^1(\mathfrak{Y}, \mathfrak{G}_{\mathfrak{Y}})$ simply by $H^1(\mathfrak{Y}, \mathfrak{G})$

Recall that a torsor $\mathfrak{E}$ over $\mathfrak{X}$ under $\mathfrak{G}$ is called *isotrivial* if it is trivialized by some *finite* étale extension of $\mathfrak{X}$, that is,

$$[\mathfrak{E}] \in H^1(\mathfrak{X}'/\mathfrak{X}, \mathfrak{G}) \subset H^1(\mathfrak{X}, \mathfrak{G})$$

for some $\mathfrak{X}'$ in $\mathfrak{X}_{fét}$. We denote by $H^1_{iso}(\mathfrak{X}, \mathfrak{G})$ the subset of $H^1(\mathfrak{X}, \mathfrak{G})$ consisting of classes of isotrivial torsors.

Proposition 2.5. *Assume that* $\mathfrak{X}$ *is noetherian and that* $\mathfrak{G}$ *is locally of finite presentation over* $\mathfrak{X}$*. Then*

$$H^1_{iso}(\mathfrak{X}, \mathfrak{G}) = \ ker\big(H^1(\mathfrak{X}, \mathfrak{G}) \to H^1(\mathfrak{X}^{sc}, \mathfrak{G})\big).$$

Proof. Assume $\mathfrak{E}$ is trivialized by $\mathfrak{X}' \in \mathfrak{X}_{f\acute{e}t}$. Since the connected components of $\mathfrak{X}'$ are also in $\mathfrak{X}_{f\acute{e}t}$[3] there exists a morphism $\mathfrak{X}_i \to \mathfrak{X}'$ for some $i \in I$. But then $\mathfrak{E} \times_{\mathfrak{X}} \mathfrak{X}_i = \mathfrak{E} \times_{\mathfrak{X}} \mathfrak{X}' \times_{\mathfrak{X}'} \mathfrak{X}_i = \mathfrak{G}_{\mathfrak{X}'} \times_{\mathfrak{X}'} \mathfrak{X}_i = \mathfrak{G}_{\mathfrak{X}_i}$ so that $\mathfrak{E}$ is trivialized by $\mathfrak{X}_i$. The image of $[\mathfrak{E}]$ on $H^1(\mathfrak{X}^{sc}, \mathfrak{G})$ is thus trivial.

Conversely assume $[\mathfrak{E}] \in H^1(\mathfrak{X}, \mathfrak{G})$ vanishes under the base change $\mathfrak{X}^{sc} \to \mathfrak{X}$. Since the $\mathfrak{X}_i$ are quasicompact and quasiseparated and $\mathfrak{G}$ is locally of finite presentation, a theorem of Grothendieck-Margaux [Mg] shows that the canonical map

$$\varinjlim H^1(\mathfrak{X}_i, \mathfrak{G}) \to H^1(\mathfrak{X}^{sc}, \mathfrak{G})$$

is bijective. Thus $\mathfrak{E} \times_{\mathfrak{X}} \mathfrak{X}_i \simeq \mathfrak{G}_{\mathfrak{X}_i}$ for some $i \in I$. $\square$

2.3 An example: Laurent polynomials in characteristic 0

We look in detail at an example that is of central importance to this work, namely the case when $\mathfrak{X} = \operatorname{Spec}(R_n)$ where $R_n = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ is the Laurent polynomial ring in n–variables with coefficients on a field k of characteristic 0.

Fix once and for all a compatible set $(\xi_m)_{m \geq 0}$ of primitive m–roots of unity in $\overline{k}$ (i.e. $\xi_{m\ell}^{\ell} = \xi_m$). Let $\{k_\lambda\}_{\lambda \in \Lambda}$ be the set of finite Galois extensions of k which are included in $\overline{k}$. Let $\Gamma_\lambda = \operatorname{Gal}(k_\lambda/k)$ and $\Gamma = \varprojlim \Gamma_\lambda$. Then Γ coincided with the algebraic fundamental group of $\operatorname{Spec}(k)$ at the geometric point $\operatorname{Spec}(\overline{k})$.

Let $\varepsilon : R_n \to k$ be the evaluation map at $t_i = 1$. The composite map $R_n \xrightarrow{\varepsilon} k \hookrightarrow \overline{k}$ defines a geometric point a of $\mathfrak{X}$ and a geometric point $\overline{a}$ of $\overline{\mathfrak{X}} = \operatorname{Spec}(\overline{R}_n)$ where $\overline{R}_n = \overline{k}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$.

Let I be the subset of $\Lambda \times \mathbb{Z}_{>0}$ consisting of all pairs (λ, m) for which k_λ contains ξ_m. Make I into a directed set by declaring that $(\lambda, \ell) \leq (\mu, n) \iff k_\lambda \subset k_\mu$ and $\ell | n$.

Each

$$R_{n,m}^{\lambda} = k_\lambda[t_1^{\pm \frac{1}{m}}, \dots, t_n^{\pm \frac{1}{m}}]$$

is a Galois extension of R_n with Galois group $\Gamma_{m,\lambda} = (\mathbb{Z}/m\mathbb{Z})^n \rtimes \Gamma_\lambda$ as follows: For $\mathbf{e} = (e_1, \dots, e_n) \in \mathbb{Z}^n$ we have ${}^{\overline{\mathbf{e}}}t_j^{\frac{1}{m}} = \xi_j^{e_j} t_j^{\frac{1}{m}}$ where $^{-} : \mathbb{Z}^n \to (\mathbb{Z}/mZ)^n$ is the canonical map, and the group Γ_λ acts naturally on $R_{n,m}^{\lambda}$ through its action on k_λ. It is immediate

[3] There exists a finite Galois connected covering $\mathfrak{F} \to \mathfrak{X}$ such that $\mathfrak{F} \times_{\mathfrak{X}} \mathfrak{X}' \cong \mathfrak{F} \sqcup \cdots \sqcup \mathfrak{F}$ (r times). If we decompose $\mathfrak{X}' = \mathfrak{Y}_1 \sqcup \cdots \sqcup \mathfrak{Y}_m$ into its connected components we have

$$\mathfrak{X}' \times_{\mathfrak{X}} \mathfrak{F} = \mathfrak{Y}_1 \times_{\mathfrak{X}} \mathfrak{F} \sqcup \cdots \sqcup \mathfrak{Y}_m \times_{\mathfrak{X}} \mathfrak{F} = \mathfrak{F} \sqcup \cdots \sqcup \mathfrak{F}.$$

It follows that each $\mathfrak{Y}_i \times_{\mathfrak{X}} \mathfrak{F}$ is a disjoint union of copies of $\mathfrak{F}$, hence $\mathfrak{Y}_i \times_{\mathfrak{X}} \mathfrak{F}$ is finite étale over $\mathfrak{F}$ for $i = 1, .., m$. Then each $\mathfrak{Y}_i$ is étale over $\mathfrak{X}$ by proposition 17.7.4.vi of [EGA IV]. By descent, each $\mathfrak{Y}_i$ is finite over $\mathfrak{X}$ [EGA IV, prop. 2.7.1 xv], so each $\mathfrak{Y}_i/\mathfrak{X}$ is finite étale.

from the definition that for $\gamma \in \Gamma_\lambda$ we have $\gamma \mathbf{e} \gamma^{-1} : t_j^{\frac{1}{m}} \mapsto ({}^\gamma\xi_j)^{e_j} t_j^{\frac{1}{m}}$. Thus if ${}^\gamma\xi_j = \xi_j^{f_j}$ then $\gamma \mathbf{e} \gamma^{-1} = \mathbf{e}'$ where $\mathbf{e}' = (f_1 e_1, \dots f_n e_n)$.

If $(\lambda, \ell) \leq (\mu, n)$ we have a canonical inclusion $R^\lambda_{n,\ell} \subset R^\mu_{n,m}$. For $i = (\lambda, \ell) \in I$ we let $\mathfrak{X}_i = \operatorname{Spec}(R^\lambda_{n,\ell})$. The above gives morphism $\varphi_{ij} : \mathfrak{X}_j \to \mathfrak{X}_i$ of $\mathfrak{X}$–schemes whenever $i \leq j$.

We have [GP3]

$$\begin{aligned} \mathfrak{X}^{sc} = \varprojlim \mathfrak{X}_i &= \operatorname{Spec}(\varinjlim \mathfrak{X}_i) \\ &= \operatorname{Spec}(\overline{R}_{n,\infty}) \end{aligned}$$

where $\overline{R}_{n,\infty} = \varinjlim_m \overline{R}_{n,m}$ with $\overline{R}_{n,m} = \overline{k}[t_1^{\pm\frac{1}{m}}, \dots, t_n^{\pm\frac{1}{m}}]$. Thus

$$\pi_1(\mathfrak{X}, a) = \varprojlim \Gamma_{m,\lambda} = \widehat{\mathbb{Z}}(1)^n \rtimes \operatorname{Gal}(k). \tag{2.5}$$

where $\widehat{\mathbb{Z}}(1)$ denotes the abstract group $\widehat{\mathbb{Z}} = \varprojlim_m \boldsymbol{\mu}_m(\overline{k})$ equipped with the natural action of the absolute Galois group $\operatorname{Gal}(k) = \operatorname{Gal}(\overline{k}/k)$.

Remark 2.6. Consider the affine k–group scheme ${}_\infty\boldsymbol{\mu} = \varprojlim_m \boldsymbol{\mu}_m$. It corresponds to the Hopf algebra

$$k[{}_\infty\boldsymbol{\mu}] = \varinjlim_m k[\boldsymbol{\mu}_m] = \varinjlim_m \frac{k[t]}{t^m - 1}.$$

Then we have ${}_\infty\boldsymbol{\mu}(\overline{k}) \simeq \widehat{\mathbb{Z}}$ and ${}_\infty\boldsymbol{\mu}(\overline{k})$ is equipped with a (canonical) structure of profinite $\operatorname{Gal}(k)$–module.

Remark 2.7. Let the notation be that of Example 2.3. Since $\mathbb{Z}^n$ is the character group of the the torus $\mathbf{G}^n_{m,k}$, we have an automorphism $\mathbf{GL}_n(\mathbb{Z}) \simeq \mathbf{Aut}_{gr}(\mathbf{G}_m^{n,k})^{op}$. This defines a left action $\mathbf{GL}_n(\mathbb{Z})$ on R_n and a right action of $\mathbf{GL}_n(\mathbb{Z})$ on the torus $\mathbf{G}^n_{m,k}$. Furthermore, by universal nonsense, this action extends uniquely to the simply connected covering $\overline{R}_{n,\infty}$ at the geometric point $\overline{a}$. The extended action on the torus $\operatorname{Spec}(\overline{R}_{n,m})$ with character group $(\frac{1}{m}\mathbb{Z})^n$ is given by the extension of the action of $\mathbf{GL}_n(\mathbb{Z})$ from $\mathbb{Z}^n$ to $(\frac{1}{m}\mathbb{Z})^n$ inside $\mathbb{Q}^n$. The group $\mathbf{GL}_n(\mathbb{Z})$ acts (on the right) on $\pi_1(R_n)$, so we can consider the semidirect product of groups $\mathbf{GL}_n(\mathbb{Z}) \ltimes \pi_1(R_n)$ which acts then on $\overline{R}_{n,\infty}$ (see §8.4 for details).

Lemma 2.8. *Let S be a connected finite étale cover of R_n. Let $L \subset S$ be the integral closure of k in S. Then there exists $g \in \mathbf{GL}_n(\mathbb{Z})$, $a_1, ..., a_n \in L^\times$ and positive integers $d_1, ..., d_n$ such that $d_1 \mid d_2 \cdots \mid d_n$ and*

$$S \otimes_{R_n} R_n^g \simeq_{R_n\text{-}alg} (R_n \otimes_k L)\big[\sqrt[d_1]{a_1 t_1}, \cdots, \sqrt[d_n]{a_n t_n}\big].$$

In particular, S is k–isomorphic to $R_n \otimes_k L$ and $\operatorname{Pic}(S) = 0$.

Proof. By taking the action of $\mathbf{GL}_n(\mathbb{Z})$ on R_n described in Remark 2.7 we can twist the R_n–module R_n by an element $g \in \mathbf{GL}_n(\mathbb{Z})$. We denote the resulting twisted R_n–algebra by R_n^g.[4] Note that R_n^g and R_n have the same units. For convenience in what follows we will for simplicity denote $(R_n)^g$ by R_n^g, $(R_{n,m})^g$ by $R_{n,m}^g$ and $S \otimes_{R_n} R_n^g$ by S^g.

By Galois theory there exists a finite Galois extension k'/k and a positive integer m such that $\mu_m(\overline{k}) \subset k'$ and $S \simeq_{R_n} (R_{n,m} \otimes_k k')^H$ where H is a subgroup of $\mathrm{Gal}(R_{n,m} \otimes_k k'/R_n) = \boldsymbol{\mu}_m(k')^n \rtimes \mathrm{Gal}(k'/k)$. Hence S is geometrically connected and S is a finite étale cover of $R_n \otimes_k L$. We can assume without loss of generality that $k = L$. To say that $L = k$ is to say that the map $H \to \mathrm{Gal}(k'/k)$ is onto. We consider the following commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \boldsymbol{\mu}_m(k')^n \cap H & \longrightarrow & H & \longrightarrow & \mathrm{Gal}(k'/k) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \boldsymbol{\mu}_m(k')^n & \longrightarrow & \mathrm{Gal}(R_{n,m} \otimes_k k'/R_n) & \longrightarrow & \mathrm{Gal}(k'/k) & \longrightarrow & 1.
\end{array}$$

Note that the action $\mathrm{Gal}(k'/k)$ on $\boldsymbol{\mu}_m(k')^n$ normalizes $\boldsymbol{\mu}_m(k')^n \cap H$. Hence $\boldsymbol{\mu}_m^n(k') \cap H$ is the group of k'–points of a split k–group $\boldsymbol{\nu}$ of multiplicative type. By considering the corresponding characters groups, we get a surjective homomorphism $(\mathbb{Z}/m\mathbb{Z})^n = \widehat{\boldsymbol{\mu}_m^n} \to \widehat{\boldsymbol{\nu}}$ of finite abelian groups. An element $g \in \mathbf{GL}_n(\mathbb{Z}) \subset \mathrm{Aut}_k(R_n)$ transforms the diagram above to yield

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \boldsymbol{\nu}(k') & \longrightarrow & H & \longrightarrow & \mathrm{Gal}(k'/k) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \boldsymbol{\mu}_m(k')^n & \longrightarrow & \mathrm{Gal}(R_{n,m} \otimes_k k'/R_n) & \longrightarrow & \mathrm{Gal}(k'/k) & \longrightarrow & 1. \\
 & & {\scriptstyle g^*}\downarrow & & {\scriptstyle g^*}\downarrow & & \| & & \\
1 & \longrightarrow & \boldsymbol{\mu}_m(k')^n & \longrightarrow & \mathrm{Gal}(R_{n,m}^g \otimes_k k'/R_n^g) & \longrightarrow & \mathrm{Gal}(k'/k) & \longrightarrow & 1. \\
 & & \uparrow & & \uparrow & & \| & & \\
1 & \longrightarrow & {}^g\boldsymbol{\nu}(k') & \longrightarrow & H^g & \longrightarrow & \mathrm{Gal}(k'/k) & \longrightarrow & 1
\end{array}$$

The action of $\mathbf{GL}_n(\mathbb{Z})$ on $(\mathbb{Z}/m\mathbb{Z})^n = \widehat{\boldsymbol{\mu}_m^n}$ is the left action provided by the homomorphism $\mathbf{GL}_n(\mathbb{Z}) \to \mathbf{GL}_n(\mathbb{Z}/m\mathbb{Z})$. By elementary facts about generators of finite abelian groups there exists $g \in \mathbf{GL}_n(\mathbb{Z})$ and positive integers $d_1, ..., d_n$ such that

[4] The multiplication on the R_n-algebras R_n^g and R_n coincide. It is the action of R_n that is different. See §4.1 for details.

$d_1 \mid d_2 \cdots \mid d_n \mid m$ for which the following holds

$$\begin{array}{ccc}
(\mathbb{Z}/m\mathbb{Z})^n = \widehat{\boldsymbol{\mu}_m^n} & \longrightarrow & \widehat{\boldsymbol{\nu}} \\
{\scriptstyle g^*}\big\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\big\downarrow \\
(\mathbb{Z}/m\mathbb{Z})^n = \widehat{\boldsymbol{\mu}_m^n} & \longrightarrow & \mathbb{Z}/(m/d_1)\mathbb{Z} \oplus \cdots \mathbb{Z}/(m/d_n)\mathbb{Z}.
\end{array}$$

This base change leads to the following commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \boldsymbol{\mu}_{m/d_1}(k') \times \cdots \times \boldsymbol{\mu}_{m/d_n}(k') & \longrightarrow & H^g & \longrightarrow & \mathrm{Gal}(k'/k) & \longrightarrow & 1 \\
 & & \big\downarrow & & \big\downarrow & & \| & & \\
1 & \longrightarrow & \boldsymbol{\mu}_m(k')^n & \longrightarrow & \mathrm{Gal}(R^g_{n,m} \otimes_k k'/R^g_n) & \longrightarrow & \mathrm{Gal}(k'/k) & \longrightarrow & 1.
\end{array}$$

We claim that S^g is equipped with a structure of an R^g_n–torsor structure under $\boldsymbol{\mu} := \boldsymbol{\mu}_{d_1} \times \cdots \boldsymbol{\mu}_{d_n}$. The diagram above provides a bijection

$$\mathrm{Gal}(R^g_{n,m} \otimes_k k'/R^g_n)/H^g \xrightarrow{\sim} \boldsymbol{\mu}(k'),$$

hence a map $\psi : \mathrm{Gal}(R^g_{n,m} \otimes_k k'/R^g_n) \longrightarrow \boldsymbol{\mu}(k')$ which is a cocycle for the standard action of $\mathrm{Gal}(R^g_{n,m} \otimes_k k'/R^g_n)$ on $\boldsymbol{\mu}(k')$ as we now check. We shall use the following two facts

(I) ψ is right H^g–invariant;

(II) the restriction of ψ to $\boldsymbol{\mu}_m(k')^n$ is a morphism of $\mathrm{Gal}(k'/k)$–modules.

We are given $\gamma_1, \gamma_2 \in \mathrm{Gal}(R^g_{n,m} \otimes_k k'/R^g_n)$. Since ${}^gH$ surjects onto $\mathrm{Gal}(k'/k)$, we can write $\gamma_i = \alpha_i h_i$ with $\alpha_i \in \boldsymbol{\mu}_m(k')^n$ and $h_i \in H^g$ for $i = 1, 2$. We have

$$\begin{aligned}
\psi(\gamma_1\gamma_2) &= \psi(\alpha_1\, h_1\, \alpha_2\, h_2) \\
&= \psi(\alpha_1\, h_1\, \alpha_2\, h_1^{-1} h_1 h_2) \\
&= \psi(\alpha_1\, h_1\, \alpha_2\, h_1^{-1}) \qquad [\textit{by (I)}\,] \\
&= \psi(\alpha_1)\, \psi(h_1\, \alpha_2\, h_1^{-1}) \qquad [\textit{by (II)}\,] \\
&= \psi(\alpha_1)\, h_1\, \psi(\alpha_2)\, h_1^{-1} \qquad [\textit{by (I)}\,] \\
&= \psi(\gamma_1)\, \gamma_1.\psi(\gamma_2).
\end{aligned}$$

Denote by $\tilde{S}$ the $\boldsymbol{\mu}$–torsor over R^g_n defined by ψ, that is

$$\tilde{S} := \Big\{\, x \in R^g_{n,m} \otimes_k k' \;\mid\; \psi(\gamma)\,.\gamma(x) = x \quad \forall \gamma \in \mathrm{Gal}(R^g_{n,m} \otimes_k k'/R^g_n) \,\Big\}.$$

Since ψ is trivial over H^g, we have $\tilde{S} \subset S^g$. But $\tilde{S}$ and S^g are both finite étale coverings of R_n^g of degree $\mid \boldsymbol{\mu}(k') \mid$, hence $\tilde{S} = S^g$. Since $\mathrm{Pic}(R_n^g) = 0$, we can use Kummer theory, namely the isomorphism

$$H^1(R_n^g, \boldsymbol{\mu}) = \prod_{j=1,\ldots,n} H^1(R_n^g, \boldsymbol{\mu}_{d_j}) \simeq \prod_{j=1,\ldots,n} R_n^{g\,\times}/(R_n^{g\,\times})^{d_j}.$$

for determining the structure of $\tilde{S}$. Since $R_n^{g\,\times} = k^\times \times \mathbb{Z}^n$, there exist scalars $a_1, ..., a_n \in k^\times$ and monomials $x_1, ..., x_n$ in the t_i such that the class of $\tilde{S}/R_n^g$ in $H^1(R_n, \boldsymbol{\mu})$ is given by $(a_1\, x_1, \cdots, a_n x_n)$. In terms of covering, this means that $\tilde{S} = k\big[\sqrt[d_1]{a_1 x_1}, \cdots, \sqrt[d_n]{a_n x_n}\big]$. Extending scalars to k', we have

$$\tilde{S} \otimes_k k' = (R_{n,m}^g \otimes_k k')^{\boldsymbol{\mu}(k')} = k'\big[\sqrt[d_1]{t_1}, \cdots, \sqrt[d_n]{t_n}\big].$$

¿From this it follows that $x_i = t_i \bmod \big((R_n^g \otimes_k k')^\times\big)^{d_i}$ and $x_i = t_i \bmod (R_n^{g\,\times})^{d_i}$. We conclude that $\tilde{S} = k\big[\sqrt[d_1]{a_1 t_1}, \cdots, \sqrt[d_n]{a_n t_n}\big]$. $\square$

2.4 Reductive group schemes: Irreducibility and isotropy

The notation that we are going to use throughout the paper deserves some comments. We will tend to use boldface characters, such as $\mathbf{G}$, for algebraic groups over k, as also for group schemes over $\mathfrak{X}$ that are obtained from groups over k. A quintessential example is $\mathbf{G}_{\mathfrak{X}} = \mathbf{G} \times_k \mathfrak{X}$. For arbitrary group schemes, or more generally group functors, over $\mathfrak{X}$ we shall tend to use german characters, such as $\mathfrak{G}$. This duality of notation will be particularly useful when dealing with twisted forms over $\mathfrak{X}$ of groups that come from k.

The concept of reductive group scheme over $\mathfrak{X}$ and all related terminology is that of [SGA3].[5]

For convenience we now recall and introduce some concepts and notation attached to a reductive group scheme $\mathfrak{H}$ over $\mathfrak{X}$. We denote by $\mathrm{rad}(\mathfrak{H})$ [resp. $\mathrm{corad}(\mathfrak{H})$] its radical (resp. coradical) torus, that is its maximal central subtorus (resp. its maximal toral quotient) of $\mathfrak{H}$ [XII.1.3].

We say that a $\mathfrak{H}$ is *reducible* if it admits a proper parabolic subgroup $\mathfrak{P}$ such that $\mathfrak{P}$ contains a Levi subgroup $\mathfrak{L}$ (see XXVI).[6] The opposite notion is *irreducible.* If $\mathfrak{X}$ is affine, the notion of reducibility for $\mathfrak{H}$ is equivalent to the existence of a proper parabolic subgroup $\mathfrak{P}$ (XXVI.2.3), so there is no ambiguity with the terminology of [CGP] and [GP2].

[5] The references to [SGA3] are so prevalent that they will henceforth be given by simply listing the Exposé number. Thus XII. 1.3, for example, refers to section 1.3 of Exposé XII of [SGA3].

[6] The concept of proper parabolic subgroup is not defined in [SGA3]. By proper we mean that $\mathfrak{P}_{\overline{s}}$ is a proper subgroup of $\mathfrak{G}_{\overline{x}}$ for all geometric points $\overline{x}$ of $\mathfrak{X}$.

By extension, if an affine group $\mathfrak{G}$ over $\mathfrak{X}$ acts on $\mathfrak{H}$, we say that the action is *reducible* if it normalizes a couple $(\mathfrak{P}, \mathfrak{L})$ where $\mathfrak{P}$ is a proper parabolic subgroup of $\mathfrak{H}$ and $\mathfrak{L}$ a Levi subgroup of $\mathfrak{P}$. The action is otherwise called *irreducible.*

We say that $\mathfrak{H}$ over is *isotropic* if $\mathfrak{H}$ admits a subgroup isomorphic to $\mathbb{G}_{m,\mathfrak{X}}$. The opposite notion is *anisotropic.*

If the base scheme $\mathfrak{X}$ is semi-local and connected (resp. normal), one can show that $\mathfrak{H}$ is anisotropic if and only if $\mathfrak{H}$ is irreducible and the torus $\mathrm{rad}(\mathfrak{H})$ [or equivalently $\mathrm{corad}(\mathfrak{H})$] is anisotropic (XXVI.2.3, resp. [Gi4]).

Similarly we say that the action of $\mathfrak{G}$ on $\mathfrak{H}$ is *isotropic* if it centralizes a split subtorus $\mathfrak{T}$ of $\mathbf{H}$ with the property that al geometric fibers of $\mathfrak{T}$ are non-trivial. Otherwise the action is *anisotropic.* One checks that this is the case if and only if the action is it is irreducible and the action of $\mathfrak{G}$ on the torus $\mathrm{rad}(\mathfrak{H})$ [or equivalently to $\mathrm{corad}(\mathfrak{H})$] is anisotropic.

3 Loop, finite and toral torsors

Throughout this section k will denote a field of arbitrary characteristic, $\mathfrak{X}$ a geometrically connected noetherian scheme over k, and $\mathfrak{X}^{sc} = \varprojlim \mathfrak{X}_i$ its simply connected cover as described in §2.1. Let $\mathbf{G}$ a group scheme over k which is locally of finite presentation. We will maintain the notation of the previous section, and assume that $\Omega = \overline{k}$. Consider the fundamental exact sequence (2.4). The geometric point a corresponds to a point of $\mathfrak{X}(\overline{k})$.

3.1 Loop torsors

Because of (2.1), the geometric points $a_i :\ \mathrm{Spec}\,(\overline{k}) \to \mathfrak{X}_i$ induce a geometric point $a^{sc} : \mathrm{Spec}\,(\overline{k}) \to \varprojlim \mathfrak{X}_i = \mathfrak{X}^{sc}$. We thus have a group homomorphism

$$\mathbf{G}(k_s) \to \mathbf{G}(\overline{k}) \overset{\mathbf{G}(a^{sc})}{\longrightarrow} \mathbf{G}(\mathfrak{X}^{sc}). \tag{3.1}$$

The group $\pi_1(\mathfrak{X}, a)$ acts on k_s, hence on $\mathbf{G}(k_s)$, via the group homomorphism $\pi_1(\mathfrak{X}, a) \to \mathrm{Gal}\,(k)$ of (2.4). This action is continuous, and together with (3.1) yields a map

$$H^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(k_s)\big) \to H^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(\mathfrak{X}^{sc})\big),$$

where we remind the reader that these H^1 are defined in the "continuous" sense (see Remark 2.4). On the other hand, by Proposition 2.3 and basic properties of torsors trivialized by Galois extensions, we have

$$\begin{aligned} H^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(\mathfrak{X}^{sc})\big) &= \varinjlim H^1\big(\mathrm{Aut}_{\mathfrak{X}}(\mathfrak{X}_i), \mathbf{G}(\mathfrak{X}_i)\big) \\ &= \varinjlim H^1(\mathfrak{X}_i/\mathfrak{X}, \mathbf{G}) \subset H^1(\mathfrak{X}, \mathbf{G}). \end{aligned}$$

By means of the foregoing observations we make the following.

Definition 3.1. A torsor $\mathfrak{E}$ over $\mathfrak{X}$ under $\mathbf{G}$ is called a *loop torsor* if its isomorphism class $[\mathfrak{E}]$ in $H^1(\mathfrak{X}, \mathbf{G})$ belongs to the image of the composite map

$$H^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(k_s)\big) \to H^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(\mathfrak{X}^{sc})\big) \subset H^1(\mathfrak{X}, \mathbf{G}). \tag{3.2}$$

We will denote by $H^1_{loop}(\mathfrak{X}, \mathbf{G})$ the subset of $H^1(\mathfrak{X}, \mathbf{G})$ consisting of classes of loop torsors. They are given by (continuous) cocycles in the image of the natural map $Z^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(k_s)\big) \to Z^1(\mathfrak{X}, \mathbf{G})$, which we call *loop cocycles.*

Examples 3.2. (a) If $\mathfrak{X} = \operatorname{Spec}(k)$ then $H^1_{loop}(\mathfrak{X}, \mathbf{G})$ is nothing but the usual Galois cohomology of k with coefficients in $\mathbf{G}$.

(b) Assume that k is separably closed. Then the action of $\pi_1(\mathfrak{X}, a)$ on $\mathbf{G}(k_s)$ is trivial, so that

$$H^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(k_s)\big) = \operatorname{Hom}\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(k_s)\big)/\operatorname{Int}\mathbf{G}(k_s)$$

where the group $\operatorname{Int}\mathbf{G}(k_s)$ of inner automorphisms of $\mathbf{G}(k_s)$ acts naturally on $\operatorname{Hom}\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(k_s)\big)$. To be precise, $\operatorname{Int}(g)(\phi) : x \to g^{-1}\phi(x)g$ for all $g \in \mathbf{G}(k_s)$, $\phi \in \operatorname{Hom}\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(k_s)\big)$ and $x \in \pi_1(\mathfrak{X}, a)$. Two particular cases are important:

(b1) $\mathbf{G}$ abelian: In this case $H^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(k_s)\big)$ is just the group of continuous homomorphisms from $\pi_1(\mathfrak{X}, a)$ to $\mathbf{G}(k_s)$.

(b2) $\pi_1(\mathfrak{X}, a) = \widehat{\mathbb{Z}}^n$: In this case $H^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(k_s)\big)$ is the set of conjugacy classes of n–tuples $\sigma = (\sigma_1, \dots, \sigma_n)$ of commuting elements of finite order of $\mathbf{G}(k_s)$. That the elements are of finite order follows from the continuity assumption.

(c) Let $\mathfrak{X} = \operatorname{Spec}(k[t^{\pm 1}])$ with k algebraically closed of characteristic 0. If $\mathbf{G}$ is a *connected* linear algebraic group over k then $H^1(\mathfrak{X}, \mathbf{G}) = 1$ ([P1, prop. 5]). We see from (b2) above that the canonical map

$$H^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(k_s)\big) \to H^1(\mathfrak{X}, \mathbf{G})$$

of (3.2) need not be injective. It need not be surjective either (take $\mathfrak{X} = \mathbb{P}^1_k$ and $\mathbf{G} = \mathbf{G}_{m,k}$).

(d) If the canonical map $\mathbf{G}(k_s) \to \mathbf{G}(\mathfrak{X}^{sc})$ is bijective, e.g. if $\mathfrak{X}$ is a geometrically integral projective variety over k (i.e. a geometrically integral closed subscheme of $\mathbb{P}^n_k$ for some n,) then $H^1_{loop}(\mathfrak{X}, \mathbf{G}) = H^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(k_s)\big)$.

Remark 3.3. The notion of loop torsor behaves well under twisting by a Galois cocycle $z \in Z^1\big(\operatorname{Gal}(k), \mathbf{G}(k_s)\big)$. Indeed the torsion map $\tau_z^{-1} : H^1(\mathfrak{X}, \mathbf{G}) \to H^1(\mathfrak{X}, {}_z\mathbf{G})$ maps loop classes to loop classes.

3.2 Loop reductive groups

Let $\mathfrak{H}$ be a reductive group scheme over $\mathfrak{X}$. Since $\mathfrak{X}$ is connected, for all $x \in \mathfrak{X}$ the geometric fibers $\mathfrak{H}_{\overline{x}}$ are reductive group schemes of the same "type" (see [SGA3, XXII.2.3]. By Demazure's theorem there exists a unique split reductive group $\mathbf{H}_0$ over k such that $\mathfrak{H}$ is a twisted form (in the étale topology of $\mathfrak{X}$) of $\mathfrak{H}_0 = \mathbf{H}_0 \times_k \mathfrak{X}$. We will call $\mathbf{H}_0$ the Chevalley k–form of $\mathfrak{H}$. The $\mathfrak{X}$–group $\mathfrak{H}$ corresponds to a torsor $\mathfrak{E}$ over $\mathfrak{X}$ under the group scheme $\mathbf{Aut}(\mathfrak{H}_0)$, namely $\mathfrak{E} = \mathbf{Isom}_{gr}(\mathfrak{H}_0, \mathfrak{H})$. We recall that $\mathbf{Aut}(\mathfrak{H}_0)$ is representable by a smooth and separated group scheme over $\mathfrak{X}$ by XXII 2.3. By definition $\mathfrak{H}$ is then the contracted product $\mathfrak{E} \wedge^{\mathbf{Aut}(\mathfrak{H}_0)} \mathfrak{H}_0$ (see [DG] III §4 n°3 for details.)

We now define one of the central concepts of our work.

Definition 3.4. We say that a group scheme $\mathfrak{H}$ over $\mathfrak{X}$ is *loop reductive* if it is reductive and if $\mathfrak{E}$ is a loop torsor.

We look more closely to the affine case $\mathfrak{X} = \mathrm{Spec}(R)$. Concretely, let $\mathbf{H}_0 = \mathrm{Spec}(k[\mathbf{H}_0])$ be a split reductive k–group and consider the corresponding R–group $\mathfrak{H}_0 = \mathbf{H}_0 \times_k R$, whose Hopf algebra is $R[\mathfrak{H}_0] = k[\mathbf{H}_0] \otimes_k R$.

Let $\mathfrak{H}$ be an R–group which is a twisted form of $\mathfrak{H}_0$ trivialized by the universal covering R^{sc}. Then to a trivialization $\mathfrak{H}_0 \times_R R^{sc} \cong \mathfrak{H} \times_R R^{sc}$, we can attach a cocycle $u \in Z^1\big(\pi_1(R,a), \mathbf{Aut}(\mathfrak{H}_0)(R^{sc})\big)$ from which $\mathfrak{H}$ can be recovered by Galois descent as we now explain in the form of a Remark for future reference.

Remark 3.5. There are two possible conventions as to the meaning of the cocycles $Z^1\big(\pi_1(R,a), \mathbf{Aut}(\mathfrak{H}_0)(R^{sc})\big)$. On the one hand $\mathfrak{H}_0$ can be thought as the affine scheme $\mathrm{Spec}(R[\mathfrak{H}_0])$, $\mathbf{Aut}(\mathfrak{H}_0)(R^{sc})$ as the (abstract) group of automorphisms of the R^{sc}–group $\mathrm{Spec}(R^{sc}[\mathfrak{H}_0])$ where $R^{sc}[\mathfrak{H}_0] = R[\mathfrak{H}_0] \otimes_R R^{sc}$, and $\pi_1(X,a)$ as the opposite group of automorphisms of $\mathrm{Spec}(R^{sc})/\,\mathrm{Spec}(R)$ acting naturally on $\mathbf{Aut}(\mathfrak{H}_0)(R^{sc})$.

We will adopt an (anti) equivalent second point of view that is much more convenient for our calculations. We view $\mathbf{Aut}(\mathfrak{H}_0)(R^{sc})$ as the group of automorphisms of the R^{sc}–Hopf algebra $R^{sc}[\mathbf{H}_0] = R[\mathfrak{H}_0] \otimes_R R^{sc} \simeq k[\mathbf{H}_0] \otimes_k R^{sc}$ on which the Galois group $\pi_1(R,a)$ acts naturally. Then the R–Hopf algebra $R[\mathfrak{H}]$ corresponding to $\mathfrak{H}$ is given by

$$R[\mathfrak{H}] = \{x \in R^{sc}[\mathbf{H}_0] : u_\gamma{}^\gamma x = a \text{ for all } \gamma \in \pi_1(R,a)\}.$$

To say then that $\mathfrak{H}$ is k–loop reductive is to say that u can be chosen so that $u_\gamma \in \mathbf{Aut}(\mathbf{H}_0)(\overline{k}) \subset \mathbf{Aut}(\mathbf{H}_0)(R^{sc}) = \mathbf{Aut}(\mathfrak{H}_0)(R^{sc})$ for all $\gamma \in \pi_1(R,a)$.

3.3 Loop torsors at a rational base point

If our geometric point a lies above a k–rational point x of $\mathfrak{X}$, then x corresponds to a section of the structure morphism $\mathfrak{X} \to \mathrm{Spec}(k)$ which maps b to a. This yields a group homomorphism $x^* : \mathrm{Gal}(k) \to \pi_1(\mathfrak{X}, a)$ that splits the sequence (2.4) above. This splitting defines an action of $\mathrm{Gal}(k)$ on the profinite group $\pi_1(\overline{\mathfrak{X}}, \overline{a})$, hence a semidirect product identification

$$\pi_1(\mathfrak{X}, a) \simeq \pi_1(\overline{\mathfrak{X}}, \overline{a}) \rtimes \mathrm{Gal}(k). \tag{3.3}$$

We have seem an example of (3.3) in Example 2.3.

Remark 3.6. By the structure of extensions of profinite groups [RZ, §6.8], it follows that $\pi_1(\overline{\mathfrak{X}}, \overline{a})$ is the projective limit of a system $\big(H_\alpha \rtimes \mathrm{Gal}(k)\big)$ where the H_α's are finite groups. The Galois action on each H_α defines a twisted finite constant k–group $\boldsymbol{\nu}_\alpha$. We define

$$\boldsymbol{\nu} = \varprojlim_{\alpha} \boldsymbol{\nu}_\alpha.$$

The $\boldsymbol{\nu}_\alpha$ are affine k–groups such that

$$\boldsymbol{\nu}(\overline{k}) = \varprojlim_{\alpha} \boldsymbol{\nu}_\alpha(\overline{k}) \;=\; \pi_1(\overline{\mathfrak{X}}, \overline{a}).$$

Note that $\boldsymbol{\nu}(k_s) = \boldsymbol{\nu}(\overline{k})$. In the case when $\mathfrak{X} = \mathrm{Spec}(R_n)$, where as before $R_n = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ with k of characteristic zero and a is the geometric point described in Example 2.3, the above construction yields the affine k–group ${}_{\infty}\boldsymbol{\mu}$ defined in Remark 2.6.

By means of the decomposition (3.3) we can think of loop torsors as being comprised of a geometric and an arithmetic part, as we now explain.

Let $\eta \in Z^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(k_s)\big)$ The restriction $\eta_{|\mathrm{Gal}(k)}$ is called the *arithmetic part* of η and its denoted by η^{ar}. It is easily seen that η^{ar} is in fact a cocycle in $Z^1\big(\,\mathrm{Gal}(k), \mathbf{G}(k_s)\big)$. If η is fixed in our discussion, we will at times denote the cocycle η^{ar} by the more traditional notation z. In particular, for $s \in \mathrm{Gal}(k)$ we write z_s instead of η^{ar}_s.

Next we consider the restriction of η to $\pi_1(\overline{\mathfrak{X}}, \overline{a})$ that we denote by η^{geo} and called the *geometric part* of η.

We thus have a map

$$\begin{array}{cccc} \Theta \; : Z^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(k_s)\big) & \longrightarrow & Z^1\big(\,\mathrm{Gal}(k), \mathbf{G}(k_s)\big) \times \mathrm{Hom}\,\big(\pi_1(\overline{\mathfrak{X}}, \overline{a}), \mathbf{G}(k_s)\big) \\ \eta & \mapsto & \big(\quad \eta^{ar} \quad , \quad \eta^{geo} \quad \big) \end{array}$$

The group $\mathrm{Gal}(k)$ acts on $\pi_1(\overline{\mathfrak{X}}, \overline{a})$ by conjugation. On $\mathbf{G}(k_s)$, the Galois group $\mathrm{Gal}(k)$ acts on two different ways. There is the natural action arising for the action of $\mathrm{Gal}(k)$ on k_s that as customary we will denote by ${}^s g$, and there is also the twisted action given by the cocycle $\eta^{ar} = z$. Following Serre we denote this last by ${}^{s'} g$. Thus

${}^{s'}g = z_s{}^s g z_s{}^{-1}$. Following standard practice to view the abstract group $\mathbf{G}(k_s)$ as a Gal(k)–module with the twisted action by z we write ${}_z\mathbf{G}(k_s)$.

For $s \in \mathrm{Gal}(k)$ and $h \in \pi_1(\overline{\mathfrak{X}}, \overline{a})$, we have

$$\begin{aligned}
\eta^{geo}_{shs^{-1}} &= \eta_{shs^{-1}} = \eta_s{}^s(\eta_{hs^{-1}}) \qquad [\eta \text{ is a cocycle}] \\
&= z_s\ {}^s(\eta_{hs^{-1}}) \qquad [\eta_s = \eta^{\mathrm{ar}}_s = z_s] \\
&= z_s\ {}^s(\eta^{geo}_h\, z_{s^{-1}}) \qquad [\eta \text{ is a cocycle and } h \text{ acts trivially on } \mathbf{G}(k_s)] \\
&= z_s\ {}^s\eta^{\mathrm{geo}}_h\, z_s^{-1} \qquad [1 z_s\, {}^s z_{s^{-1}}].
\end{aligned}$$

This shows that $\eta^{\mathrm{geo}} : \pi_1(\overline{\mathfrak{X}}, \overline{a}) \to {}_z\mathbf{G}(k_s)$ commutes with the action of Gal(k). In other words, $\eta^{geo} \in \mathrm{Hom}_{\mathrm{Gal}(k)}\big(\pi_1(\overline{\mathfrak{X}}, \overline{a}), {}_z\mathbf{G}(k_s)\big)$.

Lemma 3.7. *The map Θ defines a bijection between $Z^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(k_s)\big)$ and couples (z, η^{geo}) with $z \in Z^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(k_s)\big)$ and $\eta^{geo} \in \mathrm{Hom}_{\mathrm{Gal}(k)}\big(\pi_1(\overline{\mathfrak{X}}, \overline{a}), {}_z\mathbf{G}(k_s)\big)$.*

Proof. Since a 1–cocycle is determined by its image on generators, the map Θ is injective. For the surjectivity, assume we are given $z \in Z^1(\pi_1(\mathfrak{X}, a), \mathbf{G}(k_s))$ and $\eta^{geo} \in \mathrm{Hom}_{\mathrm{Gal}(k)}\big(\pi_1(\overline{\mathfrak{X}}, \overline{a}), {}_z\mathbf{G}(k_s)\big)$. We define then $\eta : \pi_1(\mathfrak{X}, a) \to \mathbf{G}(k_s)$ by $\eta_{hs} := \eta^{geo}_h\, z_s$ This map is continuous, its restriction to $\pi_1(\overline{\mathfrak{X}}, \overline{a})$ (resp. Gal(k)) is η^{geo} (resp. z). Finally, since η is a section of the projection map $\mathbf{G}(k_s) \rtimes \pi_1(\mathfrak{X}, a) \to \pi_1(\mathfrak{X}, a)$, it is a cocycle. □

We finish this section by recalling some basic properties of the twisting bijection. Let $\eta \in Z^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(k_s)\big)$ and consider its corresponding pair $\Theta(\eta) = (z, \eta^{geo})$. We can apply the same construction to the twisted k–group ${}_z\mathbf{G}$. This would lead to a map Θ_z that will attach to a cocycle $\eta' \in Z^1\big(\pi_1(\mathfrak{X}, a), {}_z\mathbf{G}(k_s)\big)$ a pair $(z', \eta'^{\mathrm{geo}})$ along the lines explained above. Note that by Lemma 3.7 the pair $(1, \eta^{\mathrm{geo}})$ is in the image of Θ_z. More precisely.

Lemma 3.8. *Let $\eta \in Z^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(k_s)\big)$. With the above notation, the inverse of the twisting map* [Se]

$$\tau_z^{-1} : Z^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(k_s)\big) \xrightarrow{\sim} Z^1\big(\pi_1(\mathfrak{X}, a), {}_z\mathbf{G}(k_s)\big)$$

satisfies $\Theta_z \circ \tau_z^{-1}(\eta) = (1, \eta^{\mathrm{geo}})$.

Remark 3.9. Consider the special case when the semi-direct product is direct, i.e. $\pi_1(\mathfrak{X}, a) = \pi_1(\overline{\mathfrak{X}}, \overline{a}) \times \mathrm{Gal}(k)$. In other words, the affine k–group $\boldsymbol{\nu}$ defined above is constant so that

$$\eta^{geo}_h = z_s\ {}^s\eta^{geo}_h\, z_s^{-1}$$

for all $h \in \pi_1(\overline{\mathfrak{X}}, \overline{a})$ and $s \in \mathrm{Gal}(k)$. The torsion map

$$\tau_z^{-1} : Z^1\big(\pi_1(\overline{\mathfrak{X}}, \overline{a}), \mathbf{G}(k_s)\big) \to Z^1\big(\pi_1(\overline{\mathfrak{X}}, \overline{a}), {}_z\mathbf{G}(k_s)\big)$$

maps the cocycle η to the homomorphism $\eta^{\mathrm{geo}} : \pi_1(\overline{\mathfrak{X}}, \overline{a}) \to {}_z\mathbf{G}(k_s)$.

We give now one more reason to call η^{geo} the geometric part of η.

Lemma 3.10. *Let $\boldsymbol{\nu}$ be the affine k–group scheme defined in Remark 3.6. Then for each linear algebraic k–group $\mathbf{H}$, there is a natural bijection*

$$\mathrm{Hom}_{k-gp}(\boldsymbol{\nu}, \mathbf{H}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Gal}(k)}\big(\pi_1(\overline{\mathfrak{X}}, \overline{a}), \mathbf{H}(k_s)\big).$$

Proof. First we recall that $\mathrm{Hom}_{\mathrm{Gal}(k)}\big(\pi_1(\overline{\mathfrak{X}}, \overline{a}), \mathbf{H}(k_s)\big)$ stands for the *continuous* homomorphisms from $\pi_1(\overline{\mathfrak{X}}, \overline{a})$ to $\mathbf{H}(k_s)$ that commute with the action of $\mathrm{Gal}(k)$.

Write $\boldsymbol{\nu} = \varprojlim \boldsymbol{\nu}_\alpha$ as an inverse limit of twisted constant finite k–groups. Since $\mathbf{H}$ and the $\boldsymbol{\nu}_\alpha$ are of finite presentation we have by applying [SGA3] VI_B 10.4 that

$$\mathrm{Hom}_{k-gp}(\boldsymbol{\nu}, \mathbf{H}) = \varinjlim_\alpha \mathrm{Hom}_{k-gp}(\boldsymbol{\nu}_\alpha, \mathbf{H})$$

$$= \varinjlim_\alpha \mathrm{Hom}_{\mathrm{Gal}(k)}\big(\boldsymbol{\nu}_\alpha(k_s), \mathbf{H}(k_s)\big) = \mathrm{Hom}_{\mathrm{Gal}(k)}\big(\pi_1(\overline{\mathfrak{X}}, \overline{a}), \mathbf{H}(k_s)\big).$$

□

This permits to see purely geometric k–loop torsors in terms of homomorphisms of affine k–group schemes.

3.4 Finite torsors

Throughout this section we assume that $\mathbf{G}$ is a smooth affine k–group, and $\mathfrak{X}$ a scheme over k. Let $\mathbf{G}_{\mathfrak{X}} = \mathbf{G} \times_k \mathfrak{X}$ be the $\mathfrak{X}$–group obtained from $\mathbf{G}$ by base change.

Following our convention a torsor over $\mathfrak{X}$ under $\mathbf{G}$ means under $\mathbf{G}_{\mathfrak{X}}$, and we write $H^1(\mathfrak{X}, \mathbf{G})$ instead of $H^1(\mathfrak{X}, \mathbf{G}_{\mathfrak{X}})$.

Definition 3.11. A torsor $\mathfrak{E}$ over $\mathfrak{X}$ under $\mathbf{G}$ is said to be *finite* if it admits a reduction to a finite k–subgroup $\mathbf{S}$ of $\mathbf{G}$; this is to say, the class of $\mathfrak{E}$ belongs to the image of the natural map $H^1(\mathfrak{X}, \mathbf{S}) \to H^1(\mathfrak{X}, \mathbf{G})$ for some finite subgroup $\mathbf{S}$ of $\mathbf{G}$.

We denote by $H^1_{finite}(\mathfrak{X}, \mathbf{G})$ the subset of $H^1(\mathfrak{X}, \mathbf{G})$ consisting of classes of finite torsors, that is

$$H^1_{finite}(\mathfrak{X}, \mathbf{G}) := \bigcup_{\mathbf{S} \subset \mathbf{G}} \mathrm{Im}\big(H^1(\mathfrak{X}, \mathbf{S}) \to H^1(\mathfrak{X}, \mathbf{G})\big).$$

where $\mathbf{S}$ runs over all finite k–subgroups of $\mathbf{G}$.

The case when k is of characteristic 0 is well known.

Lemma 3.12. *Assume that k is of characteristic 0. Then $H^1_{finite}(\mathfrak{X}, \mathbf{G}) \subset H^1_{loop}(\mathfrak{X}, \mathbf{G})$. If in addition k is algebraically closed, then $H^1_{finite}(\mathfrak{X}, \mathbf{G}) = H^1_{loop}(\mathfrak{X}, \mathbf{G})$.*

Proof. Let $\mathbf{S}$ be a finite subgroup of the k–group $\mathbf{G}$. Since k is of characteristic 0 the group $\mathbf{S}$ is étale. Thus $\mathbf{S}$ corresponds to a finite abstract group S together with a continuous action of $\mathrm{Gal}(k)$ by group automorphisms. More precisely (see [SGA1] or [K2] pg.184) $S = \mathbf{S}(\overline{k})$ with the natural action of $\mathrm{Gal}(k)$. Similarly the étale $\mathfrak{X}$-group $\mathbf{S}_{\mathfrak{X}}$ corresponds to S with the action of $\pi_1(\mathfrak{X}, a)$ induced from the homomorphism $\pi_1(\mathfrak{X}, a) \to \mathrm{Gal}(k)$.

By Exp. XI of [SGA1] we have

$$H^1(\mathfrak{X}, \mathbf{S}) \overset{\mathrm{def}}{=} H^1(\mathfrak{X}, \mathbf{S}_{\mathfrak{X}}) = H^1\big(\pi_1(\mathfrak{X}, a), S\big) = H^1(\pi_1(\mathfrak{X}, a), \mathbf{S}(\overline{k})) \tag{3.4}$$

which shows that $H^1(\mathfrak{X}, \mathbf{S}) \subset H^1_{loop}(\mathfrak{X}, \mathbf{G})$.

If k is algebraically closed any k-loop torsor $\mathfrak{E}$ is given by a continuous group homomorphism $f_{\mathfrak{E}} : \pi_1(\mathfrak{X}, a) \to \mathbf{G}(k)$, as explained in Example 3.2(b). Then the image of $f_{\mathfrak{E}}$ is a finite subgroup of $\mathbf{G}(k)$ which gives rise to a finite (constant) algebraic subgroup $\mathbf{S}$ of $\mathbf{G}$. By construction $[\mathfrak{E}]$ comes from $H^1(\mathfrak{X}, \mathbf{S})$. □

3.5 Toral torsors

Let k, $\mathbf{G}$ and $\mathfrak{X}$ be as in the previous section. Given a torsor $\mathfrak{E}$ over $\mathfrak{X}$ under $\mathbf{G}_{\mathfrak{X}}$ we can consider the twisted $\mathfrak{X}$–group ${}_{\mathfrak{E}}\mathbf{G}_{\mathfrak{X}} = \mathfrak{E} \wedge^{\mathbf{G}_{\mathfrak{X}}} \mathbf{G}_{\mathfrak{X}}$. Since no confusion will arise we will denote ${}_{\mathfrak{E}}\mathbf{G}_{\mathfrak{X}}$ simply by ${}_{\mathfrak{E}}\mathbf{G}$. We say that our torsor $\mathfrak{E}$ is *toral* if the twisted $\mathfrak{X}$–group ${}_{\mathfrak{E}}\mathbf{G}$ admits a maximal torus (XII.1.3). We denote by $H^1_{toral}(\mathfrak{X}, \mathbf{G}) \subset H^1(\mathfrak{X}, \mathbf{G})$ the set of classes of toral torsors.

We recall the following useful result.

Lemma 3.13. *1. Let $\mathbf{T}$ be a maximal torus of $\mathbf{G}$.[7] Then*

$$H^1_{toral}(\mathfrak{X}, \mathbf{G}) = \mathrm{Im}\Big(H^1\big(\mathfrak{X}, \mathbf{N}_{\mathbf{G}}(\mathbf{T})\big) \to H^1(\mathfrak{X}, \mathbf{G})\Big).$$

2. Let $1 \to \mathbf{S} \to \mathbf{G}' \overset{p}{\to} \mathbf{G} \to 1$ be a central extension of $\mathbf{G}$ by a k–group $\mathbf{S}$ of multiplicative type. Then the diagram

$$\begin{array}{ccc} H^1_{toral}(\mathfrak{X}, \mathbf{G}') & \subset & H^1(\mathfrak{X}, \mathbf{G}') \\ {\scriptstyle p_*}\downarrow & & {\scriptstyle p_*}\downarrow \\ H^1_{toral}(\mathfrak{X}, \mathbf{G}) & \subset & H^1(\mathfrak{X}, \mathbf{G}) \end{array}$$

[7] We remind the reader that we are abiding by [SGA3] conventions and terminology. In the expression "maximal torus of $\mathbf{G}$ " we view $\mathbf{G}$ as a k–group, namely a group scheme over $\mathrm{Spec}(k)$. In particular $\mathbf{T}$ is a k-group...

is cartesian.

Proof. (1) is established in [CGR2, 3.1].
(2) Consider first the case when $\mathbf{S}$ is the reductive center of $\mathbf{G}'$, We are given an $\mathfrak{X}$–torsor $\mathfrak{E}'$ under $\mathbf{G}'$ and consider the surjective morphism of $\mathfrak{X}$–group schemes ${}_{\mathfrak{E}'}\mathbf{G}' \to {}_{\mathfrak{E}'}\mathbf{G}$ whose kernel is $\mathbf{S} \times_k \mathfrak{X}$. By XII 4.7 there is a natural one-to-one correspondence between maximal tori of the $\mathfrak{X}$–groups ${}_{\mathfrak{E}'}\mathbf{G}'$ and ${}_{\mathfrak{E}'}\mathbf{G}$. Hence $\mathfrak{E}'$ is a toral $\mathbf{G}'$–torsor if and only if $\mathfrak{E}' \wedge^{\mathbf{G}'_{\mathfrak{X}}} \mathbf{G}_{\mathfrak{X}}$ is a toral $\mathbf{G}$–torsor. The general case follows form the fact that $\mathbf{G}'/\mathbf{Z}' \simeq \mathbf{G}/\mathbf{Z}$ where $\mathbf{Z}'$ (resp. $\mathbf{Z}$) is the reductive center of $\mathbf{G}$. □

In an important case the property of a torsor being toral is of infinitesimal nature.

Lemma 3.14. *Assume that* $\mathbf{G}$ *is semisimple of adjoint type. For a* $\mathfrak{X}$*–torsor* $\mathfrak{E}$ *under* $\mathfrak{G}$ *the following conditions are equivalent:*

1. $\mathfrak{E}$ *is toral.*
2. *The Lie algebra* $\mathcal{L}ie({}_{\mathfrak{E}}\mathbf{G})$ *admits a Cartan subalgebra.*

Proof. By XIV théorèmes 3.9 and 3.18 there exists a natural one-to-one correspondence between the maximal tori of ${}_{\mathfrak{E}}\mathbf{G}$ and Cartan subalgebras of $\mathcal{L}ie({}_{\mathfrak{E}}\mathbf{G})$. □

Recall the following result [CGR2].

Theorem 3.15. *Let* R *be a commutative ring and* $\mathbf{G}$ *a smooth affine group scheme over* R *whose connected component of the identity* $\mathbf{G}^0$ *is reductive. Assume further that one of the following holds:*

(a) R *is an algebraically closed field, or*

(b) $R = \mathbb{Z}$, $\mathbf{G}^0$ *is a Chevalley group, and the order of the Weyl group of the geometric fiber* $\mathbf{G}_{\overline{s}}$ *is independent of* $s \in \operatorname{Spec}(\mathbb{Z})$, *or*

(c) R *is a semilocal ring,* $\mathbf{G}$ *is connected, and the radical torus* $\operatorname{rad}(\mathbf{G})$ *is isotrivial.*

Then there exist a maximal torus $\mathbf{T}$ *of* $\mathbf{G}$, *and a finite* R*–subgroup* $\mathbf{S} \subset \mathbf{N}_{\mathbf{G}}(\mathbf{T})$, *such that*

1. $\mathbf{S}$ *is an extension of a finite twisted constant* R*–group by a finite* R*–group of multiplicative type,*
2. *the natural map* $H^1_{fppf}(\mathfrak{X}, \mathbf{S}) \longrightarrow H^1_{fppf}\big(\mathfrak{X}, \mathbf{N}_{\mathbf{G}}(\mathbf{T})\big)$ *is surjective for any scheme* R*–scheme* $\mathfrak{X}$ *satisfying the condition:*

   $$\operatorname{Pic}(\mathfrak{X}') = 0 \text{ for every generalized Galois cover } \mathfrak{X}'/\mathfrak{X}, \tag{3.5}$$

   where by a generalized Galois cover $\mathfrak{X}' \to \mathfrak{X}$ *we understand a* Γ*–torsor for some twisted finite constant* $\mathfrak{X}$*–group scheme* Γ. □

Corollary 3.16. *Let* $\mathbf{G}$ *be a linear algebraic* k*–group whose connected component of the identity* $\mathbf{G}^0$ *is reductive. Assume that one of the following holds:*

(i) k *is algebraically closed;*

(ii) $\mathbf{G}$ *is obtained by base change from a smooth affine* $\mathbb{Z}$*–group satisfying the hypothesis of Theorem 3.15(b);*

(iii) $\mathbf{G}$ *is reductive.*

If the k*–scheme* $\mathfrak{X}$ *satisfies condition (3.5), then*

1. $H^1_{toral}(\mathfrak{X}, \mathbf{G}) \subset H^1_{finite}(\mathfrak{X}, \mathbf{G})$.
2. *If furthermore* $\mathrm{char}(k) = 0$*, we have* $H^1_{toral}(\mathfrak{X}, \mathbf{G}) \subset H^1_{finite}(\mathfrak{X}, \mathbf{G}) \subset H^1_{loop}(\mathfrak{X}, \mathbf{G})$.

The first statement is immediate. The second one follows from Lemma 3.12. □

4 Semilinear considerations

Throughout this section $\tilde{k}$ will denote an object of k*–alg*. We will denote by Γ a subgroup of the group $\mathrm{Aut}_{k-alg}(\tilde{k})$. The elements of Γ are thus k–linear automorphisms of the ring $\tilde{k}$. For convenience we will denote the action of an element $\gamma \in \Gamma$ on an element $\lambda \in \tilde{k}$ by ${}^\gamma\lambda$.

4.1 Semilinear morphisms

Given an object R of $\tilde{k}$–alg (the category of associative unital commutative $\tilde{k}$–algebras), we will denote the action of and element $\lambda \in \tilde{k}$ on an element $r \in R$ by $\lambda_R \cdot r$, or simply $\lambda_R\, r$ or λr if no confusion is possible.

Given an element $\gamma \in \Gamma$, we denote by R^γ the object of $\tilde{k}$–alg which coincides with R as a ring, but where the $\tilde{k}$–module structure is now obtained by "twisting " by γ :

$$\lambda_{R^\gamma} \cdot r = ({}^\gamma\lambda)_R \cdot r$$

One verifies immediately that

$$(R^\gamma)^\tau = R^{\gamma\tau} \tag{4.1}$$

for all $\gamma, \tau \in \Gamma$. It is important to emphasize that (4.1) is an *equality* and not a canonical identification.

Given a morphism $\psi : A \to R$ of $\tilde{k}$–algebras and an element $\gamma \in \Gamma$ we can view ψ as a map $\psi_\gamma : A^\gamma \to R^\gamma$ (recall that $A = A^\gamma$ and $R = R^\gamma$ as rings, hence also as sets).

It is immediate to verify that ψ_γ is also a morphism of $\tilde{k}$–algebras. By (4.1) we have $(\psi_\gamma)_\tau = \psi_{\gamma\tau}$ for all $\gamma, \tau \in \Gamma$.

The map $\psi \to \psi_\gamma$ gives a natural correspondence

$$\mathrm{Hom}_{\tilde{k}-alg}(A, R) \to \mathrm{Hom}_{\tilde{k}-alg}(A^\gamma, R^\gamma). \tag{4.2}$$

In view of (4.1) we have also a natural (and equivalent) correspondence

$$\mathrm{Hom}_{\tilde{k}-alg}(A, R^\gamma) \to \mathrm{Hom}_{\tilde{k}-alg}(A^{\gamma^{-1}}, R). \tag{4.3}$$

that we record for future use.

Remark 4.1. (i) Let $\gamma, \sigma \in \Gamma$. It is clear from the definitions that the k-algebra isomorphism $\gamma : \tilde{k} \to \tilde{k}$ induces a $\tilde{k}$-algebra isomorphism $\gamma_\sigma : \tilde{k}^\sigma \to \tilde{k}^{\gamma\sigma}$. If no confusion is possible we will denote γ_σ simply by γ.

One checks that the $\tilde{k}$–algebras $(R \otimes_k \tilde{k})^\gamma$ and $R \otimes_k \tilde{k}^\gamma$ are *equal* (recall that both algebras have $R \otimes_k \tilde{k}$ as underlying sets). We thus have a $\tilde{k}$–algebra isomorphism

$$1 \otimes \gamma : R \otimes_k \tilde{k} \to R \otimes_k \tilde{k}^\gamma = (R \otimes \tilde{k})^\gamma,$$

or more generally

$$1 \otimes \gamma_\sigma : R \otimes_k \tilde{k}^\sigma \to R \otimes_k \tilde{k}^{\gamma\sigma} = (R \otimes \tilde{k})^{\gamma\sigma}.$$

(ii) If A is an object $\tilde{k}$–alg, and $\gamma \in \Gamma$, then the $\tilde{k}$–algebras A and A^γ have the same ideals.

Given a $\tilde{k}$–functor $\mathfrak{X}$, that is a functor from the category $\tilde{k}$–alg to the category of sets (see [DG] for details), and an element $\gamma \in \Gamma$ we can define a new the $\tilde{k}$–functor ${}^\gamma\mathfrak{X}$ by setting

$${}^\gamma\mathfrak{X}(R) = \mathfrak{X}(R^\gamma) \tag{4.4}$$

and ${}^\gamma\mathfrak{X}(\psi) = \mathfrak{X}(\psi_\gamma)$ where $\psi : R \to S$ is as above. The diagram

$$\begin{tikzcd} \mathfrak{X}'(R^\gamma) \arrow[r, "="] \arrow[d, "{}^\gamma\mathfrak{X}(\psi)"'] & \mathfrak{X}(R^\gamma) \arrow[d, "\mathfrak{X}(\psi_\gamma)"'] \\ {}^\gamma\mathfrak{X}'(R) \arrow[r, "="] & {}^\gamma\mathfrak{X}(R) \end{tikzcd}$$

then commutes by definition, and one can indeed easily verify that ${}^\gamma\mathfrak{X}$ is a $\tilde{k}$–functor. We call ${}^\gamma\mathfrak{X}$ the *twist of* $\mathfrak{X}$ *by* γ.

Similarly to the case of $\tilde{k}$–algebras described in (4.1) we have the *equality* of functors

$$ {}^{\gamma}({}^{\tau}\mathfrak{X}) = {}^{\gamma\tau}\mathfrak{X} \tag{4.5}$$

for all $\gamma, \tau \in \Gamma$.

A morphism $f : \mathfrak{X}' \to \mathfrak{X}$ induces a morphism ${}^{\gamma}f : {}^{\gamma}\mathfrak{X}' \to {}^{\gamma}\mathfrak{X}$ by setting ${}^{\gamma}f(R) = f(R^{\gamma})$. We thus have the commutative diagram

$$\begin{array}{ccc} \mathfrak{X}'(R^{\gamma}) & \xrightarrow{f(R^{\gamma})} & \mathfrak{X}(R^{\gamma}) \\ {\scriptstyle =}\big\downarrow & & {\scriptstyle =}\big\downarrow \\ {}^{\gamma}\mathfrak{X}'(R) & \xrightarrow{{}^{\gamma}f(R)} & {}^{\gamma}\mathfrak{X}(R) \end{array}$$

This gives a natural bijection

$$\mathrm{Hom}_{\tilde{k}-fun}(\mathfrak{X}', \mathfrak{X}) \to \mathrm{Hom}_{\tilde{k}-fun}({}^{\gamma}\mathfrak{X}', {}^{\gamma}\mathfrak{X}) \tag{4.6}$$

given by $f \mapsto {}^{\gamma}f$. This correspondence is compatible with the action of Γ, this is ${}^{\gamma}({}^{\tau}f) = {}^{\gamma\tau}f$. As before we will for future use explicitly write down an equivalent version of this last bijection, namely

$$\mathrm{Hom}_{\tilde{k}-fun}({}^{\gamma^{-1}}\mathfrak{X}', \mathfrak{X}) \to \mathrm{Hom}_{\tilde{k}-fun}(\mathfrak{X}', {}^{\gamma}\mathfrak{X}) \tag{4.7}$$

4.2 Semilinear morphisms

A $\tilde{k}$–functor morphism $f : {}^{\gamma}\mathfrak{X} \to \mathfrak{Z}$ is called a *semilinear morphism of type* γ *from* $\mathfrak{X}$ *to* $\mathfrak{Z}$. We denoted the set of such morphisms by $\mathrm{Hom}_{\gamma}(\mathfrak{X}, \mathfrak{Z})$, and set $\mathrm{Hom}_{\Gamma}(\mathfrak{X}, \mathfrak{Z}) = \cup_{\gamma\in\Gamma} \mathrm{Hom}_{\gamma}(\mathfrak{X}, \mathfrak{Z})$.[8] These are the Γ–*semilinear morphisms* from $\mathfrak{X}$ to $\mathfrak{Z}$.

If $f : {}^{\gamma}\mathfrak{X} \to \mathfrak{Y}$ and $g : {}^{\tau}\mathfrak{Y} \to \mathfrak{Z}$ are semilinear of type γ and τ respectively, then the map $gf : {}^{\tau\gamma}\mathfrak{X} \to \mathfrak{Z}$ defined by $(gf)(R) = g(R) \circ f(R^{\tau})$ according to the sequence

$${}^{\tau\gamma}\mathfrak{X}(R) = {}^{\gamma}\mathfrak{X}(R^{\tau}) \stackrel{f(R^{\tau})}{\to} \mathfrak{Y}(R^{\tau}) = {}^{\tau}\mathfrak{Y}(R) \stackrel{g(R)}{\to} \mathfrak{Z}(R) \tag{4.8}$$

is semilinear of type $\tau\gamma$.

The above considerations give the set $\mathrm{Aut}_{\Gamma}(\mathfrak{X})$ of invertible elements of $\mathrm{Hom}_{\Gamma}(\mathfrak{X}, \mathfrak{X})$ a group structure whose elements are called Γ–*semilinear automorphisms of* $\mathfrak{X}$. There is a group homomorphism $t : \mathrm{Aut}_{\Gamma}(\mathfrak{X}) \to \Gamma$ that assigns to a semilinear automorphism of $\mathfrak{X}$ its type.

[8] The alert reader may question whether the "type" is well defined. Indeed it may happen that ${}^{\gamma}\mathfrak{X}$ and $\mathfrak{X}$ are the *same* $\tilde{k}$-functor even though $\gamma \neq 1$. This ambiguity can be formally resolved by defining semilinear morphism of type γ as pairs $(f : {}^{\gamma}\mathfrak{X} \to \mathfrak{Z}, \gamma)$. We will omit this complication of notation in what follows since no confusion will be possible within our context. Note that the union of sets $\cup_{\gamma\in\Gamma} \mathrm{Hom}_{\gamma}(\mathfrak{X}, \mathfrak{Z})$ is thus disjoint by definition.

Remark 4.2. Fix a $\tilde{k}$–functor $\mathfrak{Y}$. Recall that the category of $\tilde{k}$–functors over $\mathfrak{Y}$ consists of $\tilde{k}$–functors $\mathfrak{X}$ equipped with a structure morphism $\mathfrak{X} \to \mathfrak{Y}$. This category admits fiber products: Given $f_1 : \mathfrak{X}_1 \to \mathfrak{Y}$ and $f_2 : \mathfrak{X}_2 \to \mathfrak{Y}$ then $\mathfrak{X}_1 \times_{\mathfrak{Y}} \mathfrak{X}_2$ is given by

$$(\mathfrak{X}_1 \times_{\mathfrak{Y}} \mathfrak{X}_2)(R) = \{(x_1, x_2) \in \mathfrak{X}_1(R) \times \mathfrak{X}_2(R) : f_1(R)(x_1) = f_2(R)(x_2)\}.$$

Semilinearity extend to fiber products under the right conditions. Suppose $f_1 : \mathfrak{X}_1 \to \mathfrak{Y}$ and $f_2 : \mathfrak{X}_2 \to \mathfrak{Y}$ are as above, and that the action of Γ in $\mathfrak{X}_i$ and $\mathfrak{Y}$ is compatible in the obvious way. Then for each $\gamma \in \Gamma$ the "structure morphisms" ${}^{\gamma}f_i : {}^{\gamma}\mathfrak{X}_i \to {}^{\gamma}\mathfrak{Y}$ defined above can be seen to verify

$$ {}^{\gamma}(\mathfrak{X}_1 \times_{\mathfrak{Y}} \mathfrak{X}_2) = {}^{\gamma}\mathfrak{X}_1 \times_{{}^{\gamma}\mathfrak{Y}} {}^{\gamma}\mathfrak{X}_2 \tag{4.9}$$

for all $\gamma \in \Gamma$.

4.3 Case of affine schemes

Assume that $\mathfrak{X}$ is affine, that is $\mathfrak{X} = \mathrm{Sp}_{\tilde{k}} A = \mathrm{Hom}_{\tilde{k}-alg}(A, -)$. If $\gamma \in \Gamma$ then

$$ {}^{\gamma}\mathfrak{X} = \mathrm{Sp}_{\tilde{k}} A^{\gamma^{-1}} \tag{4.10}$$

as can be seen from (4.3). In particular ${}^{\gamma}\mathfrak{X}$ is also affine. Our next step is to show that semilinear twists of schemes are also schemes.

Assume that $\mathfrak{Y}$ is an open subfunctor of $\mathfrak{X}$. We claim that ${}^{\gamma}\mathfrak{Y}$ is an open subfunctor of ${}^{\gamma}\mathfrak{X}$. We must show that for all affine functor $\mathrm{Sp}_{\tilde{k}} A$ and all morphism $f : \mathrm{Sp}_{\tilde{k}} A \to {}^{\gamma}\mathfrak{X}$ there exists an ideal I of A such that $f^{-1}({}^{\gamma}\mathfrak{Y}) = D(I)$ where

$$D(I)(R) = \{\alpha \in \mathrm{Hom}(A, R) : Rf(I) = R\}.$$

Let us for convenience denote $\mathrm{Sp}_{\tilde{k}} A$ by $\mathfrak{X}'$, and γ^{-1} by γ'. Our morphism f induces a morphism ${}^{\gamma'}f : {}^{\gamma'}\mathfrak{X}' \to \mathfrak{X}$ by the considerations described above. Because $\mathfrak{Y}$ is open in $\mathfrak{X}$ and ${}^{\gamma'}\mathfrak{X}' = \mathrm{Sp}_{\tilde{k}-alg} A^{\gamma}$ is affine there exists and ideal I of A^{γ} such that $({}^{\gamma'}f)^{-1}(\mathfrak{Y}) = D(I)$. Applying this to the $\tilde{k}$–algebra R^{γ} we obtain

$$ {}^{\gamma'}f(R^{\gamma})^{-1}(\mathfrak{Y}(R^{\gamma}) = \{\alpha \in \mathrm{Hom}_{\tilde{k}-alg}(A^{\gamma'}, R^{\gamma}) : R^{\gamma}\alpha(I) = R^{\gamma}\}. \tag{4.11}$$

On the other hand ${}^{\gamma'}f(R^{\gamma})^{-1} = f(R)^{-1}$ and $\mathfrak{Y}(R^{\gamma}) = {}^{\gamma}\mathfrak{Y}(R)$. Finally in the right hand side of (4.11) we have $\mathrm{Hom}_{\tilde{k}-alg}(A^{\gamma'}, R^{\gamma}) = \mathrm{Hom}_{\tilde{k}-alg}(A, R)$ and $R^{\gamma}\alpha(I) = R^{\gamma}$ if and only if $R\alpha(I) = R$. Since I is also an ideal of the $\tilde{k}$–algebra A this completes the proof that ${}^{\gamma}\mathfrak{Y}$ is an open subfunctor of ${}^{\gamma}\mathfrak{X}$.

If $\mathfrak{X}$ is local then so is ${}^{\gamma}\mathfrak{X}$. Indeed, given a $\tilde{k}$-algebra R and and element $f \in R$ then f can naturally be viewed as an element of R^{γ} (since R and R^{γ} coincide as rings), and it is immediate to verify that $(R_f)^\gamma = (R^\gamma)_f$. Using that it is then clear that the sequence

$$(4.12) \qquad {}^\gamma\mathfrak{X}(R) \to {}^\gamma\mathfrak{X}(R_{f_i}) \rightrightarrows {}^\gamma\mathfrak{X}(R_{f_if_j})$$

is exact whenever $1 = f_1 + \cdots + f_n$.

Since R is a field if and only if R^γ is a field it is clear that if $\mathfrak{X}$ is covered by a family of open subfunctors $(\mathfrak{Y}_i)_{i\in I}$, then ${}^\gamma\mathfrak{X}$ is covered by the open subfunctors ${}^\gamma\mathfrak{Y}_i$. From this it follows that if $\mathfrak{X}$ is a scheme then so is ${}^\gamma\mathfrak{X}$.

Remark 4.3. Let $\mathfrak{X}$ is a $\tilde{k}$–scheme defined along traditional lines (and not as a special type of $\tilde{k}$–functor), and let $\mathfrak{X}$ also denote the restriction to the category of affine $\tilde{k}$–schemes of the functor of points of $\mathfrak{X}$. If we define (again along traditional lines) ${}^\gamma\mathfrak{X} = \mathfrak{X} \times_{\mathrm{Spec}(\tilde{k})} \mathrm{Spec}(\tilde{k}^{\gamma^{-1}})$, then it can be shown that the functor of points of ${}^\gamma\mathfrak{X}$ (restricted to the category of affine $\tilde{k}$–schemes) coincides with the twist by γ of $\mathfrak{X}$ that we have defined.

Remark 4.4. We look in detail at the case when our $\tilde{k}$–scheme is an affine group scheme $\mathfrak{G}$. Thus $\mathfrak{G} = \mathrm{Sp}_{\tilde{k}}\tilde{k}[\mathfrak{G}]$ for some $\tilde{k}$–Hopf algebra $\tilde{k}[\mathfrak{G}]$ (see [DG] II §1 for details).

Let $\epsilon_{\mathfrak{G}} : \tilde{k}[\mathfrak{G}] \to \tilde{k}$ denote the counit map. As $\tilde{k}$-modules we have $\tilde{k}[\mathfrak{G}] = \tilde{k} \oplus I_{\mathfrak{G}}$ where $I_{\mathfrak{G}}$ is the kernel of $\epsilon_{\mathfrak{G}}$. Let $\gamma \in \Gamma$. As explained in (4.10) we have ${}^\gamma\mathfrak{G} = \mathrm{Sp}_{\tilde{k}}\tilde{k}[\mathfrak{G}]^{\gamma^{-1}} = \mathrm{Hom}_{\tilde{k}-alg}(\tilde{k}[\mathfrak{G}]^{\gamma^{-1}}, -)$. We leave it to the reader to verify that $\epsilon_{{}^\gamma\mathfrak{G}} : \gamma \circ \epsilon_{\mathfrak{G}}$. As an abelian group $I_{\mathfrak{G}} = I_{{}^\gamma\mathfrak{G}}$, but in this last the action of $\tilde{k}$ is obtained through the action of $\tilde{k}$ in $\tilde{k}[\mathfrak{G}]^\gamma$.

Next we make some relevant observations about Lie algebras from a functorial point of view ([DG] II §4). Recall that the group functor $\mathfrak{Lie}(\mathfrak{G})$ attaches to an object R in $\tilde{k}$-alg the kernel of the group homomorphism $\mathfrak{G}(R[\epsilon]) \to \mathfrak{G}(R)$ where $R[\epsilon]$ is the $\tilde{k}$–algebra of dual numbers of R, and the group homomorphism comes from the functorial nature of $\mathfrak{G}$ applied to the morphism $R[\epsilon] \to R$ in $\tilde{k}$–alg that maps $\epsilon \mapsto 0$. By definition $\mathcal{L}ie(\mathfrak{G}) = \mathfrak{Lie}(\mathfrak{G})(\tilde{k})$. In particular $\mathcal{L}ie(\mathfrak{G}) \subset \mathfrak{G}(\tilde{k}[\epsilon]) = \mathrm{Hom}_{\tilde{k}\text{-alg}}(\tilde{k}[\mathfrak{G}], \tilde{k}[\epsilon])$. Every element $x \in \mathcal{L}ie(\mathfrak{G})$ is given by

$$(4.13) \qquad x : a \mapsto \epsilon_{\mathfrak{G}}(a) + \delta_x(a)\epsilon$$

with $\delta_x \in \mathrm{Der}_{\tilde{k}}(\tilde{k}[\mathfrak{G}], \tilde{k})$ where $\tilde{k}$ is viewed as a $\tilde{k}[\mathfrak{G}]$–module via the counit map of $\mathfrak{G}$. In what follows we write $x = \epsilon_{\mathfrak{G}} + \delta_x\epsilon$. The map $x \mapsto \delta_x$ is in fact a $\tilde{k}$-module isomorphism $\mathcal{L}ie(\mathfrak{G}) \simeq \mathrm{Der}_{\tilde{k}}(\tilde{k}[\mathfrak{G}], \tilde{k})$. In particular if $\lambda \in \tilde{k}$ then $\lambda x \in \mathcal{L}ie(\mathfrak{G})$ is such that $\delta_{\lambda x} = \lambda\delta_x$.

Similar considerations apply to the affine $\tilde{k}$-group ${}^\gamma\mathfrak{G}$. We have $\mathcal{L}ie({}^\gamma\mathfrak{G}) = \mathrm{Der}_{\tilde{k}}(\tilde{k}[\mathfrak{G}]^{\gamma^{-1}}, \tilde{k})$. Note that if $y \in \mathcal{L}ie({}^\gamma\mathfrak{G})$ corresponds to $\delta_y \in \mathrm{Der}_{\tilde{k}}(\tilde{k}[\mathfrak{G}]^{\gamma^{-1}}, \tilde{k})$, then under the action of $\tilde{k}$ on $\mathcal{L}ie({}^{\gamma}\mathfrak{G})$ the element λy corresponds to the derivation $\lambda\delta_y$ and *not* to $({}^{\gamma}\lambda)\delta_y$: The "$\gamma$ part" is taken into consideration already by the fact that $y \in \mathcal{L}ie({}^{\gamma}\mathfrak{G})$ and that $\delta_y \in \mathrm{Der}_{\tilde{k}}(\tilde{k}[\mathfrak{G}]^{\gamma^{-1}}, \tilde{k})$.

4.4 Group functors

Let from now on $\mathfrak{G}$ denote a $\tilde{k}$-group functor. If $\mathfrak{H}$ is a subgroup functor of $\mathfrak{G}$ we let

$$\mathrm{Aut}_{\gamma}(\mathfrak{G}, \mathfrak{H}) = \Big\{ f \in \mathrm{Aut}_{\gamma}(\mathfrak{G}) \mid {}^{\gamma}\mathfrak{H} = f^{-1}(\mathfrak{H}) \Big\}.$$

It is easy to verify then that $\mathrm{Aut}_{\Gamma}(\mathfrak{G}, \mathfrak{H}) = \cup_{\gamma\in\Gamma}\mathrm{Aut}_{\gamma}(\mathfrak{G}, \mathfrak{H})$ is a subgroup of $\mathrm{Aut}_{\Gamma}(\mathfrak{G})$.

Proposition 4.5. *Let $\Pi_{\tilde{k}/k}\mathfrak{G}$ be the Weil restriction of $\mathfrak{G}$ to k (which we view as a k–group functor). There exists a canonical group homomorphism*

$$\tilde{\ } : \mathrm{Aut}_{\Gamma}(\mathfrak{G}) \to \mathrm{Aut}(\Pi_{\tilde{k}/k}\mathfrak{G}).$$

Proof. As observed in Remark 4.1 the map $\gamma : \tilde{k} \to \tilde{k}^{\gamma}$ is an isomorphism of $\tilde{k}$–alg, and for R in k–alg $(R \otimes_k \tilde{k})^{\gamma} = R \otimes_k \tilde{k}^{\gamma}$. We thus have a $\tilde{k}$–algebra isomorphism $1 \otimes \gamma : R \otimes_k \tilde{k} \to (R \otimes \tilde{k})^{\gamma}$. For a given $f \in \mathrm{Aut}_{\Gamma}(\mathfrak{G})$, the composite map

$$\tilde{f}(R) : (\Pi_{\tilde{k}/k}\mathfrak{G})(R) = \mathfrak{G}(R \otimes_k \tilde{k}) \overset{\mathfrak{G}(1\otimes\gamma)}{\longrightarrow} \mathfrak{G}\big((R \otimes_k \tilde{k})^{\gamma}\big) =$$

$$= {}^{\gamma}\mathfrak{G}(R \otimes_k \tilde{k}) \overset{f(R\otimes_k\tilde{k})}{\to} \mathfrak{G}(R \otimes_k \tilde{k}) = (\Pi_{\tilde{k}/k}\mathfrak{G})(R)$$

is an automorphism of the group $(\Pi_{\tilde{k}/k}\mathfrak{G})(R)$. One readily verifies that the family $\tilde{f} = \tilde{f}(R)_{R\in k-\mathrm{alg}}$ is functorial on R, hence an automorphism of $\Pi_{\tilde{k}/k}\mathfrak{G}$.

To check that $\tilde{\ }$ is a group homomorphism we consider two elements $f_1, f_2 \in \mathrm{Aut}_{\Gamma}(\mathfrak{G})$ of type γ_1 and γ_2 respectively. Recall that γ_2 induces a $\tilde{k}$-algebra homomorphism $1 \otimes \gamma_{2\sigma} : R \otimes \tilde{k}^{\gamma} \to R \otimes \tilde{k}^{\gamma_2\gamma}$ for all $\sigma \in \Gamma$ [see Remark 4.1(i)]. Since γ will be understood from the context we will denote this homomorphism simply by $1 \otimes \gamma_2$. By functoriality we get the following commutative diagram [see Remark 4.1(1)]

$$\begin{array}{ccc}
\mathfrak{G}(R \otimes \tilde{k}^{\gamma_1}) & \xrightarrow{\mathfrak{G}(1\otimes\gamma_2)} & \mathfrak{G}(R \otimes \tilde{k}^{\gamma_2\gamma_1}) \\
\big\downarrow{=} & & \big\downarrow{=} \\
{}^{\gamma_1}\mathfrak{G}(R \otimes \tilde{k}) & \xrightarrow{{}^{\gamma}\mathfrak{G}(1\otimes\gamma_2)} & {}^{\gamma_1}\mathfrak{G}(R \otimes \tilde{k}^{\gamma_2}) \\
\big\downarrow{f_1(R\otimes\tilde{k})} & & \big\downarrow{f_1(R\otimes\tilde{k}^{\gamma_2})} \\
\mathfrak{G}(R \otimes \tilde{k}) & \xrightarrow{\mathfrak{G}(1\otimes\gamma_2)} & \mathfrak{G}(R \otimes \tilde{k}^{\gamma_2})
\end{array}$$

Since $f_2 \circ f_1$ is of type $\gamma_2\gamma_1$, by definition we have

$$\widetilde{f_2 \circ f_1}(R \otimes \tilde{k}) = (f_2 \circ f_1)(R \otimes \tilde{k}) \circ \mathfrak{G}(1 \otimes \gamma_2\gamma_1).$$

Thus

$$\begin{aligned}\widetilde{f_2 \circ f_1}(R \otimes \tilde{k}) &= (f_2 \circ f_1)(R \otimes \tilde{k}) \circ \mathfrak{G}(1 \otimes \gamma_2 \circ 1 \otimes \gamma_1) \\ &= (f_2 \circ f_1)(R \otimes \tilde{k}) \circ \mathfrak{G}(1 \otimes \gamma_2) \circ \mathfrak{G}(1 \otimes \gamma_1) \\ &= f_2(R \otimes \tilde{k}) \circ f_1(R \otimes \tilde{k}^{\gamma_2}) \circ \mathfrak{G}(1 \otimes \gamma_2) \circ \mathfrak{G}(1 \otimes \gamma_1) \\ &= f_2(R \otimes \tilde{k}) \circ \mathfrak{G}(1 \otimes \gamma_2) \circ f_1(R \otimes \tilde{k}) \circ \mathfrak{G}(1 \otimes \gamma_1) \\ &= \tilde{f}_2(R \otimes \tilde{k}) \circ \tilde{f}_1(R \otimes \tilde{k}).\end{aligned}$$

□

Example 4.6. (a) Consider the case of the trivial $\tilde{k}$–group $\mathbf{e}_{\tilde{k}}$. Each set $\mathrm{Aut}_\gamma(\mathbf{e}_{\tilde{k}}) = \mathrm{Isom}({}^\gamma\mathbf{e}_{\tilde{k}}, \mathbf{e}_{\tilde{k}})$ consists of one element which we denote by γ_*. Then $\mathrm{Aut}_\Gamma(\mathbf{e}_{\tilde{k}}) \simeq \Gamma$. We have $\Pi_{\tilde{k}/k}\mathbf{e}_{\tilde{k}} = \mathbf{e}_k$. In particular $\mathrm{Aut}(\Pi_{\tilde{k}/k}\mathbf{e}_{\tilde{k}}) = 1$ and the homomorphism $\tilde{}\ : \mathrm{Aut}_\Gamma(\mathfrak{G}) \to \mathrm{Aut}(\Pi_{\tilde{k}/k}\mathfrak{G})$ is in this case necessarily trivial. In affine terms $\mathbf{e}_{\tilde{k}}$ is represented by $\tilde{k}$ and ${}^\gamma\mathbf{e}_{\tilde{k}}$ by $\tilde{k}^{\gamma^{-1}}$. Then the $\tilde{k}$–group isomorphism $\gamma_* : {}^\gamma\mathbf{e}_{\tilde{k}} \to \mathbf{e}_{\tilde{k}}$ corresponds to the $\tilde{k}$–Hopf algebra isomorphism $\gamma^{-1} : \tilde{k} \to \tilde{k}^{\gamma^{-1}}$.

(b) Consider the case when Γ is the Galois group of the extension $\mathbb{C}/\mathbb{R}$, and $\mathfrak{G}$ is the additive $\mathbb{C}$–group. Then $\mathrm{Aut}_\Gamma(\mathfrak{G})$ can be identified with the group of automorphisms of $(\mathbb{C}, +)$ which are of the form $z \mapsto \lambda z$ or $z \mapsto \lambda\overline{z}$ for some $\lambda \in \mathbb{C}^\times$. The Weil restriction of $\mathfrak{G}$ to $\mathbb{R}$ is the two-dimensional additive $\mathbb{R}$–group. Thus $\mathrm{Aut}(\Pi_{\tilde{k}/k}\mathfrak{G}) = \mathrm{GL}_2(\mathbb{R})$.

The above examples show that, even if $\tilde{k}/k$ is a finite Galois extension of fields and $\mathfrak{G}$ is a connected linear algebraic group over $\tilde{k}$, the homomorphism $f \mapsto \tilde{f}$ need be neither injective nor surjective

Corollary 4.7. *Assume that $\mathfrak{G} = \mathrm{Sp}_{\tilde{k}}\tilde{k}[\mathfrak{G}]$ be an affine $\tilde{k}$-group. The group $\mathrm{Aut}_\Gamma(\mathfrak{G})$ acts naturally on the groups $\mathfrak{G}(\tilde{k})$ and $\mathfrak{G}(\tilde{k}[\epsilon])$. Furthermore the action of an element $f \in \mathrm{Aut}_\gamma(\mathfrak{G})$ on $\mathfrak{G}(\tilde{k}[\epsilon])$ stabilizes $\mathcal{L}ie(\mathfrak{G}) \subset \mathfrak{G}(\tilde{k}[\epsilon])$. The induced map $\mathcal{L}ie(f) : \mathcal{L}ie(\mathfrak{G}) \to \mathcal{L}ie(\mathfrak{G})$ is an automorphism of $\mathcal{L}ie(\mathfrak{G})$ viewed as a Lie algebra over k. This automorphism is $\tilde{k}$–semilinear, i.e., $\mathcal{L}ie(f)(\lambda x) = ({}^\gamma\lambda)\mathcal{L}ie(f)(x)$ for all $\lambda \in \tilde{k}$ and $x \in \mathcal{L}ie(\mathfrak{G})$.*

Proof. We maintain the notation and use the facts presented in Remark 4.4. Let $x \in \mathcal{L}ie(\mathfrak{G})$ and write $x = \epsilon_{\mathfrak{G}} + \delta_x\epsilon$. If $\lambda \in \tilde{k}$ then $\lambda x \in \mathcal{L}ie(\mathfrak{G})$ is such that $\delta_{\lambda x} = \lambda\delta_x$.

By definition $(\Pi_{\tilde{k}/k}\mathfrak{G})(k) = \mathfrak{G}(\tilde{k})$ and $(\Pi_{\tilde{k}/k}\mathfrak{G})(k[\epsilon]) = \mathfrak{G}(\tilde{k}[\epsilon])$. The action of an element $f \in \mathrm{Aut}_\Gamma(\mathfrak{G})$ on these two groups is then given by the automorphisms $\tilde{f}_k$ and $\tilde{f}_{k[\epsilon]}$ of the previous Proposition. Thus if we let $\gamma_\epsilon : \tilde{k}[\epsilon] \to \tilde{k}[\epsilon]^\gamma$ denote the isomorphism of $\tilde{k}$–alg induced by γ the map $\tilde{f}_{k[\epsilon]}$ is then obtained by restricting to $\mathcal{L}ie(\mathfrak{G})$ the composite map

$$\mathfrak{G}(\tilde{k}[\epsilon]) = \mathrm{Hom}_{\tilde{k}-alg}(\tilde{k}[\mathfrak{G}], \tilde{k}[\epsilon]) \overset{\mathfrak{G}(\gamma_\epsilon)}{\to} \mathrm{Hom}_{\tilde{k}-alg}(\tilde{k}[\mathfrak{G}], \tilde{k}[\epsilon]^\gamma) =$$

$$= \mathrm{Hom}_{\tilde{k}-alg}(\tilde{k}[\mathfrak{G}]^{\gamma^{-1}}, \tilde{k}[\epsilon]) = {}^\gamma\mathfrak{G}(\tilde{k}[\epsilon]) \overset{f(\tilde{k}[\epsilon])}{\to} \mathrm{Hom}_{\tilde{k}-alg}(\tilde{k}[\mathfrak{G}], \tilde{k}[\epsilon]) = \mathfrak{G}(\tilde{k}[\epsilon]).$$

Using the fact that $\gamma_\epsilon \circ \epsilon_{\mathfrak{G}} = \gamma \circ \epsilon_{\mathfrak{G}} = \epsilon_{\gamma_{\mathfrak{G}}}$ it easily follows that

$$\mathcal{L}ie(f)(x) = \tilde{f}_{k[\epsilon]}(x) = f(\tilde{k}[\epsilon]) \circ \big(\epsilon_{\gamma_{\mathfrak{G}}} + (\gamma \circ \delta_x)\epsilon\big) \tag{4.14}$$

Let $y = \epsilon_{\gamma_{\mathfrak{G}}} + (\gamma \circ \delta_x)\epsilon \in \mathcal{L}ie({}^\gamma\mathfrak{G})$. If $\lambda \in \tilde{k}$ then we have

$$\begin{aligned}\mathcal{L}ie(f)(\lambda x) &= f(\tilde{k}[\epsilon]) \circ \big(\epsilon_{\gamma_{\mathfrak{G}}} + (\gamma \circ \delta_{\lambda x})\epsilon\big) \\ &= f(\tilde{k}[\epsilon]) \circ \big(\epsilon_{\gamma_{\mathfrak{G}}} + ({}^\gamma(\lambda\delta_x)\epsilon\big) \\ &= f(\tilde{k}[\epsilon]) \circ \big(\epsilon_{\gamma_{\mathfrak{G}}} + {}^\gamma\lambda(\gamma \circ \delta_x)\epsilon\big) \\ &= f(\tilde{k}[\epsilon])\big(({}^\gamma\lambda)y)\big)\end{aligned}$$

where $({}^\gamma\lambda)y$ is the action of the element ${}^\gamma\lambda \in \tilde{k}$ on the element $y \in \mathcal{L}ie({}^\gamma\mathfrak{G})$, as explained in the last paragraph of Remark 4.4. Since the restriction of $f(\tilde{k}[\epsilon])$ to $\mathcal{L}ie({}^\gamma\mathfrak{G})$ induces an isomorphism $\mathcal{L}ie({}^\gamma\mathfrak{G}) \to \mathcal{L}ie(\mathfrak{G})$ of $\tilde{k}$-Lie algebras, this restriction is in particular $\tilde{k}$-linear. It follows that

$$\mathcal{L}ie(f)(\lambda x) = f(\tilde{k}[\epsilon])\big(({}^\gamma\lambda)y\big) = ({}^\gamma\lambda)f(\tilde{k}[\epsilon])(y) = ({}^\gamma\lambda)\mathcal{L}ie(f)(x).$$

This shows that $\mathcal{L}ie(f)$ is semilinear. We leave it to the reader to verify that $\mathcal{L}ie(f)$ is an automorphism of $\mathcal{L}ie(\mathfrak{G})$ as a Lie algebra over k. ☐

Remark 4.8. There is no natural action of $\mathrm{Aut}_\Gamma(\mathfrak{G})$ on $\mathfrak{G}$.

4.5 Semilinear version of a theorem of Borel-Mostow

Throughout this section k denotes a field of characteristic 0.

Theorem 4.9. *(Semilinear Borel-Mostow) Let $\tilde{k}/k$ be a finite Galois extension of fields with Galois group Γ. Suppose we are given a quintuple $\big(\mathfrak{g}, H, \psi, \phi, (H_i)_{0\le i\le s}\big)$ where*

$\mathfrak{g}$ is a (finite dimensional) reductive Lie algebra over $\tilde{k}$,
H is a group,
ψ is a group homomorphism from H into the Galois group Γ,

ϕ is a group homomorphism from H into the group $\mathrm{Aut}_k(\mathfrak{g})$ of automorphisms of $\mathfrak{g}$ viewed as a Lie algebra over k,

$(H_i)_{1\leq i\leq s}$ is a finite family of subgroups of H for which the following two conditions hold:

(i) If we let the group H act on $\mathfrak{g}$ via ϕ and on Γ via ψ, namely ${}^h x = {}^{\phi(h)}x$ and ${}^h\lambda = {}^{\psi(h)}\lambda$ for all $h \in H$, $x \in \mathfrak{g}$, and $\lambda \in \tilde{k}$, then the action of H in $\mathfrak{g}$ is semilinear, i.e., ${}^h(\lambda x)^h = {}^h\lambda^h x$.

(ii) $\ker(\psi) = H_s \supset H_{s-1} \supset \ldots \supset H_1 \supset H_0 = 0$. Furthermore, each H_i is normal in H, the elements of $\phi(H_i)$ are semisimple,[9] and the quotients H_i/H_{i-1} are cyclic.

Then there exists a Cartan subalgebra of $\mathfrak{g}$ which is stable under the action of H.

Proof. We will reason by induction on s. If $s = 0$ we can identify by assumption (ii) H with a subgroup Γ_0 of Γ via ψ. Let $\tilde{k}_0 = \tilde{k}^{\Gamma_0}$. This yields a semilinear action of Γ_0 on $\mathfrak{g}$. By Galois descent the fixed point $\mathfrak{g}^{\Gamma_0}$ is a Lie algebra over $\tilde{k}_0$ for which the canonical map $\rho : \mathfrak{g}^{\Gamma_0} \otimes_{\tilde{k}_0} \tilde{k} \simeq \mathfrak{g}$ is a $\tilde{k}$–Lie algebra isomorphism. If $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}^{\Gamma_0}$ then $\rho(\mathfrak{h}_0 \otimes_{\tilde{k}_0} \tilde{k})$ is a Cartan subalgebra of $\mathfrak{g}$ which is H–stable as one can easily verify with the aid of assumption (i).

Assume $s \geq 1$ and consider a generator θ of the cyclic group H_1. As we have already observed the action of θ on $\mathfrak{g}$ is $\tilde{k}$–linear. If V is a $\tilde{k}$–subspace of $\mathfrak{g}$ stable under θ we will denote by V^θ the subspace of fixed points. Before continuing with we establish the following crucial fact:

Claim 4.10. *$\mathfrak{g}^\theta$ is a reductive Lie algebra over $\tilde{k}$. If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}^\theta$, then $\mathbf{z}_{\mathfrak{g}}(\mathfrak{h})$ is a Cartan subalgebra of $\mathfrak{g}$.*

Since $\phi(\theta)$ is an automorphism of the $\tilde{k}$–Lie algebra $\mathfrak{g}$ we see that $\mathfrak{g}^\theta$ is indeed a Lie subalgebra of $\mathfrak{g}$. Let $\mathfrak{g}'$ and $\mathfrak{z}$ denote the derived algebra and the centre of $\mathfrak{g}$ respectively. Because $\mathfrak{g}$ is reductive $\mathfrak{g}'$ is semisimple and $\mathfrak{g} = \mathfrak{g}' \times \mathfrak{z}$. Clearly θ induces by restriction automorphisms (also denoted by θ) of $\mathfrak{g}'$ and of $\mathfrak{z}$. By [Bbk] Ch. 8 §1 cor. to prop. 12. $(\mathfrak{g}')^\theta$ is reductive, and therefore $\mathfrak{g}^\theta = (\mathfrak{g}')^\theta \times \mathfrak{z}^\theta$ is also reductive.

Every Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}^\theta$ is of the form $\mathfrak{h} = \mathfrak{h}' \times \mathfrak{z}^\theta$ for some Cartan subalgebra $\mathfrak{h}'$ of $(\mathfrak{g}')^\theta$. Clearly $\mathbf{z}_{\mathfrak{g}}(\mathfrak{h}) = \mathbf{z}_{\mathfrak{g}'}(\mathfrak{h}') \times \mathfrak{z}$. By [P3] theorem 9 the centralizer $\mathbf{z}_{\mathfrak{g}'}(\mathfrak{h}')$ is a Cartan subalgebra of $\mathfrak{g}'$, so the claim follows.

We now return to the proof of the Theorem. Since H_1 is normal in H we have an induced action (via ϕ) of $H' = H/H_1$ on the reductive $\tilde{k}$–Lie algebra $\mathfrak{g}^\theta$. We have induced group homomorphisms $\phi' : H' \to \mathrm{Aut}_k(\mathfrak{g}^\theta)$ and $\psi' : H' \to \Gamma$ [this last since $H_1 \subset \ker(\psi)$]. For $0 \leq i < s$ define $H'_i = H_{i+1}/H_1$. We apply the induction assumption to the quintuple $\big(\mathfrak{g}^\theta, H', \psi', \phi', (H'_i)_{0\leq i\leq s-1}\big)$. This yields the existence of

[9]Because $H_i \subset \ker(\psi)$ the action of the elements of H_i on $\mathfrak{g}$ is $\tilde{k}$–linear. The assumption is that $\phi(\theta)$ be semisimple as a $\tilde{k}$–linear endomorphisms of $\mathfrak{g}$ for all $\theta \in H_i$.

a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}^\theta$ which is stable under the action of H' given by ϕ'. This means that, back in $\mathfrak{g}$, the algebra $\mathfrak{h}$ is stable under our original action of H given by ϕ. But then the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$ is also stable under this action, and we can now conclude by (4.10) . $\square$

Remark 4.11. If ψ is the trivial map the Theorem reduces to the "Main result (B)" of Borel and Mostow [BM] for $\mathfrak{g}$. The use of (4.10) allows for a slightly more direct proof of this result.

We shall use the above semilinear version of Borel-Mostow's theorem 4.12 to establish the following result which will play a crucial role in the the proof of the existence of maximal tori on twisted groups corresponding to loop torsors.

Corollary 4.12. *Let $\tilde{k}/k$ be a finite Galois extension with Galois group Γ. Let $\mathbf{G}$ be a reductive group over $\tilde{k}$. Let H be a group, and assume we are given a group homomorphism $\rho : H \to \mathrm{Aut}_\Gamma(\mathbf{G})$ for which we can find a family of subgroups $(H_i)_{0\le i\le s}$ of H as in the Theorem, that is $\ker(t\circ\rho) = H_s \supset H_{s-1} \supset \dots \supset H_1 \supset H_0 = 0$ where $t : \mathrm{Aut}_\Gamma(\mathbf{G}) \to \Gamma$ is the type morphism, each H_i is normal in H, the elements of $\rho(H_i)$ act semisimply on the $\tilde{k}$–Lie algebra $\mathcal{L}ie(\mathbf{G})$, and the quotients H_i/H_{i-1} are cyclic. Then there exists a maximal torus $\mathbf{T}$ of $\mathbf{G}$ such that ρ has values in $\mathrm{Aut}_\Gamma(\mathbf{G},\mathbf{T}) \subset \mathrm{Aut}_\Gamma(\mathbf{G})$. Namely if $h \in H$ and $(t\circ\rho)(h) = \gamma \in \Gamma$, then $\rho(h) : {}^\gamma\mathbf{G} \to \mathbf{G}$ induces by restriction an isomorphism ${}^\gamma\mathbf{T} \to \mathbf{T}$.*

Proof. Let $h \in H$. If $(t\circ\rho)(h) = \gamma$ then according to the various definitions we have the following commutative diagram.

$$\begin{array}{ccccc}
\mathbf{G}(\tilde{k}) & \xrightarrow{\mathbf{G}(\gamma)} & \mathbf{G}(\tilde{k}^\gamma) = {}^\gamma\mathbf{G}(\tilde{k}) & \xrightarrow{\rho(h)(\tilde{k})} & \mathbf{G}(\tilde{k}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{G}(\tilde{k}[\epsilon]) & \xrightarrow{\mathbf{G}(\gamma_\epsilon)} & \mathbf{G}(\tilde{k}[\epsilon]^\gamma) = {}^\gamma\mathbf{G}(\tilde{k}[\epsilon]) & \xrightarrow{\rho(h)(\tilde{k}[\epsilon])} & \mathbf{G}(\tilde{k}[\epsilon]) \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{L}ie(\mathbf{G}) & \xrightarrow{\mathfrak{Lie}(\mathfrak{G})(\gamma)} & \mathcal{L}ie(\mathbf{G}^\gamma) & \xrightarrow{\mathfrak{Lie}(\mathfrak{G})(\rho(h))} & \mathcal{L}ie(\mathbf{G}).
\end{array}$$

where we have denoted by $\gamma_\epsilon : \tilde{k}[\epsilon] \to \tilde{k}[\epsilon]^\gamma$ the $\tilde{k}$-algebra isomorphism induced by γ. For convenience in what follows we will denote $\mathcal{L}ie(\mathbf{G})$ by $\mathfrak{g}$. By Corollary 4.7 we obtain by composing ρ with the map $\tilde{}$ defined in Proposition 4.5 a group homomorphism $\phi : H \to \mathrm{Aut}_k(\mathfrak{g})$, namely $\phi(h) = \widetilde{\rho(h)}$, which together with the group homomorphism $\psi = t\circ\rho : H \to \Gamma$ and the H_i satisfy the assumptions of Theorem 4.9. It follows that there exists a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ which is stable under the action of H defined by ϕ.

Note that by definition

$$\widetilde{\rho(h)} = \rho(h)(\tilde{k}) \circ \mathbf{G}(\gamma) \tag{4.15}$$

which is nothing but the top row of our diagram above. Similarly with the notation of Corollary 4.7 we have

$$\mathcal{L}ie(\widetilde{\rho(h)}) = \widetilde{\rho_\epsilon(h)}\big|_{\mathfrak{g}} = \mathfrak{Lie}(\mathbf{G})(\rho(h)) \circ \mathfrak{Lie}(\mathbf{G})(\gamma) \tag{4.16}$$

where $\widetilde{\rho_\epsilon(h)}$ stands for the middle row of our diagram, namely $\rho(h)(\tilde{k}[\epsilon]) \circ \mathbf{G}(\gamma_\epsilon)$.

Let $\mathbf{T}$ be the maximal torus of $\mathbf{G}$ whose Lie algebra is $\mathfrak{t}$ [XIV.6.6.c]. We have $\mathbf{T} = \mathbf{Z_G}(\mathfrak{t})$ where the centralizer is taken respect to the adjoint action of $\mathbf{G}$ on $\mathfrak{g}$.[10]

Given an element $g \in \mathbf{G}(\tilde{k})$ we will denote its natural image in $\mathbf{G}(\tilde{k}[\epsilon])$ by g_ϵ. Since we are working over a base field the $\tilde{k}$-points of $\mathbf{T} = \mathbf{Z_G}(\mathfrak{t})$ can be computed in the naive way, namely

$$\mathbf{T}(\tilde{k}) = \{g \in \mathbf{G}(\tilde{k}) : g_\epsilon x {g_\epsilon}^{-1} = x \text{ for all } x \in \mathfrak{t} \subset \mathbf{G}(\tilde{k}[\epsilon])\} \tag{4.17}$$

Since $\widetilde{\rho_\epsilon(h)}$ is an automorphism of the (abstract) group $\mathbf{G}(\tilde{k}[\epsilon])$ we obtain

$$\mathbf{T}(\tilde{k}) = \{g \in \mathbf{G}(\tilde{k}) : \widetilde{\rho_\epsilon(h)}(g_\epsilon)\widetilde{\rho_\epsilon(h)}(x)\big(\widetilde{\rho_\epsilon(h)}(g_\epsilon)\big)^{-1} = \widetilde{\rho_\epsilon(h)}(x) \text{ for all } x \in \mathfrak{t}\} \tag{4.18}$$

But since $\widetilde{\rho_\epsilon(h)}$ stabilizes $\mathfrak{t}$ this last reads

$$\mathbf{T}(\tilde{k}) = \{g \in \mathbf{G}(\tilde{k}) : \widetilde{\rho_\epsilon(h)}(g_\epsilon)x\big(\widetilde{\rho_\epsilon(h)}(g_\epsilon)\big)^{-1} = x \text{ for all } x \in \mathfrak{t}\} \tag{4.19}$$

Note that by the commutativity of the top square of our diagram we have $\big(\widetilde{\rho(h)}(g)\big)_\epsilon = \widetilde{\rho_\epsilon(h)}(g_\epsilon)$. Thus from (4.19) we obtain that $\widetilde{\rho_\epsilon(h)}\big(\mathbf{T}(\tilde{k})\big) = \mathbf{T}(\tilde{k})$. On the other hand by (4.15) we have $\widetilde{\rho_\epsilon(h)}\big(\mathbf{T}(\tilde{k})\big) = \rho(h)(\tilde{k})\Big(\mathbf{G}(\gamma)\big(\mathbf{T}(\tilde{k})\big)\Big)$. But by definition $\Big(\mathbf{G}(\gamma)\big(\mathbf{T}(\tilde{k})\big)\Big) = {}^\gamma\mathbf{T}(\tilde{k})$. Thus our $\tilde{k}$-group homomorphism $\rho(h) : {}^\gamma\mathbf{G} :\to \mathbf{G}$ is such that the two tori $\rho(h)({}^\gamma\mathbf{T})$ and $\mathbf{T}$ of $\mathbf{G}$ have the same $\tilde{k}$-points. This forces $\rho(h)({}^\gamma\mathbf{T}) = \mathbf{T}$. □

Next we give a crucial application of the semilinear considerations developed thus far to the existence of maximal tori for certain loop groups.

[10] We could not find a reference for this basic fact in the literature. By [XIII 5.3] we have $\mathbf{N_G}(\mathbf{T}) = \mathbf{N_G}(\mathfrak{t})$. Since the natural homomorphism $\mathbf{N_G}(\mathbf{T})/\mathbf{T} \to \mathrm{Aut}(\mathfrak{t})$ is injective we obtain $\mathbf{T} = \mathbf{Z_G}(\mathfrak{t})$.

4.6 Existence of maximal tori in loop groups

We come back to the case of $R = R_n = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ where k is a field of characteristic zero. This is the ring that plays a central role in all applications to infinite dimensional Lie theory. It is not true in general that a reductive R_n–group admits a maximal torus; however.

Proposition 4.13. *Let $\mathfrak{G}$ be a loop reductive group scheme over R_n (see definition 3.4). Then $\mathfrak{G}$ admits a maximal torus.*

Proof. We try to recreate the situation of the semilinear Borel-Mostow theorem. We can assume that $\mathfrak{G}$ is split after base extension to the Galois covering $\tilde{R} = \tilde{k}[t_1^{\pm 1/m}, \dots, t_n^{\pm 1/m}]$ where m is a positive integers and $\tilde{k}/k$ is a finite Galois extension of fields containing all primitive m-th roots of unity of $\overline{k}$. Recall from Example 2.3 that $\tilde{R}$ is a Galois extension of R with Galois group $\tilde{\Gamma} = (\mathbb{Z}/m\mathbb{Z})^n \rtimes \Gamma$ as follows: For $\mathbf{e} = (e_1, \dots, e_n) \in \mathbb{Z}^n$ we have $\overline{\mathbf{e}}(\lambda t_j^{\frac{1}{m}}) = \lambda \xi_m^{e_j} t_j^{\frac{1}{m}}$ for all $\lambda \in \tilde{k}$, where $^- : \mathbb{Z}^n \to (\mathbb{Z}/mZ)^n$ is the canonical map, while the Galois group $\Gamma = \mathrm{Gal}(\tilde{k}/k)$ acts naturally on $\tilde{R}$ through its action on $\tilde{k}$.

Let $\mathbf{G}_0$ be the Chevalley k–form of $\mathfrak{G}$ (see §3.2). By assumption, we can assume that $\mathfrak{G}$ is the twist of $\mathfrak{G}_0 = \mathbf{G}_0 \times_k R$ by a loop cocycle

$$u : \tilde{\Gamma} \to \mathbf{Aut}(\mathbf{G}_0)(\tilde{k}).$$

The homomorphism $\psi : \tilde{\Gamma} = (\mathbb{Z}/m\mathbb{Z})^n \rtimes \Gamma \to \Gamma$ is defined to be the natural projection. For convenience we will adopt the following notational convention. The elements of $\tilde{\Gamma}$ will be denoted by $\tilde{\gamma}$, and the image under ϕ of such an element (which belongs to Γ), by the corresponding greek character: that is $\psi(\tilde{\gamma}) = \gamma$.

Consider the reductive $\tilde{k}$–group $\mathbf{G} = \mathrm{Spec}(\tilde{k}[\mathbf{G}_0])$ where, as usual, $\tilde{k}[\mathbf{G}_0]$ denotes the $\tilde{k}$–Hopf algebra $\tilde{k} \otimes_k k[\mathbf{G}_0]$. Consider for each $\tilde{\gamma}$ the map $f(\tilde{\gamma}) : \tilde{k}[\mathbf{G}] \to \tilde{k}[\mathbf{G}]$ defined by

$$f(\tilde{\gamma}) = u_{\tilde{\gamma}} \circ \gamma. \tag{4.20}$$

Since each $u_{\tilde{\gamma}}$ is an automorphism of the $\tilde{k}$–Hopf algebra $\tilde{k}[\mathbf{G}]$, it follows that $f(\tilde{\gamma})$ is in fact a $\tilde{k}$-Hopf algebra isomorphism $\tilde{k}[\mathbf{G}] \to \tilde{k}[\mathbf{G}]^{\gamma}$. As such it can be thought, by Yoneda considerations and (4.10), as an element of $\mathrm{Aut}_{\gamma}(\mathbf{G})$ of type γ^{-1} which we will denote by $\rho(\tilde{\gamma})$.

Since the restriction of the action of $\tilde{\gamma}$ on $\tilde{R}[\mathbf{G}]$ to $\tilde{k}[\mathbf{G}]$ is given by γ, the cocycle condition on u shows that for all $\tilde{\alpha}, \tilde{\beta} \in \tilde{\Gamma}$ we have

$$\rho(\tilde{\alpha}\tilde{\beta}) = \rho(\tilde{\alpha})\rho(\tilde{\beta}) \tag{4.21}$$

where this last product takes place in $\mathrm{Aut}_\Gamma(\mathbf{G})$. Thus ρ is a group homomorphism and $f(\tilde\gamma)$ can be viewed as a $\tilde k$-Hopf algebra morphism from $\tilde k[\mathbf{G}]$ to $\tilde k[\mathbf{G}]^\gamma$

¿From (4.21), the various definitions and the "anti equivalent" nature of Yoneda's correspondence it follows that the map $\tilde\gamma \to \rho(\tilde\gamma)$ can be viewed as a group homomorphism $\rho : \tilde\Gamma^{\mathrm{opp}} \to \mathrm{Aut}_\Gamma(\mathbf{G})$, where $\tilde\Gamma^{\mathrm{opp}}$ is the opposite group of $\tilde\Gamma$. Since $\rho(\tilde\gamma)$ is of type γ^{-1} we can complete the necessary semilinear picture by defining $\phi : \tilde\Gamma^{\mathrm{opp}} \to \Gamma$ to be the map $\tilde\gamma \to \gamma^{-1}$. The kernel of the composite map $t \circ \rho$ is precisely $(\mathbb{Z}/m\mathbb{Z})^n$, and the elements of this kernel act trivially on $\tilde k[\mathbf{G}]$, in particular their corresponding action on the Lie algebra of $\mathbf{G}$ is trivial, hence semisimple. We can thus apply Corollary 4.12; the role of H now being played by $\tilde\Gamma^{\mathrm{opp}}$.

Let $\mathbf{T}$ be a torus $\mathbf{G}$ such that $\rho(\tilde\gamma)({}^{\gamma^{-1}}\mathbf{T}) = \mathbf{T}$ for all $\tilde\gamma \in \tilde\Gamma$. The torus $\mathbf{T}$ corresponds to a Hopf ideal I of the Hopf $\tilde k$–algebra $\tilde k[\mathbf{G}]$ representing $\mathbf{G}$. Each $\rho(\tilde\gamma)$, which corresponds to the $\tilde k$-Hopf algebra isomorphism $f(\tilde\gamma)$ described in (4.20), induces a $\tilde k$–Hopf algebra isomorphism $\overline{f}(\tilde\gamma)$ from $\tilde k[\mathbf{T}]$ to $\tilde k[\mathbf{T}]^\gamma$ where $\tilde k[\mathbf{G}]/I = \tilde k[\mathbf{T}]$ is the Hopf algebra representing $\mathbf{T}$. For future use we observe that the resulting action of $\tilde\Gamma$ on $\tilde k[\mathbf{T}]$ is Γ–*semilinear* in the sense that if $\lambda \in \tilde k$ and $a \in \tilde k[\mathbf{T}]$ then

$$\overline{f}(\tilde\gamma)(\lambda a) = \overline{f}(\tilde\gamma)(\lambda_{\tilde k[\mathbf{T}]}.a) = \lambda_{\tilde k[\mathbf{T}]^\gamma}.\big(\overline{f}(\tilde\gamma)(a)\big) = ({}^\gamma\lambda)\overline{f}(\tilde\gamma)(a) \tag{4.22}$$

This follows immediately from the definition of $f(\tilde\gamma)$.

Consider the reductive $\tilde R$-group $\tilde{\mathfrak{G}} = \mathbf{G} \times_{\tilde k} \tilde R$ and its maximal torus $\tilde{\mathfrak{T}} = \mathbf{T} \times_{\tilde k} \tilde R$. We want to define an action of $\tilde\Gamma$ as automorphisms of the R–Hopf algebra $\tilde R[\mathfrak{T}] = \tilde k[\mathbf{T}] \otimes_{\tilde k} \tilde R$ so that the action is $\tilde\Gamma$–semilinear, this is

$$^{\tilde\gamma}(xs) = {}^{\tilde\gamma}x{}^{\tilde\gamma}s \tag{4.23}$$

for all $\tilde\gamma \in \tilde\Gamma$, $s \in \tilde R$ and $x \in \tilde R[\mathfrak{T}]$. By Galois descent this will show that the maximal torus $\tilde{\mathfrak{T}}$ of $\tilde{\mathfrak{G}}$ descends to a torus (necessarily maximal) $\mathfrak{T}$ of $\mathfrak{G}$.

To give the desired semilinear action consider, for a given fixed $\tilde\gamma \in \tilde\Gamma$, the map

$$\tilde k[\mathbf{T}] \times \tilde R \to \tilde k[\mathbf{T}] \otimes_{\tilde k} \tilde R = \tilde R[\mathfrak{T}]$$

defined by

$$(a, s) \mapsto \overline{f}(\tilde\gamma)(a) \otimes {}^{\tilde\gamma}s \tag{4.24}$$

for all $a \in \tilde k[\mathbf{T}]$ and $s \in \tilde R$. From (4.22) and the fact that ${}^{\tilde\gamma}s = {}^\gamma s$ if $s \in \tilde k \subset \tilde R$ it follows that the above map is $\tilde k$-balanced, hence that induces a morphism of $\tilde k$-spaces

$$\widehat{f}(\tilde\gamma) : \tilde k[\mathbf{T}] \otimes_{\tilde k} \tilde R = \tilde R[\mathfrak{T}] \to \tilde R[\mathfrak{T}] \tag{4.25}$$

satisfying

$$\widehat{f}(\tilde\gamma) : a \otimes s \mapsto \overline{\rho}(\tilde\gamma)(a) \otimes {}^{\tilde\gamma}s \tag{4.26}$$

for all $a \in \tilde k[\mathbf{T}]$ and $s \in \tilde R$. From (4.22) and (4.26) we then obtain an action of the group $\tilde\Gamma$ on the Hopf algebra $\tilde R[\mathfrak{T}]$ as prescribed by (4.23). □

4.7 Variations of a result of Sansuc.

We shall need the following variantion of a well-known and useful result [Sa, 1.13].

Lemma 4.14. *Assume that k is of characteristic zero. Let $\mathbf{H}$ be a linear algebraic group over k and let $\mathbf{U}$ be a normal unipotent subgroup of $\mathbf{H}$.*

1. *Let k'/k be a finite Galois extension of fields. Let Γ be a finite group acting on k'/k. Then the map*

$$H^1\big(\Gamma, \mathbf{H}(k')\big) \to H^1\big(\Gamma, (\mathbf{H}/\mathbf{U})(k')\big)$$

   is bijective.

2. *Let R be an object in k–alg. Then the map*

$$H^1(R, \mathbf{H}) \to H^1(R, \mathbf{H}/\mathbf{U})$$

   is bijective.

Proof. The k–group $\mathbf{U}$ admits a non-trivial characteristic central split unipotent subgroup $\mathbf{U}_0 \simeq \mathbf{G}_a^n$ [DG, IV.4.3.13]. We can then form the following commutative diagram of exact sequence of algebraic k–groups

$$\begin{tikzcd}
 & & & 1 \arrow[d] & \\
 & 1 \arrow[d] & & \mathbf{U}/\mathbf{U}_0 \arrow[d] & \\
1 \arrow[r] & \mathbf{U}_0 \arrow[r] \arrow[d] & \mathbf{H} \arrow[r] \arrow[d, "\cong"'] & \mathbf{H}/\mathbf{U}_0 \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbf{U} \arrow[r] \arrow[d] & \mathbf{H} \arrow[r] & \mathbf{H}/\mathbf{U} \arrow[r] \arrow[d] & 1 \\
 & \mathbf{U}/\mathbf{U}_0 \arrow[d] & & 1 & \\
 & 1 & & &
\end{tikzcd}$$

If the Lemma holds for the morphisms $\mathbf{H} \to \mathbf{H}/\mathbf{U}_0$ and $\mathbf{H}/\mathbf{U}_0 \to \mathbf{H}/\mathbf{U}$, it holds for $\mathbf{H} \to \mathbf{H}/\mathbf{U}$. Without loss of generality, we can therefore assume by devissage that $\mathbf{U} = \mathbf{G}_a^n$.

(1) Since $H^1(k', \mathbf{U}) = 0$, we have an exact sequence of Γ–groups

$$1 \to \mathbf{U}(k') \to \mathbf{H}(k') \to (\mathbf{H}/\mathbf{U})(k') \to 1.$$

For each $c \in Z^1\big(\Gamma, (\mathbf{H}/\mathbf{U})(k')\big)$, the group ${}_c\big(\mathbf{U}(k')\big)$ is a uniquely divisible abelian group, so $H^i\big(\Gamma, {}_c(\mathbf{U}(k'))\big) = 0$ for all $i > 0$. By applying a basic result on non-abelian cohomology [Se, §I.5, corollary to prop. 41], the vanishing of these H^2 implies that the map $H^1\big(\Gamma, \mathbf{H}(k')\big) \to H^1\big(\Gamma, (\mathbf{H}/\mathbf{U})(k')\big)$ is surjective. Similarly, for each $z \in Z^1\big(\Gamma, \mathbf{H}(k')\big)$, the group $0 = H^1\big(\Gamma, {}_z(\mathbf{U}(k'))\big)$ maps onto the subset of $H^1\big(\Gamma, \mathbf{H}(k')\big)$ consisting of classes of cocycles whose image in $H^1\big(\Gamma, (\mathbf{H}/\mathbf{U})(k')\big)$ coincides with that of $[z]$. We conclude that the map $H^1\big(\Gamma, \mathbf{H}(k')\big) \to H^1\big(\Gamma, (\mathbf{H}/\mathbf{U})(k')\big)$ is bijective.

(2) Let us first prove the injectivity by using the classical torsion trick. We are given a $\mathbf{H}/\mathbf{U}$–torsor $\mathfrak{E}$ over $\mathrm{Spec}(R)$. We can twist the exact sequence of R–group schemes $1 \to \mathbf{U}_R \to \mathbf{H}_R \to \mathbf{H}_R/\mathbf{U}_R \to 1$ by $\mathfrak{E}$ and get the twisted sequence $1 \to {}_\mathfrak{E}\mathbf{U} \to {}_\mathfrak{E}\mathbf{H} \to {}_\mathfrak{E}\mathbf{H}/{}_\mathfrak{E}\mathbf{U} \to 1$, where as usual we write ${}_\mathfrak{E}\mathbf{U}$ instead of ${}_\mathfrak{E}\mathbf{U}_R$ and ${}_\mathfrak{E}\mathbf{H}$ instead of ${}_\mathfrak{E}\mathbf{H}_R$. We consider the following commutative diagram of sets [Gi, III.3.3.4]

$$\begin{array}{ccccc}
 & & H^1(R, \mathbf{H}) & \longrightarrow & H^1(R, \mathbf{H}/\mathbf{U}) \\
 & & {\scriptstyle torsion}\Big\uparrow \simeq & & {\scriptstyle torsion}\Big\uparrow \simeq \\
H^1(R, {}_\mathfrak{E}\mathbf{U}) & \longrightarrow & H^1(R, {}_\mathfrak{E}\mathbf{H}) & \longrightarrow & H^1(R, {}_\mathfrak{E}\mathbf{H}/{}_\mathfrak{E}\mathbf{U})
\end{array}$$

where the bottom map is an exact sequence of pointed sets. Indeed $\mathbf{GL}_n$ is the group of automorphisms of the group scheme $\mathbf{G}_a^n$ (see Lemma 4.15 below). It follows that ${}_\mathfrak{E}\mathbf{U}$ corresponds to a locally free sheaf over $\mathrm{Spec}(R)$. By [Gr1, pp 16-17] (or [M, III.3.7]), we have $\check{\mathrm{H}}^i(R, {}_\mathfrak{E}\mathbf{U}) = 0$ for all $i > 0$.[11] So the the map $H^1(R, {}_\mathfrak{E}\mathbf{H}) \to H^1(R, {}_\mathfrak{E}\mathbf{H}/{}_\mathfrak{E}\mathbf{U})$ has trivial kernel and the fiber of $H^1(R, \mathbf{H}) \to H^1(R, \mathbf{H}/\mathbf{U})$ is only $[\mathfrak{E}]$.

For surjectivity, if we are given a $\mathbf{H}/\mathbf{U}$–torsor $\mathfrak{E}$ over $\mathrm{Spec}(R)$ then by [Gi, IV.3.6.1] there is a class

$$\Delta([\mathfrak{E}]) \in \check{\mathrm{H}}^2(R, {}_\mathfrak{E}\mathbf{U})$$

which is the obstruction to the existence of a lift of $[\mathfrak{E}]$ to $H^1(R, \mathbf{H})$. Here ${}_\mathfrak{E}\mathbf{U}$ is the R–group scheme obtained by twisting $\mathbf{G}_a^n$ by the R–torsor $\mathfrak{E}$. Since ${}_\mathfrak{E}\mathbf{U}$ corresponds to a locally free sheaf, the same reasoning used above shows that the obstruction $\Delta([\mathfrak{E}])$ vanishes as desired. $\square$

Lemma 4.15. *Let $\mathfrak{X}$ be a scheme of characteristic* 0. *Let $\mathcal{E}$ be a locally free $\mathfrak{X}$–sheaf of finite rank and let $\mathbf{V}(\mathcal{E})$ be the associated "additive" $\mathfrak{X}$–group scheme. Then the natural homomorphism of fpqc sheaves*

$$\alpha : \mathbf{GL}(\mathcal{E}) \to \mathbf{Aut}_{\mathfrak{X}-gr}(\mathbf{V}(\mathcal{E}))$$

[11] All the $\check{\mathrm{H}}^i$ that we consider coincide with the corresponding H^i defined in terms of derived functors.

is an $\mathfrak{X}$–group sheaf isomorphism. In particular, $\mathbf{Aut}_{\mathfrak{X}-gr}\big(\mathbf{V}(\mathcal{E})\big)$ *is an* $\mathfrak{X}$*–group scheme.*

Our convention is that of [DG, §2], namely $\mathbf{V}(\mathcal{E})(\mathfrak{X}') = H^0(\mathfrak{X}', \mathcal{E} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathcal{X}'})$ for every scheme $\mathfrak{X}'$ over $\mathfrak{X}$.

Proof. It is clear that α is an morphism of $\mathfrak{X}$–groups. For showing that α is an isomorphism of sheaves, we may assume that $\mathfrak{X} = \mathrm{Spec}(R)$ is affine and that $\mathcal{E} = R^n$. This in turn reduces to the case of $R = \mathbb{Q}$ and $\mathcal{E} = \mathbb{Q}^n$. By descent, it will suffice to establish the result for $R = \overline{\mathbb{Q}}$ and $\mathcal{E} = (\overline{\mathbb{Q}})^n$. Now on $\overline{\mathbb{Q}}$-schemes the functor $S \mapsto \mathbf{Aut}_{gr}(\mathbf{V}(\mathcal{E}))(S)$ is representable by a linear algebraic $\overline{\mathbb{Q}}$-group $\mathbf{H}$ according to Hochschild-Mostow's criterion [HM, th. 3.2]. Therefore we can check the fact that $\alpha : \mathbf{GL}_n \to \mathbf{H}$ is an isomorphism on $\overline{\mathbb{Q}}$–points. But this readily follows from the equivalence of categories between nilpotent Lie algebras and algebraic unipotent groups [DG, §IV §2 cor.4.5]. Since $\mathbf{GL}(\mathcal{E})$ is an $\mathfrak{X}$–group scheme, α is an isomorphism of $\mathfrak{X}$–group schemes. ☐

5 Maximal tori of group schemes over the punctured line

Let $\mathbf{G}$ be a linear algebraic k–group. One of the central results of [CGP] is the existence of maximal tori for twisted groups of the form ${}_{\mathfrak{E}}\mathbf{G}$ where $[\mathfrak{E}] \in H^1(k[t^{\pm 1}], \mathbf{G})$.[12] This result is used to describe the nature of torsors over $k[t^{\pm 1}]$ under $\mathbf{G}$. In our present work we are ultimately interested in the classification of reductive groups over Laurent polynomial rings when k is of characteristic 0, and applications to infinite dimensional Lie theory. In understanding twisted forms of $\mathbf{G}$ the relevant objects are torsors under $\mathbf{Aut}(\mathbf{G})$, and not $\mathbf{G}$. It is therefore essential to have an analogue of the [CGP] result mentioned above, but for arbitrary twisted groups, not just inner forms.[13] This is one of the crucial theorems of our paper.

Theorem 5.1. *Let $R = k[t^{\pm 1}]$ where k is a field of characteristic 0. Every reductive group scheme $\mathfrak{G}$ over R admits a maximal torus.*

Corollary 5.2. *Let k and R be as above. Let $\mathfrak{G}$ be a smooth affine group scheme over R whose connected component of the identity $\mathfrak{G}^0$ is reductive. Then*

[12]If the characteristic of k is sufficiently large.

[13]$\mathbf{Aut}(\mathbf{G})$ need not be an algebraic group. Even if it is, the fact that it need not be connected leads to considerable technical complications (stemming from the fact that, unlike the affine line, the punctured line has non-trivial geometric étale coverings). As already mentioned, these difficulties have to be dealt with if one is interested in the study of twisted forms of $\mathbf{G}_R$ or its Lie algebra.

1. $H^1_{toral}(R,\mathfrak{G}) = H^1(R,\mathfrak{G})$.

2. If $\mathfrak{G}$ *is constant, i.e.* $\mathfrak{G} = \mathbf{G}\times_k R$ *for some linear algebraic* k*–group* $\mathbf{G}$*, then*

$$H^1_{toral}(R,\mathfrak{G}) = H^1_{loop}(R,\mathfrak{G}) = H^1(R,\mathfrak{G}).$$

The first assertion is an immediate corollary of the Theorem while the second then follows from Corollary 3.16.2 and Lemma 2.8. $\square$

The proof of the Theorem relies on Bruhat-Tits twin buildings and Galois descent considerations. We begin by establishing the following useful reduction.

Lemma 5.3. *It suffices to establish Theorem 5.1 under the assumption that* $\mathfrak{G}$ *is a twisted form of a simple simply connected Chevalley* R*–group.*[14]

Proof. Assume that Theorem 5.1 holds in the simple simply connected case. By [XII.4.7.c], there is a natural one-to-one correspondence between the maximal tori of $\mathfrak{G}$, its adjoint group $\mathfrak{G}_{ad}$ and those of the simply connected covering $\tilde{\mathfrak{G}}_{ad}$ of $\mathfrak{G}_{ad}$. We can thus assume without lost of generality that $\mathfrak{G}$ is simply connected. By [XXIV.5.10] we have

$$\mathfrak{G} = \prod_{i=1,\dots,l}\ \prod_{S_i/R} \mathfrak{G}_i$$

where each S_i is a connected finite étale covering of R and each $\mathfrak{G}_i$ a simple simply connected S_i–group scheme. By Demazure's main theorem, the S_i–groups $\mathfrak{G}_i$ are twisted forms of simple simply connected Chevalley groups. Since by Lemma 2.8 S_i is a Laurent polynomial ring, our hypothesis implies that each of the S_i–groups $\mathfrak{G}_i$ admits a maximal torus $\mathfrak{T}_i$ Then our R–group $\mathfrak{G}$ admits the maximal torus $\prod_{i=1,\dots,l}\ \prod_{S_i/R} \mathfrak{T}_i$. $\square$

5.1 Twin buildings

Throughout this section k denotes a field of characteristic 0. We set $R = k[t^{\pm 1}]$, $K = k(t)$ and $\widehat{K} = K((t))$. For a "survival kit" on euclidean buildings, we recommend Landvogt's paper [L].

Let $\tilde{R}$ be a finite Galois extension of R of the form $\tilde{R} = \tilde{k}[t^{\pm\frac{1}{n}}]$ where $\tilde{k}/k$ is a finite Galois extension of k containing all n-roots of unity in $\overline{k}$. Then as we have already seen $\tilde{\Gamma} := \mathrm{Gal}(\tilde{R}/R) = \boldsymbol{\mu}_n(\tilde{k}) \rtimes \Gamma$ where $\Gamma = \mathrm{Gal}(\tilde{k}/k)$.

Set $\tilde{t} = t^{\frac{1}{n}}$ We let $L = \tilde{k}(\tilde{t}\,) = \tilde{k}(\tilde{t}^{-1})$, and consider the two completions $\widehat{L}_+ = \tilde{k}((\tilde{t}\,))$ and $\widehat{L}_- = \tilde{k}((\tilde{t}^{-1}))$ of L at 0 and ∞ respectively, as well as their corresponding valuation rings $\widehat{A}_+ = \tilde{k}[[\tilde{t}\,]]$ and $\widehat{A}_- = \tilde{k}[[\tilde{t}^{-1}]]$.

[14] The usual algebraic group literature would use the term "almost simple" in this situation. We adhere throughout to the terminology of [SGA3].

Let $\mathbf{G}$ be a split simple simply connected group over k. Let $\mathbf{T}$ be a maximal split torus of $\mathbf{G}$, $\mathbf{B}^+$ a Borel subgroup of $\mathbf{G}$ which contains $\mathbf{T}$, and $\mathbf{B}^-$ the corresponding opposite Borel subgroup (which also contains $\mathbf{T}$). We denote by $\mathbf{W} = \mathbf{N_G}(\mathbf{T})$ the corresponding Weyl group and by $\Delta_\pm$ the Dynkin diagram attached to $(\mathbf{G}, \mathbf{B}^\pm, \mathbf{T})$.

Following Tits [T3], we consider the twin building $\mathcal{B} = \mathcal{B}_+ \times \mathcal{B}_-$ of $\mathbf{G}_0 \times_k L$ with respect to the two completions $\widehat{L}_+$ and $\widehat{L}_-$. Recall that $\mathcal{B}$ comes equipped with an action of the group $\mathbf{G}(L)$, hence also of $\mathbf{G}(\tilde{R})$. The split torus $\mathbf{T}_0 \times_k L$ gives rise to a twin apartment $\mathcal{A} = \mathcal{A}_+ \times \mathcal{A}_-$ of $\mathcal{B}$. The Borel subgroups $\mathbf{B}^\pm$ define the fundamental chambers $\mathcal{C}_\pm$ of $\mathcal{A}_\pm$, each of which is an open simplex whose vertices are given by the extended Dynkin diagram $\tilde{\Delta}_\pm$ of $\Delta_\pm$.

Recall that the group functor $\mathbf{Aut}(\mathbf{G})$ is an affine group scheme. The group $\mathbf{Aut}(\mathbf{G})(L)$ acts on $\mathcal{B}$ by "transport of structure" [L] 1.3.4.[15] This leads to an action of $\mathbf{G}(L)$ on $\mathcal{B}$ via $\mathrm{Int} : \mathbf{G} \to \mathbf{Aut}(\mathbf{G})$. This action coincides with the "standard" action of $\mathbf{G}(L)$ mentioned before because $\mathbf{G}$ is semisimple. By taking into account the natural action of $\tilde{\Gamma} \simeq \mathrm{Gal}(\widehat{L}_+/\widehat{K})$ on $\mathcal{B}$ we conclude that the twin building $\mathcal{B}$ is equipped with an action of the semi-direct product $\mathrm{Aut}(\mathbf{G})(\tilde{R}) \rtimes \Gamma$ which is compatible (via the adjoint action) with the action of $\mathbf{G}(\tilde{R})$.

The hyperspecial group $\mathbf{G}(\widehat{A}_\pm)$ fixes a unique point $\phi_\pm$ of $\mathcal{A}_\pm$ [BT1, §9.1.19.c]. Recall that the hyperspecial points of $\mathcal{B}_\pm$ are $\mathbf{G}(\widehat{L}_\pm)$-conjugate to $\phi_\pm$ of $\mathcal{B}_\pm$, and can therefore be identified with the set of left cosets

$$\mathbf{G}(\widehat{L}_\pm)/\mathbf{G}(\widehat{A}_\pm) \simeq \mathbf{G}(\widehat{L}_\pm).\, \phi_\pm \subset \mathcal{B}_\pm.$$

More generally each facet of the building $\mathcal{B}_\pm$ has a type [BT1, §2.1.1] which is a subset of $\tilde{\Delta}_\pm$ and the type of a point $x \in \mathcal{B}_\pm$ is the type of its underlying facet F_x.[16] The type of the chamber $\mathcal{C}_\pm$ is $\emptyset$ and the type of an hyperspecial point is $\tilde{\Delta}_\pm \setminus \Delta_\pm$, namely the extra vertice of the affine Dynkin diagram.

5.2 Proof of Theorem 5.1

By Lemma 5.3, we can assume that $\mathfrak{G}$ is simple simply connected. By the Isotriviality Theorem [GP1, cor. 2.16], we know that our R–group $\mathfrak{G}$ is isotrivial. This means that there exists a finite Galois covering S/R and a "trivialization" $f : \mathbf{G} \times_k S \simeq \mathfrak{G} \times_R S$ where $\mathbf{G}$ is a split simple simply connected k–group. In our terminology, $\mathbf{G}$ is the Chevalley k–form of $\mathfrak{G}$.

Because of the structure of the algebraic fundamental group of R we may assume without loss of generality that $S = \tilde{R}$ is as in §5.1, and we keep all the notation

[15] The group in question acts on the set of maximal split tori, hence permutes the apartments around.

[16] Namely the smallest facet containing x in its closure.

therein. What is so special about this situation is that R and $\tilde{R}$ "look the same", namely they are both Laurent polynomial rings in one variable with coefficients in a field.

We have $\mathrm{Spec}(\tilde{R}) = \mathbb{P}^1_{\tilde{k}} \setminus \{0, \infty\}$ and the action of $\tilde{\Gamma}$ on $\tilde{R}$ extends to $\mathbb{P}^1_{\tilde{k}}$ since $\tilde{R}$ is regular of dimension 1.

For $\tilde{\gamma} \in \tilde{\Gamma}$ consider the map $z_{\tilde{\gamma}} = f^{-1} \circ {}^{\tilde{\gamma}}f : \tilde{\Gamma} \to \mathbf{Aut}(\mathbf{G})(\tilde{R})$, where $\mathbf{Aut}(\mathbf{G})$ stands for the group scheme of automorphisms of the $\mathbb{Z}$-group $\mathbf{G}$. Then $z = (z_{\tilde{\gamma}})_{\tilde{\gamma} \in \tilde{\Gamma}}$ is a cocycle in $Z^1\big(\tilde{\Gamma}, \mathbf{Aut}(\mathbf{G})(\tilde{R})\big)$ where the Galois group $\tilde{\Gamma}$ acts naturally on $\mathbf{Aut}(\mathbf{G})(\tilde{R})$ via its action on $\tilde{R}$. Descent theory tells us that $\mathbf{G}$ is isomorphic to the twisted R-group ${}_z\mathbf{G}$.[17]

The action of $\tilde{\Gamma}$ on $\mathbf{Aut}(\mathbf{G})(\tilde{R})$ allows us to consider the semidirect product group $\mathbf{Aut}(\mathbf{G})(\tilde{R}) \rtimes \tilde{\Gamma}$. We then have a group homomorphism

$$\psi_z : \tilde{\Gamma} \to \mathbf{Aut}(\mathbf{G})(\tilde{R}) \rtimes \tilde{\Gamma} \tag{5.1}$$

given by $\psi_z(\gamma) = z_\gamma\, \gamma$ which is a section of the projection map $\mathbf{Aut}(\mathbf{G})(\tilde{R}) \rtimes \tilde{\Gamma} \to \tilde{\Gamma}$.

Let $\mathbf{T}$ be a maximal split of $\mathbf{G}$. Set $L = \tilde{k}(\tilde{t})$, and let A_+ (resp. A_-) be the local ring of $\mathbb{P}^1_{\tilde{k}}$ at 0 (resp. ∞). The composite map (see §5.1)

$$\tilde{\Gamma} \xrightarrow{\psi_z} \mathbf{Aut}(\mathbf{G})(\tilde{R}) \rtimes \tilde{\Gamma} \to \mathrm{Aut}(\mathcal{B})$$

is a group homomorphism. The corresponding action of $\tilde{\Gamma}$ on $\mathcal{B}$ will be referred to as the *twisted action* of $\tilde{\Gamma}$ on the building. We now appeal to the Bruhat-Tits fixed point theorem [BT1, §3.2] to obtain a point $p = (p_+, p_-) \in \mathcal{B}$ which is fixed under the twisted action, i.e. $\psi_z(\tilde{\gamma}).p = z_{\tilde{\gamma}}\, \tilde{\gamma}(p) = p$ for all $\tilde{\gamma} \in \tilde{\Gamma}$. Abramenko's result [A, Proposition 5] states that $\mathbf{G}(\tilde{R}).\mathcal{A} = \mathcal{B}$. There thus exists $g \in \mathbf{G}(\tilde{R})$ such that p belongs to the apartment $g.\mathcal{A}$. Up to replacing $z_{\tilde{\gamma}}$ by $\mathrm{Int}(g)^{-1}\, z_{\tilde{\gamma}}\, \mathrm{Int}(\tilde{\gamma}(g))$, we can therefore assume that p belongs to $\mathcal{A}$.

We shall use several times that $\tilde{\Gamma}$ acts trivially on $\mathcal{A}$. Here since $\tilde{\gamma}(p) = p$, it follows that $z_{\tilde{\gamma}} \,.\, p = p$ for all $\tilde{\gamma} \in \tilde{\Gamma}$.

Let $F_{p_\pm}$ be the facet associated to $p_\pm$ and choose a vertex $q_\pm$ of $\overline{F}_{p_\pm}$. The transforms of $q_\pm$ by $z_{\tilde{\gamma}}$ and $\tilde{\gamma}$ are vertices of $\overline{F}_{p_\pm}$, so $\psi_z(\tilde{\gamma}).q_\pm$ belongs to $\mathcal{A}$. We define

$$x_\pm = \mathrm{Barycentre}\Big(\psi_z(\tilde{\gamma}).q_\pm, \tilde{\gamma} \in \tilde{\Gamma}\Big) \in \mathcal{A}.$$

Let d be the integer attached to $\mathbf{G}$ in [Gi1, §2], and set $m = d\,|\,\tilde{\Gamma}\,|$. Let $s \in \overline{L}$ (a fixed algebraic closure of L) be such that $s^m = \tilde{t}$. We have accordingly $s^{mn} = t$. Set

$$R' = k'[\, s^{\pm 1}\,]$$

[17] Recall that for convenience ${}_z\mathbf{G}$ is shorthand notation for ${}_z(\mathbf{G} \times_k R) = {}_z(\mathbf{G}_R)$.

where k' is a Galois extension of k which contains $\tilde{k}$ and all mn-roots of unity in $\overline{k}$. Then R' is Galois over R of Galois group

$$\Gamma' = \boldsymbol{\mu}_{mn}(k') \rtimes \mathrm{Gal}(k'/k).$$

By Galois theory the map $\upsilon : \Gamma' \to \tilde{\Gamma}$ given by

$$\upsilon : (\xi, \theta) \mapsto (\xi^m, \theta_{|\tilde{k}}) \tag{5.2}$$

is a surjective group homomorphism.

We consider the twin building $\mathcal{B}' = \mathcal{B}'_+ \times \mathcal{B}'_-$ of $\mathbf{G}$ which is constructed in the manner described above after replacing, mutatis mutandis, the relevant objects attached to $\tilde{R}$ by those of R' . We have a restriction map [Ro, §II.4] $\rho_\pm : \mathcal{B}_\pm \to \mathcal{B}'_\pm$ which gives rise to

$$\rho = (\rho_+, \rho_-) : \mathcal{B} \to \mathcal{B}'.$$

Furthermore, if $\gamma' \in \Gamma'$ and we set $\upsilon(\gamma') = \tilde{\gamma}$ then the following diagram commutes

$$(*) \qquad \begin{array}{ccc} \mathcal{B} & \xrightarrow{\rho} & \mathcal{B}' \\ {\scriptstyle\tilde{\gamma}}\big\downarrow & & {\scriptstyle\gamma'}\big\downarrow \\ \mathcal{B} & \xrightarrow{\rho} & \mathcal{B}' \end{array}$$

where the actions of $\tilde{\Gamma}$ and Γ' are the twisted actions. If we define $z' : \Gamma' \to \mathbf{Aut}(\mathbf{G})(R')$ by

$$z' : \gamma' \mapsto z'_{\gamma'} = z_{\tilde{\gamma}} \in \mathbf{Aut}(\mathbf{G})(\tilde{R}) \subset \mathbf{Aut}(\mathbf{G})(R')$$

then z' is a cocycle and the classes $[z]$ and $[z']$ in $H^1\big(R, \mathbf{Aut}(\mathbf{G})\big)$ are the same.

Lemma 5.4. *$\rho_\pm(x_\pm)$ is a hyperspecial point of $\mathcal{B}'_\pm$.*

Proof. We look at the case $\rho_+(x_+)$. We need to consider the intermediate extensions

$$\widehat{L}_+ = \tilde{k}((\,\tilde{t}\,)) \;\subset\; k'((\,\tilde{t}\,)) \;\subset\; k'((\,s^d\,)) \;\subset\; k'((\,s\,)).$$

The map ρ_+ is the composite of the corresponding maps for the intermediate fields, namely

$$\mathcal{B}_+ = \mathcal{B}_+(\mathbf{G}_{0,\,\tilde{k}((\,\tilde{t}\,))}) \;\xrightarrow{\rho_{1,+}}\; \mathcal{B}\big(\mathbf{G}_{0,\,k'((\,\tilde{t}\,))}\big) \;\xrightarrow{\rho_{d,+}}\; \mathcal{B}\big(\mathbf{G}_{0,\,k'((\,s^d\,))}\big) \;\xrightarrow{\rho_{\frac{m}{d},+}}\; \mathcal{B}\big(\mathbf{G}_{0,\,k'((\,s\,))}\big)\mathcal{B}'_+.$$

The first map does not change the type. By [Gi1, lemma 2.2.a], the image under $\rho_{d,+}$ of any vertex is a hyperspecial point. We have

$$\begin{aligned}\rho_+(x_+) &= \rho_+\Big[\text{Barycenter}\Big(\psi_z(\tilde\gamma).q_\pm, \gamma\in\tilde\Gamma\Big)\Big]\\ &= \rho_{\frac{m}{d},+}\Big[\text{Barycenter}\Big(\rho_{d,+}\circ\rho_{1,+}(\psi_z(\tilde\gamma).q_\pm), \gamma\in\tilde\Gamma\Big)\Big].\end{aligned}$$

By [Gi1, Lemma 2.3' in the errata], we know that the image under $\rho_{\frac{m}{d},+}$ of the barycentre of $\frac{m}{d}$ hyperspecial points of a common apartment (namely the one attached to the torus $\mathbf{T}$) is a hyperspecial point, so we conclude that $\rho_+(x_+)$ is a hyperspecial point. □

In view of diagram (*) above it follows that by replacing $\tilde R$ by R' we may assume without loss of generality that the points $p_\pm \in \mathcal{A}_\pm$ are hyperspecial. Note that by construction, the points $\rho_\pm(x_\pm)$ of $\mathcal{B}'_\pm$ are fixed by both actions (standard and twisted) of Γ', so that after our further extension of base ring we may assume that $\rho_\pm(x_\pm)$ of $\mathcal{B}_\pm$ are fixed by both actions of $\tilde\Gamma$.

Since $\mathbf{T}(\tilde R)$ acts transitively on the sets $\phi_\pm + (\widehat{T})^0 \subset \mathcal{A}_\pm$ of hyperspecial points of $\mathcal{A}_\pm$, there exists $g\in\mathbf{T}(\tilde R)$ and a cocharacter $\lambda\in(\widehat{\mathbf{T}})^0$ such that

$$g.(\psi_+, x_-^\lambda) = (x_+, x_-) = x.$$

where $x_-^\lambda := \phi_- + \lambda$ (recall that we have a map $(\widehat{\mathbf{T}})^0 \to \mathcal{A}_- = (\widehat{\mathbf{T}})^0\otimes_{\mathbb{Z}}\mathbb{R}$ defined by $\theta\mapsto\phi_-+\theta$). Up to replacing the cocycle z by z' where $z'_{\tilde\gamma} = \mathrm{Int}(g)^{-1} z_{\tilde\gamma}\,{}^{\tilde\gamma}\mathrm{Int}(g)$, we may assume that $\psi_z(\tilde\gamma).(\phi_+, x_-^\lambda) = z_\gamma.(\phi_+, x_-^\lambda) = (\phi_+, x_-^\lambda)$ for every $\tilde\gamma\in\tilde\Gamma$. In particular,

$$z_{\tilde\gamma}\in\mathrm{Stab}_{\mathbf{Aut}(\mathbf{G})(\widehat{L}_+)}(\phi_+) = \mathbf{Aut}(\mathbf{G})(\widehat{A}_+)$$

hence

$$z_{\tilde\gamma}\in\ \mathbf{Aut}(\mathbf{G})(\tilde R)\cap\mathbf{Aut}(\mathbf{G})(\widehat{A}_+)\mathbf{Aut}(\mathbf{G})(\tilde k[\tilde t\,])$$

for each $\tilde\gamma\in\tilde\Gamma$. Let $g_\lambda := \lambda(\tilde t\,)\in\mathbf{T}(\tilde R)\subset\mathbf{G}(\tilde R)$. We have $g_\lambda.\phi_- = x_-^\lambda$ and therefore

$$z_{\tilde\gamma}\ \in\ \mathrm{Stab}_{\mathbf{Aut}(\mathbf{G})(\widehat{L}_-)}(x_-^\lambda)$$

$$\mathrm{Int}(g_\lambda)\ \mathrm{Stab}_{\mathbf{Aut}(\mathbf{G})(\widehat{L}_-)}(\phi_-)\,\mathrm{Int}(g_\lambda)^{-1} = \mathrm{Int}(g_\lambda)\mathbf{Aut}(\mathbf{G})(\widehat{A}_-)\mathrm{Int}(g_\lambda^{-1}).$$

It follows that for each $\tilde\gamma\in\tilde\Gamma$

$$\begin{aligned} z_{\tilde\gamma}\in J_\lambda &:= \mathbf{Aut}(\mathbf{G})(\tilde k[\tilde t\,])\cap\mathrm{Int}(g_\lambda)\,\mathbf{Aut}(\mathbf{G})(\widehat{A}_-)\,\mathrm{Int}(g_\lambda^{-1})\\ &= \mathbf{Aut}(\mathbf{G})(k[\tilde t\,])\cap\mathrm{Int}(g_\lambda)\,\mathbf{Aut}(\mathbf{G})(\tilde k[\tilde t^{\,-1}])\,\mathrm{Int}(g_\lambda^{-1}).\end{aligned}$$

Note that for $\tilde\gamma\in\tilde\Gamma$ we have

$${}^{\tilde\gamma}g_\lambda = {}^{\tilde\gamma}\lambda(t') = \lambda(t')\left(\lambda(t')^{-1}\,{}^{\tilde\gamma}\lambda(t')\right)\in\lambda(t')\mathbf{T}(\tilde k)\subset\lambda(t')\mathbf{G}(\widehat{A}_-) = g_\lambda\,\mathbf{G}(\widehat{A}_-)$$

From this it follows not only that the subgroup J_λ of $\mathbf{Aut}(\mathbf{G})(\tilde{R})$ is stable under the (standard) action of $\tilde{\Gamma}$, but also that

$$[z] \in \operatorname{Im}\Big(H^1\big(\tilde{\Gamma}, J_\lambda\big) \to H^1\big(\tilde{\Gamma}, \mathbf{Aut}(\mathbf{G})(\tilde{R})\big)\Big).$$

It turns out that the structure of the group J_λ is known by a computation of Ramanathan, as we shall see in Proposition 16.2 below. We have

$$J_\lambda = \mathbf{U}_\lambda(\tilde{k}) \rtimes \mathbf{Z}_{\mathbf{Aut}(\mathbf{G})}(\lambda)(\tilde{k}) \ \subset \ \mathbf{Aut}(\mathbf{G})(\tilde{k}[\tilde{t}\,]),$$

where $\mathbf{U}_\lambda$ is a unipotent k–group. Lemma 4.14 shows that the map

$$H^1\big(\tilde{\Gamma}, \mathbf{Z}_{\mathbf{Aut}(\mathbf{G})}(\lambda)(\,\tilde{k}\,)\big) \to H^1\big(\tilde{\Gamma}, J_\lambda\big)$$

is bijective. Summarizing, we have the commutative diagram

$$\begin{array}{ccc}
H^1\big(\tilde{\Gamma}, \mathbf{Z}_{\mathbf{Aut}(\mathbf{G})}(\lambda)(\,\tilde{k}\,)\big) & \longrightarrow & H^1\big(\tilde{\Gamma}, \mathbf{Aut}(\mathbf{G})(\,\tilde{k}\,)\big) \\
\Big\downarrow \cong & & \Big\downarrow \\
H^1\big(\tilde{\Gamma}, J_\lambda\big) & \longrightarrow & H^1\big(\tilde{\Gamma}, \mathbf{Aut}(\mathbf{G})(\tilde{R})\big)
\end{array}$$

which shows that

$$[z] \in \operatorname{Im}\Big(H^1\big(\tilde{\Gamma}, \mathbf{Aut}(\mathbf{G})(\,\tilde{k}\,)\big) \to H^1\big(\tilde{\Gamma}, \mathbf{Aut}(\mathbf{G})(\tilde{R})\big)\Big).$$

This means that $[z]$ is cohomologous to a loop cocycle, and we can now conclude by Proposition 4.13 $\square$

6 Internal characterization of loop torsors and applications

We continue to assume that our base field k of characteristic zero. Let $R_n = k[t_1^{\pm 1}, ..., t_n^{\pm 1}]$ and $\mathfrak{X} = \operatorname{Spec}(R_n)$. As explained in Example 2.5 we have $\pi_1(R_n, a) \simeq \widehat{\mathbb{Z}}^n \rtimes \operatorname{Gal}(k)$, where the action of $\operatorname{Gal}(k)$ on $\widehat{\mathbb{Z}}^n$ is given by our fixed choice of compatible roots of unity in $\overline{k}$. For convenience in what follows we will denote $\pi_1(R_n, a)$ simply by $\pi_1(R_n)$.

Throughout this section $\mathbf{G}$ denotes a linear algebraic k–group.

6.1 Internal characterization of loop torsors

We first observe that loop torsors make sense over $R_0 = k$, namely $H^1_{loop}(R_0, \mathbf{G})$ is the usual Galois cohomology $H^1(k, \mathbf{G})$.

Section 3.3 shows that $Z^1\big(\pi_1(R_n), \mathbf{G}(\overline{k})\big)$ is given by couples (z, η^{geo}) where $z \in Z^1(\mathrm{Gal}(k), \mathbf{G}(\overline{k}))$ and $\eta^{geo} \in \mathrm{Hom}_{\mathrm{Gal}(k)}(\pi_1(\overline{X}, \overline{a}), {}_z\mathbf{G}(\overline{k})) = \mathrm{Hom}_{\mathrm{Gal}(k)}(\widehat{\mathbb{Z}}^n, {}_z\mathbf{G}(\overline{k})) \simeq \mathrm{Hom}_{\mathrm{Gal}(k)}({}_\infty\boldsymbol{\mu}, {}_z\mathbf{G})$. We are now ready to state and establish the internal characterization of k-loop torsors as toral torsors.

Theorem 6.1. *Assume that $\mathbf{G}^0$ is reductive. Then $H^1_{toral}(R_n, \mathbf{G}) = H^1_{loop}(R_n, \mathbf{G})$.*

First we establish an auxiliary useful result.

Lemma 6.2. *1) Let $\mathfrak{H}$ be an R_n group of multiplicative type. Then for all $i \geq 1$ the natural abstract group homomorphisms*

$$H^i\big(\pi_1(R_n), \mathfrak{H}(\overline{R}_{n,\infty})\big) \to H^i(R_n, \mathfrak{H}) \to H^i(F_n, \mathfrak{H}).$$

are all isomorphisms.

2) Let $\mathbf{T}$ be a k–torus. Let $c \in Z^1\big(\pi_1(R_n), \mathbf{Aut}(\mathbf{T})(\overline{k})\big) \subset Z^1\big(\pi_1(R_n), \mathbf{Aut}(\mathbf{T})(\overline{R}_{n,\infty})\big)$ be a cocycle, and consider the twisted R_n–torus ${}_c\mathbf{T} = {}_c(\mathbf{T} \times_k R_n)$. Consider the natural maps

$$H^i\big(\pi_1(R_n), {}_c(\mathbf{T}(\overline{k})\big) \to H^i\big(\pi_1(R_n), {}_c\mathbf{T}(\overline{R}_{n,\infty})\big) \to H^i(R_n, {}_c\mathbf{T}) \to H^i(F_n, {}_c\mathbf{T})$$

Then.

(i) If $i = 1$ then the first group homomorphism is surjective and the last one an isomorphism.

(ii) If $i > 1$ then all the maps are group isomorphisms.

Proof. 1) The second isomorphism is proposition 3.4.3 of [GP2]. As for the first isomorphism we consider the Hochschild-Serre spectral sequence $H^p\big(\pi_1(R_n), H^q(\overline{R}_{n,\infty}, \mathfrak{H})\big) \implies H^{p+q}(R_n, \mathfrak{H})$. From the fact tha the group $H^p(\overline{R}_{n,\infty}, \mathfrak{H})$ is torsion for $p \geq 1$ it follows that the map $\varinjlim_m H^p(\overline{R}_{n,\infty}, {}_m\mathfrak{H}) \to H^p(\overline{R}_{n,\infty}, \mathfrak{H})$, where ${}_m\mathfrak{H}$ stands for the kernel of the "multiplication by m" map, is surjective. By *loc. cit.* cor. 3.3 $H^p(\overline{R}_{n,\infty}, {}_m\mathfrak{H})$ vanishes for all $m \geq 1$. Hence $H^p(\overline{R}_{n,\infty}, \mathfrak{H}) = 0$. The spectral sequence degenerates and yields the isomorphisms $H^i\big(\pi_1(R_n), \mathfrak{H}(\overline{R}_{n,\infty})\big) \simeq H^i(R_n, \mathfrak{H})$ for all $i \leq 1$.

2) We begin with an observation about the notation used in the statement of the Lemma. The subgroup $\mathbf{T}(\overline{k})$ of $\mathbf{T}(\overline{R}_{n,\infty})$ is stable under the (twisted) action of $\pi_1(R_n)$ on ${}_c\mathbf{T}(\overline{R}_{n,\infty})$. To view $\mathbf{T}(\overline{k})$ as a $\pi_1(R_n)$–module with this twisted action we write ${}_c(\mathbf{T}(\overline{k}))$.

The fact that the last two maps are isomorphism for all $i \geq 1$ is a special case (1). For the first map, we first analyse the $\pi_1(R_n)$–module $A = \mathbf{T}(\overline{R}_{n,\infty})/\mathbf{T}(\overline{k})$. We have

$$\begin{aligned} A &= \varinjlim_m \mathbf{T}(\overline{R}_{n,\infty}) \,/\, \mathbf{T}(\overline{k}) \\ &= \varinjlim_m (\widehat{\mathbf{T}})^0 \otimes_{\mathbb{Z}} \overline{R}_{n,m}^{\times} \,/ (\widehat{\mathbf{T}})^0 \otimes_{\mathbb{Z}} \overline{k}^{\times} \\ &= (\widehat{\mathbf{T}})^0 \otimes_{\mathbb{Z}} \varinjlim_m \overline{R}_{n,m}^{\times} \,/\, \overline{k}^{\times} \\ &= (\widehat{\mathbf{T}})^0 \otimes_{\mathbb{Z}} \varinjlim_m (\mathbb{Z}^n)_m \text{ (where } (\mathbb{Z}^n)_m = \mathbb{Z}^n) \\ &= (\widehat{\mathbf{T}})^0 \otimes_{\mathbb{Z}} \mathbb{Q}^n \end{aligned}$$

given that the transition map $(\mathbb{Z}^n)_m \to (\mathbb{Z}^n)_{md}$ is multiplication by d. It follows that A, hence also ${}_cA$, is uniquely divisible.

We consider the sequence of continuous $\pi_1(R_n)$–modules

$$1 \to {}_c\big(\mathbf{T}(\overline{k})\big) \to {}_c\mathbf{T}(\overline{R}_{n,\infty}) \to {}_cA \to 1. \tag{6.1}$$

¿From the fact that ${}_cA$ is uniquely divisible it follows that the group homomorphisms

$$H^i\big(\pi_1(R_n),\, {}_c(\mathbf{T}(\overline{k}))\big) \to H^i\big(\pi_1(R_n),\, {}_c\mathbf{T}(\overline{R}_{n,\infty})\big) \tag{6.2}$$

are surjective for all $i \geq 1$ and bijective if $i > 1$.

□

We can proceed now with the proof of Theorem 6.1.

Proof. Let us show first show that $H^1_{loop}(R_n, \mathbf{G}) \subset H^1_{toral}(R_n, \mathbf{G})$.

Case 1: $\mathbf{G} = \mathbf{Aut}(\mathbf{H}_0)$ *where* $\mathbf{H}_0$ *is a semisimple Chevalley k–group :* Let $\phi : \pi_1(R_n) \to \mathbf{Aut}(\mathbf{H}_0)(\overline{k})$ be a loop cocycle. Consider the twisted R–group ${}_\phi\mathbf{G}$.[18] Proposition 4.13 shows that the connected component of the identity $({}_\phi\mathbf{G})^0 = {}_\phi(\mathbf{G}^0)$ of ${}_\phi\mathbf{G}$ admits a maximal R-torus. Therefore ${}_\phi\mathbf{G}$ admits a maximal R-torus, hence ϕ defines a toral R-torsor.

Case 2: $\mathbf{G} = \mathbf{Aut}(\mathbf{H})$ *where* $\mathbf{H}$ *is a semisimple k–group :* Denote by $\mathbf{H}_0$ the Chevalley k–form of $\mathbf{H}$. There exists a cocycle $z : \mathrm{Gal}(k) \to \mathbf{G}(\overline{k})$ such that $\mathbf{H}$ is isomorphic to the twisted k–group ${}_z\mathbf{H}_0$. We can assume then that $\mathbf{H} = {}_z\mathbf{H}_0$ and $\mathbf{G} = {}_z\mathbf{Aut}(\mathbf{H}_0)$. The torsion bijection $\tau_z : H^1(R, \mathbf{G}) = H^1(R, {}_z\mathbf{Aut}(\mathbf{H}_0)) \xrightarrow{\sim} H^1(R, \mathbf{Aut}(H_0))$ exchanges loop classes (resp. toral classes) according to Remark 3.3. Case 1 then yields $H^1_{loop}(R_n, \mathbf{G}) \subset H^1_{toral}(R_n, \mathbf{G})$.

[18]We recall, for the last time, that ${}_\phi\mathbf{G}$ is short hand notation for ${}_\phi(\mathbf{G}_R)$.

General case. The k–group acts by conjugacy on $\mathbf{G}^0$, its center $\mathbf{Z}(\mathbf{G}^0)$ and then on its adjoint quotient $\mathbf{G}^0_{ad}$. Denote by $f : \mathbf{G} \to \mathbf{Aut}(\mathbf{G}^0_{ad})$ this action. Let $\phi : \pi_1(R_n) \to \mathbf{G}(\overline{k})$ be a loop cocycle. We have to show that the twisted R–group scheme ${}_\phi\mathbf{G}$ admits a maximal torus. Equivalently, we need to show that $({}_\phi\mathbf{G})^0 =_\phi (\mathbf{G}^0)$ admits a maximal torus which is in turn equivalent to the fact that $({}_{f_*\phi}\mathbf{G})^0_{ad} = {}_{f_*\phi}(\mathbf{G}^0_{ad})$ admits a maximal torus [SGA3, XII.4.7]. But $f_*\phi$ is a loop cocycle for $\mathbf{Aut}(\mathbf{G}^0_{ad})$, so defines a toral R–torsor under $\mathbf{Aut}(\mathbf{G}^0)$ according to Case 2. Thus ${}_{f_*\phi}(\mathbf{G}^0_{ad})$ admits a maximal torus as desired.

To establish the reverse inclusion we consider the quotient group $\boldsymbol{\nu} = \mathbf{G}/\mathbf{G}^0$, which is finite and étale k–group. In particular $\boldsymbol{\nu} \times_k \overline{k}$ is constant and finite, and one can easily see as a consequence that the natural map $\boldsymbol{\nu}(\overline{k}) \to \boldsymbol{\nu}(\overline{R}_{n,\infty})$ is an isomorphism. We first establish the result for tori and then the general case.

$\mathbf{G}^0$ *is a torus* $\mathbf{T}$*:* We again appeal to the isotriviality theorem of [GP1] to see that $H^1(\pi_1(R_n), \mathbf{G}(\overline{R}_{n,\infty})) \xrightarrow{\sim} H^1(R_n, \mathbf{G})$. We consider the following commutative diagram of continuous $\pi_1(R_n)$–groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbf{T}(\overline{k}) & \longrightarrow & \mathbf{G}(\overline{k}) & \longrightarrow & \boldsymbol{\nu}(\overline{k}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \mathbf{T}(\overline{R}_{n,\infty}) & \longrightarrow & \mathbf{G}(\overline{R}_{n,\infty}) & \longrightarrow & \boldsymbol{\nu}(\overline{R}_{n,\infty}) & \longrightarrow & 1.
\end{array}$$

This gives rise to an exact sequence of pointed sets

$$\begin{array}{ccccccc}
1 & \longrightarrow & H^1\big(\pi_1(R_n), \mathbf{T}(\overline{k})\big) & \longrightarrow & H^1\big(\pi_1(R_n), \mathbf{G}(\overline{k})\big) & \longrightarrow & H^1\big(\pi_1(R_n), \boldsymbol{\nu}(\overline{k})\big) \\
 & & \downarrow & & \downarrow & & \| \\
1 & \longrightarrow & H^1\big(\pi_1(R_n), \mathbf{T}(\overline{R}_{n,\infty})\big) & \longrightarrow & H^1\big(\pi_1(R_n), \mathbf{G}(\overline{R}_{n,\infty})\big) & \longrightarrow & H^1\big(\pi_1(R_n), \boldsymbol{\nu}(\overline{R}_{n,\infty})\big) .
\end{array}$$

We are given a cocycle $z \in Z^1\big(\pi_1(R_n), \mathbf{G}(\overline{R}_{n,\infty})\big)$. Denote by c the image of z in $Z^1\big(\pi_1(R_n), \boldsymbol{\nu}(\overline{k})\big) = Z^1\big(\pi_1(R_n), \boldsymbol{\nu}(\overline{R}_{n,\infty})\big)$ under the bottom map. Under the top map, the obstruction to lifting $[c]$ to $H^1\big(\pi_1(R_n), \mathbf{G}(\overline{k})\big)$ is given by a class $\Delta([c]) \in H^2\big(\pi_1(R_n), {}_c(\mathbf{T}(\overline{k}))\big)$ [Se, §I.5.6]. This class vanishes in $H^2\big(\pi_1(R_n), {}_c\mathbf{T}(\overline{R}_{n,\infty}))\big)$, so Lemma 6.2 shows that $\Delta([c]) = 0$. Hence c lifts to a loop cocycle $a \in Z^1\big(\pi_1(R_n), \mathbf{G}(\overline{k})\big)$.

By twisting by a we obtain the following commutative diagram of pointed sets

$$\begin{tikzcd}
1 \arrow[r] & H^1\Big(\pi_1(R_n),\, {}_a\big(\mathbf{T}(\overline{k})\big)\Big) \arrow[r] \arrow[d] & H^1\Big(\pi_1(R_n),\, {}_a\big(\mathbf{G}(\overline{k})\big)\Big) \arrow[r] \arrow[d] & H^1\Big(\pi_1(R_n),\, {}_c\big(\boldsymbol{\nu}(\overline{k})\big)\Big) \arrow[d, equal] \\
1 \arrow[r] & H^1\Big(\pi_1(R_n),\, {}_a\big(\mathbf{T}(\overline{R}_{n,\infty})\big)\Big) \arrow[r] \arrow[d] & H^1\Big(\pi_1(R_n), {}_a\big(\mathbf{G}(\overline{R}_{n,\infty})\big)\Big) \arrow[r] & H^1\Big(\pi_1(R_n),\, {}_c\big(\boldsymbol{\nu}(\overline{R}_{n,\infty})\big)\Big) \\
 & 1 & &
\end{tikzcd}$$

where the surjectivity of the left map comes from Lemma 6.2, ${}_a(\mathbf{G}(\overline{k}))$ denotes $\mathbf{G}(\overline{k})$ as a $\pi_1(R_n)$–submodule of ${}_a\mathbf{G}(\overline{R}_{n,\infty})$, and similarly for ${}_c\big(\boldsymbol{\nu})(\overline{k})\big) = {}_c\big(\boldsymbol{\nu}(\overline{R}_{n,\infty})\big)$. We consider the torsion map

$$\tau_a : H^1\big(\pi_1(R_n), {}_a\mathbf{G}(\overline{R}_{n,\infty})\big) \xrightarrow{\sim} H^1\big(\pi_1(R_n), \mathbf{G}(\overline{R}_{n,\infty})\big).$$

Then

$$\tau_a^{-1}([z]) \in \ker\Big(H^1\big(\pi_1(R_n),\, {}_a\mathbf{G}(\overline{R}_{n,\infty})\big) \to H^1\big(\pi_1(R_n),\, {}_c\big(\boldsymbol{\nu}(\overline{R}_{n,\infty})\big)\big)\Big).$$

The diagram above shows that $\tau_a^{-1}([z])$ comes from $H^1\big(\pi_1(R_n),\, {}_a(\mathbf{G}(\overline{k}))\big)$, hence $[z]$ comes from $H^1\big(\pi_1(R_n), \mathbf{G}(\overline{k})\big)$ as desired. We conclude that $H^1(R_n, \mathbf{G})$ is covered by loop torsors.

General case : Let $\mathbf{T}$ be a maximal torus of $\mathbf{G}$. Consider the commutative diagram

$$\begin{tikzcd}
H^1_{loop}\big(R_n, \mathbf{N}_{\mathbf{G}}(T)\big) \arrow[r] \arrow[d] & H^1\big(R_n, \mathbf{N}_{\mathbf{G}}(\mathbf{T})\big) \arrow[d] \\
H^1_{loop}(R_n, \mathbf{G}) \arrow[r] & H^1_{toral}(R_n, \mathbf{G}).
\end{tikzcd}$$

The right vertical map is surjective according to Lemma 3.13.1. The top horizontal map is surjective by the previous case. We conclude that the bottom horizontal map is surjective as desired. □

Corollary 6.3. *Let $\mathfrak{G}$ be a reductive R_n–group. Then $\mathfrak{G}$ is loop reductive if and only if $\mathfrak{G}$ admits a maximal torus.*

Proof. Let $\mathbf{G}$ be the Chevalley k–form of $\mathfrak{G}$. To $\mathfrak{G}$ corresponds a class $[\mathfrak{E}] \in H^1\big(R_n, \mathbf{Aut}(\mathbf{G})\big)$. When $\mathbf{G}$ is semisimple, $\mathbf{Aut}(\mathbf{G})$ is an affine algebraic k–group and the Corollary follows from $H^1_{toral}\big(R_n, \mathbf{Aut}(\mathbf{G})\big) = H^1_{loop}\big(R_n, \mathbf{Aut}(\mathbf{G})\big)$.

We deal now with the general case. We already know by 4.13 that every loop reductive group is toral. Conversely assume that $\mathfrak{G}$ admits a maximal torus. Consider the exact sequence of smooth k–groups [XXIV.1.3.(iii)]

$$1 \to \mathbf{G}_{ad} \to \mathbf{Aut}(\mathbf{G}) \xrightarrow{p} \mathbf{Out}(\mathbf{G}) \to 1,$$

where $\mathbf{G}_{ad}$ is the adjoint group of $\mathbf{G}$ and $\mathbf{Out}(\mathbf{G})$ is a constant k–group. Since R_n is a noetherian normal domain, we know that R_n–torsors under $\mathbf{Out}(\mathbf{G})$ are isotrivial [X.6]. Furthermore by [SGA1, XI §5]

$$\mathrm{Hom}_{ct}\big(\pi_1(R_n), \mathbf{Out}(\mathbf{G})(k)\big)/\sim \;\xrightarrow{\sim}\; H^1\big(R_n, \mathbf{Out}(\mathbf{G})\big).$$

So $p_*[\mathfrak{E}]$ is given by a continous homomorphism $\pi_1(R_n) \to \mathbf{Out}(\mathbf{G})(k)$ whose image we denote by $\mathbf{Out}(\mathbf{G})^\sharp$. This is a finite group so that $\mathbf{Aut}(\mathbf{G})^\sharp := p^{-1}(\mathbf{Out}(\mathbf{G})^\sharp)$ is an affine algebraic k–group. We consider the square of pointed sets

$$\begin{array}{ccc} H^1\big(R_n, \mathbf{Aut}(\mathbf{G})^\sharp\big) & \xrightarrow{p_*^\sharp} & H^1\big(R_n, \mathbf{Out}(\mathbf{G})^\sharp\big) \\ \big\downarrow & & \big\downarrow \\ H^1\big(R_n, \mathbf{Aut}(\mathbf{G})\big) & \xrightarrow{p_*} & H^1\big(R_n, \mathbf{Out}(\mathbf{G})\big). \end{array}$$

Since $\mathbf{Aut}(\mathbf{G})/\mathbf{Aut}(\mathbf{G})^\sharp = \mathbf{Out}(\mathbf{G})/\mathbf{Out}(\mathbf{G})^\sharp$, this square is cartesian as can be seen by using the criterion of reduction of a torsor to a subgroup [Gi, III.3.2.1]. By construction, $[p_*\mathfrak{E}]$ comes from $H^1\big(R_n, \mathbf{Out}(\mathbf{G})^\sharp\big)$, hence $[\mathfrak{E}]$ comes from a class $[\mathfrak{F}] \in H^1\big(R_n, \mathbf{Aut}(\mathbf{G})^\sharp\big)$. Our assumption is that the R_n–group $\mathfrak{G} = {}_{\mathfrak{E}}\mathbf{G} = {}_{\mathfrak{F}}\mathbf{G}$ contains a maximal torus, so $\mathfrak{G}_{ad} = {}_{\mathfrak{F}}\mathbf{G}_{ad}$ contains a maximal torus and ${}_{\mathfrak{F}}\big(\mathbf{Aut}(\mathbf{G})^\sharp\big)$ contains a maximal torus. In other words, $\mathfrak{F}$ is a toral R_n–torsor under $\mathbf{Aut}(\mathbf{G})^\sharp$. ¿From the equality $H^1_{toral}\big(R_n, \mathbf{Aut}(\mathbf{G})^\sharp\big) = H^1_{loop}\big(R_n, \mathbf{Aut}(\mathbf{G})^\sharp\big)$, it follows that that $\mathfrak{F}$ is a loop torsor under $\mathbf{Aut}(\mathbf{G})^\sharp$. By applying the change of groups $\mathbf{Aut}(\mathbf{G})^\sharp \to \mathbf{Aut}(\mathbf{G})$, we conclude that $\mathfrak{E}$ is a loop torsor under $\mathbf{Aut}(\mathbf{G})$, hence that $\mathfrak{G}$ is loop reductive. □

Lemma 3.13.2 yields the following fact.

Corollary 6.4. *Let $1 \to \mathbf{S} \to \mathbf{G}' \xrightarrow{p} \mathbf{G} \to 1$ be a central extension of $\mathbf{G}$ by a k–group $\mathbf{S}$ of multiplicative type. Then the diagram*

$$\begin{array}{ccc} H^1_{loop}(R_n, \mathbf{G}') & \subset & H^1(R_n, \mathbf{G}') \\ {\scriptstyle p_*}\big\downarrow & & {\scriptstyle p_*}\big\downarrow \\ H^1_{loop}(R_n, \mathbf{G}) & \subset & H^1(R_n, \mathbf{G}) \end{array}$$

is cartesian.

Remark 6.5. For a k–group $\mathbf{G}$ satisfying the condition of Corollary 3.16, one can prove in a simpler way that toral $\mathbf{G}$–torsors over R_n are loop torsors by reducing to the case of a finite étale group.

Remark 6.6. Given an integer $d \geq 2$, the Margaux algebra (both the Azumaya and Lie versions) [GP2, 3.22 and example 5.7] provides an example of a $\mathbf{PGL}_d$-torsor over $\mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$ which is not a loop torsor. The underlying $\mathbf{PGL}_d$-torsor is therefore not toral. This means that the Margaux Azumaya algebra does not contain any (commutative) étale $\mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$-subalgebra of rank d, and that the Margaux Lie algebra, viewed as a Lie algebra over $\mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$, does not contain any Cartan subalgebras (in the sense of [SGA3]).

Remark 6.7. More generally, for each each positive integer d, we have $H^1_{toral}(R_n[x_1, ..., x_d], \mathbf{G}) = H^1_{loop}(R_n[x_1, ..., x_d], \mathbf{G})$. Since $\pi_1(R_n[x_1, ..., x_d]) \simeq \pi_1(R_n)$ and $\mathbf{N_G}(\mathbf{T})(S) = \mathbf{N_G}(\mathbf{T})(S[x_1, ..., x_d])$ for every finite étale covering S of R_n, the proof we have given works just the same in this case.

6.2 Applications to (algebraic) Laurent series.

Let $F_n = k((t_1))((t_2))...((t_n))$. In an analogous fashion to what we did in the case of R_n we define $F_{n,m} = k((t_1^{\frac{1}{m}}))((t_2^{\frac{1}{m}}))...((t_n^{\frac{1}{m}}))$ and $F_{n,\infty} = \varinjlim F_{n,m}$.

Remark 6.8. (a) If $\tilde{k}$ is a field extension of k the natural map $\tilde{k} \otimes_k F_{n,m} \to \tilde{k}((t_1^{\frac{1}{m}}))((t_2^{\frac{1}{m}}))...((t_n^{\frac{1}{m}}))$ is injective. If the extension is *finite*, then this map is an isomorphism. We will find it convenient (assuming that the field $\tilde{k}$ is fixed in our discussion) to denote $\tilde{k}((t_1^{\frac{1}{m}}))((t_2^{\frac{1}{m}}))...((t_n^{\frac{1}{m}}))$ simply by $\tilde{F}_{n,m}$.

(b) The field $\varinjlim \overline{k}((t_1^{\frac{1}{m}}))((t_2^{\frac{1}{m}}))...((t_n^{\frac{1}{m}}))$ is algebraically closed. We will denote the algebraic closure of F_n (resp. $F_{n,m}$, $F_{n,\infty}$) in this field by $\overline{F_n}$ (resp. $\overline{F_{n,m}}$, $\overline{F_{n,\infty}}$). As mentioned in (a) we have a natural injective ring homomorphsm $\overline{k} \otimes_k F_{n,\infty} \to \overline{F_n}$.

(c) There is a natural group morphism $\pi_1(R_n) \to \mathrm{Gal}(F_n)$ given by considering the Galois extensions $\tilde{R}_{n,m} = \tilde{k} \otimes_k R_{n,m}$ of R_n and $\tilde{F}_{n,m}$ of F_n respectively, where $\tilde{k} \subset \overline{k}$ is a finite Galois extension of k containing all m-roots of unity. These homomorphisms are in fact isomorphisms.[19] For by applying successively the structure theorem for local fields [GMS] §7.1 p. 17, we have $\mathrm{Gal}(F_n) = {}_\infty\boldsymbol{\mu}^n(\overline{k}) \rtimes \mathrm{Gal}(k)$. This means that

$$\mathrm{Gal}(F_n) = \varprojlim \mathrm{Gal}\big(\tilde{k}((t_1^{\frac{1}{m}}))((t_2^{\frac{1}{m}}))...((t_n^{\frac{1}{m}}))/F_n\big)$$

[19]If k is algebraically closed this was proved in [GP3] cor 2.14.

for m running over all integers and $\tilde{k}$ running over all finite Galois extensions of k inside $\overline{k}$ containing a primitive m–root of unity. Since at each step we have an isomorphism

$$\mathrm{Gal}\big(\tilde{k} \otimes R_{n,m}/R_n\big) \cong \mathrm{Gal}\big(\tilde{k}((t_1^{\frac{1}{m}}))((t_2^{\frac{1}{m}}))...((t_n^{\frac{1}{m}}))/F_n\big) \cong \boldsymbol{\mu}_m^n(\tilde{k}) \rtimes \mathrm{Gal}(\tilde{k}/k),$$

we conclude that $\pi_1(R_n) \cong \mathrm{Gal}(F_n)$.

(d) It follows from (c) that the base change $R_n \to F_n$ induces an equivalence of categories between finite étale coverings of R_n and finite étale coverings of F_n. Furthermore, if $\mathfrak{E}/R_n$ is a finite étale covering of R_n, we have $\mathfrak{E}(R_n) = \mathfrak{E}(F_n)$. Indeed, $\mathfrak{E}$ is split by some Galois covering $\tilde{R}_{n,m} = \tilde{k} \otimes_k R_{n,m}$ and $\mathfrak{E}(R_n) = \mathfrak{E}(\tilde{R}_{n,m})^{\mathrm{Gal}(\tilde{R}_{n,m}/R_n)} = \mathfrak{E}(\tilde{F}_{n,m})^{\mathrm{Gal}(\tilde{F}_{n,m}/F_n)} = \mathfrak{E}(F_n)$.

Proposition 6.9. *The canonical map*

$$H^1_{loop}(R_n, \mathbf{G}) \to H^1(F_n, \mathbf{G})$$

is surjective.

Proof. We henceforth identify $\pi_1(R_n)$ with $\mathrm{Gal}(F_n)$ as described in Remark 6.8(c). The proof is very similar to that of Theorem 6.1, and we maintain the notation therein. Again we proceed in two steps.

First case: $\mathbf{G}^0$ *is a torus* $\mathbf{T}$*:* We consider the following commutative diagram of continuous $\pi_1(R_n)$–groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbf{T}(\overline{k}) & \longrightarrow & \mathbf{G}(\overline{k}) & \longrightarrow & \boldsymbol{\nu}(\overline{k}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & || & & \\
1 & \longrightarrow & \mathbf{T}(\overline{F_n}) & \longrightarrow & \mathbf{G}(\overline{F_n}) & \longrightarrow & \boldsymbol{\nu}(\overline{F_n}) & \longrightarrow & 1.
\end{array}$$

This gives rise to an exact sequence of pointed sets

$$\begin{array}{ccccccc}
1 & \longrightarrow & H^1\big(\pi_1(R_n), \mathbf{T}(\overline{k})\big) & \longrightarrow & H^1\big(\pi_1(R_n), \mathbf{G}(\overline{k})\big) & \longrightarrow & H^1\big(\pi_1(R_n), \boldsymbol{\nu}(\overline{k})\big) \\
 & & \downarrow & & \downarrow & & || \\
1 & \longrightarrow & H^1\big(F_n, \mathbf{T}\big) & \longrightarrow & H^1\big(F_n, \mathbf{G}\big) & \longrightarrow & H^1\big(F_n, \boldsymbol{\nu}\big).
\end{array}$$

We are given a cocycle $z \in Z^1\big(\mathrm{Gal}(F_n), \mathbf{G}(\overline{F_n})\big) = Z^1\big(\pi_1(R_n), \mathbf{G}(\overline{F_n})\big)$, and consider its image $c \in Z^1\big(\pi_1(R_n), \boldsymbol{\nu}(\overline{F_n})\big)$. By reasoning as in Theorem 6.1 we see that $[z]$ comes from $H^1\big(\pi_1(R_n), \mathbf{G}(\overline{k})\big)$ as desired. We conclude that $H^1(F_n, \mathbf{G})$ is covered by k-loop torsors.

General case: Let $\mathbf{T}$ be a maximal torus of $\mathbf{G}$.

$$\begin{array}{ccc} H^1_{loop}\big(R_n, \mathbf{N}_{\mathbf{G}}(T)\big) & \longrightarrow & H^1\big(F_n, \mathbf{N}_{\mathbf{G}}(\mathbf{T})\big) \\ \big\downarrow & & \big\downarrow \\ H^1_{loop}(R_n, \mathbf{G}) & \longrightarrow & H^1(F_n, \mathbf{G}). \end{array}$$

The reasoning is again identical to the one used in Theorem 6.1. □

7 Isotropy of loop torsors

As before $\mathbf{G}$ denotes a linear algebraic group over a field k of characteristic zero. R_n and $\pi_1(R_n)$ are as in the previous section.

7.1 Fixed point statements

Let $\eta : \pi_1(R_n) \to \mathbf{G}(\overline{k})$ be a continuous cocycle. Consider as before a Galois extension $\tilde{R}_{n,m} = \tilde{k} \otimes_k R_{n,m}$ of R_n where $\tilde{k} \subset \overline{k}$ is a finite Galois extension of k containing all m–roots of unity in $\overline{k}$, chosen so that our cocycle η factors through the Galois group

$$\tilde{\Gamma}_{n,m} = \mathrm{Gal}(\tilde{R}_{n,m}/R_n) \cong \boldsymbol{\mu}_m^n(\tilde{k}) \rtimes \mathrm{Gal}(\tilde{k}/k) \tag{7.1}$$

We assume henceforth that $\mathbf{G}$ acts on a k–scheme $\mathbf{Y}$. The Galois group $\tilde{\Gamma}_{n,m}$ acts naturally on $\mathbf{Y}(\tilde{R}_{n,m})$, and we denote this action by $\gamma : y \mapsto {}^{\gamma}y$. By means of η we get a twisted action of $\tilde{\Gamma}_{n,m}$ on $\mathbf{Y}(\tilde{R}_{n,m})$ which we denote by $\gamma : y \mapsto {}^{\gamma'}y$ where

$${}^{\gamma'}y = \eta_{\gamma}\, .{}^{\gamma}y \tag{7.2}$$

By Galois descent (7.2) leads to a twisted form of the R_n–scheme $\mathbf{Y}_{R_n}$. One knows that this twisted form is up to isomorphism independent of the Galois extension $\tilde{R}_{n,m}$ chosen through which η factors. We will denote this twisted form by ${}_{\eta}\mathbf{Y}_{R_n}$, or simply by ${}_{\eta}\mathbf{Y}$ following the conventions that have been previously mentioned regarding this matter.

Let (z, η^{geo}) be the couple associated to η according to Lemma 3.7. Thus $z \in Z^1\big(\,\mathrm{Gal}(k), \mathbf{G}(\overline{k})\big)$ and $\eta^{geo} \in \mathrm{Hom}_{k-gp}({}_{\infty}\boldsymbol{\mu}^n, {}_{z}\mathbf{G})$ by taking into account Lemma 3.10. By means of z we construct a twisted form ${}_{z}\mathbf{Y}$ of the k–scheme $\mathbf{Y}$ which comes equipped with an action of ${}_{z}\mathbf{G}$. Via η^{geo}, this defines an algebraic action of the affine k-group ${}_{\infty}\boldsymbol{\mu}^n$ on ${}_{z}\mathbf{Y}$. At the level of $\overline{k}$–points of ${}_{\infty}\boldsymbol{\mu}^n$, the action is given by

$$\widehat{n}.y = \eta^{geo}(\widehat{n}).y \tag{7.3}$$

for all $\widehat{n} \in {}_\infty\boldsymbol{\mu}^n(\overline{k}) = \varprojlim_m \boldsymbol{\mu}_m^n(\overline{k})$ and $y \in {}_z\mathbf{Y}(\overline{k})$. We denote by $({}_z\mathbf{Y})^{\eta^{geo}}$ the closed fixed point subscheme for the action of ${}_\infty\boldsymbol{\mu}^n$ (see [DG] II §1 prop. 3.6.d). We have

$$({}_z\mathbf{Y})^{\eta^{geo}}(\overline{k}) = \Big\{ y \in {}_z\mathbf{Y}(\overline{k}) = \mathbf{Y}(\overline{k}) \mid y = \eta^{geo}(\widehat{n}).y \quad \forall \widehat{n} \in {}_\infty\boldsymbol{\mu}^n(\overline{k}) \Big\}$$

and in terms of rational points
(7.4)

$$({}_z\mathbf{Y})^{\eta^{geo}}(k) = {}_z\mathbf{Y}(k) \cap ({}_z\mathbf{Y})^{\eta^{geo}}(\overline{k}) = \Big\{ y \in {}_z\mathbf{Y}(k) \mid y = \eta^{geo}(\widehat{n}).y \quad \forall \widehat{n} \in {}_\infty\boldsymbol{\mu}^n(\overline{k}) \Big\}.$$

where we recall that

$${}_z\mathbf{Y}(k) = \Big\{ y \in \mathbf{Y}(\overline{k}) \mid y = z_\gamma.{}^\gamma y \quad \forall \gamma \in \mathrm{Gal}(k) \Big\}.$$

Theorem 7.1. *1. Let $\mathbf{G}$ act on $\mathbf{Y}$ as above, and assume that $\mathbf{Y}$ is projective (i.e. a closed subscheme in $\mathbb{P}_k^n$). Let $\eta : \pi_1(R_n) \to \mathbf{G}(\overline{k})$ be a (continuous) cocycle, and ${}_\eta\mathbf{Y}$ be the corresponding twisted form of $\mathbf{Y}_{R_n}$. The following are equivalent:*

(a) $({}_\eta\mathbf{Y})(R_n) \neq \emptyset$,

(b) $({}_\eta\mathbf{Y})(K_n) \neq \emptyset$,

(c) $({}_\eta\mathbf{Y})(F_n) \neq \emptyset$,

(d) $({}_z\mathbf{Y})^{\eta^{geo}}(k) \neq \emptyset$.

2. Let $\mathbf{S}$ be a closed k–subgroup of $\mathbf{G}$. Let $\mathbf{Y}$ be a smooth $\mathbf{G}$–equivariant compactification of $\mathbf{G}/\mathbf{S}$ (i.e., $\mathbf{Y}$ is projective k–variety that contains $\mathbf{G}/\mathbf{S}$ as an open dense $\mathbf{G}$-subvariety). Then the following are equivalent:

(a) $[\eta]_{K_n} \in \mathrm{Im}\big(H^1(K_n, \mathbf{S}) \to H^1(K_n, \mathbf{G})\big)$,

(b) $[\eta]_{F_n} \in \mathrm{Im}\big(H^1(F_n, \mathbf{S}) \to H^1(F_n, \mathbf{G})\big)$,

(c) $({}_z\mathbf{Y})^{\eta^{geo}}(k) \neq \emptyset$.

Proof. (1) Again we twist the action $\mathbf{G} \times \mathbf{Y} \to \mathbf{Y}$ by z to obtain an action ${}_z\mathbf{G} \times {}_z\mathbf{Y} \to {}_z\mathbf{Y}$. Lemma 3.8 enables us to assume without loss of generality that z is the trivial cocycle. We are thus left to deal with a k–homomorphism $\eta^{geo} : {}_\infty\boldsymbol{\mu}^n \to \mathbf{G}$ which factors at the finite level through $\boldsymbol{\mu}_m^n \to \mathbf{G}$ for m large enough. This allows us to reason by means of a suitable covering $\tilde{R}_{n,m}$ as in (7.1).
$(a) \implies (b) \implies (c)$ are obtained by applying the base change $R_n \subset K_n \subset F_n$.
$(c) \Longrightarrow (d)$: Each $\gamma \in \tilde{\Gamma}_{n,m}$ induces an automorphism of $\tilde{R}_{n,m} \otimes_{R_n} F_n \simeq F_{n,m} \otimes_k \tilde{k} = \tilde{F}_{n,m}$ which we also denote by γ (even though the notation $\gamma \otimes 1$ would be more

accurate.) Since $\tilde{R}_{n,m}$ trivializes ${}_\eta\mathbf{Y}$, the Galois extension $\tilde{F}_{n,m}$ of F_n (whose Galois group we identify with $\tilde{\Gamma}_{n,m}$) splits ${}_\eta\mathbf{Y}_{F_n}$. By Galois descent

$$ {}_\eta\mathbf{Y}(F_n) = \Big\{ y \in \mathbf{Y}(\tilde{F}_{n,m}) \mid \eta_\gamma \, . {}^\gamma y = y \ \ \forall \gamma \in \tilde{\Gamma}_{n,m} \Big\}. $$

Since z is trivial, this last equality reads

$$ {}_\eta\mathbf{Y}(F_n) = \Big\{ y \in \mathbf{Y}(\tilde{F}_{n,m}) \mid \eta^{geo}(\overline{\gamma}) \, . {}^\gamma y = y \ \ \forall \gamma \in \tilde{\Gamma}_{n,m} \Big\}. $$

where $\overline{\gamma}$ is the image of γ under the map $\tilde{\Gamma}_{n,m} \to \boldsymbol{\mu}_m^n(\tilde{k})$ given by (7.1). Hence we have ${}_\eta\mathbf{Y}(F_n) \subset \mathbf{Y}(F_{n,m})$ and

$$ {}_\eta\mathbf{Y}(F_n) = \Big\{ y \in \mathbf{Y}(F_{n,m}) \mid \eta^{geo}(\gamma) \, . {}^\gamma y = y \ \ \forall \gamma \in \boldsymbol{\mu}_m^n(\overline{k}) \Big\}. $$

Since $\mathbf{Y}$ is proper over k, we have

$$ {}_\eta\mathbf{Y}(F_n) = \Big\{ y \in \mathbf{Y}(F_{n-1,m}[[t_n^{\frac{1}{m}}]]) \mid \eta^{geo}(\gamma). {}^\gamma y = y \ \ \gamma \in \boldsymbol{\mu}_m^n(\overline{k}) \Big\}. $$

Our hypothesis is that this last set is not empty. By specializing at $t_n = 0$, we get that

$$ \Big\{ y \in \mathbf{Y}(F_{n-1,m}) \mid \eta^{geo}(\gamma). {}^\gamma y = y \ \ \forall \gamma \in \boldsymbol{\mu}_m^n(\overline{k}) \Big\} \neq \emptyset. \tag{7.5} $$

We write now $\boldsymbol{\mu}_m^n(\overline{k}) = \boldsymbol{\mu}_m^{n-1}(\overline{k}) \times \boldsymbol{\mu}_m(\overline{k})$ which provides a decomposing of η^{geo} into two k-homomorphisms $\eta'^{geo} : \boldsymbol{\mu}_m^{n-1} \to \mathbf{G}$ and ${\eta_n}^{geo} : \boldsymbol{\mu}_m \to \mathbf{G}$. We define $\eta' = (1, \eta'^{geo})$, $\eta_n = (1, {\eta_n}^{geo})$ and

$$ \mathbf{Y}' := \mathbf{Y}^{\eta_n^{geo}}. $$

By [DG] II §1 prop. 3.6.(d) we know that $\mathbf{Y}'$ is a closed subscheme of $\mathbf{Y}$, hence a projective k–variety. Observe that $\boldsymbol{\mu}_m^{n-1}$ acts on $\mathbf{Y}'$.

Claim 7.2. *The set (7.5) is included in ${}_{\eta'}\mathbf{Y}'(F_{n-1})$.*[20]

To look at the invariants under the action of $\boldsymbol{\mu}_m^n(\overline{k})$, we first look at the invariants under the last factor $\boldsymbol{\mu}_m(\overline{k})$, and then the first $(n-1)$-factor $\boldsymbol{\mu}_m^{n-1}(\overline{k})$ By restricting the condition to elements of the form $(1, \gamma_n)$ for $\gamma_n \in \boldsymbol{\mu}_m(\overline{k})$, we see that our set is included in

$$ \Big\{ y \in \mathbf{Y}(F_{n-1,m}) \mid \eta_n^{geo}(\gamma_n).y = y \ \ \forall \gamma_n \in \boldsymbol{\mu}_m(\overline{k}) \Big\} $$

[20]This inclusion is in fact an equality, but this stronger statement is not needed.

because $\boldsymbol{\mu}_m(\overline{k})$ acts trivially on $F_{n-1,m}$. By identity (7.4) applied to the base change of η_n^{geo} to the field F_{n-1}, this is nothing but $\mathbf{Y}^{\eta_m^{geo}}(F_{n-1,m})$. Looking now at the invariant condition for the elements of the form $(\gamma', 1)$ for $\gamma' \in \boldsymbol{\mu}_m^{n-1}(\overline{k})$, it follows that

$$\Big\{ y \in \mathbf{Y}(F_{n-1,m}) \mid \eta^{geo}(\gamma).^{\gamma}y = y \;\; \forall \gamma \in \boldsymbol{\mu}_m^n(\overline{k}) \Big\}$$

$$\subset \Big\{ y \in \mathbf{Y}^{\eta_m^{geo}}(F_{n-1,m}) \mid \eta'^{geo}(\gamma').^{\gamma'}y = y \;\; \forall \gamma' \in \boldsymbol{\mu}_m^{n-1}(\overline{k}) \Big\} = {}_{\eta'}\mathbf{Y}'(F_{n-1}).$$

By induction on n, we get that inside $({}_{\eta'}\mathbf{Y}')(F_{n-1})$ we have $\mathbf{Y}'^{\eta'_{geo}}(k) \neq \emptyset$. Thus $\mathbf{Y}(k)^{\eta^{geo}} \neq \emptyset$ as desired.
$(d) \Longrightarrow (a)$: Since

$$({}_{\eta}\mathbf{Y})(R_n) = \Big\{ y \in \mathbf{Y}(\tilde{R}_{n,m}) \mid \eta^{geo}(\overline{\gamma})\,.^{\gamma}y = y \;\; \forall \gamma \in \tilde{\Gamma}_{n,m} \Big\},$$

the inclusion $\mathbf{Y}(k) \subset \mathbf{Y}(\tilde{R}_{n,m})$ yields the inclusion

$$(\mathbf{Y}^{\eta^{geo}})(k) \subset ({}_{\eta}\mathbf{Y})(R_n).$$

Thus if $(\mathbf{Y}^{\eta^{geo}})(k) \neq \emptyset$, then $({}_{\eta}\mathbf{Y})(R_n) \neq \emptyset$.
(2) The quotient $\mathbf{G}/\mathbf{S}$ is representable by Chevalley's theorem [DG, §III.3.5]. The only non trivial implication is $(c) \Longrightarrow (a)$. Let $\mathbf{X} = (\mathbf{G}/\mathbf{S}) \times_k R_n$. By (1), we have ${}_{\eta}\mathbf{Y}(K_n) \neq \emptyset$. In other words, the K_n-homogeneous space ${}_{\eta}\mathbf{X}$ under ${}_{\eta}\mathbf{G}$ has a K_n-rational point on the compactification ${}_{\eta}\mathbf{Y}$. By Florence's theorem [F], ${}_{\eta}\mathbf{X}(K_n) \neq \emptyset$, hence (a). □

7.2 Case of flag varieties

The k–group $\mathbf{G}^0/R_u(\mathbf{G})$ is reductive. Let $\mathbf{T}$ be a maximal k–torus of $\mathbf{G}^0/R_u(\mathbf{G})$. This data permits to choose a basis Δ the root system $\Phi(\mathbf{G}^0/R_u(\mathbf{G}) \times_k \overline{k}, \mathbf{T} \times_k \overline{k})$ or in other words to choose a Borel subgroup $\mathbf{B}$ of the $\overline{k}$–group $\mathbf{G}^0/R_u(\mathbf{G}) \times_k \overline{k}$. It is well known that there is a one-to-one correspondence between the subsets of Δ and the parabolic subgroups of $\mathbf{G}^0 \times_k \overline{k}$ containing $\mathbf{B}$, which is provided by the standard parabolic subgroups $(\mathbf{P}_I)_{I \subset \Delta}$ of $\big(\mathbf{G}^0/R_u(\mathbf{G})\big) \times_k \overline{k}$ [Bo, §21.11]. We have $\mathbf{P}_\Delta = \big(\mathbf{G}^0/R_u(\mathbf{G})\big) \times_k \overline{k}$ and $\mathbf{P}_\emptyset = \mathbf{B}$. Furthermore we know that each parabolic subgroup of $\big(\mathbf{G}^0/R_u(\mathbf{G})\big) \times_k \overline{k}$ is $\big(\mathbf{G}^0/R_u(\mathbf{G})\big)(\overline{k})$–conjugate to a unique standard parabolic subgroup. This allows us to define the *type* of an arbitrary parabolic subgroup of $\mathbf{G}^0/R_u(\mathbf{G})$. It can happen that two different standard parabolic subgroups of the $\overline{k}$–group $\big(\mathbf{G}^0/R_u(\mathbf{G})\big) \times_k \overline{k}$ are conjugate under $\mathbf{G}(\overline{k})$: There are in general fewer conjugacy classes of parabolic subgroups. If $\mathbf{P}$ is a parabolic subgroup of the k–group $\mathbf{G}^0/R_u(\mathbf{G})$, we denote by $\mathbf{N}_{\mathbf{G}}(\mathbf{P})$ its normalizer for the conjugacy action of $\mathbf{G}$ on $\mathbf{G}^0/R_u(\mathbf{G})$.

Lemma 7.3. *The quotient $\mathbf{G}/\mathbf{N}_{\mathbf{G}}(\mathbf{P})$ is a projective k–variety.*

Proof. We can assume that $\mathbf{G}^0$ is reductive. Since $\mathbf{G}^0/\mathbf{P}$ is projective and is a connected component of $\mathbf{G}/\mathbf{P}$, $\mathbf{G}/\mathbf{P}$ is projective as well. The point is that the morphism $\mathbf{G}/\mathbf{P} \to \mathbf{G}/\mathbf{N}_{\mathbf{G}}(\mathbf{P})$ is a $\mathbf{N}_{\mathbf{G}}(\mathbf{P})/\mathbf{P}$–torsor. Since the affine k–group $\mathbf{N}_{\mathbf{G}}(\mathbf{P})/\mathbf{P}$ is finite, étale descent tells us that $\mathbf{G}/\mathbf{N}_{\mathbf{G}}(\mathbf{P})$ is proper [EGA IV, prop. 2.7.1]. But $\mathbf{G}/\mathbf{N}_{\mathbf{G}}(\mathbf{P})$ is quasiprojective, hence projective. $\square$

Given a loop cocycle $\eta : \pi_1(R_n) \to \mathbf{G}(\overline{k})$ with coordinates (z, η^{geo}) as before, we focus on the special case of flag varieties of parabolic subgroups of $\mathbf{G}^0/R_u(\mathbf{G})$.

Corollary 7.4. *1. Let $I \subset \Delta$. The following are equivalent:*

(a) The R_n–group ${}_\eta\big(\mathbf{G}^0/R_u(\mathbf{G})\big)$ admits a parabolic subgroup scheme of type I;

(b) The R_n–group ${}_\eta\big(\mathbf{G}^0/R_u(\mathbf{G})\big)_{R_n}$ admits a parabolic subgroup of type I;

(c) The F_n–group ${}_\eta\big(\mathbf{G}^0/R_u(\mathbf{G})\big)_{F_n}$ admits a parabolic subgroup of type I;

(d) There exists a parabolic subgroup $\mathbf{P}$ of the k–group${}_z(\mathbf{G}^0/R_u(\mathbf{G}))$ which is of type I and which is normalized by η^{geo}, i.e., η^{geo} factorizes through $\mathbf{N}_{{}_z\mathbf{G}}(\mathbf{P})$.

2. The following are equivalent:

(a) ${}_\eta\big(\mathbf{G}^0/R_u(\mathbf{G})\big)_{R_n}$ is irreducible (i.e has no proper parabolic subgroups);

(b) ${}_\eta\big(\mathbf{G}^0/R_u(\mathbf{G})\big)_{K_n}$ is irreducible;

(c) ${}_\eta\big(\mathbf{G}^0/R_u(\mathbf{G}) of type\big)_{F_n}$ is irreducible;

(d) The k–group homomorphism $\eta^{geo} : {}_\infty\boldsymbol{\mu}^n \to {}_z\mathbf{G} \to \mathbf{Aut}({}_z\mathbf{G}^0)$ is irreducible (see §2.4).

3. The following are equivalent:

(a) ${}_\eta\big(\mathbf{G}^0/R_u(\mathbf{G})\big)_{R_n}$ is anisotropic;

(b) ${}_\eta\big(\mathbf{G}^0/R_u(\mathbf{G})\big)_{K_n}$ is anisotropic;

(c) ${}_\eta\big(\mathbf{G}^0/R_u(\mathbf{G})\big)_{F_n}$ is anisotropic;

(d) The k–group homomorphism $\eta^{geo} : {}_\infty\boldsymbol{\mu}^n \to {}_z\mathbf{G} \to \mathbf{Aut}({}_z\mathbf{G}^0)$ is anisotropic (see §2.4).

Proof. Without loss of generality, we can factor out by $R_u(\mathbf{G})$ and assume that $\mathbf{G}^0$ is reductive. As in the proof of Theorem 7.1, we can assume by twisting that z is trivial and reason "at the finite level":

Claim 7.5. *There exists a positive integer m such that $[\eta] \in H^1(R_n, \mathbf{G})$ is trivialized by the base change $R_{n,m}/R_n$.*

Indeed by continuity $\eta^{geo} : {}_\infty\boldsymbol{\mu}^n \to \mathbf{G}$ factors through a morphism $f : \boldsymbol{\mu}_m^n \to \mathbf{G}$ and $[\eta] = f_*[\mathfrak{E}_{n,m}]$ where $\mathfrak{E}_{n,m} = \operatorname{Spec}(R_{n,m})/\operatorname{Spec}(R_n)$ stands for the standard $\boldsymbol{\mu}_m^n$-torsor. In particular, the class $[\eta] \in H^1(R_n, \mathbf{G})$ is trivialized by the covering $R_{n,m}/R_n$ as above.

(1) $(a) \Longrightarrow (b) \Longrightarrow (c)$: obvious.

$(c) \Longrightarrow (d)$: We assume that ${}_\eta\mathbf{G}^0{}_{F_n}$ admits a F_n-parabolic $\mathbf{Q}$ of type I. Hence ${}_\eta\mathbf{G}^0{}_{F_n} \times_{F_n} F_{n,m} = \mathbf{G}^0_{F_{n,m}}$ admits a $F_{n,m}$–parabolic of type I. Since $F_{n,m}$ is an iterated Laurent serie field over k, it implies that $\mathbf{G}^0$ admits a parabolic $\mathbf{P}$ of type I (see the proof of [CGP, lemma 5.24]). We consider the R_n–scheme $\mathbf{X} := {}_\eta\big(\mathbf{G}/\mathbf{N}_\mathbf{G}(\mathbf{P})\big) \times_k R_n$ which by descent considerations [EGA IV, 2.7.1.vii] is proper since $\mathbf{G}/\mathbf{N}_\mathbf{G}(\mathbf{P})$ is.

Claim 7.6. $\mathbf{X}(F_n) \neq \emptyset$.

The F_n–group ${}_\eta\mathbf{G}^0/F_n$ admits a subgroup $\mathbf{Q}$ such that $\mathbf{Q} \times_k \overline{F}_n$ is $\mathbf{G}^0(\overline{F}_n)$-conjugate to $\mathbf{P} \times_k \overline{F}_n \subset \mathbf{G}^0 \times_k \overline{F}_n$. Let $g \in \mathbf{G}(\overline{F}_n)$ be such that $\mathbf{Q} \times_{F_n} \overline{F}_n = g\,(\mathbf{P} \times_k \overline{F}_n)\,g^{-1}$. As in [Se, III.2, lemme 1], we check that the cocycle $\gamma \mapsto g^{-1}\,\eta_\gamma\,{}^\gamma g$ is cohomologous to η and has value in $\mathbf{N}_\mathbf{G}(\mathbf{P})(\overline{F}_n)$. In other words, the F_n–torsor corresponding to η admits a reduction to the subgroup $\mathbf{N}_\mathbf{G}(\mathbf{P})$, i.e.

$$[\eta] \in \operatorname{Im}\Big(H^1\big(F_n, \mathbf{N}_\mathbf{G}(\mathbf{P})\big) \to H^1(F_n, \mathbf{G})\Big).$$

This implies that $\mathbf{X}(F_n) \neq \emptyset$ (*ibid*, I.5, prop. 37) and the Claim is proven.

By Theorem 7.1.1, we have $(\mathbf{X}^{\eta^{geo}})(k) \neq \emptyset$, so that there exists an element $x \in (\mathbf{X}^{\eta^{geo}})(k)$. Since $H^1\big(k, \mathbf{N}_\mathbf{G}(\mathbf{P})\big)$ injects in $H^1(k, \mathbf{G})$ (see [Gi4, cor. 2.7.2]), we have $\mathbf{X}(k) = \mathbf{G}(k)/\mathbf{N}_\mathbf{G}(\mathbf{P})(k)$, i.e. $\mathbf{X}(k)$ is homogeneous under $\mathbf{G}(k)$.

Hence there exists $g \in \mathbf{G}(k)$ such that $x = g.x_0$ where x_0 stands for the image of 1 in $X(k)$. We have

$$\eta^{geo}(\boldsymbol{\mu}_m^n(\overline{k})) \subset \operatorname{Stab}_{\mathbf{G}(\overline{k})}(x) g \operatorname{Stab}_{\mathbf{G}(\overline{k})}(x_0) g^{-1} = g\mathbf{N}_\mathbf{G}(\mathbf{P})(\overline{k})g^{-1} = \mathbf{N}_\mathbf{G}(g\mathbf{P}g^{-1})(\overline{k}).$$

Thus η^{geo} normalizes a parabolic subgroup of type I of the k–group of $\mathbf{G}^0$.

$(d) \Longrightarrow (a)$: We may assume that η has values in $\mathbf{N}_\mathbf{G}(\mathbf{P})(\overline{k})$. In that case, the twisted R_n–group scheme ${}_\eta\mathbf{G}^0$ admits the parabolic subgroup ${}_\eta\mathbf{P}/R_n$.

(2) Follows of (1).

(3) Recall that a k–group $\mathbf{H}$ with reductive connected component of the identity $\mathbf{H}^0$ is anisotropic if and only if it is irreducible and its connected center $\mathbf{Z}(\mathbf{H}^0)^0$ is an anisotropic torus. Statement (3) reduces then to the case where $\mathbf{G}^0$ is a k–torus $\mathbf{T}$. We are then given a continuous action of $\pi_1(R_n)$ on the cocharacter group $\widehat{\mathbf{T}}^0(\overline{k})$. It is convenient to work with the opposite assertions to (a), (b) $(c)$and (d), which we denote by (a'), (b') (c') and (c') respectively.

$(a') \Longrightarrow (b')$: If the R_n–torus ${}_\eta\mathbf{T} := {}_\eta\mathbf{T}_{R_n}$ is isotropic, so is the K_n–torus ${}_\eta\mathbf{T} \times_{R_n} K_n$.

$(b') \Longrightarrow (c')$: If the K_n–torus ${}_\eta\mathbf{T} \times_{R_n} K_n$ is isotropic, so is the F_n–torus ${}_\eta\mathbf{T} \times_{R_n} F_n$.

$(c') \Longrightarrow (d')$: By Lemma 3.8 we have

$$\mathrm{Hom}_{F_n-gr}(\mathbf{G}_m, {}_\eta\mathbf{T}_{F_n}) = \mathrm{Hom}_{F_n-gr}(\mathbf{G}_m, \mathbf{T}_{F_n})^{\eta_{geo}}.$$

If ${}_\eta\mathbf{T}$ is isotropic, then this group is not zero and the k–group morphism $\eta^{geo} : {}_\infty\boldsymbol{\mu}^n \to {}_z\mathbf{G}$ fixes a cocharacter of $\mathbf{T} = (\mathbf{G})^0$, hence (c').

$(d') \Longrightarrow (a')$: We assume that the morphism $\eta^{geo} : {}_\infty\boldsymbol{\mu}^n \to \mathbf{G}$ fixes a cocharacter $\lambda : \mathbf{G}_m \to \mathbf{T}$. Since

$$\mathrm{Hom}_{K_n-gr}(\mathbf{G}_m, {}_\eta\mathbf{T}_{K_n}) \simeq \mathrm{Hom}_{K_n-gr}(\mathbf{G}_m, \mathbf{T}_{K_n})^{\eta_{geo}}.$$

it follows that λ provides a non-zero cocharacter of ${}_\eta\mathbf{G}_{K_n}$, hence (a').

□

As in the case of loop torsors [GP2, cor. 3.3], the Borel-Tits theorem has the following consequence.

Corollary 7.7. *The minimal elements (with respect to inclusion) of the set of parabolic subgroups of the k–group ${}_z\mathbf{G}^0$ which are normalized by η^{geo} are all conjugate under ${}_z\mathbf{G}^0(k)$. The type $I(\eta)$ of this conjugacy class is the Witt-Tits index of the F_n–group ${}_\eta\big(\mathbf{G}^0/R_u(\mathbf{G})\big) \times_{R_n} F_n$.*

7.3 Anisotropic loop torsors

For anisotropic loop classes, we have the following beautiful picture.

Theorem 7.8. *Assume that $\mathbf{G}^0$ is reductive. Let $\eta, \eta' : \pi_1(R_n) \to \mathbf{G}(\overline{k})$ be two loop cocycles. Assume that $({}_\eta\mathbf{G})_{F_n}$ is anisotropic. Then the following are equivalent:*

1. $[\eta] = [\eta'] \in H^1(R_n, \mathbf{G})$,
2. $[\eta]_{K_n} = [\eta']_{K_n} \in H^1(K_n, \mathbf{G})$,

3. $[\eta]_{F_n} = [\eta']_{F_n} \in H^1(F_n, \mathbf{G})$.

We consider first the case of purely geometric loop cocycles. Note that this is the set of *all* loop cocycles if k is algebraically closed.

Theorem 7.9. *Let $\eta, \eta' : \pi_1(R_n) \to \mathbf{G}(\overline{k})$ be two loop cocycles of the form $\eta = (1, \eta^{geo})$ and $\eta' = (1, \eta'^{geo})$. Assume that η is anisotropic. Then the following are equivalent:*

1. *η^{geo} and η'^{geo} are conjugate under $\mathbf{G}(k)$,*

2. *$[\eta] = [\eta'] \in H^1(R_n, \mathbf{G})$,*

3. *$[\eta]_{K_n} = [\eta']_{K_n} \in H^1(K_n, \mathbf{G})$,*

4. *$[\eta]_{F_n} = [\eta']_{F_n} \in H^1(F_n, \mathbf{G})$.*

Proof. Recall that $\eta^{geo}, \eta'^{geo} : {}_\infty\boldsymbol{\mu} \to \mathbf{G}$ are affine k–group homomorphisms that factor through the algebraic group $\boldsymbol{\mu}_m^n$ for m large enough. The meaning of (1) is that there exists $g \in \mathbf{G}(k)$ such that $\eta'^{geo} = \mathrm{Int}(g) \circ \eta^{geo}$.

The implications 1) $\Longrightarrow$ 2) $\Longrightarrow$ 3) $\Longrightarrow$ 4) are obvious. We shall prove the implication 4) $\Longrightarrow$ 1). Assume, therefore, that $[\eta]_{F_n} = [\eta']_{F_n} \in H^1(F_n, \mathbf{G})$.

Let $\mathbf{T}$ be a maximal torus of $\mathbf{G}^0$ and let $\mathbf{N} = \mathbf{N_G}(\mathbf{T})$ and $\mathbf{W} = \mathbf{N}/\mathbf{T}$. Since the maximal tori of $\mathbf{G}^0 \times_k \overline{k}$ are all conjugate under $\mathbf{G}^0(\overline{k})$, the map $\mathbf{N_G}(\mathbf{T}) \to \mathbf{G}/\mathbf{G}^0$ is surjective. Let $\tilde{k}$ be a finite Galois extension which contains $\boldsymbol{\mu}_m(\overline{k})$, splits $\mathbf{T}$ and such that the natural map $\mathbf{N}(\tilde{k}) \to (\mathbf{G}/\mathbf{G}^0)(\overline{k})$ is surjective. We furthermore assume without loss of generality that our choice of m and $\tilde{k}$ trivialize η and η'.

Set $\tilde{\Gamma}_{n,m} = \boldsymbol{\mu}_m^n(\overline{k}) \rtimes \mathrm{Gal}(\tilde{k}/k)$. In terms of cocycles, our hypothesis means that there exists $h_n \in \mathbf{G}(\tilde{F}_{n,m})$ such that

$$h_n^{-1}\, \eta(\gamma)\, {}^{\gamma}h_n = \eta'(\gamma) \quad \forall \gamma \in \tilde{\Gamma}_n. \tag{7.6}$$

Our goal is to show that we can actually find such an element inside $\mathbf{G}(k)$. We reason by means of a building argument, and appeal to Remark 6.8 to view $\tilde{F}_{n,m}$ as a complete local field with residue field $\tilde{F}_{n-1,m}$. Note that $F_n = (\tilde{F}_{n,m})^{\tilde{\Gamma}_{n,m}}$, and that F_n can be viewed as complete local field with residue field F_{n-1}.

Let $\mathbf{C} = \mathbf{G}^0/\mathbf{D}\mathbf{G}^0$ be the coradical of $\mathbf{G}^0$. This is a k–torus which is split by $\tilde{k}$. Recall that the (enlarged) Bruhat-Tits building $\mathcal{B}_n$ of the $\tilde{F}_{n,m}$–group $\mathbf{G} \times_k \tilde{F}_{n,m}$ [T2, §2.1] is defined by

$$\mathcal{B}_n = \mathcal{B} \times \ V$$

where $V = \widehat{\mathbf{C}}^0 \otimes_{\mathbb{Z}} \mathbb{R}$, and $\mathcal{B}$ is the building of the semisimple $\tilde{F}_{n,m}$–group $\mathbf{D}\mathbf{G}^0 \times_k \tilde{F}_{n,m}$. The building $\mathcal{B}_n$ is equipped with a natural action of $\mathbf{G}(\tilde{F}_{n,m}) \rtimes \tilde{\Gamma}_{n,m}$. By

[BT1, 9.1.19.c] the group $\mathbf{DG}^0(\tilde{F}_{n-1,m}[[t_n^{\frac{1}{m}}]])$ fixes a unique (hyperspecial) point $\phi_d \in \mathcal{B}(\mathbf{DG}^0 \times_k \tilde{F}_{n,m})$ and $\mathrm{Stab}_{\mathbf{DG}^0(\tilde{F}_{n,m})}(\phi_d) = \mathbf{DG}^0\big(\tilde{F}_{n-1,m}[[t_n^{\frac{1}{m}}]]\big)$.

Since the bounded group $\mathbf{G}(\tilde{F}_{n-1,m}[[t_n^{\frac{1}{m}}]]) \rtimes \tilde{\Gamma}_{m,n}$ fixes at least one point of the building $\mathcal{B}(\mathbf{DG}^0 \times_k \tilde{F}_{n,m})$; such a point is necessarily ϕ_d which is then fixed under $\mathbf{G}(\tilde{F}_{n-1,m}[[t_n^{\frac{1}{m}}]]) \rtimes \tilde{\Gamma}_{m,n}$.

Claim 7.10. *There exists a point $\phi = (\phi_d, v) \in \mathcal{B}_n$ such that*

1. *$\tilde{\Gamma}_{m,n}$ fixes ϕ;*

2. *$\mathrm{Stab}_{\mathbf{G}(\tilde{F}_{n,m})}(\phi) = \mathbf{G}\big(\tilde{F}_{n-1,m}[[t_n^{\frac{1}{m}}]]\big)$.*

We use the fact that $\mathbf{G}^0(\tilde{F}_{n,m})$ acts on V by translations under the map

$$\mathbf{G}^0(\tilde{F}_{n,m}) \xrightarrow{q} \mathbf{C}(\tilde{F}_{n,m}) = \widehat{\mathbf{C}}^0 \otimes_{\mathbb{Z}} \tilde{F}_{n,m}^{\times} \xrightarrow{-\mathrm{ord}_{t_n}} \widehat{\mathbf{C}}^0.$$

It follows that for each $v \in V$, we have

$$(*) \qquad \mathrm{Stab}_{\mathbf{G}^0(\tilde{F}_{n,m})}(v) = \mathrm{Stab}_{\mathbf{G}^0(\tilde{F}_{n,m})}(V) = q^{-1}\big(\mathbf{C}(\tilde{F}_{n-1,m}[[t_n^{\frac{1}{m}}]])\big).$$

Since q maps $\mathbf{G}^0(\tilde{F}_{n-1,m}[[t_n^{\frac{1}{m}}]])$ into $\mathbf{C}(\tilde{F}_{n-1,m}[[t_n^{\frac{1}{m}}]])$, it follows that $\mathbf{G}^0(\tilde{F}_{n-1,m}[[t_n^{\frac{1}{m}}]])$ fixes pointwise $\phi_d \times V$.

Let us choose now the vector v by considering the action of the group $\mathbf{N}(\tilde{k}) \rtimes \tilde{\Gamma}_{m,n}$ on V. Since this action is trivial on $\mathbf{T}(\tilde{k})$, it provides an action of the finite group $\mathbf{W}(\tilde{k}) \rtimes \tilde{\Gamma}_{m,n}$ on V. But this action is affine, so there is at least one $v \in V$ which is fixed under $\mathbf{N}(\tilde{k}) \rtimes \tilde{\Gamma}_{m,n}$. The point $\phi = (\phi_d, v)$ is $\tilde{\Gamma}_{m,n}$-invariant, hence (1). We now use that $\mathbf{N}(\tilde{k})$ surjects onto $(\mathbf{G}/\mathbf{G}^0)(\tilde{k}) = (\mathbf{G}/\mathbf{G}^0)(\tilde{F}_{n-1,m}[[t_n^{\frac{1}{m}}]]) = (\mathbf{G}/\mathbf{G}^0)(\tilde{F}_{n,m})$, hence that

$$\mathbf{G}(\tilde{F}_{n-1,m}[[t_n^{\frac{1}{m}}]]) = \mathbf{G}^0(\tilde{F}_{n-1,m}[[t_n^{\frac{1}{m}}]]) \,.\, \mathbf{N}\big(\tilde{k}\big), \;\; \mathbf{G}(\tilde{F}_{n,m}) = \mathbf{G}^0(\tilde{F}_{n,m}) \,.\, \mathbf{N}\big(\tilde{k}\big).$$

Since $\mathbf{N}\big(\tilde{k}\big)$ fixes ϕ, we have

$$\mathrm{Stab}_{\mathbf{G}(\tilde{F}_{n,m})}(\phi) = \mathrm{Stab}_{\mathbf{G}^0(\tilde{F}_{n,m})}(\phi) \,.\, \mathbf{N}\big(\tilde{k}\big),$$

and it remains to show that $\mathbf{G}^0(\tilde{F}_{n-1,m}[[t_n^{\frac{1}{m}}]]) = \mathrm{Stab}_{\mathbf{G}^0(\tilde{F}_{n,m})}(\phi)$. Since $\mathbf{T} \times_k \tilde{k}$ is split, the map $\mathbf{T} \times_k \tilde{k} \to \mathbf{C} \times_k \tilde{k}$ is split and we have the decompositions

$$\mathbf{G}^0(\tilde{F}_{n-1,m}[[t_n^{\frac{1}{m}}]]) = \mathbf{DG}^0(\tilde{F}_{n-1,m}[[t_n^{\frac{1}{m}}]]) \,.\, \mathbf{T}(\tilde{F}_{n-1,m}[[t_n^{\frac{1}{m}}]])$$

and

$$\mathbf{G}^0(\tilde F_{n,m}) = \mathbf{DG}^0(\tilde F_{n,m})\,.\,\mathbf{T}(\tilde F_{n,m}).$$

The first equality shows that $\mathbf{G}^0(\tilde F_{n-1,m}[[t_n^{\frac{1}{m}}]])$ fixes ϕ hence that $\mathbf{G}^0(\tilde F_{n-1,m}[[t_n^{\frac{1}{m}}]]) \subset \mathrm{Stab}_{\mathbf{G}^0(\tilde F_{n,m})}(\phi)$.

As for the reversed inclusion consider an element $g \in \mathrm{Stab}_{\mathbf{G}(\tilde F_{n,m})}(\phi)$. Then $q(g) \in \mathbf{C}(\tilde F_{n-1,m}[[t_n^{\frac{1}{m}}]])$. The map $\mathbf{G}(\tilde F_{n-1,m}[[t_n^{\frac{1}{m}}]]) \xrightarrow{q} \mathbf{C}(\tilde F_{n-1,m}[[t_n^{\frac{1}{m}}]])$ is surjective, hence we can assume that $g \in \mathbf{DG}^0(\tilde F_{n,m})$. Since $g.\phi_d = \phi_d$, g belongs to $\mathbf{DG}^0\big(\tilde F_{n-1,m}[[t_n^{\frac{1}{m}}]]\big)$ as desired. This finishes the proof of our claim.

We consider the twisted action of $\tilde\Gamma_{n,m}$ on $\mathcal{B}_n$ defined by

$$\gamma * x = \eta(\gamma)\,.\,{}^\gamma x.$$

The extension of local fields (with respect to t_n) $\tilde F_{n,m}/F_n$ is tamely ramified. The Bruhat-Tits-Rousseau theorem states that the Bruhat-Tits building of $({}_\eta\mathbf{G}^0)_{F_n}$ can be identified with $\mathcal{B}_n^{\tilde\Gamma_{n,m}}$, i.e. the fixed points of the building $\mathcal{B}_n$ under the twisted action ([Ro] and [Pr]). But by Corollary 7.4.3, the $F_{n-1}((t_n))$–group ${}_\eta\mathbf{G}^0 \times_{R_n} F_{n-1}((t_n))$ is anisotropic, so its building consists of a single point, which is in fact ϕ. Indeed since our loop cocycle has value in $G(\tilde k) \rtimes \Gamma_{n,m}$, ϕ is fixed under the twisted action of $\tilde\Gamma_{n,m}$. This shows that

$$\mathcal{B}_n^{\Gamma_{n,m}} = \{\phi\}.$$

We claim that $h_n.\phi = \phi$. We have

$$\begin{aligned}
\gamma * (h_n\,.\,\phi) &= \eta(\gamma)\ {}^\gamma(h_n)\,.\,{}^\gamma\phi \\
&= \eta(\gamma)\ {}^\gamma(h_n)\,.\,\phi \qquad [\phi \text{ is invariant under } \tilde\Gamma_{m,n}] \\
&= h_n\,.\,\eta'(\gamma)\ \phi \qquad [\text{relation } 7.6] \\
&= h_n\,.\,\phi \qquad [\eta'(\gamma) \in \mathbf{G}(\tilde k)] \text{ and claim 7.10}]
\end{aligned}$$

for every $\gamma \in \tilde\Gamma_{n,m}$. Hence $h_n\,.\,\phi \in \mathcal{B}_n^{\Gamma_{n,m}}$ and therefore $h_n\,.\,\phi = \phi$ as desired.

It then follows that $h_n \in \mathbf{G}(\tilde F_{n-1,m}[[t_n^{\frac{1}{m}}]])$. By specializing (7.6) at $t_n = 0$, we obtain an element $h_{n-1} \in \mathbf{G}(\tilde F_{n-1,m})$ such that

$$h_{n-1}^{-1}\,\eta(\gamma)\ {}^\gamma h_{n-1} = \eta'(\gamma) \quad \forall \gamma \in \tilde\Gamma_{n,m}. \tag{7.7}$$

Since η and η' have trivial arithmetic part, it follows that h_{n-1} is invariant under $\mathrm{Gal}(\tilde k/k)$. We apply now the relation (7.7) to the generator τ_n of $\mathrm{Gal}\big(\tilde F_{n,m}/\tilde F_{n-1,m}((t_n))\big)$. This yields

$$h_{n-1}^{-1}\,\eta(\tau_n)\,h_{n-1} = \eta'(\tau_n), \tag{7.8}$$

where $\eta(\tau_n)$, $\eta'(\tau_n) \in \mathbf{G}(\tilde{k})$ and $h_{n-1} \in \mathbf{G}(F_{n-1,m}) = \mathbf{G}(\tilde{F}_{n-1,m})^{\mathrm{Gal}(\tilde{k}/k)}$. If we denote by $\boldsymbol{\mu}_m^{(n)}$ the last factor of $\boldsymbol{\mu}_m^n$ then (7.8) implies that ${\eta^{geo}}_{|\boldsymbol{\mu}_m^{(n)}}$ and ${\eta'^{geo}}_{|\boldsymbol{\mu}_m^{(n)}}$ are conjugate under $\mathbf{G}(F_{n-1,m})$.

Claim 7.11. *${\eta^{geo}}_{|\boldsymbol{\mu}_m^{(n)}}$ and ${\eta'^{geo}}_{|\boldsymbol{\mu}_m^{(n)}}$ are conjugate under $\mathbf{G}(k)$.*

The transporter $\mathbf{X} := \{h \in \mathbf{G} \mid \mathrm{Int}(h) \circ {\eta^{geo}}_{|\boldsymbol{\mu}_n^{(n)}} = {\eta'^{geo}}_{|\boldsymbol{\mu}_n^{(n)}}\}$ is a non-empty k–variety which is a homogeneous space under the group $\mathbf{Z}_{\mathbf{G}}({\eta^{geo}}_{|\boldsymbol{\mu}_n^{(n)}})$. Since $\mathbf{X}(F_{n-1,m}) \neq \emptyset$ and $F_{n-1,m}$ is an iterated Laurent series field over k, Florence's theorem [F, §1] shows that $\mathbf{X}(k) \neq \emptyset$. The claim is thus proven.

Without loss of generality we may therefore assume that ${\eta^{geo}}_{|\boldsymbol{\mu}_m^{(n)}} = {\eta'^{geo}}_{|\boldsymbol{\mu}_m^{(n)}}$. The finite multiplicative k–group $\boldsymbol{\mu}_m^{(n)}$ acts on $\mathbf{G}$ via η^{geo}, and we let $\mathbf{G}_{n-1}$ denote the k–group which is the centralizer of this action [DG, II 1.3.7]. The connected component of the identity of $\mathbf{G}_{n-1}$ is reductive ([Ri], proposition 10.1.5). Since the action of $\boldsymbol{\mu}_m^{(n-1)}$ on $\mathbf{G}$ given by η^{geo} commutes with that of $\boldsymbol{\mu}_m^{(n)}$, the k–group morphism $\eta^{geo} : \boldsymbol{\mu}_m^n \to \mathbf{G}$ factors through $\mathbf{G}_{n-1}$. Similarly for η'^{geo}. Denote by η^{geo}_{n-1} (resp. η'^{geo}_{n-1}) the restriction of η^{geo} (resp. η'^{geo}) to the k–subgroup $\boldsymbol{\mu}_m^{n-1} = \boldsymbol{\mu}_n^{(1)} \times \cdots \times \boldsymbol{\mu}_n^{(n-1)}$ of $\boldsymbol{\mu}_m^n$. Set $\tilde{\Gamma}_{n-1,m} := \boldsymbol{\mu}_m^{n-1}(\overline{k}) \rtimes \mathrm{Gal}(\tilde{k}/k)$ and consider the loop cocycle $\eta_{n-1} : \tilde{\Gamma}_{n-1,m} \to \mathbf{G}_{n-1}(\tilde{k})$ attached to $(1, \eta^{geo}_{n-1})$, and similarly for η'^{geo}_{n-1}.

The crucial point for the induction argument we want to use is the fact that $\eta^{geo}_{n-1} : \boldsymbol{\mu}_m^{n-1} \to \mathbf{G}_{n-1}$ is anisotropic. For otherwise the k–group $\mathbf{G}^0_{n-1}$ admits a non-trivial split subtorus $\mathbf{S}$ which is normalized by $\boldsymbol{\mu}_m^{n-1}$. But then $\mathbf{S}$ is a non-trivial split subtorus of $\mathbf{G}^0$ which is normalized by $\boldsymbol{\mu}_m^n$, and this contradicts the anisotropic assumption on η^{geo}.

Inside $\mathbf{G}_{n-1}(\tilde{F}_{n-1,m})$, relation (7.7) yields that

$$h_{n-1}^{-1}\ \eta_{n-1}(\gamma)\, {}^{\gamma}h_{n-1} = \eta'(\gamma) \quad \forall \gamma \in \tilde{\Gamma}_{n-1}.$$

which is similar to (7.6). By induction on n, we may assume that η^{geo}_{n-1} is $\mathbf{G}_{n-1}(k)$–conjugate to η'^{geo}_{n-1}. Thus η^{geo} is $\mathbf{G}(k)$–conjugate to η'^{geo} as desired. $\square$

Before establishing Theorem 7.8, we need the following preliminary step.

Lemma 7.12. *Let $\mathbf{H}$ be a linear algebraic k–group. It two loop classes $[\eta]$, $[\eta']$ of $H^1\big(\pi_1(R_n), \mathbf{H}(\overline{k})\big)$ have same image in $H^1(F_n, \mathbf{H})$, then $[\eta^{ar}] = [\eta'^{ar}]$ in $H^1(k, \mathbf{H})$.*

Proof. Up to twisting $\mathbf{H}$ by η^{ar}, the standard torsion argument allows us to assume with no loss of generality that η^{ar} is trivial, i.e. that η is purely geometrical. We are thus left to deal with the case of a k–group homomorphism $\eta^{geo} : {}_{\infty}\boldsymbol{\mu} \to \mathbf{H}$ that factors through some $\boldsymbol{\mu}_m^n \to \mathbf{H}$ for m large enough. Hence $[\eta]$ is trivialized by the extension $\tilde{R}_{n,m}/R_n$ and its image in $H^1(F_n, \mathbf{H})$ by the extension $\tilde{F}_{n,m}/F_n$, where $\tilde{R}_{n,m}$ and $\tilde{F}_{n,m}$ are as above.

By further increasing m, the same reasoning allows us to assume that the image of η' in $H^1 big(R_{n,m}, \mathbf{H}(\overline{k}) big)$ is purely arithmetic. More precisely, that the map

$$Z^1\big(\pi_1(R_n), \mathbf{H}(\overline{k})\big) \to Z^1\big(\pi_1(R_{n,m}), \mathbf{H}(\overline{k})\big)$$

maps $(\eta'^{geo}, \eta'^{ar})$ to $(1, \eta'^{ar})$ where the coordinates are as in Section 3.3. But our hypothesis implies that the image of $[\eta']$ in $H^1(F_{n,m}, \mathbf{H})$ is trivial, hence

$$[\eta'^{ar}] \in \ker\big(H^1(k, \mathbf{H}) \to H^1(F_n, \mathbf{H})\big).$$

Since F_n is an iterated Laurent series field over k, this kernel is trivial (see [F, §5.4]), and we conclude that $[\eta'^{ar}] = 1 \in H^1(k, \mathbf{H})$.

$\square$

We are now ready to proceed with the proof of Theorem 7.8.

Proof. The implications 1) $\Longrightarrow$ 2) $\Longrightarrow$ 3) are obvious. We shall prove the implication 3) $\Longrightarrow$ 1) by using the previous result. By assumption $[\eta]_{F_n} = [\eta']_{F_n} \in H^1(F_n, \mathbf{G})$. It is convenient to work at finite level as we have done previously, namely with cocycles

$$\eta, \eta' : \tilde{\Gamma}_{n,m} \to \mathbf{G}(\tilde{k})$$

with $\tilde{\Gamma}_{n,m} := \boldsymbol{\mu}_m^n(\tilde{k}) \rtimes \mathrm{Gal}(\tilde{k}/k)$ where $m > 0$ large enough and $\tilde{k}/k$ is a finite Galois extension extension containing all m–roots of unity in $\overline{k}$. We associate to η its arithmetic-geometric coordinate pair (z, η^{geo}) where $z \in Z^1(\mathrm{Gal}(\tilde{k}/k), \mathbf{G}(\tilde{k}))$ and $\eta^{geo} : \boldsymbol{\mu}_m^n \to {}_z\mathbf{G}$ is a k–group homomorphism. Similar considerations apply to η', and its corresponding pair (z', η'^{geo}). By Lemma 7.12, we have $[z] = [z'] \in H^1(k, \mathbf{G})$. Without lost of generality we may assume that $z = z'$. Consider the commutative diagram

$$\begin{array}{ccc}
H^1\big(\tilde{\Gamma}_{n,m}, {}_z\mathbf{G}(\tilde{k})\big) & \longrightarrow & H^1\big(\tilde{\Gamma}_{n,m}, {}_z\mathbf{G}(\tilde{F}_{n,m})\big) \\
{\scriptstyle \tau_z}\downarrow\wr & & {\scriptstyle \tau_z}\downarrow\wr \\
H^1\big(\tilde{\Gamma}_{n,m}, \mathbf{G}(\tilde{k})\big) & \longrightarrow & H^1\big(\tilde{\Gamma}_{n,m}, \mathbf{G}(\tilde{F}_{n,m})\big).
\end{array}$$

where the vertical arrows are the torsion bijections. Thus $\tau_z^{-1}[\eta] = \tau_z^{-1}[\eta'] \in H^1\big(\tilde{\Gamma}_{n,m}, {}_z\mathbf{G}(\tilde{F}_{n,m})\big)$. By Corollary 7.4.3, $\eta^{geo} : \boldsymbol{\mu}_m^n \to {}_z\mathbf{G}$ is an anisotropic k–group homomorphism. We can thus apply Theorem 7.9 to conclude that η^{geo} and η'^{geo} are conjugate under ${}_z\mathbf{G}(k)$ hence $\tau_z^{-1}[\eta] = \tau_z^{-1}[\eta']$ in $H^1\big(\tilde{\Gamma}_{n,m}, {}_z\mathbf{G}(\tilde{k})\big)$, and therefore $[\eta] = [\eta']$ in $H^1\big(\tilde{\Gamma}_{n,m}, {}_z\mathbf{G}(\tilde{k})\big)$ as desired. $\square$

Corollary 7.13. *Let the assumptions be as in the Theorem, and let $H^1\big(\pi_1(R_n), \mathbf{G}(\overline{k})\big)_{an}$ denote the preimage of $H^1(F_n, \mathbf{G})_{an}$ under the composite map $H^1\big(\pi_1(R_n), \mathbf{G}(\overline{k})\big) \to H^1(R_n, \mathbf{G}) \to H^1(F_n, \mathbf{G})$. Then $H^1\big(\pi_1(R_n), \mathbf{G}(\overline{k})\big)_{an}$ injects into $H^1(F_n, \mathbf{G})$.* $\square$

8 Acyclicity

We have now arrived to one of the main results of our work

Theorem 8.1. *Let $\mathbf{G}$ be a linear algebraic group over a field k of characteristic* 0*. Then the natural restriction map $H^1(R_n,\mathbf{G}) \to H^1(F_n,\mathbf{G})$ induces a bijection*

$$H^1_{loop}(R_n,\mathbf{G}) \xrightarrow{\sim} H^1(F_n,\mathbf{G}).$$

In particular, the inclusion map $H^1_{loop}(R_n,\mathbf{G}) \to H^1(R_n,\mathbf{G})$ admits a canonical section.

8.1 The proof

For any k–scheme $\mathfrak{X}$, we denote by $H^1(\mathfrak{X},\mathbf{G})_{irr} \subset H^1(\mathfrak{X},\mathbf{G})$[21] the subset consisting of classes of $\mathbf{G}$–torsor $\mathfrak{E}$ over $\mathfrak{X}$ for which the twisted reductive $\mathfrak{X}$–group scheme ${}_{\mathfrak{E}}\big(\mathbf{G}^0/R_u(\mathbf{G}^0)\big)_{\mathfrak{X}}$ does not contain a proper parabolic subgroup which admits a Levi subgroup.[22] Set $H^1_{loop}(\mathfrak{X},\mathbf{G})_{irr} = H^1_{loop}(\mathfrak{X},\mathbf{G}) \cap H^1(\mathfrak{X},\mathbf{G})_{irr}$. We begin with the following special case.

Lemma 8.2. *$H^1_{loop}(R_n,\mathbf{G})_{irr}$ injects into $H^1(F_n,\mathbf{G})$.*

Proof. By Lemma 4.14, we can assume without loss of generality that $\mathbf{G}^0$ is reductive. We have an exact sequence $1 \to \mathbf{G}^0 \xrightarrow{i} \mathbf{G} \xrightarrow{p} \boldsymbol{\nu} \to 1$ where $\boldsymbol{\nu}$ is a finite étale k–group. We are given two loop cocycles η, η' in $Z^1(R_n,\mathbf{G})$ which have the same image in $H^1(F_n,\mathbf{G})$, and for which the twisted F_n–groups ${}_{\eta}\mathbf{G}^0$, ${}_{\eta'}\mathbf{G}^0$ are irreducible. Since $H^1\big(\pi_1(R_n),\boldsymbol{\nu}(\overline{k})\big) \xrightarrow{\sim} H^1(F_n,\boldsymbol{\nu})$, it follows that $p_*[\eta] = p_*[\eta']$ in $H^1\big(\pi_1(R_n),\boldsymbol{\nu}(\overline{k})\big)$. We can thus assume without loss of generality that $p_*\eta = p_*\eta'$ in $Z^1\big(\pi_1(R_n),\boldsymbol{\nu}(\overline{k})\big)$. Furthermore, as in the proof of Theorem 7.8 the standard twisting argument reduces the problem to the case of purely geometric loop torsors. In particular, the group actions of η^{geo} and η'^{geo} are irreducible according to Corollary 7.4.3.

Let $\mathbf{C}$ be the connected center of $\mathbf{G}^0$. Then $\mathbf{C}$ is a k–torus equipped with an action of $\boldsymbol{\nu}$. We consider its k–subtorus $\mathbf{C}^\sharp = (\mathbf{C}^{p\circ\eta^{geo}})^0$ and denote by $\mathbf{C}_0$ its maximal k–split subtorus which is defined by $\widehat{\mathbf{C}_0}^0 = \widehat{\mathbf{C}^\sharp}^0(k)$. By construction, $\eta^{geo} : {}_{\infty}\boldsymbol{\mu}^n \to \mathbf{G}$ centralizes $\mathbf{C}^\sharp$ and $\mathbf{C}_0$. Similarly for η'^{geo}. The k–torus $\mathbf{C}_0$ is a split subtorus of $\mathbf{C}$ centralized by η^{geo} and maximal for this property. We consider the exact sequence of k–groups

$$1 \to \mathbf{C}_0 \to \mathbf{G} \xrightarrow{q} \mathbf{G}/\mathbf{C}_0 \to 1$$

[21]We remind the reader that $H^1(\mathfrak{X},\mathbf{G})$ stands for $H^1(\mathfrak{X},\mathbf{G}_{\mathfrak{X}})$.

[22]Recall that the assumption on the existence of the Levi subgroup is superfluous whenever $\mathfrak{X}$ is affine.

Claim 8.3. *The composite $q \circ \eta^{geo} : {}_\infty\boldsymbol{\mu}^n \to \mathbf{G}/\mathbf{C}_0$ is anisotropic.*

Let us establish the claim. We are given a split subtorus $\mathbf{S}$ of the k–group $\mathbf{G}^0$ which is centralized by $q \circ \eta^{geo}$. Since $q \circ \eta^{geo}$ is irreducible, $\mathbf{S}$ is central in $\mathbf{G}^0/\mathbf{C}_0$. We consider $\mathbf{M} = q^{-1}(\mathbf{S})$, this is an extension of $\mathbf{S}$ by $\mathbf{C}_0$, so it is a split k–torus. By the semisimplicity of the category of representations of ${}_\infty\boldsymbol{\mu}^n$, we see that ${}_\infty\boldsymbol{\mu}^n$ acts trivially on $\mathbf{M}$. Then $\mathbf{M} = \mathbf{C}_0$ and $\mathbf{S} = 1$, and the claim thus holds.

Next we twist the sequence of R_n–groups

$$1 \to \mathbf{C}_0 \to \mathbf{G} \xrightarrow{q} \mathbf{G}/\mathbf{C}_0 \to 1$$

by η to obtain

$$1 \to \mathbf{C}_0 \to {}_\eta\mathbf{G} \to {}_\eta(\mathbf{G}/\mathbf{C}_0) \to 1.$$

This leads to the commutative exact diagram of pointed sets

$$\begin{array}{ccccc}
0 = H^1(R_n, \mathbf{C}_0) & \longrightarrow & H^1(R_n, \mathbf{G}) & \xrightarrow{q_*} & H^1(R_n, \mathbf{G}/\mathbf{C}_0) \\
 & & {\scriptstyle\tau_\eta}\Big\uparrow{\scriptstyle\simeq} & & {\scriptstyle\tau_\eta}\Big\uparrow{\scriptstyle\simeq} \\
0 = H^1(R_n, \mathbf{C}_0) & \longrightarrow & H^1(R_n, {}_\eta\mathbf{G}) & \longrightarrow & H^1\big(R_n, {}_\eta(\mathbf{G}/\mathbf{C}_0)\big)
\end{array}$$

where the vertical maps are the torsion bijections. Note that $H^1(R_n, \mathbf{C}_0)$ vanishes since $\mathrm{Pic}(R_n) = 0$. By diagram chasing we have $[\eta] = [\eta']$ in $H^1(R_n, \mathbf{G})$ if and only if $q_*[\eta] = q_*[\eta']$ in $H^1(R_n, \mathbf{G}/\mathbf{C}_0)$. Since $q_*[\eta]_{F_n} = q_*[\eta']_{F_n}$ in $H^1(F_n, \mathbf{G}/\mathbf{C}_0)$ it will suffice to establish the Lemma for $\mathbf{G}/\mathbf{C}_0$. The claim states that $q_*\eta^{geo}$ is anisotropic, therefore $q_*[\eta] = q_*[\eta']$ in $H^1(R_n, \mathbf{G}/\mathbf{C}_0)$ by Theorem 7.9. $\square$

We can now proceed to prove Theorem 8.1.

Proof. The surjectivity of the map $H^1_{loop}(R_n, \mathbf{G}) \to H^1(F_n, \mathbf{G})$ is a special case of Proposition 6.9. Let us establish injectivity. We are given two loop cocycles $\eta, \eta' \in Z^1\big(\pi_1(R_n), \mathbf{G}(\overline{k})\big)$ having the same image in $H^1(F_n, \mathbf{G})$. Lemma 7.12 shows that $[\eta^{ar}] = [\eta'^{ar}]$ in $H^1(k, \mathbf{G})$. Up to twisting $\mathbf{G}$ by η^{ar}, we may assume that η and η' are purely geometrical loop torsors. The proof now proceeds by reduction to the irreducible case, i.e. to the case when ${}_\eta\mathbf{G}^0 \times_{R_n} F_n$ is irreducible.

Let $\mathbf{Q}$ be a minimal F_n–parabolic subgroup of ${}_\eta\mathbf{G}^0 \times_{R_n} F_n$. Corollary 7.4 shows that the k–group $\mathbf{G}^0$ admits a parabolic subgroup $\mathbf{P}$ of the same type than $\mathbf{Q}$ which is normalized by η. The same statement shows that η' normalizes a parabolic subgroup, say $\mathbf{P}'$, of the same type than $\mathbf{P}$. Since by Borel-Tits theory $\mathbf{P}'$ is $\mathbf{G}^0(k)$–conjugate to $\mathbf{P}$ we may assume that η' normalizes $\mathbf{P}$ as well. Furthermore, $\mathbf{P}$ is minimal for

η (and η') with respect to this property. We can view then η, η' as elements of $Z^1_{loop}\big(R_n, \mathbf{N}_{\mathbf{G}}(\mathbf{P})\big)_{irr}$. We look at the following commutative diagram

$$\begin{array}{ccc} H^1_{loop}\big(R_n, \mathbf{N}_{\mathbf{G}}(\mathbf{P})\big)_{irr} & \longrightarrow & H^1(R_n, \mathbf{G}) \\ \downarrow & & \downarrow \\ H^1\big(F_n, \mathbf{N}_{\mathbf{G}}(\mathbf{P})\big)_{irr} & \xrightarrow{\sim} & H^1(F_n, \mathbf{G}) \end{array}$$

Since the bottom map is injective (see [Gi4, th. 2.15]), it will suffice to show that $H^1_{loop}\big(R_n, \mathbf{N}_{\mathbf{G}}(\mathbf{P})\big)_{irr}$ injects in $H^1\big(F_n, \mathbf{N}_{\mathbf{G}}(\mathbf{P}_I)\big)$. Since the unipotent radical $\mathbf{U}$ of $\mathbf{P}$ is a split unipotent group, we have

$$H^1\big(R_n, \mathbf{N}_{\mathbf{G}}(\mathbf{P})\big) \simeq H^1\big(R_n, \mathbf{N}_{\mathbf{G}}(\mathbf{P})/\mathbf{U}\big),$$

and similarly for F_n by Lemma 4.14. So we are reduced to showing that $H^1_{loop}\big(R_n, \mathbf{N}_{\mathbf{G}}(\mathbf{P})/\mathbf{U}\big)_{irr}$ injects in $H^1\big(F_n, \mathbf{N}_{\mathbf{G}}(\mathbf{P})/\mathbf{U}\big)$, which is covered by Lemma 8.2. This completes the proof of injectivity. $\square$

8.2 Application: Witt-Tits decomposition

By using the Witt-Tits decomposition over F_n [Gi4, th. 2.15], we get the following.

Corollary 8.4. *Assume that $\mathbf{G}^0$ is a split reductive k–group. Let $\mathbf{P}_{I_1}, \dots, \mathbf{P}_{I_l}$ be representatives of the $\mathbf{G}(\overline{k})$-conjugacy classes of parabolic subgroups of $\mathbf{G}^0$. Let $\mathbf{L}_{I_j}$ be a Levi subgroup of $\mathbf{P}_{I_j}$ for $j = 1, \dots l$. Then*

$$\bigsqcup_{j=1,\dots,l} H^1_{loop}\big(R_n, \mathbf{N}_{\mathbf{G}}(\mathbf{P}_{I_j}, \mathbf{L}_{I_j})\big)_{irr} \;\simeq\; H^1_{loop}(R_n, \mathbf{G}) \simeq H^1(F_n, \mathbf{G}).$$

Remark 8.5. It follows from the Corollary that we have a "Witt-Tits decomposition" for loop torsors. Furthermore, if we are interested in purely geometrical irreducible loop torsors, then we have a nice description in terms of k–group homomorphisms ${}_\infty\boldsymbol{\mu} \to \mathbf{G}$ as described in Theorem 7.9. This corresponds to the embedding of $\mathrm{Hom}_{k-gp}({}_\infty\boldsymbol{\mu}^n_{irr}, \mathbf{G})/\mathbf{G}(k)$ in $H^1(R_n, \mathbf{G})$.

For future use we record the connected case.

Corollary 8.6. *Assume that $\mathbf{G}$ is a split reductive group.*

1. *Let $\mathbf{P}_{I_1}, \dots, \mathbf{P}_{I_l}$ be the standard k–parabolic subgroups containing a given Borel subgroup of $\mathbf{G}/k$. Then*

$$\bigsqcup_{j=1,\dots,l} H^1_{loop}(R_n, \mathbf{P}_{I_j})_{irr} \;\simeq\; H^1_{loop}(R_n, \mathbf{G}) \simeq H^1(F_n, \mathbf{G}).$$

2. If k is algebraically closed

$$\mathrm{Hom}_{k-gp,irr}({}_\infty\boldsymbol{\mu}^n, \mathbf{G})/\mathbf{G}(k) \simeq H^1_{loop}(R_n, \mathbf{G})_{irr} \simeq H^1(F_n, \mathbf{G})_{irr}.$$

Using our choice of roots of unity (2.3), we have ${}_\infty\boldsymbol{\mu} \simeq \widehat{\mathbb{Z}}$. So the left handside is $\mathrm{Hom}_{gp}\big(\widehat{\mathbb{Z}}^n, \mathbf{G}(k)\big)_{irr}/\mathbf{G}(k)$, namely the $\mathbf{G}(k)$–conjugacy classes of finite order irreducible pairwise commuting elements $(g_1, ..., g_n)$ (irreducible in the sense that the elements do not belong to a proper parabolic subgroup).

8.3 Classification of semisimple k–loop adjoint groups

Next we discuss in detail the important case where our algebraic group is the group $\mathbf{Aut}(\mathbf{G})$ of automorphisms of a split semisimple group $\mathbf{G}$ of adjoint type. This is the situation needed to classify forms of the R_n–group $\mathbf{G} \times_k R_n$ and of the corresponding R_n–Lie algebra $\mathfrak{g} \otimes_k R_n$ where $\mathfrak{g}$ is the Lie algebra of $\mathbf{G}$. Indeed it is this particular case, and its applications to infinite dimensional Lie theory as described in [P2] and [GP2] for example, that have motivated our present work.

We fix a "Killing couple" $\mathbf{T} \subset \mathbf{B}$ of $\mathbf{G}$, as well as a base Δ of the corresponding root system. For each subset $I \subset \Delta$ we define as usual

$$\mathbf{T}_I = \Big(\bigcap_{\alpha \in I} \ker(\alpha)\Big)^0.$$

Since $\mathbf{G}$ is adjoint, we know that the roots define an isomorphism $\mathbf{T} \simeq (\mathbf{G}_m)^{|\Delta|}$, hence $\mathbf{T}_I \simeq (\mathbf{G}_m)^{|\Delta \setminus I|}$. The centralizer $\mathbf{L}_I := \mathbf{C}_\mathbf{G}(\mathbf{T}_I)$, is the standard Levi subgroup of the parabolic subgroup $\mathbf{P}_I = \mathbf{U}_I \rtimes \mathbf{L}_I$ attached to I. Since $\mathbf{G}$ is of adjoint type, we know that $\mathbf{L}_I/\mathbf{T}_I$ is a semisimple k–group of adjoint type.

We have a split exact sequence of k–groups

$$1 \to \mathbf{G} \to \mathbf{Aut}(\mathbf{G}) \to \mathbf{Out}(\mathbf{G}) \to 1$$

where $\mathbf{Out}(\mathbf{G})$ is the finite constant k–group corresponding to the finite (abstract) group $\mathrm{Out}(\mathbf{G})$ of symmetries of the Dynkin diagram of $\mathbf{G}$. [XXIV §3].[23] For $I \subset \Delta$, we need to describe the normalizer $\mathbf{N}_{\mathrm{Aut}(\mathbf{G})}(\mathbf{L}_I)$ of $\mathbf{L}_I$. Following [Sp, 16.3.9.(4)], we define the subgroup of I-automorphisms of $\mathbf{G}$ by

$$\mathbf{Aut}_I(\mathbf{G}) = \mathbf{Aut}(\mathbf{G}, \mathbf{P}_I, \mathbf{L}_I).$$

where this last is the subgroup of $\mathbf{Aut}(\mathbf{G})$ that stabilizes both $\mathbf{P}_I$ and $\mathbf{L}_I$. There is then an exact sequence

$$1 \to \mathbf{L}_I \to \mathbf{Aut}_I(\mathbf{G}) \to \mathbf{Out}_I(\mathbf{G}) \to 1,$$

[23] In [SGA3] the group $\mathbf{Out}(\mathbf{G})$ is denoted by $\mathbf{Aut}(\mathrm{Dyn})$.

where $\mathbf{Out}_I(\mathbf{G})$ is the finite constant group corresponding to the subgroup of $\mathrm{Out}(\mathbf{G})$ consisting of elements that stabilize $I \subset \Delta$. Then the preceding Corollary reads

$$\bigsqcup_{[I]\subset \Delta/\mathrm{Out}(\mathbf{G})} H^1_{loop}\big(R_n, \mathbf{Aut}_I(\mathbf{G})\big)_{irr} \ \simeq \ H^1_{loop}\big(R_n, \mathbf{Aut}(\mathbf{G})\big) \simeq H^1\big(F_n, \mathbf{Aut}(\mathbf{G})\big).$$

By [Gi4, cor. 3.5], $H^1_{loop}\big(R_n, \mathbf{Aut}_I(\mathbf{G})\big)_{irr} \simeq H^1\big(F_n, \mathbf{Aut}_I(\mathbf{G})\big)_{irr}$ can be seen as a subset of $H^1_{loop}\big(R_n, \mathbf{Aut}_I(\mathbf{G})/\mathbf{T}_I\big)_{an} \simeq H^1\big(F_n, \mathbf{Aut}_I(\mathbf{G})/\mathbf{T}_I\big)_{an}$. We come now to another of the central results of the paper.

Theorem 8.7. *Assume that k is algebraically closed and of characteristic 0. Let $\mathbf{G}$ be a simple k–group of adjoint type. Let $\mathbf{T} \subset \mathbf{B}$, I, and Δ be as above. On the set $\mathrm{Hom}_{k-gp}\big({}_\infty\boldsymbol{\mu}^n, \mathbf{Aut}_I(\mathbf{G})\big)$ define the equivalence relation $\phi \sim_I \phi'$ if there exists $g \in \mathbf{Aut}_I(\mathbf{G})(k)$ such that ϕ and $g\phi' g^{-1}$ have same image in $\mathrm{Hom}_{k-gp}\big({}_\infty\boldsymbol{\mu}^n, \mathbf{Aut}_I(\mathbf{G})/\mathbf{T}_I\big)$. Then we have a decomposition*

$$\bigsqcup_{[I]\subset \Delta/\mathrm{Out}(\mathbf{G})} \mathrm{Hom}_{k-gp}({}_\infty\boldsymbol{\mu}^n, \mathbf{Aut}_I(\mathbf{G}))_{an}/\sim_I \ \xrightarrow{\sim} \ H^1_{loop}(R_n, \mathbf{G}) \ \xrightarrow{\sim} H^1(F_n, \mathbf{G}).$$

where $\mathrm{Hom}_{k-gp}\big({}_\infty\boldsymbol{\mu}^n, \mathbf{Aut}_I(\mathbf{G})/\mathbf{T}_I\big)_{an}$ stands for the set of anisotropic group homomorphisms ${}_\infty\boldsymbol{\mu}^n \to \mathbf{Aut}_I(\mathbf{G})/\mathbf{T}_I$.

Remark 8.8. As an application of Margaux's rigidity theorem [Mg2], the right hand-side does not change by extension of algebraically closed fields. Hence $H^1_{loop}(R_n, \mathbf{G})$ does not change by extension of algebraically closed fields. This allows us in practice whenever useful to work over $\overline{\mathbb{Q}}$ or $\mathbb{C}$.

Proof. The group $(\mathbf{Aut}_I(\mathbf{G})/\mathbf{T}_I)(k)$ acts naturally on the set $\mathrm{Hom}_{k-gp}\big({}_\infty\boldsymbol{\mu}^n, \mathbf{Aut}_I(\mathbf{G})/\mathbf{T}_I\big)_{an}$ by conjugation, and we denote the resulting quotient set by $\overline{\mathrm{Hom}}_{k-gp}\big({}_\infty\boldsymbol{\mu}^n, \mathbf{Aut}_I(\mathbf{G})/\mathbf{T}_I\big)_{an}$. The commutative square

$$\begin{array}{ccc}
\mathrm{Hom}_{k-gp}\big({}_\infty\boldsymbol{\mu}^n, \mathbf{Aut}_I(\mathbf{G})\big)_{irr} & \longrightarrow & \overline{\mathrm{Hom}}_{k-gp}\big({}_\infty\boldsymbol{\mu}^n, \mathbf{Aut}_I(\mathbf{G})/\mathbf{T}_I\big)_{an} \\
\downarrow & & \downarrow \\
H^1_{loop}\big(R_n, \mathbf{Aut}_I(\mathbf{G})\big)_{irr} & \longrightarrow & H^1_{loop}\big(R_n, \mathbf{Aut}_I(\mathbf{G})/\mathbf{T}_I\big)_{an}
\end{array}$$

is well defined as one can see by taking into account Corollary 7.4. Since k is algebraically closed, loop torsors are purely geometric, hence the two vertical maps are onto. As we have seen, the bottom horizontal map is injective; this defines an equivalence relation $\sim_I'$ on the set $\mathrm{Hom}_{k-gp}\big({}_\infty\boldsymbol{\mu}^n, \mathbf{Aut}_I(\mathbf{G})\big)$ such that

$$\mathrm{Hom}_{k-gp}\big({}_\infty\boldsymbol{\mu}^n, \mathbf{Aut}_I(\mathbf{G})\big)_{irr}/\sim_I' \ \xrightarrow{\sim} \ H^1_{loop}(R_n, \mathbf{Aut}_I(\mathbf{G})).$$

It remains to establish that the equivalence relations $\sim'_I$ and $\sim_I$ coincide, and we do this by using that the right vertical map in the above diagram is injective. (Theorem 7.9). We are given $\phi_1, \phi_2 \in \mathrm{Hom}_{k-gp}\big({}_\infty\boldsymbol{\mu}^n, \mathbf{Aut}_I(\mathbf{G})\big)_{irr}$. Then $\phi_1 \sim'_I \phi_2$ if and only if the image of ϕ_1 and ϕ_2 in $\mathrm{Hom}_{k-gp}\big({}_\infty\boldsymbol{\mu}^n, \mathbf{Aut}_I(\mathbf{G})/\mathbf{T}_I\big)_{an}$ are conjugate by an element of $\big(\mathbf{Aut}_I(\mathbf{G})/\mathbf{L}_I\big)(k)$. Since the map $\mathbf{Aut}_I(\mathbf{G})(k) \to \big(\mathbf{Aut}_I(\mathbf{G})/\mathbf{L}_I\big)(k)$ is onto, it follows that $\phi_1 \sim'_I \phi_2$ if and only if $\phi_1 \sim_I \phi_2$ as desired. $\square$

Corollary 8.9. *Under the hypothesis of Theorem 8.7, the classification of loop torsors on R_n "is the same" as the classification, for each subset $I \subset \Delta$, of irreducible commuting n–uples of elements of finite order of $\mathbf{Aut}_I(\mathbf{G})(k)$ up to the equivalence relation $\sim_I$.*

8.4 Action of $\mathbf{GL}_n(\mathbb{Z})$

The assumptions are as in the previous section. We fix a pinning (epinglage) $(\mathbf{G}, \mathbf{B}, \mathbf{T})$ [XXIV §1]. This determines a section We fix a section $s : \mathbf{Out}(\mathbf{G}) \to \mathbf{Aut}(\mathbf{G})$.

The group $\mathbf{GL}_n(\mathbb{Z})$ acts on the left as automorphisms of the k–algebra R_n via

$$g = (a_{ij}) \in \mathbf{GL}_n(\mathbb{Z}) : t_i \mapsto t_1^{a_{1i}} t_2^{a_{2i}} \dots t_n^{a_{ni}} \tag{8.1}$$

We denote the resulting k–automorphism of R_n corresponding to g also by g since no confusion will arise. By Yoneda considerations g (anti)corresponds to an automorphism g^* of the k–scheme $\mathrm{Spec}(R_n)$.

Applying (8.1) where we now replace t_i by $t_1^{1/m}$ and k by $\overline{k}$ gives a left action of $\mathbf{GL}_n(\mathbb{Z})$ as automorphisms of $\overline{R}_{n,m} = \overline{k}[t^{\pm 1/m}, \dots, t_n^{\pm 1/m}]$. If we denote by g_m the automorphism corresponding to g then the diagram

$$\begin{array}{ccc} R_n & \xrightarrow{g} & R_n \\ \downarrow & & \downarrow \\ \overline{R}_{n,m} & \xrightarrow{g_m} & \overline{R}_{n,m} \end{array}$$

commutes. Passing to the direct limit on (8.4) the element g induces an automorphism g_∞ of $\overline{R}_{n,\infty} = \varinjlim \overline{k}[t^{\pm 1/m}, \dots, t_n^{\pm 1/m}]$. If no confusion is possible, we will denote g_∞ and g_m simply by g.

Recall that $\pi_1(R_n) = \widehat{\mathbb{Z}}(1)^n \rtimes \mathrm{Gal}(k)$. Our fix choice of compatible roots of unity (ξ_m) allows us to identify $\pi_1(R_n)$ with $\widehat{\mathbb{Z}}^n \rtimes \mathrm{Gal}(k)$ where the left action of $\mathrm{Gal}(k)$ on each component $\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/m\mathbb{Z}$ is as follows: If $a \in \mathrm{Gal}(k)$ and $m \geq 1$ there exists a unique $1 \leq a(m) \leq m-1$ such that $a(\xi_m) = \xi_m^{a(m)}$. This defines an automorphism

a_m of the additive group $\mathbb{Z}/m\mathbb{Z}$. Passing to the limit on each component yields the desired group automorphism $\widehat{a}$ of $\widehat{\mathbb{Z}}^n$.

View $\big(\mathbb{Z}/m\mathbb{Z}\big)^n$ as row vectors. Then $\mathbf{GL}_n(\mathbb{Z})$ acts on the right on this group by right multiplication

$$g : \mathrm{e}_m \mapsto \mathrm{e}_m^g = \mathrm{e}_m g$$

where $\big(\mathbb{Z}/m\mathbb{Z}\big)^n$ is viewed as a $\mathbb{Z}$–module in the natural way. By passing to the inverse limit we get a right action of $\mathbf{GL}_n(\mathbb{Z})$ as automorphisms of $\widehat{\mathbb{Z}}^n$ that we denote by $\mathrm{e} \mapsto \mathrm{e}^g$. We extend this to a right action on $\pi_1(R_n) = \widehat{\mathbb{Z}}^n \rtimes \mathrm{Gal}(k)$ by letting $\mathbf{GL}_n(\mathbb{Z})$ act trivially on $\mathrm{Gal}(k)$. Thus if $\gamma = (\mathrm{e}, a) \in \widehat{\mathbb{Z}}^n \rtimes \mathrm{Gal}(k)$ and $g \in \mathbf{GL}_n(\mathbb{Z})$, then $\gamma^g = (\mathrm{e}^g, a)$.[24]

By taking the foregoing discussion into consideration we can define the (right) semidirect product group $\mathbf{GL}_n(\mathbb{Z}) \ltimes \pi_1(R_n)$ with multiplication

$$\big(h, (\mathrm{e}, a)\big)\big(g, (\mathrm{f}, b)\big) = \big(hg, (\mathrm{e}^g, a)(\mathrm{f}, b)\big) = \big(hg, (\mathrm{e}^g \widehat{a}(\mathrm{f}), ab)\big) \tag{8.2}$$

for all $h, g \in \mathbf{GL}_n(\mathbb{Z})$, $\mathrm{e}, \mathrm{f} \in \widehat{\mathbb{Z}}^n$ and $a, b \in \pi_1(R_n)$. For future use we point out that under that under the natural identification of $\mathbf{GL}_n(\mathbb{Z})$ and $\pi_1(R_n)$ with subgroups of $\mathbf{GL}_n(\mathbb{Z}) \ltimes \pi_1(R_n)$ we have

$$\gamma g = g \gamma^g \tag{8.3}$$

for all $g \in \mathbf{GL}_n(\mathbb{Z})$ and $\gamma \in \pi_1(R_n)$.

By definition $\pi_1(R_n)$ acts naturally on $\overline{R}_{n,\infty}$. Under our identification $\pi_1(R_n) = \widehat{\mathbb{Z}}^n \rtimes \mathrm{Gal}(k)$ the action is given by

$$(\mathrm{e}, a) : \lambda t_i^{1/m} \mapsto a(\lambda) \xi_m^{e_{m,i}} t_i^{1/m} \tag{8.4}$$

where $\mathrm{e} = (\mathrm{e}_1, \dots, \mathrm{e}_n) \in \widehat{\mathbb{Z}}^n$, $\mathrm{e}_i = (\mathrm{e}_{m,i})_{m \geq 1}$ with $0 \leq \mathrm{e}_{m,i} < m$, and $\lambda \in k$. Using (8.2) and (8.4) it is tedious but straightforward to check that the group $\mathbf{GL}_n(\mathbb{Z}) \ltimes \pi_1(R_n)$ defined above acts on the left as automorphisms of the k–algebra $\overline{R}_{n,\infty}$ in a way which is compatible with the left actions of each of the groups, i.e.

$$(g\gamma).x = g.(\gamma.x) \tag{8.5}$$

for all $g \in \mathbf{GL}_n(\mathbb{Z})$, $\gamma \in \pi_1(R_n)$ and $x \in \overline{R}_{n,\infty}$.

In the reminder of this section we let $\mathbf{H}$ denote a linear algebraic group over k. Each element $g \in \mathbf{GL}_n(\mathbb{Z})$ viewed as an automorphism g^* of the k–scheme $\mathrm{Spec}(R_n)$ induces by functoriality a bijection, also denoted by g^*, of the pointed set $H^1(R_n, \mathbf{H})$

[24] After our identifications, this is nothing but the natural action of g^* on $\pi_1\big(\mathrm{Spec}(R_n)\big)$.

onto itself. This leads to a left action of $\mathbf{GL}_n(\mathbb{Z})$ on this pointed set which we called *base change.* Our objective is to have a precise description of this action.[25] The isotriviality theorem [GP3, th. 2.9] shows tha it will suffice to trace the base change action at the level of 1-cocycles in $Z^1\big(\pi_1(R_n), \mathbf{H}(\overline{R}_{n,\infty})\big)$. Following standard conventions for cocycles we denote the action of an element $\gamma \in \pi_1(R_n)$ on an element $h \in \mathbf{H}(\overline{R}_{n,\infty})$ by ${}^{\gamma}h$. Then (8.5) implies that

$$\gamma.h = {}^{\gamma}h. \tag{8.6}$$

Lemma 8.10. *The base change action of $\mathbf{GL}_n(\mathbb{Z})$ on $H^1(R_n, \mathbf{H})$ is induced by the action $\eta \mapsto {}^g\eta$ of $\mathbf{GL}_n(\mathbb{Z})$ on $Z^1\big(\pi_1(R_n), \mathbf{H}(\overline{R}_{n,\infty})\big)$ given by*

$$({}^g\eta)(\gamma) = g.\eta(\gamma^g)$$

for all $\gamma \in \pi_1(R_n)$ and $g \in \mathbf{GL}_n(\mathbb{Z})$.

Proof. For all $\alpha, \beta \in \pi_1(R_n)$ we have

$$\begin{array}{rcll}
({}^g\eta(\alpha\beta) & = & g.\eta\big((\alpha\beta)^g)\big) & \text{[definition]} \\
& = & g.\eta(\alpha^g\beta^g) & \\
& = & g.\big(\eta(\alpha^g)\,{}^{\alpha^g}\eta(\beta^g)\big) & [\eta \text{ a cocycle}] \\
& = & g.\Big(\eta(\alpha^g)\,\big(\alpha^g.\eta(\beta^g)\big)\Big) & \text{[by (8.6)]} \\
& = & \big(g.(\eta(\alpha^g)\big)\big(g.\alpha^g.\eta(\beta^g)\big) & \text{[by action axiom]} \\
& = & \big(g.(\eta(\alpha^g)\big)\big(\alpha.g.\eta(\beta^g)\big) & \text{[by action axiom and (8.3)]} \\
& = & {}^g\eta(\alpha)\big(\alpha.{}^g\eta(\beta)\big) & \text{[by definition]} \\
& = & {}^g\eta(\alpha)^{\alpha}\big({}^g\eta(\beta)\big) & \text{[by (8.6)].}
\end{array}$$

This shows that ${}^g\eta$ is a cocycle (which is clearly continuous since η is). That this defines a left action of $\mathbf{GL}_n(\mathbb{Z})$ on $Z^1\big(\pi_1(R_n), \mathbf{H}(\overline{R}_{n,\infty})\big)$ is easy to verify using the definitions.

Next we verify that the action factors through H^1. Assume μ is a cocycle cohomologous to η, and let $h \in \mathbf{H}(\overline{R}_{n,\infty})$ be such that $\mu(\gamma) = h^{-1}\eta\gamma{}^{\gamma}h$ for all $\gamma \in \pi_1(R_n)$. Then

[25] Our main interest is the case when $\mathbf{H} = \mathbf{Aut}(\mathbf{G})$ with $\mathbf{G}$ simple. The reason behind the importance of this case lies in the applications to infinite dimensional Lie theory.

$$\begin{aligned}
{}^g\mu(\gamma) &= g.\mu({}^g\gamma) \quad \text{[definition]}\\
&= g.\big(h^{-1}\eta(\gamma^g)^{\gamma}h\big)\\
&= g.h^{-1}\, g.\eta(\gamma^g)\, g.{}^{\gamma}h \quad \text{[action axiom]}\\
&= (g.h)^{-1}\, {}^g\eta(\gamma)\, {}^{\gamma}g\cdot{}^{\gamma}h \quad \text{[action axiom, definition, and } g = {}^{\gamma}g]\\
&= (g.h)^{-1}\, {}^g\eta(\gamma)\, {}^{\gamma}(g.h).
\end{aligned}$$

Thus ${}^g\mu$ and ${}^g\eta$ are cohomologous.

It remains to verify that the action we have defined on $H^1\big(\pi_1(R_n), \mathbf{H}(\overline{R}_{n,\infty})\big) = H^1\big(R_n, \mathbf{H}\big)$ coincides with the base change action. To see this we consider a faithfull representation $\mathbf{H} \to \mathbf{GL}_d$ and the corresponding quotient variety $\mathbf{Y} = \mathbf{GL}_d/\mathbf{H}$. Since $H^1(R_n, \mathbf{GL}_n) = 1$ by a theorem of Quillen and Suslin, we have a short exact sequence of pointed sets

$$1 \to \mathbf{H}(R_n) \to \mathbf{GL}_d(R_n) \to \mathbf{Y}(R_n) \xrightarrow{\varphi} H^1(R_n, \mathbf{H}) \to 1.$$

Therefore it is enough to verify our assertion for the image of the characteristic map φ. Given $y \in \mathbf{Y}(R_n)$, by definition $\varphi(y)$ is the class of the cocycle

$$\gamma \to \eta(\gamma) = Y^{-1}\, {}^{\gamma}Y = Y^{-1}\, \gamma.Y$$

where $Y \in \mathbf{GL}_d(\overline{R}_{n,\infty})$ is a lift of y [the last equality holds by (8.6)]. If $g \in \mathbf{GL}_d(\mathbb{Z})$ we have

$$g_*(\varphi(y)) = \varphi(g.y)$$

by the equivariance of the characteristic map relative to k–schemes. Since $g.Y$ is a lift of $g.y$, we conclude that $\varphi(g.y)$ is the class of the cocycle $(g.Y)^{-1\gamma}(g.Y)$. Using identities and compatibility of actions that have already been mentioned, we have

$$(g.Y)^{-1\gamma}(g.Y) = (g.Y)^{-1}\gamma.(g.Y) = g.Y^{-1}\, g\gamma^g.Y =$$

$$= g.Y^{-1}\, g.\gamma^g.Y = g.\big(Y^{-1}\gamma^g.Y\big) = g.\eta(\gamma^g) = {}^g\eta(\gamma)$$

as desired. □

Remark 8.11. The action of $\mathbf{GL}_n(\mathbb{Z})$ on $Z^1\big(\pi_1(R_n), \mathbf{H}(\overline{R}_{n,\infty})\big)$ stabilizes $Z^1\big(\pi_1(R_n), \mathbf{H}(\overline{k})\big)$. In particular, it preserves loop cocycles.

We pause to observe that the decomposition 8.4 is equivariant under the action of $\mathbf{GL}_n(\mathbb{Z})$ on R_n. Thus

Corollary 8.12. *With the assumptions and notation as above*

$$\bigsqcup_{j=1,\dots,l} \mathbf{GL}_n(\mathbb{Z})\backslash H^1_{loop}\big(R_n, \mathbf{Aut}(\mathbf{G}, \mathbf{P}_{I_j})\big)_{irr} \;\simeq\; \mathbf{GL}_n(\mathbb{Z})\backslash H^1_{loop}\big(R_n, \mathbf{Aut}(\mathbf{G})\big)$$

In particular, if $\mathfrak{E}$ is a loop R_n–torsor under $\mathbf{Aut}(\mathbf{G})$, the Witt-Tits index of the loop group scheme ${}_{\mathfrak{E}}\mathbf{G}/R_n$ depends only of the class of $\mathfrak{E}$ in $\mathbf{GL}_n(\mathbb{Z})\backslash H^1_{loop}\big(R_n, \mathbf{Aut}(\mathbf{G})\big)$.

Remark 8.13. Assume η is a loop cocycle. Since $\mathbf{GL}_n(\mathbb{Z})$ acts trivially on $\mathbf{H}(k)$ we have $({}^g\eta)(\gamma) = \eta(\gamma^g)$ for all $\gamma \in \pi_1(R_n)$ and $g \in \mathbf{GL}_n(\mathbb{Z})$.

Lemma 8.14. *Assume that $\mathbf{H}$ acts on a quasi-projective k–variety $\mathbf{M}$. Let $\eta \in Z^1\big(\pi_1(R_n), \mathbf{H}(\overline{k})\big)$ be a loop cocycle. Let $\Lambda_\eta \subset \mathbf{GL}_n(\mathbb{Z})$ be the stabilizer of η for the (left) action of $\mathbf{GL}_n(\mathbb{Z})$ on $Z^1\big(\pi_1(R_n), \mathbf{H}(\overline{k})\big)$.*

(1) $\Lambda_\eta = \Big\{ g \in \mathbf{GL}_n(\mathbb{Z}) \;\mid\; \eta(\gamma^g) = \eta(\gamma) \;\; \forall \gamma \in \pi_1(R_n) \Big\}$.

(2) The map

$$\big(\mathbf{GL}_n(\mathbb{Z}) \ltimes \pi_1(R_n)\big) \times \mathbf{M}(\overline{R}_{n,\infty}) \to \mathbf{M}(\overline{R}_{n,\infty}), \qquad \big((g,\gamma), x\big) \mapsto g.\eta(\gamma).\gamma.x$$

defines an action of $\Lambda_\eta \ltimes \pi_1(R_n)$ on $({}_\eta\mathbf{X})(\overline{R}_{n,\infty})$.

(3) Assume that $\mathbf{M}$ is a linear algebraic k–group on which $\mathbf{H}$ acts as group automorphisms. Let $g \in \Lambda_\eta$ and $\zeta \in Z^1\big(\pi_1(R_n), {}_\eta\mathbf{M}(\overline{R}_{n,\infty})\big)$, and set

$${}^g\zeta(\gamma) = g.\zeta(\gamma^g).$$

This defines a (left) action of Λ_η on $Z^1\big(\pi_1(R_n), {}_\eta\mathbf{M}(\overline{R}_{n,\infty})\big)$ which induces an action of Λ_η on $H^1(R_n, {}_\eta\mathbf{M})$. The action is functorial in $\mathbf{M}$. If $\mathbf{H} = \mathbf{M}$, the diagram

$$\begin{array}{ccc} H^1(R_n, {}_\eta\mathbf{H}) & \xrightarrow[\sim]{\tau_\eta} & H^1(R_n, \mathbf{H}) \\ {\scriptstyle g_*}\downarrow\wr & & {\scriptstyle g_*}\downarrow\wr \\ H^1(R_n, {}_\eta\mathbf{H}) & \xrightarrow[\sim]{\tau_\eta} & H^1(R_n, \mathbf{H}) \end{array}$$

commutes for all $g \in \mathbf{GL}_n(\mathbb{Z})$, where τ_η is the twisting bijection.

(4) Assume that in (3) $\mathbf{H}$ is finite and that $\mathbf{M}$ is of multiplicative type. For $g \in \Lambda_\eta$ and an inhomogeneous (continuous) cochain $y \in \mathcal{C}^i\big(\pi_1(R_n), {}_\eta\mathbf{M}\big)$ of degree $i \geq 0$, set

$$({}^g y)(\gamma_1, \dots, \gamma_i) = {}^g\big(y(\gamma_1^g, \dots, \gamma_i^g)\big).$$

This defines a left action of Λ_η on the chain complex $\mathcal{C}^*\big(\pi_1(R_n), {}_\eta\mathbf{M}(\overline{R}_{n,\infty})\big)$ of (contnuous) inhomogeneous cochains and on $H^*\big(\pi_1(R_n), {}_\eta\mathbf{M}(\overline{R}_{n,\infty})\big) = H^*(R_n, {}_\eta\mathbf{M})$ which is functorial with respect to short exact sequences of $\mathbf{H}$–equivariant k–groups of multiplicative type.

(5) Assume that $\mathbf{H}$ is finite and let $1 \to \mathbf{M}_1 \to \mathbf{M}_2 \to \mathbf{M}_3 \to 1$ be a an exact sequence of linear algebraic k–groups equipped with an equivariant action of $\mathbf{H}$ as group automorphism. The action of Λ_η commutes with the characteristic map ${}_\eta\mathbf{M}_3(R_n) \to H^1(R_n, {}_\eta\mathbf{M}_1)$. If $\mathbf{M}_1$ is central in $\mathbf{M}_2$, then the action of Λ_η commutes with the boundary map $\Delta : H^1(R_n, {}_\eta\mathbf{M}_3) \to H^2(R_n, {}_\eta\mathbf{M}_1)$.

(6) Assume k is algebraically closed and let d be a postive integer. The base change action of $\mathbf{GL}_n(\mathbb{Z})$ on $H^2(R_2, \boldsymbol{\mu})$ and on $\mathrm{Br}(R_2)$ is given by $g.\alpha = \det(g).\alpha$.

Proof. (1) is obvious by taking into account Remark 8.13.

In what follows we take the "Galois" point of view and notation: ${}_\eta\mathbf{M}(\overline{R}_{n,\infty})$ coincides with $\mathbf{M}(\overline{R}_{n,\infty})$ as a set, but the action of $\pi_1(R_n)$ is the twisted action, which we denote by $\star$:

$$\gamma \star x = \eta(\gamma).(\gamma.x)$$

(2) The groups $\mathbf{GL}_n(\mathbb{Z})$ and $\pi_1(R_n)$ act on $\mathbf{X}(\overline{R}_{n,\infty})$ and $\mathbf{H}(\overline{R}_{n,\infty})$ via their natural action on $\overline{R}_{n,\infty}$. For We will denote these actions by $x \mapsto g.x$, and $x \mapsto \gamma.x$ for all $g \in \mathbf{GL}_n(\mathbb{Z})$, $\gamma \in \pi_1(R_n)$ and $x \in \mathbf{X}(\overline{R}_{n,\infty})$. It follows from (8.5) and (8.6) that for all $\gamma \in \pi_1(R_n)$ we have

$$\gamma.g.x = g.\gamma^g.x$$

One also verifies using the axioms of action that

$$\gamma.(h.x) = (\gamma.h).(\gamma.x)$$

for all $h \in \mathbf{H}(\overline{R}_{n,\infty})$. The content of (2) is that

$$(g,\gamma) \star x = g.(\gamma \star x) \tag{8.7}$$

defines an action of $\Lambda_\eta \ltimes \pi_1(R_n)$ on $({}_\eta\mathbf{X})(\overline{R}_{n,\infty})$. Write for convenience $g.\gamma \star x$ instead of $g.(\gamma \star x)$ since no confusion is possible. Then

$$
\begin{aligned}
(f,\alpha)\star(g,\beta)\star x &= f.\eta(\alpha).\alpha.g.\eta(\beta).b.x \quad \text{[definition of the twisted action]}\\
&= f.\eta(\alpha).g.\alpha^g.\eta(\beta).\beta.x\\
&= f.g.\eta(\alpha).\alpha^g.\eta(\beta).\beta.x \quad [\eta \text{ is a loop cocycle}]\\
&= f.g.\eta(\alpha).(\alpha^g.\eta(\beta)).\alpha^g.(\beta.x)\\
&= f.g.\eta(\alpha^g).(\alpha^g.\eta(\beta)).\alpha^g.(\beta.x) \quad [g\in\Lambda_\eta]\\
&= fg.\eta(\alpha^g\beta).\alpha^g(\beta.x) \quad [\eta \text{ a cocycle}]\\
&= (fg,\alpha^g\beta)\star x\\
&= \big((f,\alpha)(g,\beta)\big)\star x.
\end{aligned}
$$

(3) One checks that ${}^g\eta$ is a cocycle and that two equivalent cocycles remain equivalent under this action along the same lines of Lemma 8.10.

The commutativity of the diagram takes place already at the level of cocycles. Indeed. Consider the square

$$
\begin{tikzcd}
Z^1\big(\pi_1(R_n), {}_\eta\mathbf{H}(\overline{R}_{n,\infty})\big) \arrow[r, "\tau_\eta", "\sim"'] \arrow[d, "g_*"', "\wr"] & Z^1\big(\pi_1(R_n), \mathbf{H}(\overline{R}_{n,\infty})\big) \arrow[d, "g_*"', "\wr"] \\
Z^1\big(\pi_1(R_n), {}_\eta\mathbf{H}(\overline{R}_{n,\infty})\big) \arrow[r, "\tau_\eta", "\sim"'] & Z^1\big(\pi_1(R_n), \mathbf{H}(\overline{R}_{n,\infty})\big)
\end{tikzcd}
$$

Given a cocycle $\phi \in Z^1\big(\pi_1(R_n), {}_\eta\mathbf{H}(\overline{R}_{n,\infty})\big)$ recall that $(\tau_\eta\phi)(\gamma) = \phi(\gamma)\,\eta(\gamma)$, hence

$$
{}^g\big((\tau_\eta)(\phi)\big)(\gamma) = g.\big(\tau_\eta(\phi)(\gamma^g)\big) = g.\big((\phi)(\gamma^g)\,\eta(\gamma^g)\big) \quad , \gamma \in \Lambda_\eta.
$$

$$
g.(\phi)(\gamma^g)g\,.\,\eta(\gamma^g) = {}^g\phi(\gamma)g\,.\,\eta(\gamma^g) = {}^g\phi(\gamma)\,\eta(\gamma)
$$

since $g.\eta(\gamma^g) = \eta(\gamma^g)$ because η is a loop cocycle, and $\eta(\gamma^g) = \eta(\gamma)$ because $g \in \Lambda_\eta$. On the other hand by definition of the twisting map

$$
\tau_\eta({}^g\phi)(\gamma) = {}^g\phi(\gamma)\,\eta(\gamma)
$$

so that the diagram above commutes.

(4) The continuous profinite cohomology is the direct limit of discrete group cohomology of finite quotients. Hence it is enough to establish the desired results at the "finite level", namely for a group $\Gamma = \mathrm{Gal}(\tilde{R}_{n,m}/R_n)$ where $\tilde{R}_{n,m} = \tilde{k} \otimes_k R_{n,m}$ is a finite Galois covering of R_n through wich η factors, and such that $\mathbf{H}(\tilde{k}) = \mathbf{H}(\overline{k})$. Recall that the action of $\mathbf{GL}_n(\mathbb{Z})$ on $\overline{R}_{n,\infty}$ preserves $\tilde{k} \otimes_k R_{n,m}$, so that $\mathbf{GL}_n(\mathbb{Z})$ acts on Γ.

We need to check that the given action of Λ_η on $C^*(\Gamma, {}_\eta A)$ commutes with the differentials. We are given $g \in \Lambda_\eta$ and $y \in \mathcal{C}^i(\Gamma, {}_\eta A)$. Recall that the boundary map $\partial_i : \mathcal{C}^i(\Gamma, {}_\eta A) \to \mathcal{C}^{i+1}(\Gamma, {}_\eta A)$ is given by

$$\big(\partial_i(y)\big)(\gamma_1, \dots, \gamma_{i+1}) =$$
$$\gamma_1 \,.\, \eta(\gamma_1).y(\gamma_1, \dots, \gamma_{i+1}) + \sum_{j=1}^{i} (-1)^j \, y(\gamma_1, \dots, \gamma_{j-1}, \gamma_j\gamma_{j+1}, \gamma_{j+2}, \dots \gamma_{i+1}) + (-1)^{i+1} y(\gamma_1, \dots, \gamma_i).$$

Thus

$$\begin{aligned}
\Big({}^g\big(\partial_i(y)\big)\Big)(\gamma_1, \dots, \gamma_{i+1}) \;&=\; g.\big(\partial_i(y)(\gamma_1^g, \dots, \gamma_{i+1}^g)\big) \\
&= g.\big(\gamma_1^g \,\eta(\gamma_1^g) \,.\, y(\gamma_1^g, \dots, \gamma_{i+1}^g)\big) \\
&\qquad + g.\big(\sum_{j=1}^{i} (-1)^j \; y(\gamma_1^g, \dots, \gamma_{j-1}^g, \gamma_j^g\gamma_{j+1}^g, \gamma_{j+2}^g, \dots \gamma_{i+1}^g)\big) \\
&\qquad + g.\big((-1)^{i+1} \, y(\gamma_1^g, \dots, \gamma_i^g)\big) \\
&= \gamma_1 \,\eta(\gamma_1) \,.\, {}^g y(\gamma_1, \dots, \gamma_{i+1}) \qquad [g \in \Lambda_\eta] \\
&\qquad + \sum_{j=1}^{i} (-1)^j \; {}^g y(\gamma_1, \dots, \gamma_{j-1}, \gamma_j\gamma_{j+1}, \gamma_{j+2}, \dots \gamma_{i+1}) \\
&\qquad + (-1)^{i+1} \, {}^g y(\gamma_1, \dots, \gamma_i) \\
&= \big(\partial_i({}^g y)\big)(\gamma_1, \dots, \gamma_{i+1}).
\end{aligned}$$

This shows that the action of Λ_η on $\mathcal{C}^i(\Gamma, {}_\eta A)$ commutes with the boundary maps as desired.

(5) We are given an exact sequence of linear algebraic groups $1 \to \mathbf{M}_1 \to \mathbf{M}_2 \to \mathbf{M}_3 \to 1$ equipped with an action of $\mathbf{H}$. We twist it by η to obtain $1 \to {}_\eta\mathbf{M}_1 \to {}_\eta\mathbf{M}_2 \to {}_\eta\mathbf{M}_3 \to 1$, and look at the characteristic map

$$\psi : {}_\eta\mathbf{M}_3(R_n) \to H^1(R_n, {}_\eta\mathbf{M}_1).$$

Let $x_3 \in {}_\eta\mathbf{M}_3(R_n) \subset \mathbf{M}_3(\overline{R}_{n,\infty})$. Lift x_3 to an element $x_2 \in \mathbf{M}_2(\overline{R}_{n,\infty})$. Then $\psi(x_3) = [z_\gamma]$ with $z_\gamma = x_2^{-1}\big(\eta(\gamma)\, {}^\gamma x_2\big)$. Now if $g \in \Lambda_\eta$ the element ${}^g x_2$ lifts ${}^g x_3$, hence $\psi({}^g x_3)$ is represented by the 1–cocycle

$$({}^g x_2)^{-1} \,\big(\eta(\gamma)\, {}^\gamma({}^g x_2)\big) = {}^g {x_2}^{-1}\big(\eta(\gamma)\, {}^{\gamma g} x_2\big) = g.\big({x_2}^{-1}(\eta(\gamma)\, {}^{\gamma^g} x_2)\big) = ({}^g z)_\gamma$$

by using again $\eta(\gamma^g) = \eta(\gamma)$ and the fact that g acts trivially on $\mathbf{H}(\overline{k})$. This shows that $\psi({}^g x_2) = {}^g\psi(x_2)$.

Assuming that $\mathbf{M}_1$ is central and of multiplicative type, we consider the boundary map $\Delta : H^1(R_n, {}_\eta\mathbf{M}_3) \to H^2(R_n, {}_\eta\mathbf{M}_1)$. By isotriviality, the precise nature of this map can be computed at the "finite level" by means of Galois cocycles. Let (a_γ) be a cocycle with value ${}_\eta\mathbf{M}_3(\overline{R}_{n,\infty}) = \mathbf{M}_3(\overline{R}_{n,\infty})$ and choose a lifting (b_γ) in $\mathbf{M}_2(\overline{R}_{n,\infty})$ which is trivial on an open subgroup of $\pi_1(R_n)$. Recall that $\Delta([a_\gamma]) \in H^2\big(\pi_1(R_n), {}_\eta\mathbf{M}_1(\overline{R}_{n,\infty})\big)$ is the class of the 2–cocycle [Se, I.5.6]

$$c_{\gamma,\tau} = b_\gamma \,(\eta(\gamma).{}^\gamma b_\tau)\, b_{\gamma\tau}^{-1}.$$

Similarly, the $({}^g b_{\gamma^g})$ lift the $({}^g a_{\gamma^g})$, so $\Delta(g.[a_\gamma])$ is the class of the 2–cocycle

$${}^g b_{\gamma^g}\,\big(\eta(\gamma).{}^\gamma({}^g b_{\tau^g})\big)\, {}^g b^{-1}_{\gamma^g\tau^g} = g.\Big(b_\gamma\,(\eta(\gamma).{}^{\gamma^g} b^g_\tau)\, b^{-1}_{\gamma^g\tau^g}\Big) = g.\Delta([a_\gamma])$$

as desired.

6) Since $H^2(R_2, \boldsymbol{\mu}_d)$ injects in $\mathrm{Br}(R_2) = \mathbb{Q}/\mathbb{Z}$ [GP2, 2.1], it is enough to check the formula on $\mathrm{Br}(R_2)$. Since $\mathbf{GL}_2(\mathbb{Z})$ is generated by the matrices $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, it is enough to show that the desired compatibility holds when g is one of these four elements. Consider the cyclic Azumaya R_2-algebra $A = A(1,d)$ with presentation $T_1^d = t_1, T_2^d = t_2, T_2T_1 = \zeta_d T_1 T_2$. Then for g in the above list we have $g.[A] = [A]$ (resp. $-[A]$, $[A]$, $-[A]$) respectively, so that $g.[A] = \det(g).[A]$ Since the classes of these cyclic Azumaya algebras generate $\mathrm{Br}(R_2)$ the result follows. $\square$

Remark 8.15. (a) In (4), we have $H^*\big(\pi_1(R_n), {}_\eta\mathbf{M}(\overline{R}_{n,\infty})\big) \xrightarrow{\sim} H^*(R_n, {}_\eta\mathbf{M})$ [GP3, prop 3.4], hence we have a natural action of Λ_η on $H^*(R_n, {}_\eta\mathbf{M})$. We have used an explicit description of this action in our proof, but the result can also be established in a more abstract setting. For $g \in \Lambda_\eta$, we claim that the map $g_* : A \to A, a \mapsto g.a$ applies $H^0(\Gamma, {}_\eta A)$ into itself. Indeed for $a \in H^0(\Gamma, {}_\eta A)$ and $\gamma \in \Gamma$, we compute the twisted action just as we did in (2) of the Lemma.

$$\begin{array}{rcll}
\gamma \star (g.a) & = & (\eta(\gamma)\,\gamma g)\,.a & \\
& = & (\eta(\gamma)\, g\, \gamma^g)\,.a & [\text{definition of } \gamma^g] \\
& = & (g\,\eta(\gamma)\,\gamma^g)\,.a & [\mathbf{GL}_n(\mathbb{Z}) \text{ commutes with } \mathbf{H}(\tilde{k})] \\
& = & (g\,\eta(\gamma^g)\gamma^g)\,.a & [g \in \Lambda_\eta] \\
& = & g.a \quad [a \in H^0(\Gamma, {}_\eta A)]. &
\end{array}$$

We get then a morphism of functors $g_* : F \to F$ which extends uniquely as a morphism of δ-functors [W, §2.5]. This then yields the desired natural transformations $g_* : H^i(\Gamma, {}_\eta A) \to H^i(\Gamma,_\eta A)$ for each $\mathbf{GL}_n(\mathbb{Z}) \ltimes \big(\mathbf{H}(\tilde{k}) \rtimes \Gamma\big)$-module.

(b) There is an analogous statement to (5) for homogeneous spaces.

For each class $[\mathfrak{E}] \in H^1\big(R_n, \mathbf{Out}(\mathbf{G})\big)$, we denote by $H^1\big(R_n, \mathbf{Aut}(\mathbf{G})\big)_{[\mathfrak{E}]}$ the fiber at $[\mathfrak{E}]$ of the map $H^1\big(R_n, \mathbf{Aut}(\mathbf{G})\big) \to H^1\big(R_n, \mathbf{Out}(\mathbf{G})\big)$. We then have the decomposition

$$H^1\big(R_n, \mathbf{Aut}(\mathbf{G})\big) = \bigsqcup_{[\mathfrak{E}] \in H^1(R_n, \mathbf{Out}(\mathbf{G}))} H^1\big(R_n, \mathbf{Aut}(\mathbf{G})\big)_{[\mathfrak{E}]} \tag{8.8}$$

The group $\mathbf{GL}_n(\mathbb{Z})$ acts on $H^1\big(R_n, \mathbf{Out}(\mathbf{G})\big)$ and on $H^1\big(R_n, \mathbf{Aut}(\mathbf{G})\big)$ by base change (see Lemma 8.10). It follows that $\mathbf{GL}_n(\mathbb{Z})$ permutes the subsets of the partition (8.8), and that for each class $[\mathfrak{E}] \in H^1\big(R_n, \mathbf{Out}(\mathbf{G})\big)$, its stabilizer under the action of $\mathbf{GL}_n(\mathbb{Z})$ preserves $H^1\big(R_n, \mathbf{Aut}(\mathbf{G})\big)_{[\mathfrak{E}]}$.

Let $\mathrm{Out}(\mathbf{G}) = \mathbf{Out}(\mathbf{G})(k)$. The (abstract) group $\mathrm{Out}(\mathbf{G})$ acts naturally on the right on the set of (continuous) homomorphisms $\mathrm{Hom}\big(\pi_1(R_n), \mathbf{Out}(\mathbf{G})\big)$. This action, which we denote by int, is given by $\mathrm{int}(a)(\phi)(\gamma) = \phi^a(\gamma) = a^{-1}\,\phi(\gamma)\,a$. We have $H^1\big(R_n, \mathbf{Out}(\mathbf{G})\big) = \mathrm{Hom}\,\big(\pi_1(R_n), \mathrm{Out}(\mathbf{G})\big)/\mathrm{int}\big(\mathrm{Out}(\mathbf{G})\big)$

We consider a system of representatives $([\phi_j])_{j\in J}$ of the set of double cosets $\mathbf{GL}_n(\mathbb{Z}) \setminus \mathrm{Hom}\,\big(\pi_1(R_n), \mathrm{Out}(\mathbf{G})\big)/\mathrm{int}\big(\mathrm{Out}(\mathbf{G})\big)$. Consider a fixed element $j \in J$. Denote by $\Lambda_j \subset \mathbf{GL}_n(\mathbb{Z})$ the stabilizer of $[\phi_j] \in H^1\big(R_n, \mathbf{Out}(\mathbf{G})\big)$ for the base change action of $\mathbf{GL}_n(\mathbb{Z})$ on $\mathrm{Spec}(R_n)$. An element $g \in \mathbf{GL}_n(\mathbb{Z})$ belongs to Λ_j if and only if there exists $a_g \in \mathrm{Out}(\mathbf{G})$ such that the following diagram commutes

$$\begin{array}{ccc} \phi_j : \pi_1(R_n) & \longrightarrow & \mathbf{Out}(\mathbf{G}) \\ {\scriptstyle g^*}\Big\uparrow \wr & & {\scriptstyle \mathrm{int}(a_g)}\Big\uparrow \wr \\ \phi_j : \pi_1(R_n) & \longrightarrow & \mathbf{Out}(\mathbf{G}) \end{array}$$

Note that $\Lambda_{\phi_j} \subset \Gamma_j$. We have

$$\mathbf{GL}_n(\mathbb{Z})\backslash H^1\big(R_n, \mathbf{Aut}(\mathbf{G})\big) = \bigsqcup_{j\in J} \Lambda_j \backslash H^1\big(R_n, \mathbf{Aut}(\mathbf{G})\big)_{[\phi_j]}. \tag{8.9}$$

Recall that our section $s : \mathbf{Out}(\mathbf{G}) \to \mathbf{Aut}(\mathbf{G})$ is determined by our choice of pinning of $(\mathbf{G}, \mathbf{B}, \mathbf{T})$. This allows us to trace the action of Λ_j. Indeed $[s_*(\phi_j)] \in H^1\big(R_n, \mathbf{Aut}(\mathbf{G})\big)_{[\phi_j]}$, so that the classical twisting argument (see [Gi4, lemme 1.2]) shows that the map

$$H^1(R_n, {}_{s_*(\phi_j)}\mathbf{G}) \to H^1\big(R_n, {}_{s_*(\phi_j)}\mathbf{Aut}(\mathbf{G})\big) \overset{\tau_{s_*(\phi_j)}}{\to} H^1\big(R_n, \mathbf{Aut}(\mathbf{G})\big)$$

induces a bijection

$$H^1(R_n, {}_{s_*(\phi_j)}\mathbf{G})/H^0\big(R_n, {}_{\phi_j}\mathbf{Out}(\mathbf{G})\big) \overset{\sim}{\longrightarrow} H^1\big(R_n, \mathbf{Aut}(\mathbf{G})\big)_{[\phi_j]}. \tag{8.10}$$

Note that the action of an element $a \in H^0\big(R_n, {}_{\phi_j}\mathbf{Out}(\mathbf{G})\big)$ on $H^1\big(R_n, {}_{s_*(\phi_j)}\mathbf{G}\big)$ is given by

$$H^1(R_n, {}_{s_*(\phi)}\mathbf{G}) \overset{({}_{\phi_j}s_*)(a)}{\overset{\sim}{\longrightarrow}} H^1(R_n, {}_{s_*(\phi_j)}\mathbf{G}).$$

where $({}_{\phi_j}s_*)$ is the twist of s_* by the cocycle ϕ_j. Furthermore the map (8.10) preserves toral or, what is equivalent, loop classes. Feeding this information into the decomposition (8.9), we get

(8.11)
$$\mathbf{GL}_n(\mathbb{Z})\backslash H^1\big(R_n, \mathbf{Aut}(\mathbf{G})\big) \overset{\sim}{\longrightarrow} \bigsqcup_{j\in J} \Lambda_j\backslash\Big(H^1(R_n, {}_{s_*(\phi_j)}\mathbf{G})/H^0\big(R_n, {}_{\phi j}\mathbf{Out}(\mathbf{G})\big)\Big).$$

At least in certain cases, the action of Λ_j on $H^1(R_n, {}_{s_*(\phi_j)}\mathbf{G})/H^0\big(R_n, {}_{\phi_j}\mathbf{Aut}(\mathbf{G})\big)$ can be understood quite nicely (see Remark 8.17 below).

Lemma 8.16. *(1) For each $g \in \Lambda_{\phi_j}$, the following diagrams*

$$\begin{array}{ccccc}
H^1(R_n, {}_{s_*(\phi_j)}\mathbf{G}) & \longrightarrow & H^1\big(R_n, {}_{s_*(\phi_j)}\mathbf{Aut}(\mathbf{G})\big) & \xrightarrow{\tau_{s_*(\phi_j)}} & H^1\big(R_n, \mathbf{Aut}(\mathbf{G})\big)_{[\phi_j]} \\
{\scriptstyle g_*}\big\downarrow & & {\scriptstyle g_*}\big\downarrow & & {\scriptstyle g_*}\big\downarrow \\
H^1(R_n, {}_{s_*(\phi_j)}\mathbf{G}) & \longrightarrow & H^1\big(R_n, {}_{s_*(\phi_j)}\mathbf{Aut}(\mathbf{G})\big) & \xrightarrow{\tau_{s_*(\phi_j)}} & H^1\big(R_n, \mathbf{Aut}(\mathbf{G})\big)_{[\phi_j]},
\end{array}$$

$$\begin{array}{ccccc}
H^1(R_n, {}_{s_*(\phi_j)}\mathbf{G}) & \times & H^0\big(R_n, {}_{\phi_j}\mathbf{Out}(\mathbf{G})\big) & \longrightarrow & H^1(R_n, {}_{s_*(\phi_j)}\mathbf{G}) \\
{\scriptstyle g_*}\big\downarrow & & {\scriptstyle id}\big\downarrow & & {\scriptstyle g_*}\big\downarrow \\
H^1(R_n, {}_{s_*(\phi_j)}\mathbf{G}) & \times & H^0\big(R_n, {}_{\phi_j}\mathbf{Out}(\mathbf{G})\big) & \longrightarrow & H^1(R_n, {}_{s_*(\phi_j)}\mathbf{G})
\end{array}$$

commute where the maps g_* are the base change maps defined in Lemma 8.14.

(2) The map (8.10)

$$H^1(R_n, {}_{s_*(\phi_j)}\mathbf{G}) \to H^1\big(R_n, \mathbf{Aut}(\mathbf{G})\big)_{[\phi_j]}$$

is $\Lambda_{\phi_j} \times H^0(R_n, {}_{\phi_j}\mathbf{Out}(\mathbf{G}))^{op}$–equivariant and

$$\Lambda_{\phi_j} \times H^0\big(R_n, {}_{\phi_j}\mathbf{Out}(\mathbf{G})\big)^{op} \setminus H^1\big(R_n, {}_{s_*(\phi_j)}\mathbf{G}\big) \overset{\sim}{\longrightarrow} \Lambda_{\phi_j} \setminus H^1\big(R_n, \mathbf{Aut}(\mathbf{G})\big)_{[\phi_j]}.$$

Remark 8.17. Of course (2) is useful provided that $\Lambda_{\phi_j} = \Gamma_j$. This is the case for simple groups which are not of type D_4 since $\mathbf{Out}(\mathbf{G}) = 1$ or $\mathbb{Z}/2\mathbb{Z}$.

Proof. (1) We are given $g \in \Lambda_{\phi_j}$. The left square of the first diagram commutes by the functoriality of the base change map g_*. The commutativity of the right square follows of Lemma 8.14.(3) applied to the k–group $\mathbf{Aut}(\mathbf{G})$ and the cocycle $s_*(\phi_j)$. The commutativity of the second diagram follows from the action on cocycles given in Lemma 8.10.

(2) By (1), the left action of Λ_{ϕ_j} and the right action of $H^0\big(R_{n,\,\phi_j}\mathbf{Out}(\mathbf{G})\big)$ on $H^1(R_{n,\,s_*(\phi_j)}\mathbf{G}$ commute. Hence

$$\Lambda_{\phi_j}\backslash\Big(H^1(R_{n,\,s_*(\phi_j)}\mathbf{G})/H^0\big(R_{n,\,s_*(\phi_j)}\mathbf{Aut}(\mathbf{G})\big)\Big) \xrightarrow{\sim}$$

$$\Lambda_{\phi_j} \times H^0(R_{n,\,s_*(\phi_j)}\mathbf{Out}(\mathbf{G}))^{op} \setminus H^1(R_{n,\,s_*(\phi_j)}\mathbf{G})$$

and this set maps bijectively onto $\Lambda_{\phi_j}\backslash H^1(R_n, \mathbf{Aut}(\mathbf{G}))_{[\phi_j]}$. $\square$

9 Small dimensions

9.1 The one dimensional case

By combining Theorem 8.1, Corollary 5.2 and Lemma 4.14 we get the following generalization (in characteristic 0) of theorem 2.4 of [CGP].

Corollary 9.1. *Let $\mathbf{G}$ be a linear algebraic k–group Then we have bijections*

$$H^1_{toral}(k[t^{\pm 1}], \mathbf{G}) \xrightarrow{\sim} H^1_{loop}(k[t^{\pm 1}], \mathbf{G}) \xrightarrow{\sim} H^1(k[t^{\pm 1}], \mathbf{G}) \xrightarrow{\sim} H^1(k((t)), \mathbf{G}).$$

In the case when k is algebraically closed, we also recover the original results of [P1] and [P2] that began the "cohomological approach" to classification problems in infinite dimensional Lie theory.

9.2 The two dimensional case

Throughout this section we assume that k is algebraically closed of characteristic 0 and $\mathbf{G}$ a semisimple Chevalley k–group of adjoint type. We let $\mathbf{G}^{sc} \to \mathbf{G}$ be its simply connected covering and denote by $\boldsymbol{\mu}$ its kernel.

9.2.1 Classification of semisimple loop R_2-groups.

Serre's conjecture II holds for the field F_2 by Bruhat–Tits theory [BT3, cor. 3.15], i.e. $H^1(F_2, \mathbf{H}) = 1$ for every semisimple simply connected group $\mathbf{H}$ over F_2. Furthermore, we know explicitly how to compute the Galois cohomology of an arbitrary semisimple F_2 group [CTGP, th. 2.1] and [GP2, th. 2.5]. We thus have.

Corollary 9.2. *We have a decomposition*

$$H^1_{loop}\big(R_2, \mathbf{Aut}(\mathbf{G})\big) \overset{\sim}{\longrightarrow} \bigsqcup_{[\mathfrak{E}]\in H^1(R_2,\mathbf{Out}(\mathbf{G}))} {}_{\mathfrak{E}}\mathbf{Out}(\mathbf{G})(R_2)\,\backslash H^2(R_2, {}_{\mathfrak{E}}\boldsymbol{\mu})$$

and the inner R_2–forms of $\mathbf{G}$ *are classified by the coset* $\mathbf{Out}(\mathbf{G})(R_2)\,\backslash H^2(R_2, \boldsymbol{\mu})$.

Note that the case when $\mathbf{Out}(\mathbf{G})$ is trivial recovers theorem 3.17 of [GP2]. We can thus view the last Corollary as an extension of this theorem to the case when the automorphism group of $\mathbf{G}$ is not connected.

Proof. Our choice of splitting $s : \mathbf{Out}(\mathbf{G}) \to \mathbf{Aut}(\mathbf{G})$ of the exact sequence

$$1 \to \mathbf{G} \to \mathbf{Aut}(\mathbf{G}) \to \mathbf{Out}(\mathbf{G}) \to 1$$

easily leads to the decomposition (see [Gi4, lemme 1.2])

$$H^1\big(F_2, \mathbf{Aut}(\mathbf{G})\big) \overset{\sim}{\longrightarrow} \bigsqcup_{[\mathfrak{E}]\in H^1(F_2,\mathbf{Out}(\mathbf{G}))} H^1(F_2, {}_{\mathfrak{E}}\mathbf{G})/{}_{\mathfrak{E}}\mathbf{Out}(\mathbf{G})(F_2)$$

with respect to the Dynkin-Tits invariant. On the other hand, the boundary map $H^1(F_2, {}_{\mathfrak{E}}\mathbf{G}) \to H^2(F_2, {}_{\mathfrak{E}}\boldsymbol{\mu})$ is bijective by [CTGP, th. 2.1] and [GP2, th. 2.5]. The right action of ${}_{\mathfrak{E}}\mathbf{Out}(\mathbf{G})(F_2)$ can then be transferred to $H^2(F_2, {}_{\mathfrak{E}}\boldsymbol{\mu})$, and is the opposite of the natural left action of ${}_{\mathfrak{E}}\mathbf{Out}(\mathbf{G})(F_2)$ on $H^2(F_2, {}_{\mathfrak{E}}\boldsymbol{\mu})$. Hence

$$H^1(F_2, \mathbf{Aut}(\mathbf{G})) \overset{\sim}{\longrightarrow} \bigsqcup_{[\mathfrak{E}]\in H^1(F_2,\mathbf{Out}(\mathbf{G}))} {}_{\mathfrak{E}}\mathbf{Out}(\mathbf{G})(F_2)\,\backslash H^2(F_2, {}_{\mathfrak{E}}\boldsymbol{\mu}).$$

But ${}_{\mathfrak{E}}\mathbf{Out}(\mathbf{G})$ is finite étale over R_n hence ${}_{\mathfrak{E}}\mathbf{Out}(\mathbf{G})(R_2) = {}_{\mathfrak{E}}\mathbf{Out}(\mathbf{G})(F_2)$ by Remark 6.8.(d). On the other hand, we have $H^2(R_2, {}_{\mathfrak{E}}\boldsymbol{\mu}) \overset{\sim}{\longrightarrow} H^2(F_2, {}_{\mathfrak{E}}\boldsymbol{\mu})$ since ${}_{\mathfrak{E}}\boldsymbol{\mu}$ is an R_2–group of multiplicative type [GP3, prop. 3.4]. Taking into account the acyclicity theorem for $\mathbf{Aut}(\mathbf{G})$ and $\mathbf{Out}(\mathbf{G})$, we get the square of bijections

$$\begin{array}{ccc}
H^1(F_2, \mathbf{Aut}(\mathbf{G})) & \underset{\sim}{\longrightarrow} & \displaystyle\bigsqcup_{[\mathfrak{E}]\in H^1(F_2,\mathbf{Out}(\mathbf{G}))} {}_{\mathfrak{E}}\mathbf{Out}(\mathbf{G})(F_2)\,\backslash H^2(F_2, {}_{\mathfrak{E}}\boldsymbol{\mu}). \\
\uparrow \wr & & \uparrow \wr \\
H^1_{loop}(R_2, \mathbf{Aut}(\mathbf{G})) & \underset{\sim}{\longrightarrow} & \displaystyle\bigsqcup_{[\mathfrak{E}]\in H^1(R_2,\mathbf{Out}(\mathbf{G}))} {}_{\mathfrak{E}}\mathbf{Out}(\mathbf{G})(R_2)\,\backslash H^2(R_2, {}_{\mathfrak{E}}\boldsymbol{\mu}),
\end{array}$$

and this establishes the Corollary. $\square$

Next we give a complete list of the isomorphism classes of loop R_2–forms of $\mathbf{G}$ in the case when $\mathbf{G}$ is simple of adjoint type. We have $\mathbf{Out}(\mathbf{G}) = 1$ in type A_1 B, C, E_7, E_8, F_4 and G_2, $\mathbf{Out}(\mathbf{G}) = \mathbb{Z}/2\mathbb{Z}$ in type A_n $(n \geq 2)$, D_n $(n \geq 5)$ and E_6, and $\mathbf{Out}(\mathbf{G}) = S_3$ in type D_4. [26]

In the case $\mathbf{Out}(\mathbf{G}) = 1$, then by theorem 3.17 of [GP2] we have $H^1_{loop}(R_2, \mathbf{Aut}(\mathbf{G})) \xrightarrow{\sim} H^2(R_2, \boldsymbol{\mu})$. But $\boldsymbol{\mu} = \boldsymbol{\mu}_n$ for $n = 1$ or 2. We have $H^2(R_2, \boldsymbol{\mu}_2) \cong \mathbb{Z}/2\mathbb{Z}$ [GP2, §2.1].Thus

Corollary 9.3. *(1) If $\mathbf{G}$ has type A_1, then $H^1_{loop}\big(R_2, \mathbf{Aut}(\mathbf{G})\big) \simeq \mathbb{Z}/2\mathbb{Z}$.*

(2) If $\mathbf{G}$ has type B, C or E_7, then $H^1_{loop}\big(R_2, \mathbf{Aut}(\mathbf{G})\big) \simeq \mathbb{Z}/2\mathbb{Z}$.

(3) If $\mathbf{G}$ has type E_8, F_4 or G_2, then $H^1_{loop}\big(R_2, \mathbf{Aut}(\mathbf{G})\big) = 1$. □

Remark 9.4. In Case (1) and Case (2) the non-trivial twisted groups are not quasisplit (because their "Brauer invariant" in $H^2(R_2, \boldsymbol{\mu})$ is not trivial.) In Case (1) the non-trivial twisted group is in fact anisotropic (see [GP2] for details).

In the case $\mathbf{Out}(\mathbf{G}) = \mathbb{Z}/2\mathbb{Z}$, we have $H^1(R_2, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. The $\mathbb{Z}/2\mathbb{Z}$-Galois extensions of R_2 under consideration are $R_2 \times R_2$, $R_2[\sqrt{t_1}]$, $R_2[\sqrt{t_2}]$ and $R_2[\sqrt{t_1 t_2}]$ which correspond to the elements $(0,0), (1,0), (0,1)$ and $(1,1)$ respectively. These can also be thought as $\mathbb{Z}/2\mathbb{Z}$-torsors over R_2 that we will denote by $\mathfrak{E}_{0,0}$, $\mathfrak{E}_{1,0}$, $\mathfrak{E}_{0,1}$ and $\mathfrak{E}_{1,1}$ respectively. In the first case the generator of the Galois group acts by permuting the two factors, while in the other three is of the form $\sqrt{x} \mapsto -\sqrt{x}$.

Since ${}_{\mathfrak{E}}\mathbf{Out}(\mathbf{G}) \cong \mathbf{Out}(\mathbf{G}) = \mathbb{Z}/2\mathbb{Z}$, for any of our four torsors we have

$$H^1_{loop}\big(R_2, \mathbf{Aut}(\mathbf{G})\big) \xrightarrow{\sim} \mathbb{Z}/2\mathbb{Z} \backslash H^2(R_2, \boldsymbol{\mu}) \bigsqcup_{\mathfrak{E}=\mathfrak{E}_{1,0},\mathfrak{E}_{0,1},\mathfrak{E}_{1,1}} \mathbb{Z}/2\mathbb{Z} \backslash H^2(R_2, {}_{\mathfrak{E}}\boldsymbol{\mu}).$$

This leads to a case by case discussion.

Corollary 9.5. *(1) For $\mathbf{G}$ of type A_{2n} $(n \geq 1)$*

$$H^1_{loop}\big(R_2, \mathbf{Aut}(\mathbf{G})\big) \simeq \{\pm 1\} \backslash \Big(\mathbb{Z}/(2n+1)\mathbb{Z}\Big) \bigsqcup_{\mathfrak{E}=\mathfrak{E}_{1,0},\mathfrak{E}_{0,1},\mathfrak{E}_{1,1}} \{{}_{\mathfrak{E}}\mathbf{G}\}.$$

There are $n+1$ inner and three outer loop R_2–forms of $\mathbf{G}$. All outer forms are quasisplit.

(2) For $\mathbf{G}$ of type A_{2n-1} $(n \geq 2)$

$$H^1_{loop}\big(R_2, \mathbf{Aut}(\mathbf{G})\big) \simeq \{\pm 1\} \backslash \Big(\mathbb{Z}/2n\mathbb{Z}\Big) \bigsqcup_{\mathfrak{E}=\mathfrak{E}_{1,0},\mathfrak{E}_{0,1},\mathfrak{E}_{1,1}} \{{}_{\mathfrak{E}}\mathbf{G}^{\pm}\}.$$

[26] Of course here $1, \mathbb{Z}/2\mathbb{Z}$ and S_3 are here viewed as constant R_2 groups or finite (abstract) groups as the situation requires.

There are $n+1$ inner and six outer loop R_2–forms of $\mathbf{G}$. The outer forms come in three pairs. Each pair has one form which is quasisplit and one which is not.

(3) For $\mathbf{G}$ of type D_{2n-1} $(n \geq 3)$

$$H^1_{loop}\big(R_2, \mathbf{Aut}(\mathbf{G})\big) \simeq \{\pm 1\} \backslash \Big(\mathbb{Z}/4\mathbb{Z}\Big) \quad \bigsqcup_{\mathfrak{E}=\mathfrak{E}_{1,0},\mathfrak{E}_{0,1},\mathfrak{E}_{1,1}} \{{}_{\mathfrak{E}}\mathbf{G}^{\pm}\}.$$

There are three inner and six outer loop R_2–forms of $\mathbf{G}$. The outer forms come in three pairs. Each pair has one form which is quasisplit and one which is not.

(4) For $\mathbf{G}$ of type D_{2n} $(n \geq 3)$

$$H^1_{loop}\big(R_2, \mathbf{Aut}(\mathbf{G})\big) \simeq \mathrm{switch} \backslash \Big(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}\Big) \quad \bigsqcup_{\mathfrak{E}=\mathfrak{E}_{1,0},\mathfrak{E}_{0,1},\mathfrak{E}_{1,1}} \{{}_{\mathfrak{E}}\mathbf{G}^{\pm}\}.$$

There are three inner and six outer loop R_2–forms of $\mathbf{G}$. The outer forms come in three pairs. Each pair has one form which is quasisplit and one which is not.

(5) For $\mathbf{G}$ of type E_6

$$H^1_{loop}(R_2, \mathbf{Aut}(\mathbf{G})) \simeq \{\pm 1\} \backslash \Big(\mathbb{Z}/3\mathbb{Z}\Big) \quad \bigsqcup_{\mathfrak{E}=\mathfrak{E}_{1,0},\mathfrak{E}_{0,1},\mathfrak{E}_{1,1}} \{{}_{\mathfrak{E}}\mathbf{G}\}.$$

There are two inner and three outer loop R_2–forms of $\mathbf{G}$. All outer forms are quasisplit

Proof. (1) We have $\boldsymbol{\mu} = \boldsymbol{\mu}_{2n+1} = \ker\big(\boldsymbol{\mu}^2_{2n+1} \stackrel{\prod}{\to} \boldsymbol{\mu}_{2n+1}\big)$ and the action of $\mathbb{Z}/2\mathbb{Z}$ switches the two factors. We have $H^2(R_2, \boldsymbol{\mu}) \simeq \mathbb{Z}/(2n+1)\mathbb{Z}$ and the outer action of $\mathbb{Z}/2\mathbb{Z}$ is by signs.[27]

Let $\mathfrak{E} = \mathfrak{E}_{(1,0)}$. It follows that ${}_{\mathfrak{E}}\boldsymbol{\mu} = \ker\big(\prod_{R_2[\sqrt{t_1}]/R_2} \boldsymbol{\mu}_{2n+1} \stackrel{norm}{\to} \boldsymbol{\mu}_{2n+1}\big)$. Since $2n+1$ is odd, the norm is split and Shapiro lemma yields

$$H^2(R_2,\, {}_{\mathfrak{E}}\boldsymbol{\mu}) = \ker\big(H^2(R_2[\sqrt{t_1}], \boldsymbol{\mu}_{2n+1}) \stackrel{\mathrm{Cores}}{\to} H^2(R_2, \boldsymbol{\mu}_{2n+1})\big).$$

This reads $\ker\big(\mathbb{Z}/(2n+1)\mathbb{Z} \stackrel{id}{\to} \mathbb{Z}/(2n+1)\mathbb{Z}\big) = 0$ by taking into account proposition 2.1 of [GP3]. The same calculation holds for $E_{(0,1)}$ and $E_{(1,1)}$ and we obtain the desired decomposition. In particular there are $n+1$ inner forms and three outer forms. The outer forms are all quasiplit.

[27] Strictly speaking we are looking, here and in what follows, at the action of $\mathrm{Out}(\mathbf{G})$ on R_2–groups or cohomology of R_2–groups which are of multiplicative type. Since we have an equivalence of categories between R_2 and F_2 groups of multiplicative type ([GP3]. See Remark 6.8(d)) we can carry all relevant calculations at the level of fields, in which case the situation is well understood. See for example the table in page 332 of [PR].

(2) We have $\boldsymbol{\mu} = \boldsymbol{\mu}_{2n} = \ker\big(\boldsymbol{\mu}_{2n}^2 \overset{\prod}{\to} \boldsymbol{\mu}_{2n}\big)$ and the action of $\mathbb{Z}/2\mathbb{Z}$ switches the two factors. The coset $\mathbb{Z}/2\mathbb{Z}\backslash H^2(R, \boldsymbol{\mu}_{2n})$ is as before $\{\pm 1\}\backslash(\mathbb{Z}/2n\mathbb{Z})$. However, the computation of $H^2(R_2, {}_{\mathfrak{E}}\boldsymbol{\mu})$ is different. The exact sequence

$$1 \to {}_{\mathfrak{E}}\boldsymbol{\mu}_{2n} \to \prod_{R_2[\sqrt{t_1}]/R_2} \boldsymbol{\mu}_{2n} \overset{norm}{\to} \boldsymbol{\mu}_{2n} \to 1$$

gives rise to the long exact sequence of étale cohomology

$$\cdots \to H^1(R_2[\sqrt{t_1}], \boldsymbol{\mu}_{2n}) \overset{norm}{\to} H^1(R_2, \boldsymbol{\mu}_{2n}) \overset{\delta}{\to} H^2(R_2, {}_{\mathfrak{E}}\boldsymbol{\mu}) \to H^1(R_2[\sqrt{t_1}], \boldsymbol{\mu}_{2n}) \overset{norm}{\to} H^2(R_2, \boldsymbol{\mu}_{2n}).$$

The norm map appearing on the righthand side is the identity map $id : \mathbb{Z}/2n\mathbb{Z} \to \mathbb{Z}/2n\mathbb{Z}$, so δ is onto. By the choices of coordinates $\sqrt{t_1}$ and t_2 on $R_2[\sqrt{t_1}]$ and t_1, t_2 on R_2, the beginning of the exact sequence decomposes as

$$\mathbb{Z}/2n\mathbb{Z} \oplus \mathbb{Z}/2n\mathbb{Z} \overset{(id,\times 2)}{\longrightarrow} \mathbb{Z}/2n\mathbb{Z} \oplus \mathbb{Z}/2n\mathbb{Z}.$$

So $H^2(R_2, {}_{\mathfrak{E}}\boldsymbol{\mu}) \simeq \mathbb{Z}/2\mathbb{Z}$ and the action of $\mathbb{Z}/2\mathbb{Z}$ on $H^2(R_2, {}_{\mathfrak{E}}\boldsymbol{\mu})$ is therefore necessarily trivial. Thus $\mathfrak{E}$ leads to two distinct twisted forms ${}_{\mathfrak{E}}\mathbf{G}^{\pm}$. More precisely ${}_{\mathfrak{E}}\mathbf{G}^{+} = {}_{\mathfrak{E}}\mathbf{G}$ (which is quasiplit), while ${}_{\mathfrak{E}}\mathbf{G}^{-}$ is not quasiplit (since its "Brauer invariant" in $H^2(R_2, {}_{\mathfrak{E}}\boldsymbol{\mu})$ is not trivial). Similarly for $\mathfrak{E}_{(0,1)}$ and $\mathfrak{E}_{(1,1)}$. This gives the desired decomposition. There are $n+1$ inner forms and six outer forms (three of which are quasisplit).

(3) In this case $\boldsymbol{\mu} = \boldsymbol{\mu}_4$. The computation of the H^2 are exactly as in case (2) for $n = 2$. There are three inner forms and six outer forms (three of which are quasisplit).

(4) This case is rather different since $\boldsymbol{\mu} = \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$ where $\mathbb{Z}/2\mathbb{Z}$ switches the two summands. We have $H^2(R_2, \boldsymbol{\mu}) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ where again $\mathbb{Z}/2\mathbb{Z}$ acts by switching the two summands

Given that ${}_{\mathfrak{E}}\boldsymbol{\mu} = \prod_{R_2[\sqrt{t_1}]/R_2} \boldsymbol{\mu}_2$, we have $H^2(R_2, {}_{\mathfrak{E}}\boldsymbol{\mu}) \overset{\sim}{\longrightarrow} H^2(R_2[\sqrt{t_1}], \boldsymbol{\mu}_2) = \mathbb{Z}/2\mathbb{Z}$. Similarly for $E_{(0,1)}$ and $E_{(1,1)}$, whence our decomposition. Again we have three inner forms and six outer forms (three of which are quasisplit).

(5) This is exactly as in case (1) for $n = 1$. There are two inner forms and three outer forms (all three of them quasisplit). $\square$

It remains to look at the case when $\mathbf{G}$ is of type D_4. The set $H^1(R_2, S_3)$ classifies all degree 3 étale extensions S of R_2. Then S is a direct product of connected extensions. There are tree cases: $S = R_2 \times R_2 \times R_2$ (the split case), $S = S' \times R_2$ with S'/R_2 of degree 2, and the connected case.

The case of $S' \times R_2$ is already understood: They correspond to a 1-cocycle $\phi : \pi_1(R_n) \to \mathbb{Z}/2\mathbb{Z} \subset S_3$, where we view $\mathbb{Z}/2\mathbb{Z}$ as a subgroup of S_3 generated by a permutation. Note that up to conjugation by S_3, there are exactly three such maps ϕ. These are three non-isomorphic quadratic extensions which were denoted by $\mathfrak{E}_{(i,j)}$ above for $(i,j) \neq (0,0)$. We shall denote them by $\mathfrak{E}_2^{(i,j)}$ in the present situation to avoid confusion.

In the connected case there are four cubic extensions of R_2. They correspond to adjoining to R_2 a cubic root in $R_{2,\infty}$ of t_1, t_2, t_1t_2 and $t_1^2t_2$ respectively. We will denote the corresponding four S_3–torsors by $\mathfrak{E}_3^{(i,j)}$ with the obvious values for (i,j). The cubic case, which a priori appears as the most complicated, ends up being quite simple due to vanishing cohomological reasons, as we shall momentarily see.

According to Corollary 9.2 we have the decomposition

$$\begin{aligned} H^1_{loop}(R_2, \mathbf{Aut}(\mathbf{G})) \;\simeq\;\; & S_3 \backslash H^2(R_2, \boldsymbol{\mu}) \\ & \bigsqcup_{\mathfrak{E}_2^{(i,j)}} \big({}_{\mathfrak{E}_2^{(i,j)}}S_3\big)(R_2) \,\backslash H^2(R_2, {}_{\mathfrak{E}_2^{(i,j)}}\,\boldsymbol{\mu}) \\ & \bigsqcup_{\mathfrak{E}_3^{(i,j)}} \big({}_{\mathfrak{E}_3^{(i,j)}}S_3\big)(R_2) \,\backslash H^2(R_2, {}_{\mathfrak{E}_3^{(i,j)}}\,\boldsymbol{\mu}). \end{aligned} \tag{9.1}$$

The centre is $\boldsymbol{\mu} = \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 = \ker\big(\boldsymbol{\mu}_2^3 \overset{\prod}{\to} \boldsymbol{\mu}_2\big)$ and S_3 acts by permutation on $\boldsymbol{\mu}_2^3$. Hence $H^2(R_2,\boldsymbol{\mu}) = \ker\big(H^2(R_2,\boldsymbol{\mu}_2)^3 \to H^2(R_2,\boldsymbol{\mu}_2)\big) \subset H^2(R_2,\boldsymbol{\mu}_2)^3 \simeq (\mathbb{Z}/2\mathbb{Z})^3$. There are two orbits for the action of S_3 on $H^2(R_2,\boldsymbol{\mu})$, namely $(0,0,0)$ and $(1,1,0)$.

For simplicity we will denote $\mathfrak{E}_2^{(1,0)}$ by $\mathfrak{E}_2$ and $\mathfrak{E}_3^{(1,0)}$ by $\mathfrak{E}_3$. Since the group $\mathbf{GL}_2(\mathbb{Z})$ acts transitively on the set of quadratic and cubic extensions of R_2 we may consider only the case of $\mathfrak{E}_2 := \mathfrak{E}_2^{1,0}$ [resp. $\mathfrak{E}_3 := \mathfrak{E}_3^{1,0}$] for the purpose of determining the coset $\big({}_{\mathfrak{E}_2^{(i,j)}}S_3\big)(R_2)\backslash H^2(R_2, {}_{\mathfrak{E}_2^{(i,j)}}\boldsymbol{\mu})$ [resp. $\big({}_{\mathfrak{E}_3^{(i,j)}}S_3\big)(R_2)\backslash H^2(R_2, {}_{\mathfrak{E}_3^{(i,j)}}\boldsymbol{\mu})$]. that all the twists of $\boldsymbol{\mu}$ and S_3 by quadratic or cubic torsors are of the form ${}_{\mathfrak{E}_i}\boldsymbol{\mu}$ and ${}_{\mathfrak{E}_i}S_3$ for $i=2$ (resp. $i=3$) in the quadratic (resp. cubic) case.

We have ${}_{\mathfrak{E}_2}\boldsymbol{\mu} = \ker\big(\prod_{R_2[\sqrt{t_1}]/R_2} \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \overset{norm\times id}{\to} \boldsymbol{\mu}_2\big)$, hence

$$H^2(R_2, {}_{\mathfrak{E}_2}\boldsymbol{\mu}) = \ker\big(H^2(R_2[\sqrt{t_1}],\boldsymbol{\mu}_2) \oplus H^2(R_2,\boldsymbol{\mu}_2) \to H^2(R_2,\boldsymbol{\mu}_2)\big) \cong H^2(R_2[\sqrt{t_1}],\boldsymbol{\mu}_2) = \mathbb{Z}/2\mathbb{Z}.$$

Since $\mathbb{Z}/2\mathbb{Z}$ has trivial automorphism group, we get three copies of $\mathbb{Z}/2\mathbb{Z}$ in the second summand of the decomposition (9.1).

In the cubic case we have ${}_{\mathfrak{E}_3}\boldsymbol{\mu} = \ker\big(\prod_{R_2[\sqrt[3]{t_1}]/R_2} \boldsymbol{\mu}_2 \overset{norm}{\to} \boldsymbol{\mu}_2\big)$. Since 2 is prime to 3, the norm is split and

$$H^2(R_2, {}_{\mathfrak{E}_3}\boldsymbol{\mu}) = \ker\big(H^2(R_2[\sqrt[3]{t_1}],\boldsymbol{\mu}_2) \overset{\text{Cores}}{\longrightarrow} H^2(R_2,\boldsymbol{\mu}_2)\big) = \ker\big(\mathbb{Z}/2\mathbb{Z} \overset{id}{\to} \mathbb{Z}/2\mathbb{Z}\big) = 0.$$

Finally we observe, with the aid of Remark 6.8(d), that $({}_{\mathfrak{e}_2}S_3)(R_2) \cong \mathbb{Z}/2\mathbb{Z}$ and $({}_{\mathfrak{e}_3}S_3)(R_2) \cong \mathbb{Z}/3\mathbb{Z}$.

Looking at (9.1) we obtain.

Corollary 9.6. *For* $\mathbf{G}$ *of type* D_4 *there are twelve loop* R_2*–forms, Two inner and ten outer. Six of the outer forms are "quadratic", and come divided into three pairs, where each pair contains exactly one quasiplit group. The remaining four outer forms are "cubic" and are all quasiplit.* □

9.2.2 Applications to the classification of EALAs in nullity 2.

The Extended Affine Lie Algebras (EALAs), as their name implies, are a class of Lie algebras which generalize the affine Kac-Moody Lie algebras. To an EALA $\mathcal{E}$ one can attached its so called centreless core, which is usually denoted by $\mathcal{E}_{\mathrm{cc}}$. This is a Lie algebra over k (in general infinite dimensional) which satisfies the axioms of a Lie torus.[28] Neher has shown that all Lie torus arise as centreless cores of EALAs, and conversely. He has also given an explicit procedure that constructs all EALAs having a given Lie torus $\mathcal{L}$ as their centreless cores. To some extent this reduces many central questions about EALAs (such as their classification) to that of Lie tori.

The centroid of a Lie tori $\mathcal{L}$ is always of the form R_n. This gives a natural R_n–Lie algebra structure to $\mathcal{L}$. If $\mathcal{L}$ as an R_n–module is of finite type, then $\mathcal{L}$ is necessarily a multiloop algebra $L(\mathfrak{g}, \boldsymbol{\sigma})$ as explained in the Introduction. Let $\mathbf{G}$ be a Chevalley k–group of adjoint type with Lie algebra $\mathfrak{g}$. Since $\mathbf{Aut}(\mathfrak{g}) \simeq \mathbf{Aut}(\mathbf{G})$ the n–loop algebras based on $\mathfrak{g}$ (as R_n–Lie algebras) are in bijective correspondence with the loop R_n–forms of $\mathbf{G}$. Indeed, they are precisely the Lie algebras of the loop R_n–groups. The subtlety comes from the fact that in infinite dimensional Lie theory one is interested in these Lie algebras as Lie algebras over k, *and not* R_n. In the present context the "centroid trick" (see [GP2, §4.1]) translates into the $\mathbf{GL}_n(\mathbb{Z})$ action on $H^1\big(R_n, \mathbf{Aut}(\mathfrak{g})\big)$ we have defined. This allows us to describe, in terms of orbits, all the isomorphism classes of R_n–multiloop algebras $L(\mathfrak{g}, \boldsymbol{\sigma})$ that become isomorphic when viewed as Lie algebras over k.

In what follows "loop algebras based on $\mathfrak{g}$" will be though as Lie algebras *over* k.

In the case $\mathbf{Out}(\mathbf{G}) = 1$ we have seen that $H^1_{loop}\big(R_2, \mathbf{Aut}(\mathbf{G})\big) \simeq H^2(R_2, \boldsymbol{\mu})$, and this last H^2 is either trivial or $\mathbb{Z}/2\mathbb{Z}$. In both cases the action of $\mathbf{GL}_2(\mathbb{Z})$ on $H^1_{loop}\big(R_2, \mathbf{Aut}(\mathbf{G})\big)$ is necessarily trivial. In particular.

[28]This terminology is due to Neher and Yoshii. It may seem strange to call a Lie algebra a Lie torus (since tori have already a meaning in Lie theory). The terminology was motivated by the concept of Jordan tori, which are a class of Jordan algebras.

Corollary 9.7. *(1) If $\mathfrak{g}$ has type A_1 B, C or E_7, there exists two isomorphism classes of 2–loop algebras based on $\mathfrak{g}$ denoted by $\mathfrak{g}_0$ (the split case) and $\mathfrak{g}_1$.*[29]

(2) All 2–loop algebras based on $\mathfrak{g}$ of type E_8, F_4 or G_2 are trivial, i.e isomorphic as k–Lie algebras to $\mathfrak{g}_0 = \mathfrak{g} \otimes_k R_2$.

In the case $\mathbf{Out}(\mathbf{G}) = \mathbb{Z}/2\mathbb{Z}$, we have $H^1(R_2, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and the action of $\mathbf{GL}_2(\mathbb{Z})$ on $H^1(R_2, \mathbb{Z}/2\mathbb{Z})$ is given by the linear action mod 2. Since $\mathbf{SL}_2(\mathbb{Z}/2\mathbb{Z}) = \mathbf{GL}_2(\mathbb{Z}/2\mathbb{Z})$ and $\mathbf{SL}_2(\mathbb{Z}/2\mathbb{Z})$ is generated by elementary matrices, the reduction map $\mathbf{GL}_2(\mathbb{Z}) \to \mathbf{GL}_2(\mathbb{Z}/2\mathbb{Z})$ is onto. Hence there are two orbits for the action of $\mathbf{GL}_2(\mathbb{Z})$ on $H^1(R_2, \mathbb{Z}/2\mathbb{Z})$, namely the trivial one and $H^1(R_2, \mathbb{Z}/2\mathbb{Z}) \setminus \{0\}$. The last one is represented by the quadratic Galois extension $R_2[\sqrt{t_1}]/R_2$, denoted by $\mathfrak{E}_{(1,0)}$ in the previous section, which we will again denote simply by $\mathfrak{E}$ in what follows. The action of $\mathbf{GL}_2(\mathbb{Z})$ we have just described shows that in all cases the outer forms, which came in three families (each with one or two classes) in the case of loop R_2–groups, collapse into a single family. This single family consists of either a single class, namely the quasi-split algebra ${}_{\mathfrak{E}}\mathfrak{g} = {}_{\mathfrak{E}}\mathfrak{g}^+$, or two classes ${}_{\mathfrak{E}}\mathfrak{g}^+$ and ${}_{\mathfrak{E}}\mathfrak{g}^-$. The algebra ${}_{\mathfrak{E}}\mathfrak{g}^-$ is not quasisplit.

Corollary 9.8. *If $\mathbf{Out}(\mathbf{G}) = \mathbb{Z}/2\mathbb{Z}$ the classification of isomorphism classes of 2–loop algebras based on $\mathfrak{g}$ is as follows:*

(1) In type A_{2n} $(n \geq 1)$

$$\mathbf{GL}_2(\mathbb{Z}) \backslash H^1_{loop}(R_2, \mathbf{Aut}(\mathfrak{g})) \simeq \{\pm 1\} \backslash \Big(\mathbb{Z}/(2n+1)\mathbb{Z}\Big) \bigsqcup \{{}_{\mathfrak{E}}\mathfrak{g}\}.$$

There are $n+1$ inner forms, denoted by $\mathfrak{g}_q$ with $0 \leq q \leq n$, and one outer form (which is quasisplit).

(2) In type A_{2n-1} $(n \geq 2)$

$$\mathbf{GL}_2(\mathbb{Z}) \backslash H^1_{loop}(R_2, \mathbf{Aut}(\mathfrak{g})) \simeq \{\pm 1\} \backslash \Big(\mathbb{Z}/2n\mathbb{Z}\Big) \bigsqcup \{{}_{\mathfrak{E}}\mathfrak{g}^+\} \bigsqcup \{{}_{\mathfrak{E}}\mathfrak{g}^-\}.$$

There are $n+1$ inner forms, denoted by $\mathfrak{g}_q$ with $0 \leq q \leq n$, and two outer forms (one of them quasiplit, the other one not).

(3) In type D_{2n-1} $(n \geq 3)$

$$\mathbf{GL}_2(\mathbb{Z}) \backslash H^1_{loop}(R_2, \mathbf{Aut}(\mathfrak{g})) \simeq \{\pm 1\} \backslash \Big(\mathbb{Z}/4\mathbb{Z}\Big) \bigsqcup \{{}_{\mathfrak{E}}\mathfrak{g}^+\} \bigsqcup \{{}_{\mathfrak{E}}\mathfrak{g}^-\}.$$

[29] As pointed out in [GP2], the case of E_7 has an amusing story behind it. The existence of a k–Lie algebra $L(\mathfrak{g}, \sigma_1, \sigma_2)$ which is not isomorphic to $\mathfrak{g} \otimes_k R_2$ was first established by van de Leur with the aid of a computer. In nullity 1 inner automorphisms always lead to trivial loop algebras. van de Leur's example shows that this fails already in nullity two.

There are 3 *inner forms, denoted by* $\mathfrak{g}_{0,1,2}$*, and two outer forms (one of them quasiplit, the other one not).*

(4) In type D_{2n} $(n \geq 3)$*, we have*

$$\mathbf{GL}_2(\mathbb{Z}) \backslash H^1_{loop}(R_2, \mathbf{Aut}(\mathfrak{g})) \simeq \text{switch} \backslash \Big(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}\Big) \bigsqcup \{\mathfrak{e}\mathfrak{g}^+\} \bigsqcup \{\mathfrak{e}\mathfrak{g}^-\}.$$

There are 3 *inner forms, denoted by* $\mathfrak{g}_{0,1,2}$*, and two outer forms (one of them quasiplit, the other one not).*

(5) In type E_6

$$\mathbf{GL}_2(\mathbb{Z}) \backslash H^1_{loop}(R_2, \mathbf{Aut}(\mathfrak{g})) \simeq \{\pm 1\} \backslash \Big(\mathbb{Z}/3\mathbb{Z}\Big) \bigsqcup \{\mathfrak{e}\mathfrak{g}\}.$$

There are 2 *inner forms,* $\mathfrak{g}_0$ *and* $\mathfrak{g}_1$*, and one outer form (which is quasisplit).*

Proof. The nature of the collapse of outer forms when passing from R_2 to k was explained before the statement of the Corollary. It remains to understand the inner cases. According to Corollary 9.2 and (8.9), we need to trace the action of $\mathbf{GL}_2(\mathbb{Z})$ on $\mathbb{Z}/2\mathbb{Z}\backslash H^2(R_2, \boldsymbol{\mu})$ and use the fact that this action lifts to an action of $\mathbf{GL}_2(\mathbb{Z})$ on $H^2(R_2, \boldsymbol{\mu})$ which commutes with that of $\mathbb{Z}/2\mathbb{Z}$.

(1) We have $\boldsymbol{\mu} = \boldsymbol{\mu}_{2n+1} = \ker\big(\boldsymbol{\mu}^2_{2n+1} \overset{\prod}{\to} \boldsymbol{\mu}_{2n+1}\big)$ and the action of $\mathbb{Z}/2\mathbb{Z}$ switches the two factors. We have $H^2(R_2, \boldsymbol{\mu}) \cong \mathbb{Z}/(2n+1)\mathbb{Z}$ and the action of $\mathbf{GL}_2(\mathbb{Z})$ on $H^2(R_2, \boldsymbol{\mu})$ is given by the determinant (Lemma 8.14.6), that of $\mathbb{Z}/2\mathbb{Z}$ is given by signs. Thus $\mathbf{GL}_2(\mathbb{Z})$ acts trivially on $\mathbb{Z}/2\mathbb{Z}\backslash H^2(R_2, \boldsymbol{\mu})$ and the result follows.

(2) We have $\boldsymbol{\mu} = \boldsymbol{\mu}_{2n} = \ker\big(\boldsymbol{\mu}^2_{2n} \overset{\prod}{\to} \boldsymbol{\mu}_{2n}\big)$ and the action of $\mathbb{Z}/2\mathbb{Z}$ switches the two factors. The action of $\mathbf{GL}_2(\mathbb{Z})$ on $H^2(R, \boldsymbol{\mu})$ is given by the determinant, hence the set of cosets $(\mathbf{GL}_2(\mathbb{Z}) \times \mathbb{Z}/2\mathbb{Z})) \backslash H^2(R, \boldsymbol{\mu}_{2n})$ can still be identified with $\{\pm 1\} \backslash (\mathbb{Z}/2n\mathbb{Z})$.

(3) In this case $\boldsymbol{\mu} = \boldsymbol{\mu}_4$. The computation of H^2 and reasoning are exactly as in (2) above for $n = 2$.

(4) This case is rather different since $\boldsymbol{\mu} = \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$ where $\mathbb{Z}/2\mathbb{Z}$ switches the two summands. We have $H^2(R_2, \boldsymbol{\mu}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ with respect to the switch action. Again $\mathbf{GL}_2(\mathbb{Z})$ acts by $g.\alpha = \det(g).\alpha$, hence trivially.

(5) This is exactly as in case (1) for $n = 1$. □

It remains to look at the case when $\mathbf{G}$ is of type D_4.

Lemma 9.9. *There are three orbits for the action of* $\mathbf{GL}_2(\mathbb{Z})$ *on* $H^1(R_2, S_3)$ *:*

- *the trivial class;*
- $\Big\{\mathfrak{E}_2^{0,1}, \mathfrak{E}_2^{1,0}, \mathfrak{E}_2^{1,1}\Big\}$*;*

- $\left\{\mathfrak{E}_3^{1,0}, \mathfrak{E}_3^{0,1}, \mathfrak{E}_3^{1,1}, \mathfrak{E}_3^{2,1}\right\}$

where the notations is as in Corollary 9.6 supra.

Proof. The three classes above correspond to case of the split étale cubic R_2-algebra, the case $S' \times R_2$ where S'/R_2 is quadratic and the cubic case. Obviously each of the above sets is $\mathbf{GL}_2(\mathbb{Z})$-stable, so we need to check that there is a single orbit. The quadratic case was dealt with in Corollary 9.8. In the cubic case, we have $\mathfrak{E}_3^{1,0} = R_2[\sqrt[3]{t_1}]$. By applying the base change corresponding to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ we obtain $\mathfrak{E}_3^{0,1}$, $\mathfrak{E}_3^{1,1}$ and $\mathfrak{E}_3^{2,1}$ respectively. $\square$

Corollary 9.10. *Up to k-isomorphism there are five 2-loop algebras based on $\mathfrak{g}$ of type D_4: Two inner forms, denoted by $\mathfrak{g}_0$ and $\mathfrak{g}_1$; two "quadratic" algebras, ${}_{\mathfrak{E}_2^{1,0}}\mathfrak{g}^+$ (which is quasisplit) and ${}_{\mathfrak{E}_2^{1,0}}\mathfrak{g}^-$ (which is not quasiplit); and one "cubic" algebra ${}_{\mathfrak{E}_3^{1,0}}\mathfrak{g}$ (which is quasisplit).*

Proof. By Lemma 9.9, the quadratic (resp. cubic) classes of Corollary 9.6 are in the $\mathbf{GL}_2(\mathbb{Z})$–orbit of those having Dynkin-Tits invariant $\mathfrak{E}_2^{1,0}$ (resp. $\mathfrak{E}_3^{1,0}$). So the cubic case is done. In the quadratic case, there are then one or two non-isomorphic "quadratic" 2-loop algebras. Since one of these R_2–algebras is quasisplit and the other one is not, Corollary 8.12 shows that they remain non-isomorphic as k–algebras. Finally, there are two orbits for the action of S_3 on $H^2(R_2, \boldsymbol{\mu})$, namely $(0,0,0)$ and $(1,1,0)$, and these correspond to the two inner R_2-forms. The action of $\mathbf{GL}_2(\mathbb{Z})$ is trivial in this set, so the algebras remain non-isomorphic over k. $\square$

9.2.3 Rigidity in nullity 2 outside type A.

The following theorem extends results of Steinmetz from classical types [SZ, th. 6.4] (which involves certain small rank restrictions) to all types. This establishes Conjecture 6.4 of [GP2].[30]

Theorem 9.11. *Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over k which is not of type A. Let $\mathcal{L}$ and $\mathcal{L}'$ be two 2-loop algebras based on $\mathfrak{g}$. The following are equivalent:*

(1) $\mathcal{L}$ and $\mathcal{L}'$ are isomorphic (as Lie algebras over k);

(2) $\mathcal{L}$ and $\mathcal{L}'$ have the same Witt-Tits index.

[30] An even stronger version of this Conjecture will be established in the next section.

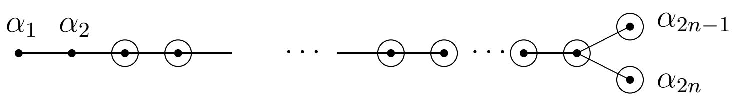
Proof. Of course, we use that $\mathcal{L}$ (resp. $\mathcal{L}'$) arise as the Lie algebra of R_2–loop adjoint groups $\mathbf{H}$ (resp. $\mathbf{H}'$) which are forms of $\mathbf{G} = \mathrm{Aut}(\mathfrak{g})^0$.[31] Then condition (1) reads that $[\mathbf{H}] = [\mathbf{H}']$ in $\mathbf{GL}_2(\mathbb{Z}) \backslash H^1_{loop}\big(R_2, \mathbf{Aut}(\mathfrak{g})\big)$ and condition (2) reads that $\mathbf{H} \times_{R_2} K_2$ and $\mathbf{H}' \times_{R_2} K_2$ have the same Witt-Tits index.

(1) $\Longrightarrow$ (2) : This is is the simple part of the the equivalence (and it is not necessary to exclude type A). Let $\mathbf{G}$ be the corresponding adjoint group. If $[\mathbf{H}]$ and $[\mathbf{H}']$ are equal in $\mathbf{GL}_2(\mathbb{Z}) \backslash H^1_{loop}\big(R_2, \mathbf{Aut}(\mathbf{G})\big)$, it is obvious that their Dynkin-Tits invariant coincide in $\mathbf{GL}_2(\mathbb{Z}) \backslash H^1_{loop}\big(R_2, \mathbf{Out}(\mathbf{G})\big)$, and also that their Tits index over K_2 coincide by Corollary 8.12.

(2) $\Longrightarrow$ (1) : Without loss of generality we can assume that $\mathbf{H}$ and $\mathbf{H}'$ have same Dynkin-Tits invariant in $H^1\big(R_2, \mathbf{Out}(\mathfrak{g})\big)$. The proof is given by a case-by-case discussion. The cases of type E_8, F_4 and G_2 follow directly from Corollary 9.7.2. Types B, C and E_7 are also straightforward since (over R_2) there is only one class of non-split 2–loop algebras. For obvious reasons, this non-split Lie algebras necessarily remain non-isomorphic to the split Lie algebra $\mathfrak{g} \otimes_k R_2$ when viewed as Lie algebras over k.

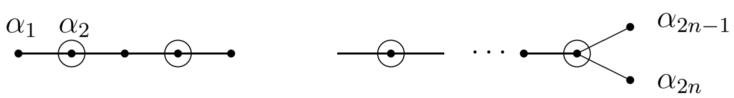
Type D_{2n}, $n \geq 3$: If the Dynkin-Tits invariant is non-trivial, then the summand of $\mathbf{GL}_2(\mathbb{Z}) \backslash H^1_{loop}(R_2, \mathbf{Aut}(\mathfrak{g}))$ corresponding to $[\mathfrak{E}_1]$ has only one non quasi-split class, so $[\mathbf{H}]$ and $[\mathbf{H}']$ are equal in $\mathbf{GL}_2(\mathbb{Z}) \backslash H^1_{loop}\big(R_2, \mathbf{Aut}(\mathfrak{g})\big)$. Corollary 9.8 states that the inner part of $\mathbf{GL}_2(\mathbb{Z}) \backslash H^1_{loop}(R_2, \mathrm{Aut}(\mathfrak{g}))$ is $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ modulo the switch action, so is represented by $(0,0)$, $(1,0)$ and $(1,1)$. It has then three elements, the split one and two others. It is then enough to explicitly describe these two other elements and distinguish them by their Witt-Tits index by means of Tits tables [T1]. The first one is the R_2-loop group $\mathbf{PSO}(q)$ with $q = \langle 1, t_1, t_2, t_1 t_2 \rangle \perp (2n-1)\langle 1, -1 \rangle$. Its Witt-Tits K_2-index is

(9.2)



The other one is $\mathrm{PSU}(A, h)$ where $A = A(2,1)$ is the R_2-quaternion algebra $T_1^2 = t_1$, $T_2^2 = t_2$, $T_1 T_2 + T_2 T_1 = 0$ and h is the hyperbolic hermitian form over A^{2n} with respect to the quaternionic involution $q \mapsto \overline{q}$. Indeed $\mathrm{PSU}(A, h)$ is an adjoint inner loop R_2-group of type D_{2n} and its Witt-Tits K_2-index is

(9.3)

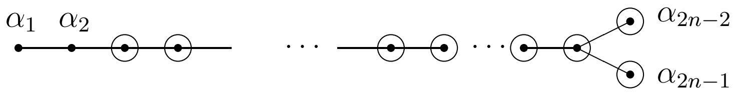


So $[\mathbf{H}]$ and $[\mathbf{H}']$ are equal in $\mathbf{GL}_2(\mathbb{Z}) \backslash H^1_{loop}(R_2, \mathrm{Aut}(\mathfrak{g}))$ to the split form or one of these two forms.

[31] Strictly speaking "...the Lie algebra..." is an R_2-Lie algebra, but we view this in a natural away as a k-Lie algebra.

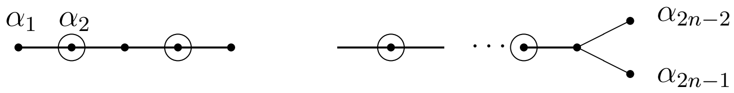
Type D_{2n-1}, $n \geq 3$: As in the preceding case, we need to discuss only the inner case and it is enough to provide two non-split R_2-loop groups with distinct K_2-Witt-Tits index. The first one is the R_2-loop group $\mathbf{PSO}(q)$ with $q = \langle 1, t_1, t_2, t_1t_2 \rangle \perp (2n-2)\langle 1, -1 \rangle$. Its Witt-Tits K_2-index is

(9.4)



The other one is $\mathrm{PSU}(A, h)$ where h is the hyperbolic hermitian form over A^{2n-1} which is the orthogonal sum of $\langle 1 \rangle$ and the hyperbolic form over A^{2n-2}. Its Witt-Tits K_2-index is

(9.5)



Type D_4*:* Follows from Corollary 9.10.

Type E_6*:* This case is straightforward because there is only one class of 2–loop algebras which is not quasi-split. □

Remark 9.12. There is some redundancy in the statement of the Theorem. It is well known, by descent considerations, that if $\mathcal{L}$ and $\mathcal{L}'$ are isomorphic as Lie algebras over k, then their absolute type coincide, i.e. they are both 2–loop algebras based on the same $\mathfrak{g}$ (see [ABP2.5] for further details). It will thus suffice to assume in the Theorem that neither $\mathcal{L}$ nor $\mathcal{L}'$ are of absolute type A.

9.2.4 Tables

The following table summarizes the classification on 2-loop algebras that are not of absolute type A. The table includes the Cartan-Killing (absolute) type $\mathfrak{g}$, its name, the Witt-Tits index (with Tits' notations) of an R_n–representative of the k–Lie algebra in question, and the type of the relative root system. For example, van de Leur's algebra has absolute type E_7, Tits index $E^9_{7,4}$ and relative index F_4. The way in which the Witt-Tits index are determined was illustrated in the previous section. The procedure of how to obtain the relative type from the index is described by Tits.

In all cases the "trivial" loop algebra $\mathfrak{g} \otimes k[t_1^{\pm}, t_2^{\pm}]$ is denoted by $\mathfrak{g}_0$. When the relative type is A_0, the loop algebra in question is anisotropic. For example, in absolute type A_1 the Lie algebra $\mathfrak{g}_0$ is $\mathfrak{sl}_2([t_1^{\pm}, t_2^{\pm}])$. The Lie algebra $\mathfrak{g}_1$ is the derived algebra of the Lie algebra that corresponds to the quaternion algebra over $k[t_1^{\pm}, t_2^{\pm}]$ with relations $T_1T_2 = -T_2T_1$ and $T_i^2 = t_i$. This rank 3 free Lie algebra over $k[t_1^{\pm}, t_2^{\pm}]$ is anisotropic, and is a twisted form of $\mathfrak{sl}_2 \otimes k[t_1^{\pm}, t_2^{\pm}]$ split by the quadratic extension $k[t_1^{\pm}, t_2^{\pm}](T_1)$.

Cartan-Killing type $\mathfrak{g}$	Name	Tits index	Relative root system
A_1	$\mathfrak{g}_0$	${}^1A_{1,1}^{(1)}$	A_1
A_1	$\mathfrak{g}_1$	${}^1A_{1,0}^{(2)}$	A_0
A_{2n} ($n \geq 1$)	$\mathfrak{g}_q$	${}^1A_{2n,r-1}^{(\frac{2n+1}{r})}$ $r = \gcd(q, 2n+1)$	A_{r-1}
A_{2n} ($n \geq 1$)	$\mathfrak{e}\mathfrak{g}$	${}^2A_{2n,n}^{(1)}$	BC_n
A_{2n-1} ($n \geq 2$)	$\mathfrak{g}_q$	${}^1A_{2n-1,r-1}^{(\frac{2n}{r})}$ $r = \gcd(q, 2n)$	A_{r-1}
A_{2n-1} ($n \geq 2$)	$\mathfrak{e}\mathfrak{g}^+$	${}^2A_{2n-1,n}^{(1)}$	C_n
A_{2n-1} ($n \geq 2$)	$\mathfrak{e}\mathfrak{g}^-$	${}^2A_{2n-1,n-1}^{(1)}$	BC_{n-1}
B_n $(n \geq 2)$	$\mathfrak{g}_0$	$B_{n,n}$	B_n
B_n $(n \geq 2)$	$\mathfrak{g}_1$	$B_{n,n-1}$	B_{n-1}
C_n $(n \geq 3)$	$\mathfrak{g}_0$	$C_{n,n}^{(1)}$	C_n
C_{2n+1} $(n \geq 1)$	$\mathfrak{g}_1$	$C_{2n+1,n}^{(2)}$	BC_n
C_{2n} $(n \geq 2)$	$\mathfrak{g}_1$	$C_{2n,n}^{(2)}$	C_n
D_4	$\mathfrak{g}_0$	${}^1D_{4,4}^{(1)}$	D_4
D_4	$\mathfrak{g}_1$	${}^1D_{4,2}^{(1)}$	B_2
D_4	$\mathfrak{e}_2\mathfrak{g}^+$	${}^2D_{4,3}^{(1)}$	B_3
D_4	$\mathfrak{e}_2\mathfrak{g}^-$	${}^2D_{4,1}^{(2)}$	BC_1
D_4	$\mathfrak{e}_3\mathfrak{g}$	${}^3D_{4,2}^{2}$	G_2
D_{2n-1} $(n \geq 3)$	$\mathfrak{g}_0$	${}^1D_{2n-1,2n-1}^{(1)}$	D_{2n-1}
D_{2n-1} $(n \geq 3)$	$\mathfrak{g}_1$	${}^1D_{2n-1,2n-3}^{(1)}$	B_{2n-3}
D_{2n-1} $(n \geq 3)$	$\mathfrak{g}_2$	${}^1D_{2n-1,n-2}^{(2)}$	BC_{n-2}
D_{2n-1} $(n \geq 3)$	$\mathfrak{e}\mathfrak{g}^+$	${}^2D_{2n-1,2n-2}^{(1)}$	B_{2n-2}
D_{2n-1} $(n \geq 3)$	$\mathfrak{e}\mathfrak{g}^-$	${}^2D_{2n-1,n-2}^{(2)}$	BC_{n-2}
D_{2n} $(n \geq 3)$	$\mathfrak{g}_0$	${}^1D_{2n,2n}^{(1)}$	D_{2n}
D_{2n} $(n \geq 3)$	$\mathfrak{g}_1$	${}^1D_{2n,2n-2}^{(1)}$	B_{2n-2}
D_{2n} $(n \geq 3)$	$\mathfrak{g}_2$	${}^1D_{2n,n}^{(2)}$	C_n
D_{2n} $(n \geq 3)$	$\mathfrak{e}\mathfrak{g}^+$	${}^2D_{2n,2n-1}^{(1)}$	B_{2n-1}
D_{2n} $(n \geq 3)$	$\mathfrak{e}\mathfrak{g}^-$	${}^2D_{2n,n-1}^{(2)}$	BC_{n-1}
E_6	$\mathfrak{g}_0$	${}^1E_{6,6}^{0}$	E_6
E_6	$\mathfrak{g}_1$	${}^1E_{6,2}^{16}$	G_2
E_6	$\mathfrak{e}\mathfrak{g}$	${}^2E_{6,4}^{2}$	F_4
E_7	$\mathfrak{g}_0$	$E_{7,7}^{0}$	E_7
E_7	$\mathfrak{g}_1$	$E_{7,4}^{9}$	F_4
E_8	$\mathfrak{g}_0$	$E_{8,8}^{0}$	E_8
F_4	$\mathfrak{g}_0$	$F_{4,4}^{0}$	F_4
G_2	$\mathfrak{g}_0$	$G_{2,2}^{0}$	G_2

By taking Remark 9.12 into consideration, an inspection of the Table shows that a stronger version of Theorem 9.11 holds.

Theorem 9.13. *Let $\mathcal{L}$ and $\mathcal{L}'$ be two 2-loop algebras neither of which is of absolute type A. The following are equivalent:*

(1) $\mathcal{L}$ and $\mathcal{L}'$ are isomorphic (as Lie algebras over k);

(2) $\mathcal{L}$ and $\mathcal{L}'$ have the same absolute and relative type.

Remark 9.14. This result was established, also by inspection, in Cor.13.3.3 of [ABP3]. In this paper the classification of nullity 2 multiloop algebras over k is achieved by considering loop algebras of the affine algebras. More precisely, it is shown that every multiloop algebra of nulllity2 is isomorphic as a Lie algebra over k to a Lie algebra of the form $L(\mathfrak{g} \otimes k[t_1^{\pm 1}], \pi)$ where π is a diagram automorphism of the untwisted affine Lie algebra $\mathfrak{g} \otimes k[t_1^{\pm 1}]$. For example, van de Leur's algebra appears by taking $\mathfrak{g}$ of type E_7 and considering the diagram automorphism of order two of the corresponding extended Coxeter-Dynkin.

Note that in the present work we have outline a general procedure to classify loop adjoint groups and algebras *over* R_n, and that the classification of multiloop algebras *over* k follows by $\mathbf{GL}_n$-considerations from that over R_n. This is not the case in [ABP3]. The nullity 2 classification relies on the structure of the affine algebras and only yields results over k.

10 The case of orthogonal groups

These groups are related to quadratic forms, which allows for a very precise understanding of their nature based on our results.

We consider the example of the split orthogonal group $\mathbf{O}(d)$ for $d \geq 1$. If $d = 2m$ (resp. $d = 2m+1$), this is the orthogonal group corresponding to the quadratic form $\sum\limits_{i=1}^{m} X_i X_{2m+1-i}$ (resp. $\sum\limits_{i=1}^{m} X_i X_{2m+1-i} + X_{2m+1}^2$). Since R_n-projective modules of finite type are free, we know that $H^1\big(R_n, \mathbf{O}(d)\big)$ classifies regular quadratic forms over R_n^d [K2, §4.6]. We have $H^1_{loop}\big(R_n, \mathbf{O}(d)\big) \xrightarrow{\sim} H^1\big(F_n, \mathbf{O}(d)\big)$. By iterating Springer's theorem for quadratic forms over $k((t))$ [Sc, §6.2], the classification of F_n-quadratic forms reads as follows: For each subset $I \subset \{1, ..., n\}$, we put $t_I = \prod_{i \in I} t_i$ with the convention $1 = t_\emptyset$; we denote by $\mathbf{H}$ the hyperbolic plane, that is the rank two split form. The isometry classes of d-dimensional R_n–forms are then of the form

$$\perp_{I \subset \{1,\ldots,n\}} \, t_I \, q_I \quad \perp \quad \mathbf{H}^v$$

where the q_I's are anisotropic quadratic k-forms and v a non negative integer such that $\sum\limits_{I \subset \{1,\dots,n\}} \dim_k(q_I) + 2v = d$.

Corollary 10.1. *The set $H^1_{loop}\big(R_n, \mathbf{O}(d)\big)$ is parametrized by the quadratic forms*

$$\perp_{I \subset \{1,\dots,n\}} t_I\, q_I \quad \perp \quad \mathbf{H}^v$$

where the q_I are anisotropic quadratic k-forms and v a non-negative integer such that $\sum\limits_{I \subset \{1,\dots,n\}} \dim_k(q_I) + 2v = 2d$. □

We denote by $\mathcal{P}(n)$ the set of subsets of $\{1, ..., n\}$ and by $\mathcal{P}^{even}_{\leq d}(n) \subset \mathcal{P}(n)$ the set of subsets of $\{1, ..., n\}$ of even cardinal $\leq d$. In a similar fashion we define $\mathcal{P}^{odd}_{\leq d}(n)$.

Corollary 10.2. *Assume that k is quadratically closed.*

(1) If $d = 2m$, then the map

$$\begin{array}{ccc} \mathcal{P}^{even}_{\leq d}(n) & \longrightarrow & H^1_{loop}(R_n, \mathbf{O}(d)) \\ S & \mapsto & \perp_{I \subset S} \langle t_I \rangle \ \perp \ \mathbf{H}^{m - \frac{|S|}{2}} \end{array}$$

is a bijection.

(2) If $d = 2m+1$, then the map

$$\begin{array}{ccc} \mathcal{P}^{odd}_{\leq d}(n) & \longrightarrow & H^1_{loop}(R_n, \mathbf{O}(d)) \\ S & \mapsto & \perp_{I \subset S} \langle t_I \rangle \ \perp \ \mathbf{H}^{m + \frac{1-|S|}{2}} \end{array}$$

is a bijection.

Corollary 10.3. *Assume that k is quadratically closed. Inside $\mathbf{O}'_d = \mathbf{O}(\langle 1, \dots, 1\rangle) \simeq \mathbf{O}_d$. there is a single $\mathbf{O}'_d(k)$-conjugacy class of maximal anisotropic abelian constant subgroup of $\mathbf{O}'(d)$, that of the diagonal subgroup $\boldsymbol{\mu}_2^d$. In particular anisotropic abelian subgroups of $\mathbf{O}'(d)$ are 2-elementary.*

Proof. Let $\mathbf{A}$ be a finite abelian constant group of $\mathbf{O}'_d$. There exist an even integer $m \geq 1$ and a surjective homomorphism $\phi : (\mathbb{Z}/m\mathbb{Z})^n \to \mathbf{A}(k)$. Then the corresponding loop torsor $[\phi] \in H^1(R_n, \mathbf{O}'_d)$ is anisotropic. Indeed the map $H^1(R_n, \boldsymbol{\mu}_2^d) \to H^1(R_n, \mathbf{O}'_d)$ is surjective. Hence there exists $\psi : (\mathbb{Z}/m\mathbb{Z})^n \to \boldsymbol{\mu}_2^d$ such that $[\phi] = [\psi] \in H^1(R_n, \mathbf{O}'_d)$. Theorem 7.9 shows that ϕ and ψ are $\mathbf{O}'_d(k)$-conjugate. By considering their images, we conclude that $\mathbf{A}(k)$ is $\mathbf{O}'_d(k)$-conjugate to a subgroup of $\boldsymbol{\mu}_2^d(k)$. □

Remark 10.4. (1) All anisotropic abelian constant subgroups of $\mathbf{O}'_d$ are related to codes, and these are not explicitly enumerated (see [Gs] for details).

(2) Under the hypothesis of the Corollary, let $f : \mathbf{Spin}'_d \to \mathbf{SO}'_d$ be the universal covering of $\mathbf{O}'_d$. Since the image of a finite abelian constant anisotropic subgroup of $\mathbf{Spin}'_d$ in $\mathbf{O}'_d$ is still anisotropic, it follows that an anisotropic finite constant abelian subgroup of $\mathbf{Spin}'_d$ is of rank $\leq d$ and has 4-torsion.

11 Groups of type G_2

We denote by $\mathbf{G}_2$ the split Chevalley group of type G_2 over k. If F is a field of characteristic zero containing k, we know that $H^1(F_n, \mathbf{G}_2)$ classifies octonion F-algebras or alternatively 3-Pfister forms [Se2, §8.1]. This follows from the fact that the Rost invariant [GMS]

$$r_F : H^1(F, \mathbf{G}_2) \to H^3(F, \mathbb{Z}/2\mathbb{Z})$$

is injective and sends the class of an octonion algebra to the Arason invariant of its norm form.

Consider the standard non-toral constant abelian subgroup $f : (\mathbb{Z}/2\mathbb{Z})^3 \subset \mathbf{G}_2$. Then the composite map

$$(F^\times / F^{\times 3}) \cong H^1(F, (\mathbb{Z}/2\mathbb{Z})^3) \xrightarrow{f_*} H^1(F, \mathbf{G}_2) \xrightarrow{r_F} H^3(F, \mathbb{Z}/2\mathbb{Z}).$$

sends an element $\big((a), (b), (c)\big)$ to the cup product $(a).(b).(c) \in H^3(F, \mathbb{Z}/2\mathbb{Z})$ [GiQ, §6]. For $n \geq 0$, we consider the mapping

$$(R_n^\times / R_n^{\times 2})^3 \simeq H^1\big(R_n, (\mathbb{Z}/2\mathbb{Z})^3\big) \xrightarrow{f_*} H^1(R_n, \mathbf{G}_2).$$

For a class $\big((x), (y), (z)\big) \in \big(R_3^\times / (R_3^\times)^2\big)^3$ we write only (x, y, z).

Corollary 11.1. *The map above surjects onto $H^1_{loop}(R_n, \mathbf{G}_2)$.*

Proof. By the Acyclicity Theorem, it suffices to observe that the analogous statement holds for $H^1_{loop}(F_n, \mathbf{G}_2)$. □

By using the Rost invariant, we get a full classification of the multiloop algebras based on on the split Lie algebra of type G_2.

Corollary 11.2. *Assume that k is quadratically closed. Assume that $n \geq 3$.*

1) $H^1_{loop}(R_n, \mathbf{G}_2) \setminus \{1\}$ consists in the images by f_* of the $\Big(t_{I_1}, t_{I_2}, t_{I_3}\Big)$ where I_1, I_2, I_3 are non-empty subsets of $\{1, .., n\}$ such that $i_1 < i_2 < i_3$ for all $(i_1, i_2, i_3) \in I_1 \times I_2 \times I_3$.

2) $\mathbf{GL}_n(\mathbb{Z}) \backslash \big(H^1_{loop}(R_n, \mathbf{G}_2) \setminus \{1\}\big)$ consists of the image by f_* of (t_1, t_2, t_3).

Proof. (1) Again by Aciclicity it suffices to establish the analogous result over F_n. Since k is quadratically closed, we have $R_n^\times / (R_n^\times)^{\times 2} \cong F_n^\times / (F_n^\times)^{\times 2} \cong (\mathbb{Z}/2\mathbb{Z})^n$. Hence $H^1(F_n, \mathbf{G}_2)$ consists of the image of $f_*(t_{I_1}, t_{I_2}, t_{I_3})$ for I_1, I_2, I_3 running over the subsets of $\{1, .., n\}$. The Rost invariant of such a class is $(t_{I_1}).(t_{I_2}).(t_{I_3}) \in H^3(F_n, \mathbb{Z}/2\mathbb{Z})$. Since $(t_i).(t_i) = 0$ and $(t_i).(t_j) = (t_j)(t_i) \in H^3(F_n, \mathbb{Z}/2\mathbb{Z})$, it follows that $H^1(F_n, \mathbf{G}_2)$

consists of the trivial class and the images by f_* of the $\Big(t_{I_1}, t_{I_2}, t_{I_3}\Big)$ where I_1, I_2, I_3 are non-empty subsets of $\{1, .., n\}$ such that $i_1 < i_2 < i_3$ for each $(i_1, i_2, i_3) \in I_1 \times I_2 \times I_3$. The last classes are non-trivial pairwise distinct elements since the $(t_{I_1}).(t_{I_2}).(t_{I_3}) \in H^3(F_n, \mathbb{Z}/2\mathbb{Z})$ are distinct pairwise elements by residue considerations (see for example prop. 3.1.1 of [GP3]).

(2) Follows easily from (1). □

The following corollary refines Griess' classification in the G_2-case [Gs].

Corollary 11.3. *Assume that k is algebraically closed. Let $\mathbf{A}$ be an anisotropic constant abelian subgroup of $\mathbf{G}_2$. Then $\mathbf{A}$ is $\mathbf{G}_2(k)$-conjugate to the standard non-toral subgroup $(\mathbb{Z}/2\mathbb{Z})^3$.*

Proof. Let $\mathbf{A}$ be a finite abelian constant anisotropic subgroup of $\mathbf{G}_2$. We reason as before. There exist an even integer $m \geq 1$ and a surjective homomorphism $\phi : (\mathbb{Z}/m\mathbb{Z})^n \to \mathbf{A}(k)$ so that the corresponding loop torsor $[\phi] \in H^1(R_n, \mathbf{G}_2)$ is anisotropic. By part (1) of Corollary 11.1 there exists $\psi : (\mathbb{Z}/m\mathbb{Z})^n \to (\mathbb{Z}/2\mathbb{Z})^3$ such that $[\phi] = [\psi] \in H^1(R_n, \mathbf{G}_2)$. Theorem 7.9 shows that ϕ and ψ are $\mathbf{G}_2(k)$conjugate. By taking the images, we conclude that $\mathbf{A}(k)$ is $\mathbf{G}_2(k)$–conjugate to the standard $(\mathbb{Z}/2\mathbb{Z})^3$. □

12 Case of groups of type F_4, E_8 and simply connected E_7 in nullity 3

In this section, we assume that k is algebraically closed. We denote by $\mathbf{F}_4$, and $\mathbf{E}_8$ the split algebraic k–group of type F_4 and E_8 respectively, and by $\mathbf{E}_7$ the split simply connected k–group of type E_7 For either of these three groups we know that $\mathbf{G} = \mathbf{Aut}(\mathbf{G})$ and that $H^1(R_2, \mathbf{G}) = 1$ [GP2, th. 2.7]. The goal is then to compute $H^1_{loop}(R_3, \mathbf{G})$, or at least the anisotropic classes.

Since we want to use Borel-Friedman-Morgan's classification of rank zero (i.e. with finite centralizer) abelian subgroups and triples of the corresponding compact Lie group [BFM, §5.2], we will assume that $k = \mathbb{C}$. Note that there is no loss of generality in doing this as explained in Remark 8.8.[32]

Denote by $\mathbf{G}_0$ the anisotropic real form of $\mathbf{G}$ (viewed as algebraic group over $\mathbb{R}$) and let $K = \mathbf{G}_0(\mathbb{R})$. This is a compact Lie group.

In the F_4 and E_7 case K has a single conjugacy class of rank zero abelian subgroup of rank 3. In the E_8 case, K has two conjugacy classes of rank zero abelian subgroup

[32] All the results that we need about rank zero abelian groups and triples can also be found in [KS].

of rank 3, $(\mathbb{Z}/5\mathbb{Z})^3$ and $(\mathbb{Z}/6\mathbb{Z})^3$. To translate this to the complex case we establish the following fact.

Lemma 12.1. *Let $\mathbf{H}$ be a complex affine algebraic group whose connected component of the identity is reductive. Denote by $\mathbf{H}_0$ its anisotropic real form, viewed as algebraic group over $\mathbb{R}$ (see [OV, §5.2, th. 12]). Set $K_H = \mathbf{H}_0(\mathbb{R})$.*

(1) Let A is a finite abelian subgroup of K_H and denote by $\mathbf{A}$ the underlying constant subgroup of the algebraic $\mathbb{R}$-group $\mathbf{H}_0$. Then A is a rank zero subgroup of K_H if and only if $\mathbf{A} \times_{\mathbb{R}} \mathbb{C}$ is an anisotropic subgroup of $\mathbf{H}$.

(2) Let $\mathbf{A}$ be an anisotropic abelian constant subgroup of $\mathbf{H}$ and put $A = \mathbf{A}(\mathbb{C})$. Then there exists $h \in \mathbf{H}(\mathbb{C})$ such that ${}^hA \subset K_H$ and $N_{K_H}({}^hA) = \mathbf{N_H}({}^h\mathbf{A})(\mathbb{C})$, both groups being finite. Furthermore $Z_{K_H}({}^hA) = \mathbf{Z_H}({}^h\mathbf{A})(\mathbb{C})$.

Recall that K is a maximal subgroup of $\mathbf{H}(\mathbb{C})$ and that maximal compact subgroups are conjugate under $\mathbf{H}^0(\mathbb{C})$.

Proof. (1) Let $\mathbf{C}$ denote the connected component of the identity of the centralizer $\mathbf{Z}_{\mathbf{H}_0}(\mathbf{A})$. It is a real reductive group [BMR, 10.1.5]. If $\mathbf{A}$ is an anisotropic subgroup of $\mathbf{H}$, then the maximal tori of $\mathbf{Z_H}(A)$ are trivial and $\mathbf{C} = 1$. Hence $\mathbf{C}(\mathbb{C})$ is finite and $Z_{K_H}(A)$ is finite, i.e. A is a rank zero subgroup of K_H. Conversely, if A is a rank zero subgroup of K_H then $\mathbf{C}(\mathbb{R})$ is finite. Since $\mathbf{C}(\mathbb{R})$ is Zariski dense in the connected group $\mathbf{C}$, we see that $\mathbf{C} = 1$, and $\mathbf{A} \times_{\mathbb{R}} \mathbb{C}$ is an anisotropic constant abelian subgroup of $\mathbf{H}$.
(2) We are given a finite anisotropic constant subgroup $\mathbf{A}$ of $\mathbf{H}$. Since $1 = \mathbf{Z_H}(\mathbf{A})^0 = \mathbf{N_H}(\mathbf{A})^0$, $\mathbf{N_H}(\mathbf{A})$ is a finite algebraic group and $\mathbf{N_H}(A)(\mathbb{C})$ is finite. Since $\mathbf{N_H}(\mathbf{A})(\mathbb{C})$ is included in a maximal compact group of $\mathbf{H}(\mathbb{C})$, we know that there exists $h \in \mathbf{H}(\mathbb{C})$ such that $A \subset \mathbf{N_H}(\mathbf{A})(\mathbb{C}) \subset {}^{h^{-1}}K_H$. We have then ${}^hA \subset \mathbf{N_H}({}^h\mathbf{A})(\mathbb{C}) \subset K_H$, hence $N_{K_H}({}^hA) = \mathbf{N_H}({}^h\mathbf{A})(\mathbb{C})$. It follows that $Z_{K_H}({}^hA) = \mathbf{Z_H}({}^h\mathbf{A})(\mathbb{C})$.

□

Lemma 12.2. *(1) The group $\mathbf{F}_4$ has a single conjugacy class of anisotropic finite abelian (constant) subgroups of rank* 3, *denoted by $f_3 : (\mathbb{Z}/3\mathbb{Z})^3 \subset \mathbf{F}_4$. Furthermore*

$$\mathbf{N}_{\mathbf{F}_4}\big((\mathbb{Z}/3\mathbb{Z})^3\big)/\mathbf{Z}_{\mathbf{F}_4}\big((\mathbb{Z}/3\mathbb{Z})^3\big) \simeq \mathbf{SL}_3(\mathbb{Z}/3\mathbb{Z}).$$

(2) The group $\mathbf{E}_7$ has a single conjugacy class of anisotropic finite abelian (constant) subgroups of rank 3, *denoted by $f_4 : (\mathbb{Z}/4\mathbb{Z})^3 \subset \mathbf{E}_7$. The finite group f_4 is a subgroup the maximal subgroup $\mathbf{SL}_8/\boldsymbol{\mu}_2$. Furthermore $\mathbf{N}_{\mathbf{E}_7}\big((\mathbb{Z}/4\mathbb{Z})^3\big)/\mathbf{Z}_{\mathbf{E}_7}\big((\mathbb{Z}/4\mathbb{Z})^3\big) \simeq \mathbf{SL}_3(\mathbb{Z}/4\mathbb{Z})$.*

(3) The group $\mathbf{E}_8$ has two conjugacy classes of anisotropic finite abelian (constant) subgroups of rank 3, *denoted by f_5 and f_6. We have.*

(a) $f_5 : (\mathbb{Z}/5\mathbb{Z})^3 \subset \mathbf{E}_8$ *and* $\mathbf{N}_{\mathbf{E}_8}\big((\mathbb{Z}/5\mathbb{Z})^3\big)/\mathbf{Z}_{\mathbf{E}_8}\big((\mathbb{Z}/5\mathbb{Z})^3\big) \simeq \mathbf{SL}_3(\mathbb{Z}/5\mathbb{Z})$.

(b) $f_6 : (\mathbb{Z}/6\mathbb{Z})^3 \subset \mathbf{E}_8$ *is a subgroup of the subgroup* $(\mathbf{SL}_2 \times \mathbf{SL}_3 \times \mathbf{SL}_6)/\boldsymbol{\mu}_6$. *Furthermore* $\mathbf{N}_{\mathbf{E}_8}\big(\mathbb{Z}/6\mathbb{Z})^3\big)/\mathbf{Z}_{\mathbf{E}_8}\big((\mathbb{Z}/6\mathbb{Z})^3\big) \simeq \mathbf{SL}_3(\mathbb{Z}/6\mathbb{Z})$.

Remark 12.3. The finite subgroups 1) and 3 (a) are described precisely in [GiQ, §6]. That the third one $f_6 : (\mathbb{Z}/6\mathbb{Z})^3 \subset \mathbf{E}_8$ sits inside the subgroup $(\mathbf{SL}_2 \times \mathbf{SL}_3 \times \mathbf{SL}_6)/\boldsymbol{\mu}_6$ is well-known from work of Borel and de Siebenthal.

Proof. As explained above we may assume that $k = \mathbb{C}$. The previous Lemma 12.1 shows that any rank 0 finite abelian constant subgroup $\mathbf{A}$ of $\mathbf{G}$ arises from rank 0 abelian subgroup A of K, so the list of Borel-Friedman-Morgan [BFM, §5.2] provides all relevant conjugacy classes, and this yields the inclusions f_3, f_4, f_5 and f_6 described above. Given a two rank 0 finite abelian constant subgroups $\mathbf{A}$ and $\mathbf{A}'$ of $\mathbf{G}$ arising respectively from rank 0 abelian subgroups A, A' of K, it remains to check that $\mathbf{A}(\mathbb{C})$ and $\mathbf{A}'(\mathbb{C})$ are $\mathbf{G}(\mathbb{C})$–conjugate if and only if A and A' are K-conjugate. But this is obvious since the subgroups from the list are distinct as groups. We investigate now the normalizers and centralizers.

Claim 12.4. *Let $A \subset K$ be a rank zero subgroup. Then* $N_K(A) = \mathbf{N}_{\mathbf{G}}(\mathbf{A})(\mathbb{C})$, $Z_K(A) = \mathbf{Z}_{\mathbf{G}}(\mathbf{A})(\mathbb{C})$.

Indeed Lemma 12.1.(2) shows the existence of an element $g \in \mathbf{G}(\mathbb{C})$ such that ${}^gA \subset K$ and

$$N_K({}^gA) = \mathbf{N}_{\mathbf{G}}({}^g\mathbf{A})(\mathbb{C}), \;\; Z_K(A) = \mathbf{Z}_{\mathbf{G}}({}^g\mathbf{A})(\mathbb{C}).$$

But A and ${}^gA$ are K–conjugate by Borel-Friedman-Morgan's theorem, so the same fact holds for $g = 1$.

It is then enough to know the quotient "normalizer/centralizer" in the compact group case. For each relevant d, we have an exact sequence of groups

$$1 \to Z_K\big((\mathbb{Z}/d\mathbb{Z})^3\big) \to N_K\big(\mathbb{Z}/d\mathbb{Z})^3\big) \xrightarrow{\theta} \mathbf{GL}_3(\mathbb{Z}/d\mathbb{Z})$$

and we want to determine the image of θ. Denote by $\mathcal{S}_d$ the set of K-conjugacy classes of rank zero triples of K of order d. Since such a triple generates a rank zero abelian subgroup of order d^3 of K, the set $\mathcal{S}_d$ is covered by rank zero triples inside $(\mathbb{Z}/d\mathbb{Z})^3$, namely $\mathbf{GL}_3(\mathbb{Z}/d\mathbb{Z})$-conjugates of the standard triple $(1,1,1)$. So we have $\mathbf{GL}_3(\mathbb{Z}/d\mathbb{Z})/\mathrm{Im}(\theta) \cong \mathcal{S}_d$. Proposition 5.1.5 of [BFM] states that the K-conjugacy classes of rank zero triples of K of order d consists of the classes $f_d(1,1,i)$ for $i = 1, .., d-1$ with i prime to d. Hence the image of θ in $\mathbf{GL}_3(\mathbb{Z}/d\mathbb{Z})$ is exactly $\mathbf{SL}_3(\mathbb{Z}/d\mathbb{Z})$ as desired. $\square$

Given $f_d : (\mathbb{Z}/d\mathbb{Z})^3 \to \mathbf{G}$ as above consider the map

$$f_{d,*} : \big(R_3^\times/(R_3^\times)^d\big)^3 \simeq H^1\big(R_3, (\mathbb{Z}/d\mathbb{Z})^3\big) \to H^1(R_3, \mathbf{G}).$$

A class $\big((x),(y),(z)\big) \in \big(R_3^\times/(R_3^\times)^d\big)^3$ will for convenience simply be written as (x,y,z).

Corollary 12.5. *(1) The set* $H^1_{loop}(R_3,\mathbf{F}_4)_{an}$ *consists of the classes of* $f_{3,*}(t_1,t_2,t_3)$ *and* $f_{3,*}(t_1,t_2,t_3^2)$.

(2) The set $H^1_{loop}(R_3,\mathbf{E}_7)_{an}$ *consists of the classes of* $f_{4,*}(t_1,t_2,t_3)$, $f_{4,*}(t_1,t_2,t_3^3)$.

(3) The set $H^1_{loop}(R_3,\mathbf{E}_8)_{an}$ *consists in the classes of* $f_{5,*}(t_1,t_2,t_3^i)$ *for* $i=1,2,3,4$, $f_{6,*}(t_1,t_2,t_3)$ *and* $f_{6,*}(t_1,t_2,t_3^5)$.

Proof. We do in detail the case of F_4, the other cases being similar. The set $H^1_{loop}(R_3,\mathbf{F}_4)_{an}$ is covered by the image of the anisotropic loop cocycles $\phi : \pi_1(R_3) \to \mathbf{F}_4(\mathbb{C})$. The image of such a ϕ is an anisotropic finite abelian subgroup of $\mathbf{F}_4$, so Lemma 12.2.1 allows us to assume that its image is the subgroup $(\mathbb{Z}/3\mathbb{Z})^3$. Furthermore, we know that two such homomorphisms ϕ and ϕ' have the same image in $H^1_{loop}(R_3,\mathbf{F}_4)_{an}$ if and only if there exists $g \in \mathbf{F}_4(\mathbb{C})$ such that $g\phi g^{-1} = \phi'$, or equivalently if there exists $g \in \mathbf{N}_{\mathbf{F}_4}\big((\mathbb{Z}/3\mathbb{Z})^3\big)(\mathbb{C})$ such that $g\phi g^{-1} = \phi'$. Note the importance of the isomorphism $\mathbf{N}_{\mathbf{F}_4}((\mathbb{Z}/3\mathbb{Z})^3)/\mathbf{Z}_{\mathbf{F}_4}((\mathbb{Z}/3\mathbb{Z})^3) \simeq \mathbf{SL}_3(\mathbb{Z}/3\mathbb{Z})$.

Rephrasing what has been said in terms of the mapping $f_{3,*}$, we see that $H^1_{loop}(R_3,\mathbf{F}_4)_{an}$ is the image under $f_{3,*}$ of the classes (x,y,z) where $x,y,z \in R_3^\times$ are such that (x,y,z) generates $R_3^\times/(R_3^\times)^3$; furthermore, two such classes (x,y,z) and (x',y',z') have the same image in $H^1_{loop}(R_3,\mathbf{F}_4)_{an}$ if and only if there exists $\tau \in \mathbf{SL}_3(\mathbb{Z}/3\mathbb{Z})$ such that $(x',y',z') = \tau_*\big((x,y,z)\big)$. We conclude that $H^1_{loop}(R_3,\mathbf{F}_4)_{an}$ consists of the classes of $f_{3,*}(t_1,t_2,t_3)$ and $f_{3,*}(t_1,t_2,t_3^2)$. $\square$

Corollary 12.6. *(1) The set* $\mathbf{GL}_3(\mathbb{Z})\backslash H^1_{loop}(R_3,\mathbf{F}_4)_{an}$ *consists of the class of* $f_{3,*}(t_1,t_2,t_3)$.

(2) The set $\mathbf{GL}_3(\mathbb{Z})\backslash H^1_{loop}(R_3,\mathbf{E}_7)_{an}$ *consists of the class* $f_{4,*}(t_1,t_2,t_3)$.

(3) The set $\mathbf{GL}_3(\mathbb{Z})\backslash H^1_{loop}(R_3,\mathbf{E}_8)_{an}$ *consists of the classes of* $f_{5,*}(t_1,t_2,t_3)$ *and* $f_{6,*}(t_1,t_2,t_3)$.

Remark 12.7. The above Corollary gives the full classification of nullity 3 anisotropic multiloop algebras of absolute type F_4 or E_8.

13 The case of $\mathbf{PGL}_d$

13.1 Loop Azumaya algebras

For any base scheme $\mathfrak{X}$, the set $H^1(\mathfrak{X},\mathbf{PGL}_d)$ classifies the isomorphism classes of Azumaya $\mathcal{O}_{\mathfrak{X}}$-algebras A of degree d, i.e. $\mathcal{O}_{\mathfrak{X}}$-algebras which are locally isomorphic for the étale topology to the matrix algebra $M_d(\mathcal{O}_{\mathfrak{X}})$ [Gr2] and [K2, §III].

The exact sequence $1 \to \mathbf{G}_m \to \mathbf{GL}_d \xrightarrow{p} \mathbf{PGL}_d \to 1$ induces the sequence of pointed sets

$$\mathrm{Pic}(\mathfrak{X}) \to H^1(\mathfrak{X}, \mathbf{GL}_d) \to H^1(\mathfrak{X}, \mathbf{PGL}_d) \xrightarrow{\delta} H^2(\mathfrak{X}, \mathbf{G}_m) = \mathrm{Br}(\mathfrak{X}).$$

We denote again by $[A] \in \mathrm{Br}(\mathfrak{X})$ the class of $\delta([A])$ in the cohomological Brauer group.

By [GP3, 3.1], we have an isomorphism $\mathrm{Br}(R_n) \cong \mathrm{Br}(F_n)$. We look now at the diagram

$$\begin{array}{ccc} H^1_{loop}(R_n, \mathbf{PGL}_d) & \xrightarrow{\delta} & \mathrm{Br}(R_n) \\ \cong \downarrow & & \cong \downarrow \\ H^1(F_n, \mathbf{PGL}_d) & \xrightarrow{\delta} & \mathrm{Br}(F_n) \end{array}$$

where the bottom map is injective [GS, §4.4] and the left map is bijective because of Theorem 8.1. We thus have

Corollary 13.1. *The boundary map $H^1_{loop}(R_n, \mathbf{PGL}_d) \to \mathrm{Br}(R_n)$ is injective.* □

Azumaya R_n–algebras whose classes are in $H^1_{loop}(R_n, \mathbf{PGL}_d)$ are called *loop Azumaya algebras.* They are isomorphic to twisted form of M_d by a loop cocycle. One can rephrase the last Corollary by saying that loop Azumaya algebras of degree d are classified by their "Brauer invariant".

Similarly, Wedderburn's theorem [GS, 2.1] for F_n–central simple algebras has its counterpart.

Corollary 13.2. *Let A be a loop Azumaya R_n–algebra of degree d. Then there exists a unique positive integer r dividing d and a loop Azumaya R_n–algebra B (unique up to R_n–algebras isomorphism) of degree d/r such that $A \simeq M_r(B)$ and $B \otimes_{R_n} F_n$ is a division algebra.* □

This reduces the classification of loop Azumaya R_n-algebras to the "anisotropic" case, namely to the case of loop Azumaya R_n–algebras A such that $A \otimes_{R_n} F_n$ is a division algebra.

In the same spirit, the Brauer decomposition [GS, 4.5.16] for central F_n–division algebras yields the following.

Corollary 13.3. *Write $d = p_1^{m_1} \cdots p_l^{m_l}$. Let A be an anisotropic loop Azumaya R_n-algebra of degree d. Then there exists a unique decomposition*

$$A \simeq A_1 \otimes_{R_n} \cdots \otimes_{R_n} A_l$$

where A_i is an anisotropic k–loop Azumaya R_n-algebra of degree $p_i^{m_i}$ for $i = 1, .., l$. □

The two preceding Corollaries show that the classification of loop Azumaya R_n–algebra reduces the the classification of anisotropic loop Azumaya R_n-algebras of degree p^m. Though the Brauer group of R_n and F_n are well understood, the understanding of $H^1_{loop}(R_n, \mathbf{PGL}_d)_{an}$ is much more delicate.

We are given a loop cocycle $\phi = (\phi^{geo}, z)$ with values in $\mathbf{PGL}_d(\overline{k})$. Set $A = {}_z(M_d)$. This is a central simple k–algebra such that ${}_z\,\mathbf{PGL}_d = \mathbf{PGL}_1(A)$. Recall that ϕ^{geo} is given by a k–group homomorphism $\phi^{geo} : \boldsymbol{\mu}_m^n \to \mathbf{PGL}_1(A)$. To say that ϕ is anisotropic is to to say that $\phi^{geo} : \boldsymbol{\mu}_m^n \to \mathbf{PGL}_1(A)$ is anisotropic.

We discuss in detail the following two special cases : the one dimensional case, and the geometric case (i.e. k is algebraically closed).

13.2 The one dimensional case

If k is algebraically closed $H^1(R_1, \mathbf{PGL}_d)$ is trivial. The interesting new case is when k is not algebraically closed, e.g. the case of real numbers. Since the map $H^1(F_1, \mathbf{PGL}_d) \to \mathrm{Br}(F_1)$ is injective, as a consequence of Corollary 9.1, we have $H^1(R_1, \mathbf{PGL}_d) \simeq H^1(F_1, \mathbf{PGL}_d)$ and the map

$$H^1(R_1, \mathbf{PGL}_d) \to \mathrm{Br}(R_1) = \mathrm{Br}(k) \oplus H^1(k, \mathbb{Q}/\mathbb{Z}) \tag{13.1}$$

is injective.

Theorem 13.4. *The image of the map 13.1 consists of all pairs $[A_0] \oplus \chi$ where A_0 is a central simple algebra of degree d and $\chi : \mathrm{Gal}(k_s/k) \to \mathbb{Q}/\mathbb{Z}$ a character for which that there exists an étale algebra K/k of degree d inside A_0 such that $\chi_K = 0$.*

Remark 13.5. The indices of such algebras over F_1 are known ([Ti, prop. 2.4] in the prime exponent case, and [FSS, Lemma 4.6] in the general case). The index of a F_1–algebra of invariant $[A_0] \oplus \chi$ is $\deg(\chi) \times \mathrm{ind}_{k_\chi}(A \otimes_k k_\chi)$ where k_χ/k stands for the cyclic extension associated to χ.

The proof needs some preparatory material from homological algebra based on Cartier duality for groups of multiplicative type. More precisely, the dual of an extension of k–groups of multiplicative type

$$1 \to \mathbf{G}_m \to \mathbf{E} \to \boldsymbol{\mu}_m \to 1$$

is the exact sequence

$$0 \to \mathbb{Z}/m\mathbb{Z} \to \widehat{E} \to \mathbb{Z} \to 1.$$

We have then an isomorphism

$$Ext^1_{k-gr}(\boldsymbol{\mu}_m, \mathbf{G}_m) \simeq Ext^1_{\mathrm{Gal}(k)}(\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) = H^1(k, \mathbb{Z}/m\mathbb{Z})$$

which permits to attach to the first extension a character. Up to isomorphism, there exists a unique extension $\mathbf{E}_\chi$ of $\boldsymbol{\mu}_m$ by $\mathbf{G}_m$ of class $[\chi]$.

Lemma 13.6. *Let $\chi : \mathrm{Gal}(k) \to \mathbb{Z}/m\mathbb{Z}$ be a character for some $m \geq 1$.*

1. *The boundary map*

$$k^\times/(k^\times)^m \xrightarrow{\sim} H^1(k, \boldsymbol{\mu}_m) \to H^2(k, \mathbf{G}_m) = \mathrm{Br}(k)$$

   is given by $(x) \mapsto \chi \cup (x)$.

2. *Let K/k be an étale algebra. The following are equivalent:*

   (a) There exists a morphism of extensions $\mathbf{E}_\chi \to R_{K/k}(\mathbf{G}_m)$ rendering the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbf{G}_m & \longrightarrow & \mathbf{E}_\chi & \longrightarrow & \boldsymbol{\mu}_m & \longrightarrow & 1 \\ & & \big\downarrow{\cong} & & \big\downarrow & & \big\downarrow & & \\ 1 & \longrightarrow & \mathbf{G}_m & \longrightarrow & R_{K/k}(\mathbf{G}_m) & \longrightarrow & R_{K/k}(\mathbf{G}_m)/\mathbf{G}_m & \longrightarrow & 1; \end{array}$$

   commutative.

   (b) $\chi_K = 0$.

Proof. (1) The cocharacter group $\widehat{E}_\chi$ is $\mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}$ together with the Galois action $\gamma(\alpha, \beta) = (\alpha + \chi(\gamma), \beta)$. The Galois action on $\mathbf{E}_\chi(\overline{k}) \simeq \overline{k}^\times \times \boldsymbol{\mu}_m(\overline{k})$ is then given by

$$\gamma(y, \zeta) = \Big(\gamma(y)\, \zeta^{\chi(\gamma)}, \gamma(\zeta)\Big)$$

for every $\gamma \in \mathrm{Gal}(k)$. The class $(x) \in H^1(k, \boldsymbol{\mu}_m)$ is represented by the cocycle $c_\gamma = \gamma(\sqrt[m]{x})/\sqrt[m]{x}$. The element $b_\gamma = (1, c_\gamma) \in \mathbf{E}_\chi(\overline{k})$ lifts c_γ. The boundary $\partial\big((x)\big) \in H^2(k, \overline{k}^\times)$ is then represented by the 2–cocycle

$$a_{\gamma,\tau} = b_\gamma \times \gamma(b_\tau)\, b_{\gamma\tau}^{-1} = c_\tau^{\chi(\gamma)} \chi(\gamma) \,.\, c_\tau \ .$$

(2) We decompose $K = k_1 \times \cdots \times k_l$ as a product of field extensions and denote by M_j the cocharacter module of $R_{k_j/k}(\mathbf{G}_m)$. Then the character module of $R_{K/k}(\mathbf{G}_m)$ is $M = \oplus M_j$. By dualizing we are interested in morphism of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & M & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow{\simeq} & & \\ 0 & \longrightarrow & \mathbb{Z}/m\mathbb{Z} & \longrightarrow & \hat{E}_\chi & \longrightarrow & \mathbb{Z} & \longrightarrow & 0. \end{array}$$

By Shapiro's lemma $Ext^1(M_j,\mathbb{Z}/m\mathbb{Z}) = H^1(k_j,\mathbb{Z}/m\mathbb{Z})$ and the map $Ext^1(\mathbb{Z},\mathbb{Z}/m\mathbb{Z}) \to Ext^1(M_j,\mathbb{Z}/m\mathbb{Z})$ yields the restriction map $H^1(k,\mathbb{Z}/m\mathbb{Z}) \to H^1(k_j,\mathbb{Z}/m\mathbb{Z})$. It follows that the bottom extension above is killed by the pull-back $M_j \to \mathbb{Z}$, and therefore that $\chi_{k_j} = 0$ for $j = 1,..,l$. This shows that $(a) \Longrightarrow (b)$.
$(b) \Longrightarrow (a)$: We assume that $\chi_K = 0$, namely $\chi_{k_j} = 0$ for $j = 1,..,l$. Hence $\widehat{E}_\chi$ belongs to the kernel of $Ext^1(\mathbb{Z},\mathbb{Z}/m\mathbb{Z}) \to Ext^1(M_j,\mathbb{Z}/m\mathbb{Z})$ for for $j = 1,..,l$ so $\widehat{E}_\chi$ belongs to the kernel of $Ext^1(\mathbb{Z},\mathbb{Z}/m\mathbb{Z}) \to Ext^1(M,\mathbb{Z}/m\mathbb{Z})$. This means that the map $M \to \mathbb{Z}$ of Galois modules lifts to $\widehat{E}_\chi \to \mathbb{Z}$ as desired. $\square$

We can proceed now with the proof of Theorem 13.4.

Proof. We show first that the image of ∂ consists of pairs with the desired properties Again by Corollary 9.1, we have $H^1_{loop}(R_1,\mathbf{PGL}_d) = H^1(R_1,\mathbf{PGL}_d)$ and we are reduced to twisted algebras given by loop cocycles $\phi = (\phi^{geo}, z)$ with value in $\mathbf{PGL}_d(\overline{k})$. Recall that $A_0 = {}_z(M_d)$ and that we have then a k–group homomorphism $\phi^{geo} : \boldsymbol{\mu}_m \to \mathbf{PGL}_1(A_0)$. We may assume that ϕ^{geo} is injective. We pull back the central extension $1 \to \mathbf{G}_m \to \mathbf{GL}_1(A_0) \xrightarrow{p} \mathbf{PGL}(A_0) \to 1$ by ϕ^{geo} and get a central extension of algebraic k–groups

$$1 \to \mathbf{G}_m \to \mathbf{E} \xrightarrow{p'} \boldsymbol{\mu}_m \to 1$$

such that $\mathbf{E}$ is a k–subgroup of $\mathbf{GL}_1(A_0)$. By extending the scalars to $\overline{k}$, we see that $\mathbf{E}$ is commutative, hence a k–group of multiplicative type. Then $\mathbf{E}$ is contained in a maximal torus of the k–group $\mathbf{GL}_1(A_0)$ and is of the form $R_{K/k}(\mathbf{G}_m)$ where $K \subset A_0$ is an étale algebra of degree d. We have then the commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbf{G}_m & \longrightarrow & \mathbf{E} & \longrightarrow & \boldsymbol{\mu}_m & \longrightarrow & 1 \\
 & & \downarrow \simeq & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbf{G}_m & \longrightarrow & R_{K/k}(\mathbf{G}_m) & \longrightarrow & R_{K/k}(\mathbf{G}_m)/\mathbf{G}_m & \longrightarrow & 1.
\end{array}$$

Lemma 13.6.2 tells us that $\chi_K = 0$. We compute the Brauer class of this loop algebra by taking into account the commutative diagram

$$\begin{array}{ccc}
H^1(F_1,\mathbf{PGL}_d) & \longrightarrow & \mathrm{Br}(F_1) \\
{\scriptstyle \tau_z}\uparrow \wr & & \wr \uparrow {\scriptstyle ?+[A_0]} \\
H^1\big(F_1,\mathbf{PGL}(A_0)\big) & \xrightarrow{\partial} & \mathrm{Br}(F_1) \\
\uparrow & & \uparrow \\
H^1(F_1,\boldsymbol{\mu}_m) & \xrightarrow{\partial} & \mathrm{Br}(F_1).
\end{array}$$

The commutativity of the upper square is that of the torsion map [Se, §I.5.7], while that of the bottom square is trivial. The image of $(t_1) \in F_1^\times/(F_1^\times)^\times$ is $\chi \cup (t_1)$ by Lemma 13.6.1. The diagram yields the formula $\partial([\phi]) = [A_0] \oplus \chi$ which has the required properties.

Conversely, let K/k be an étale algebra of degree d inside A_0 and let χ be a character such that $\chi_K = 0$. Let m be the order of χ; by restriction-corestriction considerations m divides d. Lemma 13.6.2 shows that there exists a morphism of extensions $E_\chi \to R_{K/k}(\mathbf{G}_m)$. This yields a morphism $\psi^{geo} : \boldsymbol{\mu}_m \to R_{K/k}(\mathbf{G}_m)/\mathbf{G}_m \to \mathbf{PGL}_1(A_0)$. The previous computation shows that the loop torsor (ψ^{geo}, z) has Brauer invariant $[A_0] \oplus \chi$.

□

As an example, we consider the real case.

Corollary 13.7. *Assume that $k = \mathbb{R}$. Then the image of the injective map*

$$H^1(R_1, \mathbf{PGL}_d) \to \mathrm{Br}(R_1) = \mathrm{Br}(\mathbb{R}) \oplus H^1(\mathbb{R}, \mathbb{Q}/\mathbb{Z})$$

is as follows:

1. $0 \oplus 0$ *if d is odd;*
2. $0 \oplus 0$, $0 \oplus \chi_{\mathbb{C}/\mathbb{R}}$, $[(-1,-1)] \oplus 0$ *and* $[(-1,-1)] \oplus \chi_{\mathbb{C}/\mathbb{R}}$ *if d is even.* □

Remark 13.8. In the case $d = 2$, the four classes under consideration corresponds to the quaternion algebras $(1,1)$, $(1,t)$, $(-1,-1)$, $(-1,t)$.

13.3 The geometric case

We assume that k is algebraically closed. According to Corollary 8.6.2, our goal is to extract information from the bijections

$$\mathrm{Hom}_{gp}\left(\widehat{\mathbb{Z}}^n, \mathbf{PGL}_d(k)\right)_{irr} / \mathbf{PGL}_d(k) \xrightarrow{\sim} H^1_{loop}(R_n, \mathbf{PGL}_d)_{irr} \xrightarrow{\sim} H^1(F_n, \mathbf{PGL}_d)_{irr}.$$

The right hand set is known from the work of Amitsur [A], Tignol-Wadsworth [TiW] and Brussel [Br],[33] the left hand-side is known by a classification due to Mumford [Mu, Prop. 3]

[33] See also [L], [Ne], [RY1, §8] and [RY2]. These last two references relate to finite abelian constant subgroups of $\mathbf{PGL}_d$ which have been investigated by Reichstein-Youssin in their construction of division algebras with large essential dimension. [Ne] is more interested in the "quantum tori" point of view and its relation to EALAs of absolute type A.

As a byproduct of our main result, we can provide then a proof of Mumford's classification of irreducible finite abelian constant subgroups of $\mathbf{PGL}_d$ from the knowledge of the Brauer group of the field F_n. Let us state first Mumford's classification. If $d = s_1 .... s_l \hat{s}$ with $s_1 \mid s_2 \cdots \mid s_l$ and $s_1 \geq 2$, we consider the embedding

$$\mathbf{PGL}_{s_1} \times \cdots \times \mathbf{PGL}_{s_l} \to \mathbf{PGL}_d$$

and define the subgroup $\mathbf{H}(s_1, ..., s_l)$ to be the image of the product of the standard anisotropic subgroups $\mathbf{H}(s_j) = (\mathbb{Z}/s_j\mathbb{Z})^2$ of $\mathbf{PGL}_{s_j}$ for $j = 1, .., l$ defined by the generators

$$(13.2) \quad a_j = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & & 0 \\ & \cdots & & 1 & 0 \end{pmatrix}, \qquad b_j = \begin{pmatrix} 1 & 0 & \cdots & & \\ 0 & \zeta_{s_j} & 0 & \cdots & 0 \\ 0 & \cdots & & & 0 \\ 0 & \cdots & & 0 & \zeta_{s_j}^{s_j - 1} \end{pmatrix}.$$

Remark 13.9. The way of expressing the group $\mathbf{H}(s_1, s_2)$ in the form $\mathbf{H}(s_1) \times \mathbf{H}(s_2)$ is not unique when s_1 and s_2 are coprime. There is then a unique way to arrange such a group $\mathbf{H}$ as $\mathbf{H}(s'_1, ..., s'_{l'})$ with $s'_1 \mid s'_2 \cdots \mid s'_l$ and $s'_1 \geq 2$. Note that $\operatorname{rank}\big(\mathbf{H}(s'_1, ..., s'_l)(k)\big) = 2l'$.

We can now state and establish the classification of irreducible finite abelian groups of the projective linear group.

Theorem 13.10. [Mu, Prop. 3] (see also [BL, §6], [GM, Th. 8.28]).

1. *$d = s_1 \times ... \times s_l$ if and only $\mathbf{H}(s_1, ..., s_l)$ is irreducible in $\mathbf{PGL}_d$.*
2. *If $\mathbf{H}$ is an irreducible finite abelian constant subgroup of $\mathbf{PGL}_d$, then $\mathbf{H}$ is $\mathbf{PGL}_d(k)$–conjugate to a unique $\mathbf{H}(s_1, ..., s_l)$ with $d = s_1 ... s_l$, $s_1 \mid s_2 \cdots \mid s_l$ and $s_1 \geq 2$.*

As mentioned above, our proof makes use of Galois cohomology results over R_n for $n \geq 1$ (or equivalently F_n) collected from our previous paper [GP3].

Our convention on the cyclic algebra $(t_i, t_j)_p^q$ is that of Tate[34] for the Azumaya R_n–algebra with presentation

$$X^q = t_i, \; Y^q = t_j^p, \; YX = \zeta_q \, XY.$$

[34] This is the opposite convention than that of [Br] and [GP3], but consistent with that of [GS] which we use in the proof.

Let us consider now Brussel R_n–algebras. Given sequences of length $l \leq [\frac{n}{2}]$

$$2 \leq s_1 \cdots \leq s_l, \qquad r_1, \cdots, r_l$$

we define

$$A(r_1, s_1, ..., r_l, s_l) : (t_1, t_2)^{r_1}_{s_1} \otimes_{R_n} \cdots \otimes_{R_n} (t_{2l-1}, t_{2l})^{r_l}_{s_l}.$$

Lemma 13.11. *With the notation as above, set $d = s_1 ... s_l$ and define*

$$\phi : \widehat{\mathbb{Z}}^n \to \mathbf{H}(s_1)(k) \times \cdots \times \mathbf{H}(s_l)(k) = \mathbf{H}(s_1, ..., s_l)(k) \ \subset \ \mathbf{PGL}_d(k),$$

$$(e_1, e_2, ..., e_{2l-1}, e_{2l}) \mapsto (-a_1, -r_1 b_1, ..., -a_l, -r_l \overline{b}_l)$$

1. *Then ${}_\phi(M_d) \xrightarrow{\sim} A(r_1, s_1, ..., r_l, s_l)$ as R_n–algebras.*
2. *The following are equivalent:*
   - *(a) $A(r_1, s_1, ..., r_l, s_l) \otimes_{R_n} F_n$ is division F_n–algebra;*
   - *(b) ϕ is irreducible;*
   - *(c) $\mathbf{H}(s_1, ..., s_l)$ is irreducible in $\mathbf{PGL}_d$ and $(r_j, s_j) = 1$ for $j = 1, ..., l$.*

Proof. (1) This is done for R_2 and each $\mathbf{H}(s_i)$ in [GP2, proof of Th. 3.17]. This "extends" in an additive way to yield the general case.
(2) The equivalence $(a) \iff (b)$ is a special case of [GP3, 3.1].
$(b) \implies (c)$: Since ϕ is irreducible, its image $\mathrm{Im}(\phi)$ is an irreducible subgroup of $\mathbf{PGL}_d$. This image is a product of the $\mathrm{Im}(\phi_i)$ which are then irreducible in $\mathbf{PGL}_{s_i}$. According to [GP3, 3.13], we have then $(r_j, s_j) = 1$ for $j = 1, ..., l$. Hence $\mathrm{Im}(\phi) = \mathbf{H}(s_1, ..., s_l)$ is irreducible in $\mathbf{PGL}_d$.
$(c) \implies (a)$: Since $\mathbf{H}(s_1, ..., s_l)$ is irreducible in $\mathbf{PGL}_d$, we have $d = s_1 .... s_l$. The condition $(r_j, s_j) = 1$ for $j = 1, ..., l$ implies that the algebra $A(r_1, s_1, ..., r_l, s_l) \otimes_{R_n} F_n$ is division [Am, th. 3]. □

We can now proceed with the proof of Theorem 13.10.

Proof. (1) If $d = s_1 ... s_l$ then $A(1, s_1, ..., 1, s_l) \otimes_{R_n} F_n$ is a division F_n–algebra [Am, th. 3], so Lemma 13.11 shows that $\mathbf{H}(s_1, ..., s_n)$ is irreducible in $\mathbf{PGL}_d$. If $d \neq s_1 ... s_l$, then this algebra is not division and $\mathbf{H}(s_1, ..., s_n)$ is reducible.
(2) If $\mathbf{H}(s_1, ..., s_l)$ is $\mathbf{PGL}_d(k)$–conjugate to some $\mathbf{H}(s'_1, ..., s'_{l'})$, then $\mathbf{H}(s_1, ..., s_l)$ is isomorphic to $\mathbf{H}(s'_1, ..., s'_{l'})$ as finite abelian group. So the divisibility conditions yield $l = l'$ and $s_j = s'_j$ for $j = 1, ..., l$.

The delicate points are existence and conjugacy. Let $\mathbf{H}$ be a finite abelian constant irreducible subgroup of $\mathbf{PGL}_d$. Denote by n the rank of $\mathbf{H}(k)$ and by m its exponent.

Let us prove first that $\mathbf{H}$ is $\mathbf{PGL}_d(k)$–conjugate to some $\mathbf{H}(s_1, ..., s_l)$. We view $\mathbf{H}(k)$ as the image of an irreducible group homomorphism $\psi : (\mathbb{Z}/m\mathbb{Z})^n \to \mathbf{PGL}_d(k)$. Since k is algebraically closed ψ is a cocycle. The loop construction then defines an Azumaya R_n–algebra of degree d such that $A \otimes_{R_n} F_n$ is division (i.e. the group $\mathbf{PGL}_1(A)_{F_n}$ is anisotropic). Up to base change by a suitable element of $\mathbf{GL}_n(\mathbb{Z})$, Theorem 4.7 of [GP3] provides an element $g \in \mathbf{GL}_n(\mathbb{Z})$ such that

$$g^*(A) \cong A(r_1, s_1, ..., r_l, s_l)$$

with $(r_j, s_j) = 1$ for $j = 1, .., l$.

By Lemma 6.1.1, $A(r_1, s_1, ..., r_l, s_l)$ is the loop Azumaya algebra defined by the morphism

$$\phi : \hat{\mathbb{Z}}^n \to \mathbf{H}(s_1)(k) \times \cdots \times \mathbf{H}(s_l)(k) = \mathbf{H}(s_1, ..., s_l)(k) \subset \mathbf{PGL}_d(k),$$

$$(e_1, e_2, ..., e_{2l-1}, e_{2l}) \mapsto (-a_1, -b_1, ..., -a_l, -r_l b_l).$$

which is then irreducible by the second statement of the same lemma. Theorem 7.9 tells us that ϕ and ψ are $\mathbf{PGL}_d(k)$–conjugate, hence $\mathbf{H}(k)$ is $\mathbf{PGL}_d(k)$–conjugate to $\mathbf{H}(s_1, ..., s_l)(k) = \mathrm{Im}(\psi)$. Since $n = \mathrm{rank}(\mathbf{H}(s_1, ..., s_l)(k))$, we have $s_1 \mid s_2 ... \mid s_l$. □

We can now go back to Azumaya algebras.

Theorem 13.12. *Let A be an anisotropic loop Azumaya R_n–algebra of degree d.*

1. *There exists a sequence $s_1, ..., s_l$ and an integer r_1 prime to s_1 satisfying*

   $$s_1 \mid \cdots \mid s_l, \quad 2 \geq s_1, \quad d = s_1 \cdots s_l, \quad (r_1, s_1) = 1$$

   and an element $g \in \mathbf{GL}_n(\mathbb{Z})$ such that

   $$g^*(A) \cong A(r_1, s_1, 1, s_2, 1, s_3, \cdots, 1, s_l) \cong A(-r_1, s_1, 1, s_2, 1, s_3, \cdots, 1, s_l).$$

   Such a sequence $s_1, ..., s_l$ is unique.

2. *If $n = 2l$, $\pm r_1$ is unique modulo s_1.*

3. *If $n > 2l$, $g^*(A) \cong A(1, s_1, 1, s_2, 1, s_3, \cdots, 1, s_l)$.*

Proof. (1) By definition, A is the twist of $M_d(k)$ by a morphism $\phi : (\mathbb{Z}/m\mathbb{Z})^n \to \mathbf{PGL}_d(k)$. Since $A \otimes_{R_n} F_n$ is division, ϕ is irreducible [GP2, th. 3.1]. Theorem 13.10 shows that there exists a unique sequence $s_1, ..., s_l$ such that $s_1 \mid \cdots \mid s_l, \quad 2 \geq s_1$ and $\mathrm{Im}(\phi)$ is $\mathbf{PGL}_d(k)$–conjugate to $H(s_1, ..., s_l) := \mathbf{H}(s_1, ..., s_l)(k)$. Without lost of generality, we can assume that $\mathrm{Im}(\phi) = H(s_1, ..., s_l)$.

Recall that $a_1, b_1, ..., a_l, b_l$ stand for the standard generators of $\mathbf{H}(s_1, ..., s_l)$. We shall use that $\Lambda^l\big(H(s_1, ..., s_l)\big) \simeq \mathbb{Z}/s_1\mathbb{Z}$ generated by $a_1 \wedge b_1 \cdots a_l \wedge b_l$ [RY2, Lemma 2.1], as well as the following invariant of ϕ (*ibid*, 2.5)

$$\delta(\phi) = \phi(e_1) \wedge \phi(e_2) \wedge \cdots \wedge \phi(e_n) \, \in \, \Lambda^n\big(\mathbf{H}(s_1, ..., s_l)(k)\big)$$

This invariant has the remarkable property that a homomorphism $\phi' : (\mathbb{Z}/m\mathbb{Z})^n \to \mathbf{H}(s_1, ..., s_l)(k)$ is $\mathbf{GL}_n(\mathbb{Z})$–conjugate to ϕ if and only if $\delta(\phi) = \pm\delta(\phi')$.

We shall prove (1) together with (2) [resp. (3)] in the case $n = 2l$ (resp. $n > 2l$).

First case. $n = 2l$: The family $(\phi(e_1), ..., \phi(e_n))$ generates $H(s_1, ..., s_l)$, and we consider the class

$$[r_1^\sharp] := \phi(e_1) \wedge \phi(e_2) \wedge \cdots \wedge \phi(e_n) \in (\mathbb{Z}/s_1\mathbb{Z})^\times.$$

Let r_1 be an inverse of $r_1^\sharp$ modulo s_1. We have

$$\phi(r_1 e_1) \wedge \phi(e_2) \cdots \wedge \phi(e_n) = a_1 \wedge b_1 \cdots a_l \wedge b_l$$

so there exists $g \in \mathbf{GL}_n(\mathbb{Z})$ such that (*ibid*, 2.5)

$$\psi(r_1 e_1) = a_1, \psi(e_2) = b_2, \cdots, \phi(e_{n-1}) = a_l, \, \phi(e_n) = b_l$$

where $\psi = \phi \circ g$. In terms of algebras, this means that

$$g^*(A) \simeq A(r_1, s_1, 1, s_2, 1, s_3, \cdots, 1, s_l).$$

Let us first prove the unicity assertions The unicity of $(s_1, .., s_l)$ follows from Theorem 13.10, hence (1) is proven. For (2), we are given then $r_1' \in \mathbb{Z}$ coprime to s_1, and an element $h \in \mathbf{GL}_n(\mathbb{Z})$ such that

$$h^*\big(A(r_1, s_1, 1, s_2, 1, s_3, \cdots, 1, s_l)\big) \simeq A(r_1', s_1, 1, s_2, 1, s_3, \cdots, 1, s_l).$$

Denote by $\psi' : (\mathbb{Z}/m\mathbb{Z})^n \to H(s_1, .., s_n)$ the group homomorphism defined by $\psi'(e_1) = r_1' a_1$, $\psi'(e_2) = b_1, \cdots, \psi(e_n) = b_{2l}$. Since the ($F_n$–anisotropic) loop algebras attached to $h^*\psi$ and ψ' are isomorphic, Theorem 7.9 provides an element $u \in \mathbf{PGL}_d(k)$ such that

$$\psi \circ h = u \circ \psi'.$$

Since $H(s_1, ..., s_l) = \mathrm{Im}(\psi) = \mathrm{Im}(\psi')$, it follows that $u \in \mathbf{N}_{\mathbf{PGL}_d(k)}\big(H(s_1, ...s_l)\big)$. But the map $u : \mathbf{H}(s_1, ..., s_l)(k) \to \mathbf{H}(s_1, ...s_l)(k)$ preserves the symplectic pairing $\mathbf{H}(s_1, ..., s_l)(k) \times \mathbf{H}(s_1, ..., s_l)(k) \to k^\times$ arising by taking the commutator of lifts in $\mathbf{GL}_d(k)$. It follows that $\Lambda^n(u) = id$ (*ibid*, 2.3.a) hence

$$\delta(\psi) = \pm\delta(\psi \circ h) = \pm\delta(u \circ \psi') = \pm\delta(u \circ \psi').$$

Thus $r_1 = \pm r_1' \in \mathbb{Z}/s_1\mathbb{Z}$ as prescribed in (2) .
Second case. $n > 2l$*:* For $i = 2l+1, ..., n$ we set $c_i = 0 \in H(s_1, .., s_l)$. Both families $\big(\phi(e_1), \dots \phi(e_n)\big)$ and $(r_1\, a_1), b_1 \cdots, a_l, b_l, c_{2l+1}, \cdots, c_n)$ generate $H(s_1, ..., s_l)$ and satisfy

$$\phi(e_1) \wedge \phi(e_2) \cdots \wedge \phi(e_n) = (r_1\, a_1) \wedge b_1 \cdots a_l \wedge b_l \wedge c_{2l+1} \wedge \cdots \wedge c_n \in \Lambda^n\big(H(s_1, ..., s_l)\big) = 0.$$

The same fact [RY2, Lemma 2.5] shows that there exists $g \in \mathbf{GL}_n(\mathbb{Z})$ such that $(g^*\phi)(e_1) = r_1 a_1$, $(g^*\phi)(e_2) = b_1$, $(g^*\phi)(e_{2i-1}) = a_i$ and $(g^*\phi)(e_{2i}) = b_i$ for $i = 2, .., l$ and $(g^*\phi)(e_i) = c_i$ for $i = 2l+1, ..., n$. Therefore the preceding case with $2l$ variables yields the existence and the unicity of the s_i's. It remains to prove (3), namely that we can assume that $r_1 = 1$. But this follows along the same lines of the argument given above since $(r_1\, a_1) \wedge b_1 \cdots a_l \wedge b_l \wedge c_{2l+1} \cdots \wedge c_n = (a_1) \wedge b_1 \cdots a_l \wedge b_l \wedge c_{2l+1} \cdots \wedge c_n \in \Lambda^n\big(H(s_1, ..., s_l)\big)$. □

13.4 Loop algebras of inner type A

To the Azumaya R_n–algebra $A(r_1, s_1, ..., r_l, s_l)$ we can attach (using the commutator $[x, y] = xy - yx$) a Lie algebra over R_n. We denote by $L(r_1, s_1, ..., r_l, s_l)$ the derived algebra of this Lie algebra. It is a twisted form of $\mathfrak{sl}_d(R_n)$ where $d = s_1 \dots s_l$.

Corollary 13.13. *Let d be a positive integer. Let L be a nullity n loop algebra of inner (absolute) type A_{d-1}.*

1. *If L is not split, it is k–isomorphic to $L(r_1, s_1, 1, s_2, \cdots, 1, s_l)$ where*

$$s_1 \mid \cdots \mid s_l, \quad 2 \geq s_1, \quad d = s_1 \cdots s_l, \quad (r_1, s_1) = 1 \quad \textit{and} \quad l \leq \left[\frac{n}{2}\right]$$

and such a sequence $s_1, ..., s_l$ is unique.

2. *If $n = 2l$, r_1 is unique modulo s_1 and up to the sign.*

3. *If $n > 2l$, L is k–isomorphic to $L(1, s_1, 1, s_2, \cdots, 1, s_l)$*

Proof. The classification in question is given by considering the image of the natural map

$$H^1_{loop}(R_n, \mathbf{PGL_d}) \to \mathbf{GL}_n(\mathbb{Z})\backslash H^1(R_n, \mathbf{Aut}(\mathbf{PGL}_d))$$

The image can be identified with $(\mathbb{Z}/2\mathbb{Z} \times \mathbf{GL}_n(\mathbb{Z}))\backslash H^1_{loop}(R_n, \mathbf{PGL_d})$ where $\mathbb{Z}/2\mathbb{Z}$ acts by the opposite Azumaya algebra construction. Corollary 8.6 reduces the problem to the "anisotropic case". Theorem 13.12 determines the set $\mathbf{GL}_n(\mathbb{Z})\backslash H^1_{loop}(R_n, \mathbf{PGL_d})$, and as it turns out the action of $\mathbb{Z}/2\mathbb{Z}$ is trivial. Therefore the desired classification is also provided by $\mathbf{GL_n}(\mathbb{Z})\backslash H^1_{loop}(R_n, \mathbf{PGL}_d)$ and we can now appeal to Theorem 13.12 to obtain the Corollary. □

14 Invariants attached to EALAs and multiloop algebras

Both the finite dimensional simple Lie algebras over $\overline{k}$ (nullity 0) and their affine counterparts (nullity 1) have Coxeter-Dynkin diagrams attached to them that contain a considerable amount of information about the algebras themselves. It has been a long dream to find a meaningful way of attaching some kind of diagram to multiloop, or at least EALAs of arbitrary nullity (perhaps with as many nodes as the nullity). Our work shows that this can indeed be done and in a very natural way.

Let us recall (see the Introduction for more details) the multiloop algebras based on a finite dimensional simple Lie algebra $\mathfrak{g}$ over an algebraically closed field k of characteristic 0. Consider an n–tuple $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ of commuting elements of $\mathrm{Aut}(\mathfrak{g})$ satisfying $\sigma_i^m = 1$. For each n–tuple $(i_1, \dots, i_n) \in \mathbb{Z}^n$ we consider the simultaneous eigenspace $\mathfrak{g}_{i_1 \dots i_n} = \{x \in \mathfrak{g} : \sigma_j(x) = \xi_m^{i_j} x \text{ for all } 1 \leq j \leq n\}$. The multiloop algebra $L(\mathfrak{g}, \boldsymbol{\sigma})$ corresponding to $\boldsymbol{\sigma}$ is defined by

$$L(\mathfrak{g}, \boldsymbol{\sigma}) = \bigoplus_{(i_1, \dots, i_n) \in \mathbb{Z}^n} \mathfrak{g}_{i_1 \dots i_n} \otimes t^{\frac{i_1}{m}} \dots t_n^{\frac{i_n}{m}} \subset \mathfrak{g} \otimes_k R_{n,m} \subset \mathfrak{g} \otimes_k R_\infty$$

Recall that $L(A, \boldsymbol{\sigma})$, which does not depend on the choice of common period m, is not only a k–Lie algebra (in general infinite dimensional), but also naturally an R–algebra. It is when $L(\mathfrak{g}, \boldsymbol{\sigma})$ is viewed as an R–algebra that Galois cohomology and the theory of torsors enter into the picture. Indeed a rather simple calculation shows that

$$L(\mathfrak{g}, \boldsymbol{\sigma}) \otimes_{R_n} R_{n.m} \simeq \mathfrak{g} \otimes_k R_{n,m} \simeq (\mathfrak{g} \otimes_k R_n) \otimes_{R_n} R_{n,m}.$$

Thus $L(\mathfrak{g}, \boldsymbol{\sigma})$ corresponds to a torsor $\mathfrak{E}_{\boldsymbol{\sigma}}$ over $\mathrm{Spec}(R)$ under $\mathbf{Aut}(\mathfrak{g})$. It is, however, the k–Lie algebra structure that is of interest in infinite dimensional Lie theory and Physics.

Let $\mathbf{G}$ be the k–Chevalley group of adjoint type corresponding to $\mathfrak{g}$. Since $\mathbf{Aut}(\mathbf{G})$ and $\mathbf{Aut}(\mathfrak{g})$ coincide we can also consider the twisted R_n–group ${}_{\mathfrak{E}}\mathbf{G}_{R_n}$. By functoriality and the definition of Lie algebra of a group functor in terms of dual numbers we see that $\mathfrak{Lie}({}_{\mathfrak{E}}\mathbf{G}_{R_n}) = L(\mathfrak{g}, \boldsymbol{\sigma})$. By the Aciclicity Theorem to ${}_{\mathfrak{E}}\mathbf{G}_{R_n}$ we can attach a Witt-Tits index, and this is the "diagram" that we attach to $L(\mathfrak{g}, \boldsymbol{\sigma})$ as *as Lie algebra over k*. Note that by Corollary 8.12 this is well defined. The diagram carries the information about the absolute and relative type of $L(\mathfrak{g}, \boldsymbol{\sigma})$.[35]

[35] The relative type as an invariant of $L(\mathfrak{g}, \boldsymbol{\sigma})$ is defined in in §3 of [ABP3] by means of the central closure. If C is the centroid of $L(\mathfrak{g}, \boldsymbol{\sigma})$ and $\widetilde{C}$ denotes its field of quotients, then $L(\mathfrak{g}, \boldsymbol{\sigma}) \otimes_C \widetilde{C}$ is a finite dimensional central simple algebra over $\widetilde{C}$. As such it has an absolute and relative type. This construction applies to an arbitrary prime perfect Lie algebra which is finitely generated over its centroid.

To reassure ourselves that this is the correct point of view we can look at the nullity one case. EALAs of nullity one are the same than the affine Kac-Moody Lie algebras. If one uses Tits methods to compute simple adjoint algebraic groups over the field $k((t))$ one obtains precisely the diagrams of the affine algebras.

15 Appendix 1: Pseudo-parabolic subgroup schemes

We extend the definition of pseudo-parabolic subgroups[36] of affine algebraic groups (Borel-Tits [BoT2], see also [Sp, §13.4]) to the case of a group scheme $\mathfrak{G}$ which is of finite type and affine over a fixed base scheme $\mathfrak{X}$. We begin by establishing some notation.

We will denote by $\mathbf{G}_{m,\mathfrak{X}}$ and $\mathbf{G}_{a,\mathfrak{X}}$ the multiplicative and additive $\mathfrak{X}$-groups. The underlying schemes of these groups will be denoted by $\mathbb{A}^{\times}_{\mathfrak{X}}$ and $\mathbb{A}_{\mathfrak{X}}$ respectively. After applying a base change $\mathfrak{X} \to \mathfrak{X}'$ we obtain corresponding $\mathfrak{X}'$-groups and schemes that we denote by $\mathbf{G}_{m,\mathfrak{X}'}$, $\mathbf{G}_{a,\mathfrak{X}'}$, $\mathbb{A}^{\times}_{\mathfrak{X}}$ and $\mathbb{A}_{\mathfrak{X}'}$.

The structure morphism of the $\mathfrak{X}'$–scheme $\mathbb{A}^{\times}_{\mathfrak{X}'}$ gives by functoriality a group homomorphism

$$\eta_{\mathfrak{X}'} : \mathfrak{G}(\mathfrak{X}') \to \mathfrak{G}(\mathbb{A}^{\times}_{\mathfrak{X}'}) \tag{15.1}$$

Let $\lambda : \mathbf{G}_{m,\mathfrak{X}} \to \mathfrak{G}$ be a cocharacter. By applying $\lambda_{\mathfrak{X}'}$ to the identity map $\mathrm{id}_{\mathbb{A}^{\times}_{\mathfrak{X}'}} \in \mathbf{G}_{m,\mathfrak{X}'}(\mathbb{A}^{\times}_{\mathfrak{X}'})$ we obtain an element of $\lambda_{\mathfrak{X}'}(\mathrm{id}_{\mathbb{A}^{\times}_{\mathfrak{X}'}}) \in \mathfrak{G}(\mathbb{A}^{\times}_{\mathfrak{X}'})$.

We have a natural group homomorphism $\mathfrak{G}(\mathbb{A}_{\mathfrak{X}'}) \to \mathfrak{G}(\mathbb{A}^{\times}_{\mathfrak{X}'})$. Given an element $x' \in \mathfrak{G}(\mathbb{A}^{\times}_{\mathfrak{X}'})$ we will write $x' \in \mathfrak{G}(\mathbb{A}_{\mathfrak{X}'})$ if x' is in the image of this map.

After these preliminary definitions we are ready to define the three group functors that are relevant to the definition of pseudo-parabolic subgroups.

Let $\mathbf{Z}_{\mathfrak{G}}(\lambda)$ denote the centralizer of λ. Recall that this is the $\mathfrak{X}$-group functor that to a scheme $\mathfrak{X}'$ over $\mathfrak{X}$ attaches the group

$$\mathbf{Z}_{\mathfrak{G}}(\lambda)(\mathfrak{X}') = \{x' \in \mathfrak{G}(\mathfrak{X}') : x'' \text{ commutes with } \lambda\big(\mathbf{G}_{m,\mathfrak{X}}(\mathfrak{X}'')\big) \subset \mathfrak{G}(\mathfrak{X}'')\} \tag{15.2}$$

where $\mathfrak{X}''$ is a scheme over $\mathfrak{X}'$ and x'' denotes the image of x' under the natural group homomorphism $\mathfrak{G}(\mathfrak{X}') \to \mathfrak{G}(\mathfrak{X}'')$.

We consider the two following $\mathfrak{X}$–functors

$$\mathbf{P}(\lambda)(\mathfrak{X}') = \Big\{g \in \mathfrak{G}(\mathfrak{X}') \;\mid\; \lambda_{\mathfrak{X}'}(\mathrm{id}_{\mathbb{A}^{\times}_{\mathfrak{X}'}})\, \eta_{\mathfrak{X}'}(g)\, \big(\lambda_{\mathfrak{X}'}(id_{\mathbb{A}^{\times}_{\mathfrak{X}'}})\big)^{-1} \in \mathfrak{G}(\mathbb{A}_{\mathfrak{X}'})\Big\}$$

[36] In [CoGP] the groups that we are about to define are called *limit subgroups*. We have decided, since we are only dealing with analogues of pseudo-parabolic subgroups over fields, to abide by this terminology. This material can in part be recovered from their work, but we have decided to include it in the form that we needed for the sake of completeness.

and

$$\mathbf{U}(\lambda)(\mathfrak{X}') = \Big\{ g \in \mathfrak{G}(\mathfrak{X}') \ | \ \lambda_{\mathfrak{X}'}(\mathrm{id}_{\mathbb{A}^\times_{\mathfrak{X}'}})\, \eta_{\mathfrak{X}'}(g)\, \big(\lambda_{\mathfrak{X}'}(id_{\mathbb{A}^\times_{\mathfrak{X}'}})\big)^{-1} \in \ker\big(\mathfrak{G}(\mathbb{A}_{\mathfrak{X}'}) \to \mathfrak{G}(\mathfrak{X}')\big) \Big\}$$

for every $\mathfrak{X}$-scheme $\mathfrak{X}'$. The centralizer $\mathbf{Z}_{\mathfrak{G}}(\lambda)$ is an $\mathfrak{X}$–subgroup functor of $\mathbf{P}(\lambda)$ which normalizes $\mathbf{U}(\lambda)$.

We look at the previous definitions in the case when $\mathfrak{X} = \mathrm{Spec}(R)$ and $\mathfrak{X}' = \mathrm{Spec}(R')$ are both affine.[37] We have $\mathbb{A}_{R'} = \mathrm{Spec}(R'[x])$ and $\mathbb{A}^\times_{R'} = \mathrm{Spec}(R'[x^{\pm 1}])$. Then $x \in R'[x^{\pm 1}]^\times = \mathbf{G}_{m,R'}(R'[x^{\pm 1}]) = \mathbf{G}_{m,R'}(\mathbb{A}^\times_{R'})$, and by applying our cocharacter we obtain an element $\lambda_{R'}(x) \in \mathfrak{G}(R')$. Under Yoneda's correspondence $\mathbf{G}_{m,R'}(R'[x^{\pm 1}]) \simeq \mathrm{Hom}_{R'}(R'[x^{\pm 1}], R'[x^{\pm 1}])$ our element x corresponds to the identity map, namely to the element $\mathrm{id}_{\mathbb{A}^\times_{R'}} \in \mathbf{G}_{m,R'}(\mathbb{A}^\times_{R'})$ if we rewrite our ring theoretical objects in terms of schemes. We thus have

$$\mathbf{P}(\lambda)(R') = \Big\{ g \in \mathfrak{G}(R') \ | \ \lambda_{R'}(x)\, \eta_{R'}(g)\, \big(\lambda_{R'}(x)\big)^{-1} \in \mathfrak{G}(R'[x]) \Big\}$$

and

$$\mathbf{U}(\lambda)(R') = \Big\{ g \in \mathfrak{G}(R') \ | \ \lambda_{R'}(x)\, \eta_{R'}(g)\, \big(\lambda_{R'}(x)\big)^{-1} \in \ker\big(\mathfrak{G}(R'[x]) \to \mathfrak{G}(R')\big) \Big\}$$

where $\eta_{R'}(g)$ is the natural image of $g \in \mathfrak{G}(R')$ in $\mathfrak{G}(R'[x^{\pm 1}])$, and the group homomorphism $\mathfrak{G}(R'[x]) \to \mathfrak{G}(R')$ comes from the ring homomorphism $R'[x] \to R'$ that maps $x \mapsto 0$.

15.1 The case of $\mathbf{GL}_{n,\mathbb{Z}}$.

Assume $S =$ Spec$(\mathbb{Z})$ and let $\mathbf{G}$ denote the general linear group $\mathbf{GL}_{n,\mathbb{Z}}$ over $\mathbb{Z}$. We let $\mathbf{T}$ denote the standard maximal torus of $\mathbf{G}$. Let $\lambda : \mathbf{G}_{m,\mathbb{Z}} \to \mathbf{T} \hookrightarrow \mathbf{G}$ be a cocharacter of $\mathbf{G}$ that factors through $\mathbf{T}$. We review the structure of the groups $\mathbf{Z}_{\mathfrak{G}}(\lambda)$, $\mathbf{P}(\lambda)$ and $\mathbf{U}(\lambda)$.

After replacing λ by $\mathrm{int}(\theta) \circ \lambda$ for some suitable $\theta \in \mathbf{G}(\mathbb{Z})$ we may assume that there exists (unique) integers $1 \le \ell_1 < \ell_2 < \cdots < \ell_j \le n$ and $e_1, \ldots, e_n$ such that

$$\begin{aligned} &e_i = e_j \quad \text{if} \quad \ell_k \le i,\ j < \ell_{k+1} \quad \text{for some} \quad k \\ &e_{\ell_k + 1} > e_{\ell_k} \quad \text{for all} \quad 1 \le k \le j \end{aligned}$$

so that the functor of points of our map $\lambda : \mathbf{G}_{m,\mathbb{Z}} \to \mathbf{T}$ is given by

$$\lambda_R : \mathbf{G}_{m,\mathbb{Z}}(R) \longrightarrow \mathbf{T}(R)$$

[37]As customary we write $\mathbf{G}_{m,R}$ instead of $\mathbf{G}_{m,\mathfrak{X}}$...

$$(*) \qquad r \mapsto \begin{pmatrix} r^{e_1} & & 0 \\ & \ddots & \\ 0 & & \\ & & r^{e_n} \end{pmatrix}$$

for any (commutative) ring R and for all $r \in \mathbf{G}_{m,\mathbb{Z}}(R) = R^\times$.

At the level of coordinate rings if $\mathbf{G}_{m,\mathbb{Z}} = \operatorname{Spec}\big(\mathbb{Z}[x^{\pm 1}]\big)$ and $\mathbf{T} = \operatorname{Spec}\big(\mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]\big)$, then λ corresponds (under Yoneda) to the ring homomorphism

$$\lambda^* : \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \longrightarrow \mathbb{Z}[x^{\pm 1}]$$

given by

$$\lambda^* : t_i \mapsto x^{e_i}.$$

¿From this it follows that $\mathbf{Z}_\mathbf{G}(\lambda)(R)$ consists of block diagonal matrices inside $\mathbf{GL}_n(R)$ of size $\ell_1, \dots, \ell_j$. Note that one "cannot see" this by looking at the centralizer of $\lambda\big(\mathbf{G}_{m,\mathbb{Z}}(R)\big)$ inside $\mathbf{G}(R)$. This is clear, for example, if $n = 2$, $R = \mathbb{Z}$, $j = 1$, $\ell_1 = 1$ and $1 = e_1 < e_2 = 3$. The easiest way to eliminate "naive" contralizers in $\mathbf{Z}_\mathbf{G}(\lambda)(R)$ is to look at their image in $\mathbf{G}\big(R[x^{\pm 1}]\big)$. In fact

Lemma 15.1. *With the above notation we have*

$$\mathbf{Z}_\mathbf{G}(\lambda)(R) = \big\{\, A \in \mathbf{G}(R) \subset \mathbf{G}(R[x^{\pm 1}]) : A \textit{ commutes with } \lambda_R\big(\mathbf{G}_{m,\mathbb{Z}}(R[x^{\pm 1}])\big) \,\big\}.$$

Proof. The inclusion $\subset$ follows from the definition of $\mathbf{Z}_\mathbf{G}(\lambda)$. Conversely suppose that $A \in \mathbf{G}(R)$ is not an element of $\mathbf{Z}_\mathbf{G}(\lambda)(R)$. Then there exists a ring homomorphism $R \to S$ and an element $s \in S^\times$ such that the image of A in $\mathbf{G}(S)$ does not commute with the diagonal matrix

$$\lambda_S(s) = \begin{pmatrix} s^{e_1} & & \\ & \ddots & \\ & & s^{e_n} \end{pmatrix}.$$

But then A, viewed now as an element of $\mathfrak{G}\big(R[x^{\pm 1}]\big)$ cannot commute with

$$\lambda_{R[x^{\pm 1}]}(x) = \begin{pmatrix} x^{e_1} & & \\ & \ddots & \\ & & x^{e_n} \end{pmatrix}.$$

For if it did, we would reach a contradiction by functoriality considerations applied to the (natural) ring homomorphism $R[x^{\pm 1}] \to S$ that takes $x \mapsto s$. $\square$

Returning to our example we see that there are two extreme cases for $\mathbf{Z_G}(\lambda)$. If $j = 1$ and $\ell_1 = n$ then $\mathbf{Z_G}(\lambda) = \mathbf{G}$. At the other extreme if $j = n$ then λ is regular and $\mathbf{Z_G}(\lambda) = \mathbf{T}$. In all cases we see from the diagonal block description that $\mathbf{Z_G}(\lambda)$ is a closed subgroup of $\mathbf{G}$ (in particular affine).

We now turn our attention to $\mathbf{P}(\lambda)$ and $\mathbf{U}(\lambda)$. By using $(*)$ one immediately sees that

$$\mathbf{P}(\lambda)(R) = \{A = (a_{ij}) \in \mathfrak{G}(R) : a_{ji} = 0 \text{ if } e_i > e_j\}.$$

Thus the $\mathbf{P}(\lambda)$ are the standard parabolic subgroups of $\mathbf{G}$.

Example 15.2. We illustrate with the case $n = 5$ with $j = 2$ and $\ell_1 = 2,\ \ell_2 = 5,\ e_1 = 1,\ e_2 = 3$. Then $A = (a_{ij}) \in \mathbf{GL}_5(R)$ is of the form

$$A = \left(\begin{array}{c|c} \times & + \\ \hline - & \times \end{array}\right).$$

We have two blocks, the top left of size 2 and the bottom right of size 3. Given $A = (a_{ij}) \in \mathbf{GL}_5(R) \subset \mathbf{GL}_5(R[x^{\pm 1}])$ define P by

$$\begin{pmatrix} x^1 & & & 0 & \\ & x & & & \\ & & x^3 & & \\ & 0 & & x^3 & \\ & & & & x^3 \end{pmatrix} A \begin{pmatrix} x^{-1} & & & 0 & \\ & x^{-1} & & & \\ & & x^{-3} & & \\ & 0 & & x^{-3} & \\ & & & & x^{-3} \end{pmatrix} = P$$

that is

$$\lambda(x) A \lambda(x)^{-1} = P$$

where $P = (p_{ij})$ and $p_{ij} = \sum p_{ijk} x^k \in R[x^{\pm 1}]$. To belong to $\mathbf{P}(\lambda)$ the element A must be such that $p_{ijk} = 0$ for $k < 0$. This forces all entries in the 3×2 block marked with a $-$ to vanish. For the elements in $\mathbf{Z_G}(\lambda)(R)$ both blocks $-$ and $+$ must vanish.

It is easy to determine that if $A \in \mathbf{P}(\lambda)$ the matrix P is such that the $p_{ij} = a_{ij} \in R$ whenever $1 \le i,\ j \le 2$ or $2 \le i,\ j \le 5$. If, on the other hand, $i \le 2 < j$ then $p_{ij} = a_{ij} x^2$.

This makes the meaning of $\mathbf{U}(\lambda)$ quite clear in general. If $\lambda(x) A \lambda(x)^{-1} \in \mathbf{G}(R[x])$ is mapped to the identity element of $\mathbf{G}(R)$ under the map $R[x] \to R$ which sends $x \mapsto 0$ then $a_{ii} = 1$ and $a_{ij} = 0$ if $e_i > e_j$. That is

$$\mathbf{U}(\lambda)(R) = \{A = (a_{ij}) \in \mathbf{P}(\lambda) : a_{ii} = 1 \text{ and } a_{ij} = 0 \text{ if } e_i > e_j\}.$$

In particular $\mathbf{U}(\lambda)$ is an unipotent subgroup of $\mathbf{P}(\lambda)$.

15.2 The general case

Lemma 15.3. *Assume that there exists locally for the fpqc-topology a closed embedding of $\mathfrak{G}$ in a linear group scheme.*[38] *Then*

1. *the $\mathfrak{X}$–functor* $\mathbf{U}(\lambda)$ *(resp.* $\mathbf{P}(\lambda)$*, resp.* $\mathbf{Z}_{\mathfrak{G}}(\lambda)$*) is representable by a closed subgroup scheme of $\mathfrak{G}$ which is affine over $\mathfrak{X}$.*
2. *The geometric fibers of* $\mathbf{U}(\lambda)$ *are unipotent.*
3. $\mathbf{P}(\lambda) = \mathbf{U}(\lambda) \rtimes \mathbf{Z}_{\mathfrak{G}}(\lambda)$.
4. $\mathbf{Z}_{\mathfrak{G}}(\lambda) = \mathbf{P}(\lambda) \times_{\mathfrak{G}} \mathbf{P}(-\lambda)$.
5. $\mathbf{P}(\lambda) = \mathbf{N}_{\mathfrak{G}}\big(\mathbf{P}(\lambda)\big)$.

Proof. The case of $\mathbf{GL}_{n,S}$*:* The question is local with respect to the *fpqc* topology, so we can assume then that $\mathfrak{X}$ is the spectrum of a local ring R. Since all maximal split[39] tori of the R–group $\mathbf{GL}_{n,R}$ are conjugate under $\mathbf{GL}_n(R)$ [SGA3, XXVI.6.16], we can assume that $\lambda : \mathbf{G}_{m,R} \to \mathbf{T}_R < \mathbf{GL}_{n,R}$ where $\mathbf{T}$ is the standard maximal torus of $\mathbf{GL}_{n,\mathbb{Z}}$. Since $\mathrm{Hom}_{\mathbb{Z}}(\mathbf{G}_m, \mathbf{T}) \simeq \mathrm{Hom}_R(\mathbf{G}_{m,R}\mathbf{T}_R)$, we can reduce our problem to the case when $R = \mathbb{Z}$, which has been already done in Example 15.1.

General case:

By *fpqc*-descent, we can assume that $\mathfrak{X}$ is the spectrum of a ring R, and that we are given a R–group scheme homomorphism $\rho : \mathfrak{G} \to \mathfrak{G}' = \mathbf{GL}_{n,R}$ which is a closed immersion.

(1) Denote by $\mathbf{P}'(\lambda)$ and $\mathbf{U}'(\lambda)$ the R–subfunctors of $\mathfrak{G}'$ attached to the cocharacter $\rho \circ \lambda$. The identities $\mathbf{P}(\lambda) = \mathbf{P}'(\lambda) \times_{\mathfrak{G}'} \mathfrak{G}$ and $\mathbf{U}(\lambda) = \mathbf{U}'(\lambda) \times_{\mathfrak{G}'} \mathfrak{G}$ can be established by reducing to the case of $\mathfrak{G}' = \mathbf{GL}_{n,R}$. This reduces the representability questions to the case when $\mathfrak{G} = \mathbf{GL}_{n,R}$ considered above.

(2) This follows as well from the $\mathbf{GL}_{n,R}$ case.

(3) We know that the result holds for $\mathfrak{G}'$. Let R' be a ring extension of R and let $g \in \mathfrak{G}(R')$. Then $g = uz$ with $u \in \mathbf{U}'(\lambda)(R')$ and $z \in \mathbf{P}'(\lambda)(R')$. We have

$$\lambda(x)\, g\, \lambda(x)^{-1} \lambda(x)\, u\, \lambda(x)^{-1}\, z \in \mathfrak{G}(\mathbb{A}_{R'}).$$

By specializing at 0, we get that $z \in \mathfrak{G}(R')$. Thus $g \in \mathbf{Z}_{\mathfrak{G}}(\lambda)(R')$ and $u \in \mathbf{U}(\lambda)(R')$. We conclude that $\mathbf{P}(\lambda) = \mathbf{U}(\lambda) \rtimes \mathbf{Z}_{\mathfrak{G}}(\lambda)$.

(4) and (5) follows from the $\mathbf{GL}_{n,R}$ case. □

[38] This condition is satisfied if $\mathfrak{X}$ is locally noetherian of dimension ≤ 1 [BT2, §1.4], and also for reductive $\mathfrak{X}$–group schemes.

[39] Trivial, in the terminology of [SGA3].

Definition 15.4. An $\mathfrak{X}$–subgroup of $\mathfrak{G}$ is *pseudo-parabolic* if it is of the form $\mathbf{P}(\lambda)$ for some $\mathfrak{X}$–group homomorphism $\mathbf{G}_{m,X} \to \mathfrak{G}$.

Proposition 15.5. *Let $\mathfrak{G}$ be a reductive group scheme over $\mathfrak{X}$.*

1. *Let $\lambda : \mathbf{G}_{m,\mathfrak{X}} \to \mathfrak{G}$ be a cocharacter. Then $\mathbf{P}(\lambda)$ is a parabolic subgroup scheme of $\mathfrak{G}$ and $\mathbf{Z}_{\mathfrak{G}}(\lambda)$ is a Levi subgroup of the $\mathfrak{X}$–group scheme $\mathbf{P}(\lambda)$.*

2. *Assume that $\mathfrak{X}$ is semi-local, connected and non-empty. Then the pseudo parabolic subgroup schemes of $\mathfrak{G}$ coincide with the parabolic subgroup schemes of $\mathfrak{G}$.*

We shall use that this fact is known for reductive groups over fields [Sp, §15.1].

Proof. We can assume that $\mathfrak{X} = \mathrm{Spec}(R)$ is affine.
(1) The geometric fibers of $\mathbf{P}(\lambda)$ are parabolic subgroups. By definition [SGA3, §XXVI.1], it remains to show that $\mathbf{P}(\lambda)$ is smooth. The question is then local with respect to the *fpqc* topology, so that we can assume that R is local and that $\mathfrak{G}$ is split. By Demazure's theorem [SGA3, XXIII.4], we can assume that $\mathfrak{G}$ arises by base change from a (unique) split Chevalley group $\mathfrak{G}_0$ over $\mathbb{Z}$.

We now reason along similar lines than the ones used in studying the $\mathbf{GL}_{n,\mathbb{Z}}$ case above. Let $\mathfrak{T} \subset \mathfrak{G}_0$ be a maximal split torus. Since all maximal split tori of $\mathfrak{G}$ are conjugate under $\mathfrak{G}(R)$, we can assume that our cocharacter λ factors through $\mathfrak{T}_R$. Since $\mathrm{Hom}_{\mathbb{Z}}(\mathbf{G}_{m,\mathbb{Z}}, \mathfrak{T}) \cong \mathrm{Hom}_R(\mathbf{G}_{m,R}, \mathfrak{T}_R)$, the problem again reduces to the case when $R = \mathbb{Z}$ and of $\mathfrak{G} = \mathfrak{G}_0$, and $\lambda : \mathfrak{G}_{m,\mathbb{Z}} \to \mathfrak{T}$. By the field case, the morphism $\mathbf{P}(\lambda) \to \mathrm{Spec}(\mathbb{Z})$ is equidimensional. Since $\mathbb{Z}$ is a normal ring and the geometric fibers are smooth, we can conclude by [SGA1, prop. II.2.3] that $\mathbf{P}(\lambda)$ is smooth and is a parabolic subgroup scheme of the $\mathbb{Z}$–group $\mathfrak{G}_0$.

The geometric fibers of $\mathbf{P}(\lambda) \times_S \mathbf{P}(-\lambda)$ are Levi subgroups. By applying [SGA3, th. XXVI.4.3.2], we get that $\mathbf{Z}_{\mathfrak{G}}(\lambda) = \mathbf{P}(\lambda) \times_{\mathfrak{G}} \mathbf{P}(-\lambda)$ is a Levi S-subgroup scheme of $\mathbf{P}(\lambda)$.
(2) Using the theory of relative root systems [SGA3, §XXVI.7], the proof is the same than in the field case. □

16 Appendix 2: Global automorphisms of G–torsors over the projective line

In this appendix there is no assumption on the characteristic of the base field k. Let $\mathbf{G}$ be a linear algebraic k–group such that $\mathbf{G}^0$ is reductive. One way to state Grothendieck-Harder's theorem is to say that the natural map

$$\mathrm{Hom}_{gp}(\mathbf{G}_m, \mathbf{G})/\mathbf{G}(k) \to H^1_{Zar}(\mathbb{P}^1, \mathbf{G})$$

which maps a cocharacter $\lambda : \mathbf{G}_m \to \mathbf{G}$ to the $\mathbf{G}$-torsor $\mathfrak{E}_\lambda := (-\lambda)_*\big(\mathcal{O}(-1)\big)$ over $\mathbb{P}^1_k$ where $\mathcal{O}(-1)$ stands for the Hopf bundle $\mathbf{A}^2_k \setminus \{0\} \to \mathbb{P}^1_k$, is bijective.[40]

We fix now a cocharacter $\lambda : \mathbf{G}_m \to \mathbf{G}$. We are interested in the twisted $\mathbb{P}^1_k$–group scheme $\mathfrak{E}_\lambda(\mathbf{G}) = \underline{\mathrm{Isom}}_{\mathbf{G}}(\mathfrak{E}_\lambda, \mathfrak{E}_\lambda)$, as well as the abstract group $\mathfrak{E}_\lambda(\mathbf{G})(\mathbb{P}^1_k)$. This group is the group of global automorphisms of the $\mathbf{G}$–torsor $\mathfrak{E}_\lambda$ over $\mathbb{P}^1_k$. It has a concrete description.

Lemma 16.1. *$\mathfrak{E}_\lambda(\mathbf{G})(\mathbb{P}^1_k) = \mathbf{G}(k[t]) \cap \lambda(t)\mathbf{G}(k[t^{-1}])\lambda(t^{-1})$.*

Proof. We recover $\mathbb{P}^1_k$ by two affine lines $\mathbf{U}_0 = \mathrm{Spec}(k[t])$ and $\mathbf{U}_1 = \mathrm{Spec}(k[t^{-1}])$. The Hopf bundle is isomorphic to the twist of $\mathbf{G}_m$ by the cocycle $z \in Z^1(\mathbf{U}_0 \sqcup \mathbf{U}_1/\mathbb{P}^1_k, \mathbf{G}_m)$ where $z_{0,0} = 1$, $z_{0,1} = t^{-1}$, $z_{1,0} = t$, $z_{1,1} = 1$. Then $\lambda(z) \in Z^1(\mathbf{U}_0 \sqcup \mathbf{U}_1/\mathbb{P}^1_k, \mathbf{G}_m)$ is the cocycle of $\mathfrak{E}_\lambda$. Hence

$$\begin{aligned} \mathfrak{E}_\lambda(\mathbf{G})(\mathbb{P}^1_k) &= \Big\{(g_0, g_1) \in \mathbf{G}(\mathbf{U}_0) \times \mathbf{G}(\mathbf{U}_1) \mid \lambda^{-1}(z_{0,1}).g_1 = g_0 \Big\} \\ &= \mathbf{G}(k[t]) \cap \lambda(t)\mathbf{G}(k[t^{-1}])\lambda(t^{-1}). \end{aligned}$$

□

In the split connected case, this group has been computed by Ramanathan [Ra, prop. 5.2] and by the first author in the split case (see proposition II.2.2.2 of [Gi0]). We provide here the general case by computing the Weil restriction

$$\mathbf{H}_\lambda = \prod_{\mathbb{P}^1_k/k} \mathfrak{E}_\lambda(\mathbf{G}),$$

which is known to be a representable by an algebraic affine k–group. Let $\mathbf{P}(\lambda) = \mathbf{U}(\lambda) \rtimes \mathbf{Z}_{\mathbf{G}}(\lambda) \subset \mathbf{G}$ be the parabolic subgroup attached to λ (lemma 15.3).

Denote by $\mathbf{Z}(\lambda)$ the center of $\mathbf{Z}_{\mathbf{G}}(\lambda)$. Then λ factors through $\mathbf{Z}(\lambda)$ and this allows us to define the $\mathbf{Z}(\lambda)$–torsor $\mathbf{S}_\lambda := (-\lambda)_*\big(\mathcal{O}(-1)\big)$ over $\mathbb{P}^1_k$. We can twist the morphism $\mathbf{Z}_{\mathbf{G}}(\lambda) \to \mathbf{G}$ by $\mathbf{S}_\lambda$, so we get a morphism $\mathbf{Z}_{\mathbf{G}}(\lambda) \times_k \mathbb{P}^1_k \to \mathfrak{E}_\lambda(\mathbf{G})$ and then a morphism $\mathbf{Z}_{\mathbf{G}}(\lambda) \to \mathbf{H}_\lambda$.

Proposition 16.2. *The homomorphisms of k–groups*

$$\Big(\prod_{\mathbb{P}^1_k/k} \mathbf{S}_\lambda(\mathbf{U}(\lambda))\Big) \rtimes \mathbf{Z}_{\mathbf{G}}(\lambda) \;\to\; \prod_{\mathbb{P}^1_k/k} \mathbf{S}_\lambda\big(\mathbf{P}(\lambda)\big) \;\to\; \mathbf{H}_\lambda.$$

are isomorphisms. Furthermore, $\prod_{\mathbb{P}^1_k/k} \mathbf{S}_\lambda(\mathbf{U}(\lambda))$ *is a unipotent k–group.*

[40] This is note the usual way to state the theorem (see [Gi0, II.2.2.1]), but it is easy to derive the formulation that we are using.

Proof. Write $\mathbf{P}$, $\mathbf{U}$ for $\mathbf{P}(\lambda)$, $\mathbf{U}(\lambda)$.

$$\prod_{\mathbb{P}^1_k/k} \mathbf{S}_\lambda(\mathbf{P}) = \prod_{\mathbb{P}^1_k/k} \mathbf{S}_\lambda(\mathbf{U}) \rtimes \prod_{\mathbb{P}^1_k/k} \mathbf{Z}_\mathbf{G}(\lambda) = \prod_{\mathbb{P}^1_k/k} \mathbf{S}_\lambda(\mathbf{U}) \ \rtimes \mathbf{Z}_\mathbf{G}(\lambda)$$

so it remains to show that $\prod\limits_{\mathbb{P}^1_k/k} \mathbf{S}_\lambda(\mathbf{P}) \xrightarrow{\sim} \mathbf{H}_\lambda$. Consider a faithfull representation $\rho : \mathbf{G} \to \mathbf{G}' = \mathbf{GL}_n$. Denote by $\mathbf{P}'$ the parabolic subgroup of $\mathbf{G}'$ attached to λ. We have $\mathbf{P} = \mathbf{G} \times_{\mathbf{G}'} \mathbf{P}'$, hence $\mathbf{S}_\lambda(\mathbf{P}) = \mathfrak{C}_\lambda(\mathbf{G}) \times_{\mathfrak{C}_\lambda(\mathbf{G}')} \mathbf{S}_\lambda(\mathbf{P}')$. It follows that

$$\prod_{\mathbb{P}^1_k/k} \mathbf{S}_\lambda(\mathbf{P}) \ = \prod_{\mathbb{P}^1_k/k} \mathfrak{C}_\lambda(\mathbf{G}) \times_{\prod\limits_{\mathbb{P}^1_k/k} \mathfrak{C}_\lambda(\mathbf{G}')} \prod_{\mathbb{P}^1_k/k} \mathbf{S}_\lambda(\mathbf{P}')$$

as can be seen by reducing to the case of $\mathbf{GL}_n$ already done in Example 15.1. This case also shows that $\prod\limits_{\mathbb{P}^1_k/k} \mathbf{S}_\lambda(\mathbf{U}(\lambda))$ is a unipotent k–group. $\square$

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