CONJUGACY THEOREMS FOR LOOP REDUCTIVE GROUP SCHEMES AND LIE ALGEBRAS

V. CHERNOUSOV, P. GILLE, AND A. PIANZOLA

Abstract. The conjugacy of split Cartan subalgebras in the finite dimensional simple case (Chevalley) and in the symmetrizable Kac-Moody case (Peterson-Kac) are fundamental results of the theory of Lie algebras. Among the Kac-Moody Lie algebras the affine algebras stand out. This paper deals with the problem of conjugacy for a class of algebras –extended affine Lie algebras– that are in a precise sense higher nullity analogues of the affine algebras. Unlike the methods used by Peterson-Kac, our approach is entirely cohomological and geometric. It is deeply rooted on the theory of reductive group schemes developed by Demazure and Grothendieck, and on the work of J. Tits on buildings.
Keywords: Reductive group scheme, torsor, Laurent polynomials, non-abelian cohomology, building
MSC 2000 11E72, 14L30, 14E20.

1. Introduction

Let $\mathbf{g}$ be a split simple finite dimensional Lie algebra over a field k of characteristic 0. From the work of Cartan and Killing one knows that $\mathbf{g}$ is determined by its root system. The problem, of course, is that a priori the type of the root system may depend on the choice of split Cartan subalgebra. One of the most elegant ways of establishing that this does not happen, hence that the type of the root system is an invariant of $\mathbf{g}$, is the conjugacy theorem of split Cartan subalgebras due to Chevalley: All split Cartan subalgebras of $\mathbf{g}$ are conjugate under the adjoint action of $\mathbf{G}(k)$ where $\mathbf{G}$ is the split simply connected group corresponding to $\mathbf{g}$.

Variations of this theme are to be found on the seminal work of Peterson and Kac on conjugacy of "Cartan subalgebras" for symmetrizable Kac-Moody Lie algebras [PK]. Except for the toroidal case, nothing is known about conjugacy for extended affine Lie algebras (EALAs for short); a fascinating class of algebras which can be thought as higher nullity analogues of the affine algebras. The philosophy that we follow is motivated by two assumptions:

(1) The affine Kac-Moody and extended affine Lie algebras are among the most relevant infinite dimensional Lie algebras today.

V. Chernousov was partially supported by the Canada Research Chairs Program and an NSERC research grant.

A. Pianzola wishes to thank NSERC and CONICET for their continuous support.

(2) Since the affine and extended affine algebras are closely related to finite dimensional simple Lie algebras, a proof of conjugacy ought to exist that is faithful to the spirit of finite dimensional Lie theory.

That this much is true for toroidal Lie algebras (which correspond to the "untwisted case" in this paper) has been shown in [P1]. The present work is much more ambitious. Not only it tackles the twisted case, but it does so in arbitrary nullity. Some of the algebras covered by our result are related to extended affine Lie algebras, but our work depicts a more global point of view. It builds a bridge between "Cartan subalgebras" of twisted forms of semisimple Lie algebras over a normal ring R (viewed as infinite dimensional Lie algebras over the base field), and split tori of the corresponding reductive group schemes over R.

Using the natural one-to-one correspondence between "Cartan subalgebras" and maximal split tori in question shown in Theorem 9.1 we establish their conjugacy in Theorem 15.1. Fundamental applications to the structure theory of extended affine Lie algebras are given in §16. The main ingredient of the proof of conjugacy is the classification of loop reductive torsors over Laurent polynomial rings given by Theorem 17.1, a result that we believe is of its own interest.

2. Generalities and statement of the fundamental objective

2.1. **Notation and conventions.** Throughout this work, with the exception of the Appendix, k will denote a field of characteristic 0 and $\overline{k}$ an algebraic closure of k. For integers $n \geq 0$ and $m > 0$ we set

$$R_n = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}], \quad K_n = k(t_1, \dots, t_n), \quad F_n = k((t_1)) \cdots ((t_n)),$$

and

$$R_{n,m} = k[t_1^{\pm \frac{1}{m}}, \dots, t_n^{\pm \frac{1}{m}}], \;\; K_{n,m} = k(t_1^{\frac{1}{m}}, \dots, t_n^{\frac{1}{m}}), \;\; F_{n,m} = k((t_1^{\frac{1}{m}})) \cdots ((t_n^{\frac{1}{m}})).$$

The category of commutative associate unital algebras over k will be denoted by *k–alg*. If $\mathfrak{X}$ is a scheme over $\mathrm{Spec}(k)$, by an *$\mathfrak{X}$–group* we will understand a group scheme over $\mathfrak{X}$. When $\mathfrak{X} = \mathrm{Spec}(R)$ for some object R of *k–alg*, we use the expression *R–group.*

2.2. **Forms.** Le $\mathbf{g}$ be a finite dimensional split semisimple Lie algebra over k. Recall that a Lie algebra $\mathcal{L}$ over R is called a *form* of $\mathbf{g} \otimes_k R$ (or simply a form of $\mathbf{g}$) if there exists a faithfully flat and finitely presented R–algebra $\widetilde{R}$ such that

$$\mathcal{L} \otimes_R \widetilde{R} \simeq (\mathbf{g} \otimes_k R) \otimes_R \widetilde{R} \simeq \mathbf{g} \otimes_k \widetilde{R}, \tag{2.2.1}$$

where all the above are isomorphisms of Lie algebras over $\widetilde{R}$. Since $\mathbf{g}$ is finite dimensional the assumption that $\widetilde{R}/R$ be finitely presented is superflous whenever R is noetherian. The set of isomorphism classes of such forms is measured by the pointed set

$$H^1_{fppf}\big(\mathrm{Spec}(R), \mathbf{Aut}(\mathbf{g})_{\mathrm{R}}\big)$$

where $\mathbf{Aut}(\mathbf{g})_{\mathrm{R}}$ is the R–group obtained by base change from the k–linear algebraic group $\mathbf{Aut}(\mathbf{g})$. We have a split exact sequence of k–groups

$$1 \longrightarrow \mathbf{G}_{ad} \longrightarrow \mathbf{Aut}(\mathbf{g}) \longrightarrow \mathbf{Out}(\mathbf{g}) \longrightarrow 1 \tag{2.2.2}$$

where $\mathbf{G}_{ad}$ is the adjoint group corresponding to $\mathbf{g}$ and $\mathbf{Out}(\mathbf{g})$ is the constant k–group corresponding to the finite (abstract) group of symmetries of the Coxeter-Dynkin diagram of $\mathbf{g}$. By base change we obtain an analogous sequence over R.

In what follows we will follow standard practice and for convenience denote $H^1_{fppf}\big(\mathrm{Spec}(R), \mathbf{Aut}(\mathbf{g})_{\mathrm{R}}\big)$ simply by $H^1_{fppf}\big(R, \mathbf{Aut}(\mathbf{g})\big)$ when no confusion is possible. Similarly for the Zariski and étale topologies, as well as for k–groups other than $\mathbf{Aut}(\mathbf{g})$.

2.3. **Remark.** Since $\mathbf{Aut}(\mathbf{g})$ is smooth and affine over $\mathrm{Spec}(R)$

$$H^1_{ét}\big(R, \mathbf{Aut}(\mathbf{g})\big) \simeq \mathrm{H}^1_{fppf}\big(\mathrm{R}, \mathbf{Aut}(\mathbf{g})\big).$$

2.4. **Remark.** Let $R = R_n$ be as in §2.1. By the Isotriviality Theorem of [GP2] the trivializing algebra $\widetilde{R}$ in (2.2.1) may be taken to be of the form

$$\widetilde{R} := R_{n,m} \otimes_k \tilde{k} = \tilde{k}[t_1^{\pm\frac{1}{m}}, \dots, t_n^{\pm\frac{1}{m}}]$$

for some m and some Galois extension $\tilde{k}$ of k containing all m-th roots of unity of $\overline{k}$. The extension $\widetilde{R}/R$ is Galois (see §2.5 below)

2.5. **Multiloop algebras.** Assume now that k is algebraically closed. We fix a compatible set of primitive m–th roots of unity ξ_m, namely such that $\xi_{me}^e = \xi_m$ for all $e > 0$. Let $R = R_n$ and $\widetilde{R} = R_{n,m}$. Then $\widetilde{R}/R$ is Galois. Via our choice of roots of unity, we can identify $\mathrm{Gal}(\widetilde{R}/R)$ with $(\mathbb{Z}/m\mathbb{Z})^n$ as follows: For each $\mathbf{e} = (e_1, \dots, e_n) \in \mathbb{Z}^n$ the corresponding element $\overline{\mathbf{e}} = (\overline{e}_1, \cdots, \overline{e}_n) \in \mathrm{Gal}(\widetilde{R}/R)$ acts on $\widetilde{R}$ via ${}^{\overline{\mathbf{e}}}(t_i^{\frac{1}{m}}) = \xi_m^{e_i} t_i^{\frac{1}{m}}$.

The primary example of forms $\mathcal{L}$ of $\mathbf{g} \otimes_k R$ which are trivialized by a Galois extension $\widetilde{R}/R$ as above are the multiloop algebras based on $\mathbf{g}$. These are defined as follows. Consider an n–tuple $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ of commuting elements of $\mathrm{Aut}_k(\mathbf{g})$ satisfying $\sigma_i^m = 1$. For each n–tuple $(i_1, \dots, i_n) \in \mathbb{Z}^n$ we consider the simultaneous eigenspace

$$\mathbf{g}_{i_1 \dots i_n} = \{x \in \mathbf{g} : \sigma_j(x) = \xi_m^{i_j} x \text{ for all } 1 \leq j \leq n\}.$$

Then $\mathbf{g} = \sum \mathbf{g}_{i_1 \dots i_n}$, and $\mathbf{g} = \bigoplus \mathbf{g}_{i_1 \dots i_n}$ if we restrict the sum to those n–tuples $(i_1, \dots, i_n)$ for which $0 \leq i_j < m_j$, where m_j is the order of σ_j.

The *multiloop algebra based on* $\mathbf{g}$ *corresponding to* $\boldsymbol{\sigma}$, commonly denoted by $L(\mathbf{g}, \boldsymbol{\sigma})$, is defined by

$$L(\mathbf{g}, \boldsymbol{\sigma}) = \bigoplus_{(i_1, \dots, i_n) \in \mathbb{Z}^n} \mathbf{g}_{i_1 \dots i_n} \otimes t_1^{\frac{i_1}{m}} \dots t_n^{\frac{i_n}{m}} \subset \mathbf{g} \otimes_k \widetilde{R} \subset \mathbf{g} \otimes_k \overline{R}_\infty$$

where $\overline{R}_\infty = \varinjlim \overline{k}[t_1^{\pm\frac{1}{m}}, \dots, t_n^{\pm\frac{1}{m}}]$.[1] Note that $L(\mathbf{g},\boldsymbol{\sigma})$, which does not depend on the choice of common period m, is not only a k–algebra (in general infinite dimensional), but also naturally an R–algebra. A rather simple calculation shows that

$$L(\mathbf{g},\boldsymbol{\sigma}) \otimes_R \widetilde{R} \simeq \mathbf{g} \otimes_k \widetilde{R} \simeq (\mathbf{g} \otimes_k R) \otimes_R \widetilde{R}.$$

Thus $L(\mathbf{g},\boldsymbol{\sigma})$ corresponds to a torsor over $\mathrm{Spec}(R)$ under $\mathbf{Aut}(\mathbf{g})$ (see §4 for details). The crucial point in the classification of forms of $\mathbf{g} \otimes_k R$ by cohomological methods is the exact sequence of pointed sets obtained from (2.2.2)

$$H^1_{ét}\big(R, \mathbf{G}_{ad}\big) \to H^1_{ét}\big(R, \mathbf{Aut}(\mathbf{g})\big) \to H^1_{ét}\big(R, \mathbf{Out}(\mathbf{g})\big) \to 1 \tag{2.5.1}$$

Grothendieck's theory of the algebraic fundamental group allows us to identify $H^1_{ét}\big(R, \mathbf{Out}(\mathbf{g})\big)$ with the set of conjugacy classes of n–tuples of commuting elements of the corresponding finite (abstract) group $\mathrm{Out}(\mathbf{g})$ (recall that k is algebraically closed). This will be explained when we introduce loop torsors in §6.1. This conjugacy class is an important cohomological invariant attached to any twisted form of $\mathbf{g} \otimes_k R$.

It is worth to point out that the cohomological information is always about the twisted forms viewed as algebras over R (and *not* k). In practice, as the affine Kac-Moody case illustrates, one is interested in understanding these algebras as objects over k (and *not* R). We find in Theorem 9.1 a bridge between these two very different and contrasting kinds of mathematical worlds.

3. Some terminology

If $\mathfrak{G}$ is an $\mathfrak{X}$–group, the pointed set of non-abelian Čech cohomology on the flat (resp. étale, resp. Zariski) site of $\mathfrak{X}$ with coefficients in $\mathfrak{G}$, is denoted by $H^1_{fppf}(\mathfrak{X},\mathfrak{G})$ [resp. $H^1_{ét}(\mathfrak{X},\mathfrak{G})$, resp. $H^1_{Zar}(\mathfrak{X},\mathfrak{G})$]. These pointed sets measure the isomorphism classes of sheaf torsors over R under $\mathfrak{G}$ with respect to the chosen topology (see [M1, Ch. IV §1] and [DG] for basic definitions and references). If $\mathfrak{X} = \mathrm{Spec}(R)$, following customary usage and depending on the context, we also use the notation $H^1_{fppf}(R,\mathfrak{G})$ instead of $H^1_{fppf}(\mathfrak{X},\mathfrak{G})$. Similarly for the étale and Zariski site.

If $\mathfrak{G}$ is flat and locally of finite presentation over $\mathfrak{X}$, then $\mathfrak{G}$ is necessarily smooth over $\mathfrak{X}$.[2] If furthermore $\mathfrak{G}$ is affine over $\mathfrak{X}$, by faithfully flat descent all of our sheaf torsors are representable. They are thus *torsors* in the usual sense. Furthermore, the smoothness of $\mathfrak{G}$ yields that all torsors are locally trivial for the étale topology. In particular, $H^1_{ét}(\mathfrak{X},\mathfrak{G}) = H^1_{fppf}(\mathfrak{X},\mathfrak{G})$. These assumptions on $\mathfrak{G}$ hold in most of the situations that arise in our work.

[1] The ring $\overline{R}_\infty$ is a useful artifice that allows us to see *all* multiloop algebras based on a given $\mathbf{g}$ as subalgebras of one Lie algebra.

[2] Since $\mathfrak{X}$ is of "characteristic zero", the geometric fibers of $\mathfrak{G}$ are smooth by Cartier's theorem [DG, II.6.1.1]. The smoothness criterion on fibers [EGA4, 17.8.2], shows that $\mathfrak{G}$ is indeed smooth over $\mathfrak{X}$.

Given an $\mathfrak{X}$–group $\mathfrak{G}$ and a morphisms $\mathfrak{Y} \to \mathfrak{X}$ of k–schemes, we let $\mathfrak{G}_{\mathfrak{Y}}$ denote the $\mathfrak{Y}$–group $\mathfrak{G} \times_{\mathfrak{X}} \mathfrak{Y}$ obtained by base change. For convenience, we will under these circumstances denote most of the times $H^1_{ét}(\mathfrak{Y}, \mathfrak{G}_{\mathfrak{Y}})$ by $H^1_{ét}(\mathfrak{Y}, \mathfrak{G})$.

If $S = \mathrm{Spec}(L)$ where L is a field extension of k and $\mathfrak{G}$ is an algebraic k–group we we shall always write $H^1(L, \mathfrak{G})$ instead of $H^1_{ét}(L, \mathfrak{G})$. This is the "usual" Galois cohomology of the field L and group $\mathfrak{G}$.

The expression *linear algebraic group (defined) over* k, is to be understood in the sense of Borel [Bor]. For a k–group $\mathbf{G}$, this is equivalent to requiring that $\mathbf{G}$ be affine of finite type ([SGA3, VI_B, 11.11]) because such a group is smooth by Cartier's theorem. The connected component of the identity of $\mathbf{G}$, will be denoted by $\mathbf{G}^\circ$. If R is an object in k*–alg* we will denote the corresponding multiplicative and additive groups by $\mathbf{G}_{m,R}$ and $\mathbf{G}_{a,R}$.

As it has probably become evident to the reader by now, we will use bold roman characters, e.g. $\mathbf{G}$, $\mathbf{g}$ to denote k–groups and their Lie algebras. The notation $\mathfrak{G}$ and $\mathfrak{g}$ will be reserved for R–groups (which are usually not obtained from a k–group by base change) and their Lie algebras.

A *reductive* $\mathfrak{X}$–group is to be understood in the sense of [SGA3]. In particular, a reductive k–group is a reductive *connected* algebraic group defined over k in the sense of Borel.

We recall now two fundamental notions about reductive $\mathfrak{X}$–groups.

3.1. **Definition.** *Let $\mathfrak{G}$ be a reductive $\mathfrak{X}$–group. We say that $\mathfrak{G}$ is* reducible *if $\mathfrak{G}$ admits a proper parabolic subgroup $\mathfrak{P}$ which has a Levi subgroup, and* irreducible *otherwise.*

We say that $\mathfrak{G}$ is isotropic *if $\mathfrak{G}$ admits a subgroup isomorphic to $\mathbf{G}_{m,X}$. Otherwise we say that $\mathfrak{G}$ is* anisotropic.

We denote by $\mathbf{Par}(\mathfrak{G})$ the $\mathfrak{X}$–scheme of parabolic subgroup of $\mathfrak{G}$. This scheme is smooth and projective over $\mathfrak{X}$ [SGA3, XXVI, 3.5]. Since by definition $\mathfrak{G}$ is a parabolic subgroup of $\mathfrak{G}$, when $\mathfrak{X}$ is connected, to say that $\mathfrak{G}$ admits a proper parabolic subgroup is to say that $\mathbf{Par}(\mathfrak{G})(\mathfrak{X}) \neq \{\mathfrak{G}\}$.

3.2. **Remark.** If $\mathfrak{X}$ is connected, to each parabolic subgroup $\mathfrak{P}$ of $\mathfrak{G}$ corresponds a "type" $\mathbf{t} = \mathbf{t}(\mathfrak{P})$ which is a subset of the corresponding Coxeter-Dynkin diagram. Given a type $\mathbf{t}$, the scheme $\mathbf{Par}_{\mathbf{t}}(\mathfrak{G})$ of parabolic subgroups of $\mathfrak{G}$ of type $\mathbf{t}$ is also smooth and projective over $\mathfrak{X}$ (*ibid.* cor.3.6).

4. Preliminary I: The algebraic fundamental group.

The following is a brief but nonetheless necessary summary of (mostly background) material from [SGA1] mentioned in [GP3]. Through this section $\mathfrak{X}$ will denote a *connected and locally noetherian* scheme over k. The generalities about the fundamental group are followed by a detailed analysis of the case of $\mathfrak{X} = \mathrm{Spec}(R_n)$. This knowledge is essential for the concept of loop reductive groups which is central to the present work.

4.1. **The algebraic fundamental group.** Fix a geometric point a of $\mathfrak{X}$ i.e. a morphism $a:\ \mathrm{Spec}(\Omega)\to\mathfrak{X}$ where Ω is an algebraically closed field.

Let $\mathfrak{X}_{fet}$ be the category of finite étale covers of $\mathfrak{X}$, and F the covariant functor from $\mathfrak{X}_{fet}$ to the category of finite sets given by

$$F(\mathfrak{X}')=\{\text{geometric points of } \mathfrak{X}' \text{ above } a).$$

That is, $F(\mathfrak{X}')$ consists of all morphisms $a':\ \mathrm{Spec}\,(\Omega)\to\mathfrak{X}'$ for which the diagram.

$$\begin{tikzcd} & \mathfrak{X}' \arrow[d] \\ \mathrm{Spec}\,(\Omega) \arrow[ur, "a'"] \arrow[r, "a"] & \mathfrak{X} \end{tikzcd}$$

commutes. The group of automorphism of the functor F is called the *algebraic fundamental group of* $\mathfrak{X}$ *at* a, and is denoted by $\pi_1(\mathfrak{X},a)$. The functor F is *pro-representable*: There exists a directed set I, objects $(\mathfrak{X}_i)_{i\in I}$ of $\mathfrak{X}_{fet}$, surjective morphisms $\varphi_{ij}\in\mathrm{Hom}_{\mathfrak{X}}(\mathfrak{X}_j,\mathfrak{X}_i)$ for $i\le j$ and geometric points $a_i\in F(\mathfrak{X}_i)$ such that

$$a_i=\varphi_{ij}\circ a_j,$$

The canonical map $f:\varinjlim \mathrm{Hom}_{\mathfrak{X}}(\mathfrak{X}_i,\mathfrak{X}')\to F(\mathfrak{X}')$ *is bijective.*

Since the $\mathfrak{X}_i$ are finite and étale over $\mathfrak{X}$ the morphisms φ_{ij} are affine. Thus the inverse limit

$$\mathfrak{X}^{sc}=\varprojlim\ \mathfrak{X}_i$$

exists in the category of schemes over $\mathfrak{X}$ [EGA4] §8.2. For any scheme $\mathfrak{X}'$ over $\mathfrak{X}$ we thus have a canonical map

$$\mathrm{Hom}_{Pro-\mathfrak{X}}(\mathfrak{X}^{sc},\mathfrak{X}')\overset{\mathrm{def}}{=}\varinjlim\ \mathrm{Hom}_{\mathfrak{X}}(\mathfrak{X}_i,\mathfrak{X}')\simeq F(\mathfrak{X}')$$

obtained by considering the canonical morphisms $\varphi_i:\mathfrak{X}^{\mathrm{sc}}\to\mathfrak{X}_i$.

In computing $\mathfrak{X}^{sc}=\varprojlim\mathfrak{X}_i$ we may replace $(\mathfrak{X}_i)_{i\in I}$ by any cofinal family. This allows us to assume that the $\mathfrak{X}_i$ are (connected) Galois, i.e. the $\mathfrak{X}_i$ are connected and the (left) action of $\mathrm{Aut}_{\mathfrak{X}}(\mathfrak{X}_i)$ on $F(\mathfrak{X}_i)$ is transitive. We then have

$$F(\mathfrak{X}_i)\simeq\ \mathrm{Hom}_{Pro-\mathfrak{X}}(\mathfrak{X}^{sc},\mathfrak{X}_i)\simeq\ \mathrm{Hom}_{\mathfrak{X}}(\mathfrak{X}_i,\mathfrak{X}_i)=\mathrm{Aut}_{\mathfrak{X}}(\mathfrak{X}_i).$$

Thus $\pi_1(\mathfrak{X},a)$ can be identified with the group $\varprojlim\ \mathrm{Aut}_{\mathfrak{X}}(\mathfrak{X}_i)^{opp}$. Each $\mathrm{Aut}_{\mathfrak{X}}(\mathfrak{X}_i)$ is finite, and this endows $\pi_1(\mathfrak{X},a)$ with the structure of a profinite topological group.

Suppose now that our $\mathfrak{X}$ is a geometrically connected k–scheme. We will denote $\mathfrak{X}\times_k\overline{k}$ by $\overline{\mathfrak{X}}$. Fix a geometric point $\overline{a}:\mathrm{Spec}(\overline{k})\to\overline{\mathfrak{X}}$. Let a (resp. b) be the geometric point of $\mathfrak{X}$ [resp. $\mathrm{Spec}(k)$] given by the composite maps $a:\mathrm{Spec}(\overline{k})\overset{\overline{a}}{\to}\overline{\mathfrak{X}}\to\mathfrak{X}$ [resp. $b:\mathrm{Spec}(\overline{k})\overset{\overline{a}}{\to}\mathfrak{X}\to\mathrm{Spec}(k)$]. Then by [SGA1, théorème IX.6.1] $\pi_1\big(\mathrm{Spec}(k),b\big)\simeq\mathrm{Gal}(k):=\mathrm{Gal}(\overline{k}/k)$ and the sequence

$$1\to\pi_1(\overline{\mathfrak{X}},\overline{a})\to\pi_1(\mathfrak{X},a)\to\mathrm{Gal}(k)\to 1 \tag{4.1.1}$$

is exact.

4.2. **Case of a normal integral scheme.** Assume now that our $\mathfrak{X}$ is normal and integral. Denote by $k(\mathfrak{X})$ its fraction field and let $k(\mathfrak{X})_{sep}$ be a fixed separable closure of $k(\mathfrak{X})$. The structure of the fundamental group in this case goes back to Lang and Serre [LS]. A connected finite étale cover $\mathfrak{Y}$ of $\mathfrak{X}$ is normal [SGA1, 9.11] and integral (*ibid*, 10.1). According to [EGA4, 18.1.12], the functor $\mathfrak{Y} \to k(\mathfrak{Y})$ provides an equivalence between the categories of finite étale connected covers of $\mathfrak{X}$ and finite separable field extensions of $k(\mathfrak{X})$ inside $k(\mathfrak{X})_{sep}$ which are unramified over $\mathfrak{X}$ (i.e. the normalization of $\mathfrak{X}$ in L is unramified over $\mathfrak{X}$); the quasi-inverse functor maps such field extension $L/k(\mathfrak{X})$ to the integral closure of $\mathfrak{X}$ in L.

The passage to the limit is done in [SGA1, V.8.2]. The finite separable extensions of $k(\mathfrak{X})$ inside $k(\mathfrak{X})_{sep}$ which are unramified over $\mathfrak{X}$ form a distiguished class of subextensions so it makes sense to talk about the maximal subextension $LS_{\mathfrak{X}}$ of $k(\mathfrak{X})_{sep}/k(\mathfrak{X})$ which is unramified over X. The field $LS_{\mathfrak{X}}$ is the union of all finite subextensions unramified over $\mathfrak{X}$. By passing to the limit, the simply connected covering of $\mathfrak{X}$ is the integral closure of $\mathfrak{X}$ in $LS_{\mathfrak{X}}$. We take as base point $a : \mathrm{Spec}\big(k(\mathfrak{X})_{sep}\big) \to \mathrm{Spec}\big(k(\mathfrak{X})\big)$. Then the profinite group $\pi_1(X,a) = \mathrm{Gal}(LS_{\mathfrak{X}}/k(\mathfrak{X}))$ occurs as a quotient of $\mathrm{Gal}\big(k(\mathfrak{X})_{sep}/k(\mathfrak{X})\big)$. Note that in the case $\mathfrak{X} = \mathrm{Spec}(R)$ with R local the simply connected covering $\mathfrak{X}^{sc}$ of $\mathfrak{X}$ is $\mathrm{Spec}(R^{sh})$ where R^{sh} is the strict henselisation of R (see [Ra, §X.2]).

4.3. **The algebraic fundamental group of R_n.** We look in detail at an example that is of central importance to this work, namely the case when $\mathfrak{X} = \mathrm{Spec}\,(R_n)$ where $R_n = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ is the Laurent polynomial ring in n–variables with coefficients in k. We refer the reader to [GP2] and [GP3] for details.

The simply connected cover R_n^{sc} of R_n is

$$\overline{R}_{n,\infty} = \varinjlim \overline{R}_{n,m}$$

with $\overline{R}_{n,m} = \overline{k}[t_1^{\pm\frac{1}{m}}, \dots, t_n^{\pm\frac{1}{m}}]$. The “evaluation at 1” provides a geometric point that we denote by a. The algebraic fundamental group is best described as

$$\pi_1(\mathfrak{X}, a) = \widehat{\mathbb{Z}}(1)^n \rtimes \mathrm{Gal}\,(k). \tag{4.3.1}$$

where $\widehat{\mathbb{Z}}(1)$ denotes the abstract group $\varprojlim_m \boldsymbol{\mu}_m(\overline{k})$ equipped with the natural action of the absolute Galois group $\mathrm{Gal}(k)$.

5. Preliminaries II: Reductive group schemes

Let $\mathfrak{H}$ denote a reductive $\mathfrak{X}$–group. If $\mathfrak{T}$ is a subgroup of $\mathfrak{H}$ the expression “$\mathfrak{T}$ is a maximal torus of $\mathfrak{H}$” has a precise meaning ([SGA3, XII, Définition 1.3]). A maximal torus may or may not be split. If it is, we say that $\mathfrak{T}$ is a *split maximal torus.* This is in contrast with the concept of *maximal split torus* which we also need. This is a closed subgroup of $\mathfrak{H}$ which is a split torus and which is not properly included in any other split torus of

$\mathfrak{H}$. Note that split maximal tori (even maximal tori) need not exist, while maximal split tori always do exist if R is noetherian. A good example to clarify the difference of these two notions is any reductive k–group which is not split. Of course any split maximal torus is a maximal split torus (but not conversely, as we have just observed).

If $\mathfrak{S} < \mathfrak{G}$ are $\mathfrak{X}$–groups and $\mathfrak{s} \subset \mathfrak{g}$ are their respective Lie algebras[3], we will denote by $Z_{\mathfrak{G}}(\mathfrak{S})$ [resp. $Z_{\mathfrak{g}}(\mathfrak{s})$] the centralizer of $\mathfrak{S}$ in $\mathfrak{G}$ [resp. of $\mathfrak{s}$ in $\mathfrak{g}$].

We now recall and establish for future reference some basic useful facts.

5.1. **Lemma.** *Let $\mathfrak{H}$ be a reductive $\mathfrak{X}$–group and $\mathfrak{T}$ a maximal torus of $\mathfrak{H}$.*

(1) *If $\mathfrak{X}$ is connected then $\mathfrak{T}$ contains a unique maximal split subtorus $\mathfrak{T}_d$. Moreover, if $\mathfrak{T}$ is isotrivial then for $\mathfrak{T}_d$ to be non-trivial it is necessary and sufficient that there exists a non-trivial group homomorphism $\mathfrak{T} \to \mathbf{G}_{m,\mathfrak{X}}$.*

(2) *Let $\mathfrak{S} \subset \mathfrak{H}$ be a split torus and let $\mathfrak{C} = Z_{\mathfrak{H}}(\mathfrak{S})$ be its centralizer in $\mathfrak{H}$. Then $\mathfrak{C}$ is a closed reductive subgroup of $\mathfrak{H}$.*

(3) *Let $\mathfrak{C} = Z_{\mathfrak{H}}(\mathfrak{S})$ be as in* (2). *Then there exists a parabolic subgroup $\mathfrak{P} \subset \mathfrak{H}$ such that $\mathfrak{C}$ is a Levi subgroup in $\mathfrak{P}$.*

Proof. (1) This is [SGA3, XXVI, 6.5, 6.6].

(2) See [SGA3, XIX, 2.2].

(3) See [SGA3, XXVI, cor. 6.2]. □

5.2. **Remark.** Let $\mathfrak{C} = Z_{\mathfrak{H}}(\mathfrak{S})$ be as above. Let $\mathfrak{T}$ be the radical of $\mathfrak{C}$.[4] Assume that $\mathfrak{X}$ is connected, and let $\mathfrak{T}_d \subset \mathfrak{T}$ be the maximal split subtorus of $\mathfrak{T}$. Since $\mathfrak{S}$ is split we have $\mathfrak{S} \subset \mathfrak{T}_d$. Note that if we are given additionally that $\mathfrak{S}$ is a maximal split torus then $\mathfrak{S} = \mathfrak{T}_d$.

5.3. **Lemma.** *Let $\mathfrak{S}$ be a split torus of $\mathfrak{H}$, and let $\mathfrak{T}$ be the radical torus of the reductive group $\mathfrak{C} = Z_{\mathfrak{H}}(\mathfrak{S})$. Assume that $\mathfrak{X}$ is connected . Then $Z_{\mathfrak{H}}(\mathfrak{T}_d) = \mathfrak{C}$.*

Proof. Since $\mathfrak{T}$ is the centre of $\mathfrak{C}$ we have $\mathfrak{C} \subset Z_{\mathfrak{H}}(\mathfrak{T})$. Also, the inclusions $\mathfrak{S} \subset \mathfrak{T}_d \subset \mathfrak{T}$ yield

$$Z_{\mathfrak{H}}(\mathfrak{T}) \subset Z_{\mathfrak{H}}(\mathfrak{T}_d) \subset Z_{\mathfrak{H}}(\mathfrak{S}) = \mathfrak{C}.$$

Altogether, we obtain

$$\mathfrak{C} \subset Z_{\mathfrak{H}}(\mathfrak{T}) \subset Z_{\mathfrak{H}}(\mathfrak{T}_d) \subset \mathfrak{C},$$

whence the result. □

5.4. **Proposition.** *Let $\mathfrak{H}$ be a reductive group scheme over $\mathfrak{X}$. Assume $\mathfrak{X}$ is connected. Let $\mathfrak{S}$ be a split subtorus of $\mathfrak{H}$ and let $\mathfrak{P}$ be a parabolic subgroup of $\mathfrak{H}$ containing $Z_{\mathfrak{H}}(\mathfrak{S})$ as Levi subgroup.*

a) The following are equivalent:

[3] For a discussion of Lie algebras see §8.

[4] Recall that the radical of a reductive $\mathfrak{X}$–group is the unique maximal torus of its centre [SGA3, XXII, 4.3.6].

1) $\mathfrak{S}$ *is maximal split in* $\mathfrak{H}$*;*

2) $\mathfrak{S}$ *is maximal split in* $Z_{\mathfrak{H}}(\mathfrak{S})$*;*

3) The reductive group scheme $Z_{\mathfrak{H}}(\mathfrak{S})/\mathfrak{S}$ *is anisotropic.*

b) The following are equivalent:

4) The reductive group scheme $Z_{\mathfrak{H}}(\mathfrak{S})$ *has no proper parabolic subgroups.*

5) $\mathfrak{P}$ *is a minimal parabolic subgroup of* $\mathfrak{H}$*.*

c) We have $(3) \Longrightarrow (4)$*.*

d) If $\mathfrak{S}$ *is the maximal split subtorus of the radical of* $Z_{\mathfrak{H}}(\mathfrak{S})$*, we have* $(1) \Longleftrightarrow (2) \Longleftrightarrow (3) \Longleftrightarrow (4) \Longleftrightarrow (5)$*.*

Note that Lemma 5.3 will allows us to assume in practice that we are under case *d*) of the Proposition, so that all five conditions are equivalent.

Proof. a) $(1) \Longrightarrow (2)$ is obvious.

$(2) \Longrightarrow (1)$. Let $\mathfrak{S}_0$ be a split subtorus of $\mathfrak{H}$ containing $\mathfrak{S}$. Then $\mathfrak{S} \subset \mathfrak{S}_0 \subset Z_{\mathfrak{H}}(\mathfrak{S})$, hence $\mathfrak{S} = \mathfrak{S}_0$.

$(2) \Longrightarrow (3)$. Let $\mathfrak{T}_0$ be a split subtorus of $Z_{\mathfrak{H}}(\mathfrak{S})/\mathfrak{S}$. Then its preimage in $Z_{\mathfrak{H}}(\mathfrak{S})$ is a split subtorus of $Z_{\mathfrak{H}}(\mathfrak{S})$, so is $\mathfrak{S}$ by the hypothesis. Thus $\mathfrak{T}_0$ is trivial and $Z_{\mathfrak{H}}(\mathfrak{S})/\mathfrak{S}$ is anisotropic.

$(3) \Longrightarrow (2)$. Assume that $\mathfrak{S}$ is not maximal, so that there exists a split subtorus $\mathfrak{S}_0$ properly containing $\mathfrak{S}$. Then $\mathfrak{S}_0/\mathfrak{S}$ is a a non-trivial split subtorus of the group scheme $Z_{\mathfrak{H}}(\mathfrak{S})/\mathfrak{S}$ which is therefore isotropic.

b) According to [SGA3, XXVI.1.20], there is a one-to-one correspondence

$$\Big\{ \text{parabolics } \mathfrak{Q} \text{ of } \mathfrak{H} \text{ included in } \mathfrak{P} \Big\} \quad <-- > \quad \Big\{ \text{parabolics } \mathfrak{M} \text{ of } Z_{\mathfrak{H}}(\mathfrak{S}) \Big\}$$

Thus the left handside consists of one element if and only if the right handside consists of one element, which shows that (4) and (5) are equivalent.

c) If $Z_{\mathfrak{H}}(\mathfrak{S})/\mathfrak{S}$ is anisotropic it is is a fortiori irreducible. But then $Z_{\mathfrak{H}}(\mathfrak{S})$ is irreducible as well.

d) Now we assume that $\mathfrak{S}$ is the maximal split subtorus of the radical of $Z_{\mathfrak{H}}(\mathfrak{S})$ and we shall prove $(4) \Longrightarrow (3)$. Our assumption is the irreduciblity of $Z_{\mathfrak{H}}(\mathfrak{S})$, or equivalently, that of $Z_{\mathfrak{H}}(\mathfrak{S})/\mathfrak{S}$. Let $\mathfrak{T}_0$ be a split subtorus of $Z_{\mathfrak{H}}(\mathfrak{S})/\mathfrak{S}$. Since the semisimple part of $Z_{\mathfrak{H}}(\mathfrak{S})/\mathfrak{S}$ is anisotropic $\mathfrak{T}_0$ is contained in its radical. Then its preimage in $Z_{\mathfrak{H}}(\mathfrak{S})$ is a split torus of the radical of $Z_{\mathfrak{H}}(\mathfrak{S})$ which contains $\mathfrak{S}$, so it is $\mathfrak{S}$. Thus $\mathfrak{T}_0 = 1$ and $Z_{\mathfrak{H}}(\mathfrak{S})/\mathfrak{S}$ is anisotropic as desired. □

5.5. **Proposition.** *Let* $\mathfrak{H}$ *be a reductive group scheme over* $\mathfrak{X}$*. Assume* $\mathfrak{X}$ *is connected. If* $\mathfrak{H}$ *contains a split subtorus* $\mathfrak{S}$ *with the property that the fiber* $\mathfrak{S}_x$ *is a non-central torus of* $\mathfrak{H}_x$ *for some* x *of* $\mathfrak{X}$*, then* $\mathfrak{H}$ *contains a proper parabolic subgroup.*

Proof. Let $\mathfrak{S}$ be a split torus of $\mathfrak{H}$ as in the Proposition, and let $\mathfrak{C} = Z_{\mathfrak{H}}(\mathfrak{S})$ be its centralizer in $\mathfrak{H}$. By Lemma 5.1(2),(3) $\mathfrak{C}$ is a closed reductive subgroup of $\mathfrak{H}$ and there exists a parabolic subgroup $\mathfrak{P} \subset \mathfrak{H}$ such that $\mathfrak{C}$ is a Levi subgroup of $\mathfrak{P}$. Since $\mathfrak{S}_x$ is non-central, we necessarily have $\mathfrak{C} \neq \mathfrak{H}$. By Remark 3.2 $\mathfrak{P}$ is proper. □

5.6. **Proposition.** *Let $\mathfrak{G}$ be an affine and smooth group scheme over $\mathfrak{X}$, $\mathfrak{S}$ a subtorus of $\mathfrak{G}$, and let $\mathfrak{g}$ and $\mathfrak{s}$ denote their respective Lie algebras.*

(1) *There is a natural inclusion $Z_{\mathfrak{G}}(\mathfrak{S}) \subset Z_{\mathfrak{G}}(\mathfrak{s})$. Both of these functors commute with base change.*

(2) *$Z_{\mathfrak{G}}(\mathfrak{S})$ is a smooth group scheme and $Z_{\mathfrak{G}}(\mathfrak{s})$ a closed subgroup of $\mathfrak{G}$ (in particular a group scheme).*

(3) $\mathrm{Lie}\big(Z_{\mathfrak{G}}(\mathfrak{s})\big) = Z_{\mathfrak{g}}(\mathfrak{s})$.

(4) *If $\mathfrak{G}$ is reductive and $\mathfrak{X}$ is connected then $Z_{\mathfrak{G}}(\mathfrak{S}) = Z_{\mathfrak{G}}(\mathfrak{s})$ and both of these are Levi subgroups of $\mathfrak{G}$.*

Proof. (1) This is easy to verify. See [SGA3, I and II 5.3.3].

(2) $Z_{\mathfrak{G}}(\mathfrak{S})$ is smooth by [SGA3, XI cor 5.3]. Since $\mathfrak{S}$ is smooth we can apply [SGA3, II cor. 4.11.8(ii] to conclude that $\mathfrak{s}$ is locally a direct summand of $\mathfrak{g}$. By [DG, II §2 prop 1.4] $Z_{\mathfrak{G}}(\mathfrak{s})$ is a closed subgroup of $\mathfrak{G}$.

(3) This is a particular case of [SGA3, Exp.II theo. 5.3.1(i)].

(4) We first look at the case of a base field.

$\mathfrak{X} = \mathrm{Spec}(k)$ *and* $\mathfrak{G}$ *simply connected*: Then this is a result of Steinberg. See [St75, 3.3 and 3.8] and [St75, 0.2].

$\mathfrak{X} = \mathrm{Spec}(k)$ *and* $\mathfrak{G}$ *reductive*: Embed $\mathfrak{G}$ into $\mathbf{SL}_n$ for a suitable n. Then

$$Z_{\mathfrak{G}}(\mathfrak{S}) = \mathfrak{G} \cap Z_{\mathbf{SL_n}}(\mathfrak{S}) \ \text{ and } \ Z_{\mathfrak{G}}(\mathfrak{s}) = \mathfrak{G} \cap Z_{\mathbf{SL_n}}(\mathfrak{s}).$$

and we are reduced to the previous case

In general, because of (1) and (2), we can proceed by étale descent. This reduces the problem to the case $\mathfrak{S} \subset \mathfrak{T} \subset \mathfrak{G}$ where $\mathfrak{G}$ is a Chevalley group, $\mathfrak{T}$ its standard split maximal torus and $\mathfrak{S}$ is split. This sequence is obtained by base change to $\mathfrak{X}$ from a similar sequence over k by [SGA3, VII cor. 1.6]. Over k our equality holds. Since both centralizers commute with base change the result follows. □

6. Loop torsors and loop reductive group schemes

Throughout this section $\mathfrak{X}$ will denote a connected and noetherian scheme over k and $\mathbf{G}$ a k–group which is locally of finite presentation.[5]

6.1. **Loop torsors.** Because of the universal nature of $\mathfrak{X}^{sc}$ we have a natural group homomorphism

$$\mathbf{G}(\overline{k}) \longrightarrow \mathbf{G}(\mathfrak{X}^{sc}). \tag{6.1.1}$$

[5]The case most relevant to our work is that of the group of automorphism of a reductive k–group.

The group $\pi_1(\mathfrak{X}, a)$ acts on $\overline{k}$, hence on $\mathbf{G}(\overline{k})$, via the group homomorphism $\pi_1(\mathfrak{X}, a) \to \mathrm{Gal}\,(k)$ of (4.1.1). This action is continuous, and together with (6.1.1) yields a map

$$H^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(\overline{k})\big) \to H^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(\mathfrak{X}^{sc})\big),$$

where we remind the reader that these H^1 are defined in the "continuous" sense (see Remark 6.2 immediately below). On the other hand, by [GP3, prop.2.3] and basic properties of torsors trivialized by Galois extensions, we have

$$\begin{aligned} H^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(\mathfrak{X}^{sc})\big) &= \varinjlim H^1\big(\mathrm{Aut}_{\mathfrak{X}}(\mathfrak{X}_i), \mathbf{G}(\mathfrak{X}_i)\big) \\ &= \varinjlim H^1_{\acute{e}t}(\mathfrak{X}_i/\mathfrak{X}, \mathbf{G}) \subset H^1_{\acute{e}t}(\mathfrak{X}, \mathbf{G}). \end{aligned}$$

6.2. **Remark.** Here and elsewhere when a profinite group A acts discretely on a module M the corresponding cohomology $H^1(A, M)$ is the *continuous* cohomology as defined in [Se1]. Similarly, if a group H acts in both A and M, then $\mathrm{Hom}_H(A, M)$ stands for the continuos group homomorphism of A into M that commute with the action of H.

By means of the foregoing observations we make the following.

6.3. **Definition.** *A torsor $\mathfrak{E}$ over $\mathfrak{X}$ under $\mathbf{G}$ is called a* loop torsor *if its isomorphism class $[\mathfrak{E}]$ in $H^1_{\acute{e}t}(\mathfrak{X}, \mathbf{G})$ belongs to the image of the composite map*

$$H^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(\overline{k})\big) \to H^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(\mathfrak{X}^{sc})\big) \subset H^1_{\acute{e}t}(\mathfrak{X}, \mathbf{G}).$$

We will denote by $H^1_{loop}(\mathfrak{X}, \mathbf{G})$ the subset of $H^1_{\acute{e}t}(\mathfrak{X}, \mathbf{G})$ consisting of classes of loop torsors. They are given by (continuous) cocycles in the image of the natural map $Z^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(\overline{k})\big) \to Z^1_{\acute{e}t}(\mathfrak{X}, \mathbf{G})$, which we call *loop cocycles.*

This fundamental concept is used in the definition of loop reductive groups which we will recall momentarily. As we shall see, loop reductive groups play a central role in our conjugacy theorem.[6]

The following examples illustrate the immensely rich class of objects that fit within the language of loop torsors.

6.4. **Examples.** (a) If $\mathfrak{X} = \mathrm{Spec}\,(k)$ then $H^1_{loop}(\mathfrak{X}, \mathbf{G})$ is nothing but the usual Galois cohomology of k with coefficients in $\mathbf{G}$.

(b) Assume that k is algebraically closed. Then the action of $\pi_1(\mathfrak{X}, a)$ on $\mathbf{G}(\overline{k})$ is trivial, so that

$$H^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(\overline{k})\big) = \mathrm{Hom}\,\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(\overline{k})\big)/\mathrm{Int}\,\mathbf{G}(\overline{k})$$

where the group $\mathrm{Int}\,\mathbf{G}(\overline{k})$ of inner automorphisms of $\mathbf{G}(\overline{k})$ acts naturally on the right on $\mathrm{Hom}\,\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(\overline{k})\big)$. To be precise, $\mathrm{Int}(g)(\phi) : x \to g^{-1}\phi(x)g$

[6] The concept of loop torsors and loop reductive group were introduced in [GP3] among other things to gain better understanding of the "right" concept of EALAs over non closed fields.

for all $g \in \mathbf{G}(\overline{k})$, $\phi \in \operatorname{Hom}\big(\pi_1(\mathfrak{X},a), \mathbf{G}(\overline{k})\big)$ and $x \in \pi_1(\mathfrak{X},a)$. Two particular cases are important:

(b1) $\mathbf{G}$ abelian: In this case $H^1\big(\pi_1(\mathfrak{X},a), \mathbf{G}(\overline{k})\big)$ is just the group of continuous homomorphisms from $\pi_1(\mathfrak{X},a)$ to $\mathbf{G}(\overline{k})$.

(b2) $\pi_1(\mathfrak{X},a) = \widehat{\mathbb{Z}}(1)^n$: In this case $H^1\big(\pi_1(\mathfrak{X},a), \mathbf{G}(\overline{k})\big)$ is the set of conjugacy classes of n–tuples $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ of commuting elements of finite order of $\mathbf{G}(\overline{k})$.[7]

This last example is exactly the setup of multiloop algebras, and the motivation for the "loop torsor" terminology.

6.5. **Geometric and arithmetic part of a loop cocycle.** By means of the decomposition (4.1.1) we can think of loop cocycles as being comprised of a geometric and an arithmetic part, as we now explain. This material will be needed to establish the "density results" used in the proof of the main conjugacy theorem. It is included to facilitate the reading of the paper.[8]

Let $\eta \in Z^1\big(\pi_1(\mathfrak{X},a), \mathbf{G}(\overline{k})\big)$. The restriction $\eta_{|\mathrm{Gal}(k)}$ is called the *arithmetic part* of η and it is denoted by η^{ar}. It is easily seen that η^{ar} is in fact a cocycle in $Z^1\big(\mathrm{Gal}(k), \mathbf{G}(\overline{k})\big)$. If η is fixed in our discussion, we will at times denote the cocycle η^{ar} by the more traditional notation z. In particular, for $s \in \mathrm{Gal}(k)$ we write z_s instead of η^{ar}_s.

Next we consider the restriction of η to $\pi_1(\overline{\mathfrak{X}}, \overline{a})$ that we denote by η^{geo} and called the *geometric part* of η.

We thus have a map

$$\begin{array}{ccccc} \Theta \ : Z^1\big(\pi_1(\mathfrak{X},a), \mathbf{G}(\overline{k})\big) & \longrightarrow & Z^1\big(\mathrm{Gal}(k), \mathbf{G}(\overline{k})\big) \times \operatorname{Hom}\big(\pi_1(\overline{\mathfrak{X}}, \overline{a}), \mathbf{G}(\overline{k})\big) \\ \eta & \mapsto & \big(\ \eta^{ar}\ ,\ \eta^{geo}\ \big) \end{array}$$

The group $\mathrm{Gal}(k)$ acts on $\pi_1(\overline{\mathfrak{X}}, \overline{a})$ by conjugation. On $\mathbf{G}(\overline{k})$, the Galois group $\mathrm{Gal}(k)$ acts on two different ways. There is the natural action arising for the action of $\mathrm{Gal}(k)$ on $\overline{k}$, and there is also the twisted action given by the cocycle $\eta^{ar} = z$. Following standard practice to view the abstract group $\mathbf{G}(\overline{k})$ as a $\mathrm{Gal}(k)$–module with the twisted action by z we write ${}_z\mathbf{G}(\overline{k})$.

6.6. **Lemma.** *The map Θ described above yields a bijection between $Z^1\big(\pi_1(\mathfrak{X},a), \mathbf{G}(\overline{k})\big)$ and couples (z, η^{geo}) with $z \in Z^1\big(\pi_1(\mathfrak{X},a), \mathbf{G}(\overline{k})\big)$ and $\eta^{geo} \in \operatorname{Hom}_{\mathrm{Gal}(k)}\big(\pi_1(\overline{\mathfrak{X}}, \overline{a}), {}_z\mathbf{G}(\overline{k})\big)$.*

Proof. See Lemma 3.7 of [GP3]. □

6.7. **Remark.** Assume that $\mathfrak{X} = \mathrm{Spec}(R_n)$. It is easy to verify that η^{geo} arises from a unique k–group homomorphism

$${}_\infty\boldsymbol{\mu} = \Big(\varprojlim \boldsymbol{\mu}_m\Big)^n \to {}_z\mathbf{G}$$

[7] That the elements are of finite order follows from the continuity assumption.

[8] The reader is referred to [GP3] for more details

We finish this section by recalling some basic properties of the twisting bijection (or torsion map) $\tau_z : H^1(\mathfrak{X}, \mathfrak{G}) \to H^1(\mathfrak{X}, {}_z\mathfrak{G})$. Let $\eta \in Z^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(\overline{k})\big)$ and consider its corresponding pair $\Theta(\eta) = (z, \eta^{geo})$. We can apply the same construction to the twisted k–group ${}_z\mathbf{G}$. This would lead to a map Θ_z that will attach to a cocycle $\eta' \in Z^1\big(\pi_1(\mathfrak{X}, a), {}_z\mathbf{G}(\overline{k})\big)$ a pair $(z', \eta'^{\mathrm{geo}})$ along the lines explained above.

6.8. **Lemma.** *Let $\eta \in Z^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(\overline{k})\big)$. With the above notation, the inverse of the twisting map* [Se1]

$$\tau_z^{-1} : Z^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(\overline{k})\big) \xrightarrow{\sim} Z^1\big(\pi_1(\mathfrak{X}, a), {}_z\mathbf{G}(\overline{k})\big)$$

satisfies $\Theta_z \circ \tau_z^{-1}(\eta) = (1, \eta^{\mathrm{geo}})$. $\square$

6.9. **Remark.** The notion of loop torsor behaves well under twisting by a Galois cocycle $z \in Z^1\big(\mathrm{Gal}(k), \mathbf{G}(\overline{k})\big)$. Indeed the torsion map τ_z^{-1} : $H^1_{ét}(\mathfrak{X}, \mathbf{G}) \to H^1_{et}(\mathfrak{X}, {}_z\mathbf{G})$ maps loop classes to loop classes.

6.10. **Loop reductive groups.** Let $\mathfrak{H}$ be a reductive group scheme over$\mathfrak{X}$. Since $\mathfrak{X}$ is connected, for all $x \in \mathfrak{X}$ the geometric fibers $\mathfrak{H}_{\overline{x}}$ are reductive group schemes of the same "type" [SGA3, XXII, 2.3]. By Demazure's theorem there exists a unique split reductive group $\mathbf{H}_0$ over k such that $\mathfrak{H}$ is a twisted form (in the étale topology of $\mathfrak{X}$) of $\mathfrak{H}_0 = \mathbf{H}_0 \times_k \mathfrak{X}$. We will call $\mathbf{H}_0$ the *Chevalley k–form of* $\mathfrak{H}$. The $\mathfrak{X}$–group $\mathfrak{H}$ corresponds to a torsor $\mathfrak{E}$ over $\mathfrak{X}$ under the group scheme $\mathbf{Aut}(\mathfrak{H}_0)$, namely $\mathfrak{E} = \mathbf{Isom}_{gr}(\mathfrak{H}_0, \mathfrak{H})$. We recall that $\mathbf{Aut}(\mathfrak{H}_\mathbf{0})$ is representable by a smooth and separated group scheme over $\mathfrak{X}$ by [SGA3, XXII, 2.3]. It is well-known that $\mathfrak{H}$ is then the contracted product $\mathfrak{E} \wedge^{\mathbf{Aut}(\mathfrak{H}_0)} \mathfrak{H}_0$ (see [DG] III §4 n°3 for details). Given $\mathfrak{H}$, in what follows we may denote $\mathbf{H}_0$ simply by $\mathbf{H}$.

We now recall one of the central concepts needed for our work.

6.11. **Definition.** *We say that a group scheme $\mathfrak{H}$ over $\mathfrak{X}$ is* loop reductive *if it is reductive and if $\mathfrak{E}$ is a loop torsor.*

7. Preliminaries III: Reductive group schemes over a normal noetherian base

We begin with a useful variation of Lemma 5.3 under some extra assumptions on our connected base k–scheme $\mathfrak{X}$. Let $\mathfrak{P} \subset \mathfrak{H}$ be a parabolic subgroup which admits a Levi subgroup $\mathfrak{L} \subset \mathfrak{P}$.[9] As above we denote by $\mathfrak{T}$ the radical of $\mathfrak{L}$ and by $\mathfrak{T}_d \subset \mathfrak{T}$ the maximal split subtorus of $\mathfrak{T}$.

7.1. **Lemma.** *Assume that $\mathfrak{X}$ is normal noetherian and integral. Let $\mathfrak{H}$ be a reductive $\mathfrak{X}$–group. Then there exists an étale cover $(\mathfrak{U}_i)_{i=1,..,l} \to \mathfrak{X}$ such that :*

(i) $\mathfrak{H} \times_{\mathfrak{X}} \mathfrak{U}_i$ is a split reductive $\mathfrak{U}_i$–group scheme,

(ii) $\mathfrak{U}_i = \mathrm{Spec}(R_i)$ with R_i a normal noetherian domain.

[9]The existence of $\mathfrak{L}$ is automatic if the base scheme is affine by [SGA3, XXVI.2.3]

Proof. Since $\mathfrak{X}$ is normal noetherian, $\mathfrak{H}$ is a locally isotrivial group scheme [SGA3, XXIV.4.1.6]. We can thus cover $\mathfrak{X}$ by affine Zariski open subsets $\mathfrak{X}_1, \dots, \mathfrak{X}_l$ such that there exists a finite étale cover $\mathfrak{V}_i \to \mathfrak{X}_i$ for $i = 1, .., l$ which splits $\mathfrak{H}$. For each i, choose a connected component $\mathfrak{U}_i$ of $\mathfrak{V}_i$. According to the classification of étale maps over $\mathfrak{X}$ (see [EGA4, 18.10.12] and also §4.2), we know that $\mathfrak{U}_i$ is a finite étale cover of $\mathfrak{X}_i$ and that $\mathfrak{U}_i = \mathrm{Spec}(R_i)$ where R_i is a normal domain. Since R_i is finite over the noetherian ring $H^0(\mathfrak{X}_i, O_{\mathfrak{X}})$, it is noetherian as well. □

7.2. **Remark.** If $\mathfrak{X}$ is local, one single $\mathfrak{U}_i$ suffices, i.e. we may assume that $l = 1$.

7.3. **Proposition.** *Assume that $\mathfrak{X}$ is normal and noetherian. Let $\mathfrak{H}$ be a reductive $\mathfrak{X}$–group, $\mathfrak{P} \subset \mathfrak{H}$ be a parabolic subgroup and $\mathfrak{L} \subset \mathfrak{P}$ a Levi subgroup. Let $\mathfrak{T}$ be the radical of $\mathfrak{L}$ and $\mathfrak{T}_d \subset \mathfrak{T}$ its maximal split subtorus. Then $Z_{\mathfrak{H}}(\mathfrak{T}_d) = \mathfrak{L}$.*

Proof. The existence of $\mathfrak{T}_d$ follows from Lemma 5.1(1). Since $\mathfrak{T}$ is the centre of $\mathfrak{L}$ we have $\mathfrak{L} \subset Z_{\mathfrak{H}}(\mathfrak{T})$. The inclusion $\mathfrak{T}_d \subset \mathfrak{T}$ yields $Z_{\mathfrak{H}}(\mathfrak{T}) \subset Z_{\mathfrak{H}}(\mathfrak{T}_d)$. Thus we have $\mathfrak{L} \subset Z_{\mathfrak{H}}(\mathfrak{T}_d)$. By the Lemma below and by [SGA3, XXVI, prop. 6.8] the above inclusion is an equality locally in the Zariski topology, whence equal globally. □

7.4. **Lemma.** *Assume $\mathfrak{X} = \mathrm{Spec}(R)$ is affine and as in the Proposition.*[10] *Let $x \in \mathfrak{X}$ and consider the localized ring R_x. Then $(\mathfrak{T}_d)_{R_x}$ is the maximal split subtorus of $\mathfrak{T}_{R_x}$. In particular, if K denotes the quotient field of R then $\mathfrak{T}_d \times_R K$ is the maximal split subtorus of $\mathfrak{T} \times_R K$.*

Proof. It suffices to show that $(\mathfrak{T}_d)_K$ is the maximal split subtorus of $\mathfrak{T}_K$. We may assume the R is local. By Remark 7.2 there exists a Galois extension $\widetilde{R}/R$ that splits $\mathfrak{T}$. Recall that $\mathfrak{T}$ is determined by its lattice of characters $X(\mathfrak{T})$ equipped with an action of $\mathrm{Gal}\,(\widetilde{R}/R)$, and that $\mathfrak{T}_d$ corresponds to the maximal sublattice in $X(\mathfrak{T})$ stable (elementwise) with respect to $\mathrm{Gal}\,(\widetilde{R}/R)$. Similar considerations apply to $\mathfrak{T}_K$. It remains to note that $\mathfrak{T}_K$ and $\mathfrak{T}$ have the same lattices of characters and that $\mathrm{Gal}\,(\widetilde{R}/R) \simeq \mathrm{Gal}\,(\widetilde{K}/K)$ by [Bbk, Ch5 §2.2 theo.2]). □

7.5. **Proposition.** *Let $\mathfrak{G}$ be a reductive group over a normal ring R containing a proper parabolic subgroup $\mathfrak{P}$. Then $\mathfrak{G}$ contains a split non-central subtorus $\mathbf{G}_{m,R}$.*

Proof. We may assume that $\mathfrak{G}$ is semisimple. Since the base is affine $\mathfrak{P}$ contains a Levi subgroup $\mathfrak{L}$ as we have already observed.

Let $\mathfrak{T}$ be the radical of $\mathfrak{L}$. By [SGA3, XXVI, lemme 6.7] there exists a nontrivial morphism $\mathfrak{T} \to \mathbf{G}_m$. Since R is normal *ibid.* lemme 6.6 shows that the torus $\mathfrak{T}$ contains a split subtorus $\mathbf{G}_{m,R}$. □

[10] All of our normal rings are hereon assumed to be integral domains.

7.6. **Corollary.** *For a reductive group scheme $\mathfrak{G}$ over a normal ring R to contain a proper parabolic subgroup it is necessary and sufficient that it contain a non-central split subtorus.*

Proof. This follows from Propositions 5.5 and 7.5. □

7.7. **Lemma.** *Let $\mathfrak{W}$ be a finite étale R-group with R normal. Let K be the field of quotients of R. Then*
(1) *The canonical map*

$$\chi : H^1_{ét}(R, \mathfrak{W}) \longrightarrow H^1(K, \mathfrak{W}_K)$$

is injective.
(2) $H^1_{Zar}(R, \mathfrak{W}) = 1$.

Proof. (1) Because of the assumptions on $\mathfrak{W}$ we can compute $H^1_{ét}(R, \mathfrak{W})$ as the limit of $H^1_{ét}(S/R, \mathfrak{W})$ with S a connected finite Galois extension of R. Let $\Gamma = \mathrm{Gal}(S/R)$. It is well-known that $\mathfrak{W}$ corresponds to a finite group W together with an action of the algebraic fundamental group of R, and that $H^1_{ét}(S/R, \mathfrak{W}) = H^1\big(\Gamma, \mathfrak{W}(S)\big)$ (see [SGA1, XI §5]).

If L denotes the field of quotients of S then L/K is also Galois with Galois group naturally isomorphic to Γ as explained in [Bbk, Ch.5 §2.2 theo. 2].

Our map $\chi : H^1_{ét}(R, \mathfrak{W}) \longrightarrow H^1_{ét}(K, \mathfrak{W}_K)$ is obtained by the base change K/R. By the above considerations the problem reduces to the study of the map

$$\chi : H^1\big(\Gamma, \mathfrak{W}(S)\big) \longrightarrow H^1\big(\Gamma, \mathfrak{W}(S \otimes_R K)\big)$$

when passing to the limit over S. Since R is normal by [EGA4, 18.10.8 and 18.10.9] we have $S \otimes_R K = L$. On the other hand for S sufficiently large $\mathfrak{W}(S) = W = \mathfrak{W}(L)$. The compatibility of the two Galois actions gives the desired injectivity.

(2) It is clear that $H^1_{Zar}(R, \mathfrak{W})$ is in the kernel of χ. □

7.8. **Remark.** Recall that if $\mathrm{Pic}(R) = 1$ a reductive group scheme $\mathfrak{G}$ over R of rank ℓ is split if and only if it contains a split torus of rank ℓ by [SGA3, XXII, prop. 2.2].

7.9. **Proposition.** *Let $\mathfrak{G}$ be a split reductive group scheme over R. Assume that R has trivial Picard group. Then any two split maximal tori $\mathfrak{T}$ and $\mathfrak{T}'$ of $\mathfrak{G}$ are conjugate under $\mathfrak{G}(R)$.*

Proof. Consider the transporter functor $\mathbf{Trans}_{\mathfrak{G}}(\mathfrak{T}, \mathfrak{T}')$. It is represented by a closed subscheme of $\mathfrak{G}$ ([SGA3, XXII, theo. 5.3.9]) which is a $N_{\mathfrak{G}}(\mathfrak{T})$–torsor ([SGA3, XXII, cor 5.3.11]). Let $\xi \in H^1(R, N_{\mathfrak{G}}(\mathfrak{T}))$ be the corresponding element and let $\mathfrak{W} = N_{\mathfrak{G}}(\mathfrak{T})/\mathfrak{T}$ (the Weyl group, which is a finite constant R–group since $\mathfrak{T}$ is split). Locally (for the Zariski topology) $\mathfrak{T}$ and $\mathfrak{T}'$ are conjugate by [SGA3, XXVI, prop. 6.16]. Hence ξ is locally trivial. By Lemma 7.7 its image under the canonical map $H^1_{Zar}(R, N_{\mathfrak{G}}(\mathfrak{T})) \to H^1_{Zar}(R, \mathfrak{W})$ is trivial. Hence ξ comes from $H^1(R, \mathfrak{T})$. Since $\mathrm{Pic}(R) = 1$ and since $\mathfrak{T}$ is R–split we conclude $\xi = 1$. It follows that $\mathbf{Trans}_{\mathfrak{G}}(\mathfrak{T}, \mathfrak{T}')(R) \neq \emptyset$. $\square$

7.10. **Proposition.** *Let $\mathfrak{G}$ be a split reductive R–group. Assume that R has the property that $H^1_{Zar}(R, \mathfrak{L}) = 1$ for all split reductive groups $\mathfrak{L}$.*[11] *Let $\mathfrak{S} \subset \mathfrak{G}$ be a split torus. Then there exist split maximal torus of $\mathfrak{G}$ containing $\mathfrak{S}$.*

Proof. Let $\mathfrak{T} \subset \mathfrak{G}$ be a split maximal torus and let $\mathfrak{C} = Z_{\mathfrak{G}}(\mathfrak{S})$. We know from Lemma 5.1(2),(3) that $\mathfrak{C}$ is a Levi subgroup of a parabolic subgroup, say $\mathfrak{P}$. From Lemma 5.3 it follows that there exists a subtorus $\mathfrak{T}' \subset \mathfrak{T}$ such that $Z_{\mathfrak{G}}(\mathfrak{T}')$ is a Levi subgroup of a standard parabolic subgroup, say $\mathfrak{P}'$, containing $\mathfrak{T}$ and having the same type as $\mathfrak{P}$.

We first claim that $\mathfrak{P}$ and $\mathfrak{P}'$ are conjugate under $\mathfrak{G}(R)$. Indeed, the functor $\mathbf{Trans}_{\mathfrak{G}}(\mathfrak{P}', \mathfrak{P})$ is represented by a scheme which is a principal homogeneous space under $N_{\mathfrak{G}}(\mathfrak{P}') = \mathfrak{P}'$, hence it corresponds to an element $[\xi] \in H^1_{\acute{e}t}(R, \mathfrak{P}')$. By ([SGA3, XXVI, cor. 5.5]) for every prime ideal $\mathfrak{p} \subset R$ after the base change $R \to R_{\mathfrak{p}}$ the groups $\mathfrak{P}'_{\mathfrak{p}}$ and $\mathfrak{P}_{\mathfrak{p}}$ are conjugate by an element in $\mathfrak{G}(R_{\mathfrak{p}})$. This implies that $[\xi] \in H^1_{Zar}(R, \mathfrak{P}')$. Since $\mathfrak{P}'$ is the semidirect product of its (split) unipotent radical and the Levi subgroup $\mathfrak{L} = Z_{\mathfrak{G}}(\mathfrak{T}')$, we conclude that ξ is equivalent to a locally trivial torsor under $\mathfrak{L}$. Since $\mathfrak{L}$ is a split reductive group we have $H^1_{Zar}(R, \mathfrak{L}) = 1$ by assumption, hence the claim.

Without loss of generality we may thus assume that $\mathfrak{S} \subset \mathfrak{P}'$. It then follows that $Z_{\mathfrak{G}}(\mathfrak{S}) \subset \mathfrak{P}'$ is a Levi subgroup in $\mathfrak{P}'$. By [SGA3, XXVI, cor. 1.8], $Z_{\mathfrak{G}}(\mathfrak{S})$ and $Z_{\mathfrak{G}}(\mathfrak{T}')$ are conjugate under $\mathfrak{G}(R)$. Thus, up to conjugacy, we may assume that $\mathfrak{S}$ is a central subtorus of $Z_{\mathfrak{G}}(\mathfrak{T}')$. This of course implies that $\mathfrak{S}$ is contained in $\mathfrak{T}$. $\square$

8. AD and MAD subalgebras

Let R be an object in k–*alg* and $\mathfrak{G}$ be an R–group, i.e a group scheme over R. Recall (see [DG] II §4.1) that to $\mathfrak{G}$ we can attach an R–functor on Lie algebras $\mathfrak{Lie}(\mathfrak{G})$ which attaches to an object S of R–*alg* the kernel of the natural map $\mathfrak{G}(S[\epsilon]) \to \mathfrak{G}(S)$ where $S[\epsilon]$ is the algebra of dual number over S. Let $\mathrm{Lie}(\mathfrak{G}) = \mathfrak{Lie}(\mathfrak{G})(R)$. This is an R–Lie algebra that will be denoted by $\mathfrak{g}$ in what follows.

8.1. **Remark.** If $\mathfrak{G}$ is smooth, the additive group of $\mathrm{Lie}(\mathfrak{G})$ represents $\mathfrak{Lie}(\mathfrak{G})$, that is $\mathfrak{Lie}(\mathfrak{G})(S) = \mathrm{Lie}(\mathfrak{G}) \otimes_R S$ as S–Lie algebras (this equality is strictly speaking a functorial family of canonical isomorphisms).

If S is in R–*alg*, $g \in \mathfrak{G}(S)$ and $x \in \mathfrak{Lie}(\mathfrak{G})(S)$, then $gxg^{-1} \in \mathfrak{Lie}(\mathfrak{G})(S)$ This last product is computed in the group $\mathfrak{G}(S[\epsilon])$ where g is viewed as an element of $\mathfrak{G}(S[\epsilon])$ by functoriality. The above defines an action of $\mathfrak{G}$ on

[11] For example $R = R_n$. See [GP2, Cor. 2.3].

$\mathfrak{Lie}(\mathfrak{G})(S)$, called the adjoint action and denoted by $g \mapsto \mathrm{Ad}(g)$. This action in fact induces an R–group homomorphism

$$\mathrm{Ad} : \mathfrak{G} \to \mathfrak{Aut}\big((\mathfrak{Lie}(\mathfrak{G})\big)$$

whose kernel is the centre of $\mathfrak{G}$.

Given a k–subspace V of $\mathfrak{g}$ consider the R–group functor $Z_{\mathfrak{G}}(V)$ defined by

(8.1.1) $$Z_{\mathfrak{G}}(V) : S \to \{g \in \mathfrak{G}(S) : \mathrm{Ad}(g)(v_S) = v_S \text{ for every } \ v \in V\}$$

for all S in R*–alg*, where v_S denotes the image of v in $\mathfrak{g} \otimes_R S$.

We will denote by RV the R-span of V inside $\mathfrak{g}$, i.e. RV is the R-submodule of $\mathfrak{g}$ generated by V.

8.2. **Remark.** Note that $Z_{\mathfrak{G}}(V) = Z_{\mathfrak{G}}(RV)$. This follows from the fact that the adjoint action of $\mathfrak{G}$ on $\mathfrak{g}$ is "linear" (in a functorial way).

We now introduce some of the central concepts of this work.

A subalgebra $\mathfrak{m}$ of the k-Lie algebra $\mathfrak{g}$ is called an AD *subalgebra* if $\mathfrak{g}$ admits a k–basis consisting of simultaneous eigenvectors of $\mathfrak{m}$, i.e. there exists a family (λ_i) of functionals $\lambda_i \in \mathfrak{m}^*$, and a k–basis $\{ v_i \}_{i \in I}$ of $\mathfrak{g}$ such that

$$[h, v_i] = \langle \lambda_i, h \rangle \, v_i \ \text{ for all } \ h \in \mathfrak{m}.$$

A maximal AD subalgebra of $\mathfrak{g}$, namely one which is not properly included in any other AD subalgebra of $\mathfrak{g}$ is called a MAD of $\mathfrak{g}$.[12]

8.3. **Example.** Let $\mathbf{G}$ be a semisimple Chevalley k–group and $\mathbf{T}$ its standard maximal split torus. Let $\mathbf{h}$ be the Lie algebra of $\mathbf{T}$; it is a split Cartan subalgebra of $\mathbf{g}$. For all R we have $\mathfrak{g} := \mathrm{Lie}(\mathbf{G}_R) = \mathbf{g} \otimes_k R$. Assume that R is *connected*. Then $\mathfrak{m} = \mathbf{h} \otimes 1$ is a MAD of $\mathfrak{g}$ by [P1, cor. to theo.1(i)]. We have $Z_{\mathbf{G}_R}(\mathfrak{m}) = \mathbf{T}_R$.

Note that $\mathfrak{m}$ is not its own normalizer. Indeed $N_{\mathfrak{g}}(\mathfrak{m}) = Z_{\mathfrak{g}}(\mathfrak{m}) = \mathbf{h} \otimes_k R$. Thus $\mathbf{h} \otimes 1$ is not a Cartan subalgebra of $\mathfrak{g}$ in the usual sense. However, in infinite dimensional Lie theory–for example, in the case of Kac-Moody Lie algebras–these type of subalgebras do play the role that the split Cartan subalgebras play in the classical theory. This is our motivation for studying conjugacy questions related to MADs.

8.4. **Remark.** Let $\mathfrak{s}$ be an abelian Lie subalgebra of $\mathfrak{g}$. Let $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be two subalgebras of $\mathfrak{s}$ which are AD subalgebras of $\mathfrak{g}$. Because $\mathfrak{s}$ is abelian their sum $\mathfrak{m}_1 + \mathfrak{m}_2$ is also an AD subalgebra of $\mathfrak{g}$. By considering the sum of all such subalgebras we see that $\mathfrak{s}$ contains a *unique* maximal subalgebra $\mathfrak{m}(\mathfrak{s})$ which is an AD subalgebra of $\mathfrak{g}$. Of course this AD subalgebra need not be a MAD of $\mathfrak{g}$.

We will encounter this situation when $\mathfrak{s}$ is the Lie algebra of a torus $\mathfrak{S}$ inside a reductive group scheme $\mathfrak{G}$. In this case we denote $\mathfrak{m}(\mathfrak{s})$ by $\mathfrak{m}(\mathfrak{S})$.

[12]It is not difficult to see that any such $\mathfrak{m}$ is necessarily abelian, so AD can be thought as shorthand for abelian k–diagonalizable or ad k–diagonalizable.

8.5. **Remark.** Let $\mathfrak{m}$ be an AD subalgebra of $\mathfrak{g}$. Then for any extension S/R in k–alg the image $\mathfrak{m}\otimes 1$ of $\mathfrak{m}$ in $\mathfrak{g}\otimes_R S$ is an AD subalgebra of $\mathfrak{g}\otimes_R S$. Indeed if $x \in \mathfrak{m}$ and $v \in \mathfrak{g}$ are such that $[x,v] = \lambda v$ for some $\lambda \in k$, then $[x\otimes 1, v\otimes s] = v\otimes \lambda s = \lambda(v\otimes s)$ for all $s \in S$. Thus $\mathfrak{g}\otimes_R S$ is spanned as a k–space by common eigenvectors of $\mathfrak{m}\otimes 1$. Note that if the map $\mathfrak{g} \to \mathfrak{g}\otimes_R S$ is injective, for example if S/R is faithfully flat, then we can identify $\mathfrak{m}$ with $\mathfrak{m}\otimes 1$ and view $\mathfrak{m}$ as an AD subalgebra of $\mathfrak{g}\otimes_R S$.

The main thrust of this work is to investigate the question of conjugacy of MAD subalgebras of $\mathfrak{g}$ when $\mathfrak{g}$ is a twisted form of $\mathbf{g}\otimes_k R_n$. The result we aim for is in the spirit of Chevalley's work, as explained in the Introduction. In the "untwisted case" the result is as expected.

8.6. **Theorem.** *All MADs of $\mathbf{g}\otimes_k R_n$ are conjugate to $\mathbf{h}\otimes 1$ under $\mathbf{G}(R_n)$.* $\square$

This is a particular case of Theorem 1 of [P1] by taking Cor 2.3 of [GP2] into consideration. The proof is cohomological in nature, which is also the approach that we will pursue here. As we shall see, the general twisted case holds many surprises in place.

We finish by stating and proving a simple result for future use.

8.7. **Lemma.** *Let $\mathbf{G}$ be a semisimple algebraic group over a field L of characteristic 0. Let $\mathbf{T} \subset \mathbf{G}$ be a torus and $\mathbf{T}_d$ be the (unique) maximal split subtorus of $\mathbf{T}$. Set $\mathbf{g} = \mathrm{Lie}\,(\mathbf{G})$, $\mathfrak{t} = \mathrm{Lie}\,(\mathbf{T})$ and $\mathfrak{t}_d = \mathrm{Lie}\,(\mathbf{T}_d)$. Then*

(i) *The adjoint action of $\mathbf{T}_d$ on $\mathbf{g}$ is L–diagonalizable. In particular, $\mathfrak{t}_d$ is an* AD *subalgebra of $\mathbf{g}$.*

(ii) *$\mathfrak{t}_d$ is the largest subalgebra of $\mathfrak{t}$ satisfying the condition given in* (i).

Proof. Part (i) is clear. As for (ii) we may assume that $\mathbf{G}$ is semisimple adjoint. Let $\mathbf{T}_a$ be the largest anisotropic subtorus of $\mathbf{T}$. The product morphism $\mathbf{T}_d \times \mathbf{T}_a \to \mathbf{T}$ is a central isogeny, hence $\mathfrak{t} = \mathfrak{t}_d \oplus \mathfrak{t}_a$ where $\mathfrak{t}_a = \mathrm{Lie}\,(\mathbf{T}_a)$. We must show that $\mathfrak{t}_a$ does not contain any nonzero element whose adjoint action on $\mathbf{g}$ is L–diagonalizable. Let h be such an element. Fix a basis $\{\, v_1, \dots, v_n \,\}$ of $\mathbf{g}$ and scalars $\lambda_i \in L$ such that

$$[\,h, v_i\,] = \lambda_i v_i \quad \forall\ 1 \le i \le n.$$

By means of this basis we identify $\mathbf{GL}\,(\mathfrak{g})$ with $\mathbf{GL}_{n,L}$. Consider the adjoint representation diagrams

$$\mathbf{T} \hookrightarrow \mathbf{G} \xrightarrow{\mathrm{Ad}} \mathbf{GL}(\mathfrak{g}) \simeq \mathbf{GL}_{n,L}$$

and

$$\mathfrak{t} \hookrightarrow \mathfrak{g} \xrightarrow{\mathrm{ad}} \mathfrak{gl}\,(\mathfrak{g}) \simeq \mathfrak{gl}_{n,L}.$$

Since $\mathbf{G}$ is of adjoint type Ad is injective, so that we can identify $\mathbf{T}$ with a subtorus, say $\widetilde{\mathbf{T}}$, of $\mathbf{GL}_{n,L}$. Similarly for $\mathbf{T}_d$ and $\mathbf{T}_a$. Since $\mathbf{T} \simeq \widetilde{\mathbf{T}}$ we see that $\widetilde{\mathbf{T}}_d$ and $\widetilde{\mathbf{T}}_a$ are the maximal split and anisotropic parts of $\widetilde{\mathbf{T}}$.

Let $\mathbf{D}_n$ be the diagonal subgroup of $\mathbf{GL}_{n,L}$. By construction we see that

$$\mathrm{ad}_{\mathfrak{g}}(h) \in \mathrm{Lie}\,(\mathbf{D}_n) \cap \mathrm{Lie}\,(\widetilde{\mathbf{T}}_a) = \mathrm{Lie}\,(\mathbf{D}_n \cap \widetilde{\mathbf{T}}_a),$$

this last by [Hu, Theorem 12.5] since k is of characteristic 0. Thus $\mathbf{D}_n \cap \widetilde{\mathbf{T}}_a$ has dimension > 0. But then the connected component of the identity of $\mathbf{D}_n \cap \widetilde{\mathbf{T}}_a$ is a non-trivial split torus which contradicts the fact that $\widetilde{\mathbf{T}}_a$ is anisotropic.

□

9. The correspondence between MADs and maximal split tori

Throughout this section R will denote an object of k–alg such that $\mathfrak{X} = \mathrm{Spec}(R)$ is integral and noetherian. The purpose of this section is to establish the following fundamental correspondence.

9.1. **Theorem.** *Let $\mathfrak{G}$ be a semisimple simply connected R–group and $\mathfrak{g}$ its Lie algebra. Assume that R is normal integral and noetherian.*

(1) *Let $\mathfrak{m}$ be a* MAD *subalgebra of $\mathfrak{g}$. Then $Z_{\mathfrak{G}}(\mathfrak{m})$ is a reductive R–group and its radical contains a unique maximal split torus $\mathfrak{S}(\mathfrak{m})$ of $\mathfrak{G}$.*

(2) *Let $\mathfrak{S}$ is a maximal split torus of $\mathfrak{G}$, and let $\mathfrak{m}(\mathfrak{S})$ be the unique maximal subalgebra of Lie algebra* $\mathrm{Lie}\,(\mathfrak{S})$ *which is an* AD *subalgebra of $\mathfrak{g}$ (see Remark 8.4). Then $\mathfrak{m}(\mathfrak{S})$ is a* MAD *subalgebra of $\mathfrak{g}$.*

(3) *The process $\mathfrak{m} \to \mathfrak{S}(\mathfrak{m})$ and $\mathfrak{S} \to \mathfrak{m}(\mathfrak{S})$ described above gives a bijection between the set of* MAD *subalgebras of $\mathfrak{g}$ and the set of maximal split tori of $\mathfrak{G}$.*

(4) *If $\mathfrak{m}$ and $\mathfrak{m}'$ are two* MAD *subalgebras of $\mathfrak{g}$, then for $\mathfrak{m}$ and $\mathfrak{m}$ to be conjugate under the adjoint action of $\mathfrak{G}(R)$ it is necessary and sufficient that the maximal split tori $\mathfrak{S}(\mathfrak{m})$ and $\mathfrak{S}(\mathfrak{m}')$ be conjugate under the adjoint action of $\mathfrak{G}(R)$ on $\mathfrak{g}$.*

9.2. **Remark.** Since $\mathfrak{S}$ is split we have $\mathrm{Lie}\,(\mathfrak{S}) = X(\mathfrak{S})^{\mathrm{o}} \otimes_{\mathbb{Z}} R$ where $X(\mathfrak{S})^{\mathrm{o}}$ is the cocharacter group of $\mathfrak{S}$. As we shall see in the proof of Lemma 9.5 $\mathfrak{m}(\mathfrak{S}) = X(\mathfrak{S})^{\mathrm{o}} \otimes_{\mathbb{Z}} k$.

The proof of the Theorem will be given at the end of this section after a long list of preparatory results.

We begin with some general observations and fixing some notation that will be used throughout the proofs of this section. Since $\mathfrak{X}$ is connected all geometric fibers of $\mathfrak{G}$ are of the same type. Let $\mathbf{G}$ be the corresponding Chevalley group over k and $\mathbf{g}$ its Lie algebra.

9.3. **Lemma.** *Let $\mathfrak{m}$ be an AD subalgebra of $\mathfrak{g}$. Then*

(1) $\dim_k(\mathfrak{m}) \leq \mathrm{rank}\,(\mathbf{g})$. *In particular any AD subalgebra of $\mathfrak{g}$ is included inside a MAD subalgebra of $\mathfrak{g}$.*

(2) *The natural map $\mathfrak{m} \otimes_k R \to R\,\mathfrak{m}$ is an R–module isomorphism. In particular $R\,\mathfrak{m}$ is a free R–module of rank $= \dim_k(\mathfrak{m})$.*

(3) *Let* $\{v_1, \dots, v_m\}$ *be a* k*–basis of* $\mathfrak{m}$. *For every* $x \in \mathfrak{X}$ *the elements* $v_i \otimes 1 \in \mathfrak{g} \otimes_R R_x$ *are* R_x*–linearly independent. Similarly if we replace* R_x *by* K *or any field extension of* K.

Proof. The three assertions are of local nature, so we can assume that R is local. We will establish the Lemma by first reducing the problem to the split case. According to Remark 7.2 there exists a finite étale extension $\widetilde{R}/R$ with the following properties

(i) $\widetilde{R}$ is integral and normal,

(ii) $\mathfrak{G} \times_R \widetilde{R} \simeq \mathbf{G}_{\widetilde{R}}$,

Note the following facts:

(iii) the canonical map $\mathfrak{g} \to \mathfrak{g} \otimes_R \widetilde{R} \simeq \mathbf{g} \otimes_k \widetilde{R}$ is injective,

(iv) if $\{v_1, \dots, v_m\}$ are k–linearly independent elements of $\mathfrak{m}$ which are R–linearly dependent, then the image of the elements $\{v_1, \dots, v_m\}$ on $\mathrm{Lie}(\mathfrak{G} \times_R \widetilde{R}) \simeq \mathbf{g} \otimes_k \widetilde{R}$ are k–linearly independent and are $\widetilde{R}$–linearly dependent.

Let $\widetilde{K}$ be the field of fractions of $\widetilde{R}$. By Remark 8.5 the image of $\mathfrak{m}$ under the injection $\mathfrak{g} \hookrightarrow \mathfrak{g} \otimes_R \widetilde{R} \simeq \mathbf{g} \otimes_k \widetilde{R}$ is an AD subalgebra of $\mathbf{g} \otimes_k \widetilde{R}$. By [P1, theo.1.(i)] the dimension of $\mathfrak{m}$ is at most the rank of $\mathbf{g}$. This establishes (1).

As for (2) and (3), the crucial point–as explained in [P1, Prop. 4]–lies in the fact that the image $\widetilde{\mathfrak{m}}$ of $\mathfrak{m}$ under the injection

$$\mathfrak{g} \hookrightarrow \mathfrak{g} \otimes_R \widetilde{R} \simeq \mathbf{g} \otimes_k \widetilde{R} \hookrightarrow \mathbf{g} \otimes \widetilde{K}$$

sits inside a split Cartan subalgebra $\mathcal{H}$ of the split semisimple $\widetilde{K}$-algebra $\mathbf{g} \otimes_k \widetilde{K}$. Consider the basis $\{\check{\omega}_1, \dots, \check{\omega}_\ell\}$ of $\mathcal{H}$ consisting of fundamental coweights for a base $\alpha_1, \dots, \alpha_\ell$ of the root system of $(\mathbf{g} \otimes_k \widetilde{K}, \mathcal{H})$. Let $1 \leq n \leq m$ be such that $\{\widetilde{v}_1, \dots \widetilde{v}_n\}$ is a maximal set of $\widetilde{K}$–linearly independent elements of $\mathbf{g}(\widetilde{K})$. To establish (2) and (3) it will suffice to show that $n = m$.

Assume on the contrary that $n < m$. Write $\widetilde{v}_i = \sum c_{ji} \check{\omega}_j$ with $c_{1i}, \dots c_{\ell i}$ in $\widetilde{K}$. The fact that the eigenvalues of $ad_{\mathbf{g}(\widetilde{K})}(\widetilde{v}_i)$ belongs to k show that the c_{ji} necessarily belong to k. Indeed $\widetilde{v}_i$ acts on $\mathbf{g}(\widetilde{K})_{\alpha_j}$ as multiplication by the scalar c_{ji}.

Let $v = v_{n+1}$. Write $\widetilde{v} = \sum a_i \widetilde{v}_i$ with $a_1, \dots, a_n$ in $\widetilde{K}$. Let $c_{jn+1} = \lambda_j$. Then $< \alpha_j, \widetilde{v} >= \lambda_j$ and

$$\widetilde{v} = \sum_j (\sum_i a_i c_{ji}) \check{\omega}_j = \sum_j \lambda_j \check{\omega}_j$$

and therefore for all $1 \leq j \leq \ell$

$$\sum_i a_i c_{ji} = \lambda_j.$$

Write $\widetilde{K} = k \oplus W$ as a k–space and use this decomposition to write $a_i = d_i + w_i$. Then

$$\sum_i d_i c_{ji} = \lambda_j.$$

A straightforward calculation shows that $< \alpha_j, \widetilde{v} - \sum_i d_i \widetilde{v_i} >= 0$ for all j. This forces

$$v_{n+1} = v = \sum_i d_i v_i$$

which contradicts the linear independence of the v_i's over k. □

9.4. **Remark.** Let $\mathfrak{S} < \mathfrak{G}$ be a split torus. Then there exist characters $\lambda_i : \mathfrak{S} \to \mathbf{G}_{m,R}$ for $1 \le i \le l$ such that

$$\mathfrak{g} = \bigoplus_{i=1}^{l} \mathfrak{g}_{\lambda_i}$$

where

$$\mathfrak{g}_{\lambda_i} = \{\, v \in \mathfrak{g} : \ \mathrm{Ad}(g)v = \lambda_i(g)v \ \forall g \in \mathfrak{S}(R) \,\}.$$

At the Lie algebra level the situation is as follows. Let $\mathfrak{s} = \mathrm{Lie}\,(\mathfrak{S}) \subset \mathfrak{g}$. Then $\mathfrak{s} \subset \mathfrak{S}(R[\varepsilon])$. We avail ourselves of the useful convention that if $s \in \mathfrak{s}$ then to view s as an element of $\mathfrak{S}(R[\varepsilon])$ we write $e^{s\varepsilon}$. There exist unique R–linear functionals $d\lambda_i : \mathfrak{s} \to R$ such that

$$\lambda_i(e^{s\varepsilon}) = 1 + d\lambda_i(s)\varepsilon \in R[\varepsilon]^\times = \mathbf{G}_{m,R}(R[\varepsilon]).$$

Then for $s \in \mathfrak{s}$ and $v \in \mathfrak{g}_{\lambda_i}$ we have the following equality in $\mathfrak{g}$

$$[s, v] = d\lambda_i(s)v. \tag{9.4.1}$$

9.5. **Lemma.** *Consider the restriction* $\mathrm{Ad}_{\mathfrak{S}} : \mathfrak{S} \to \mathbf{Gl}\,(\mathfrak{g})$ *of the adjoint representation of* $\mathfrak{G}$ *to* $\mathfrak{S}$. *There exists a finite number of characters* $\lambda_1, \dots, \lambda_l$ *of* $\mathfrak{S}$ *such that*

$$\mathfrak{g} = \bigoplus_{i=1}^{l} \mathfrak{g}_{\lambda_i}$$

The λ_i *are unique and*

$$\mathfrak{m}(\mathfrak{S}) = \{\, s \in \mathrm{Lie}\,(\mathfrak{S}) \subset \mathfrak{S}\,(R[\varepsilon]) \ : \ d\lambda_i(s) \in k \,\}.$$

Furthermore

$$\dim_k\big(\mathfrak{m}(\mathfrak{S})\big) = \mathrm{rank}(\mathfrak{S}) = \mathrm{rank}_{R-mod}(R\mathfrak{m}(\mathfrak{S}))$$

and $\mathrm{Lie}(\mathfrak{S}) = R\mathfrak{m}(\mathfrak{S})$.

Proof. We appeal to the explanation given in Remark 9.4. Let

$$\mathfrak{n} = \{\, s \in \mathfrak{s} \ : \ d\lambda_i(s) \in k \ \forall i \,\}.$$

Then (9.4.1) shows not only that $\mathfrak{n} \subset \mathfrak{s}$ is an AD subalgebra of $\mathfrak{g}$, but in fact that $\mathfrak{m}(\mathfrak{S}) \subset \mathfrak{n}$. By maximality we have $\mathfrak{m}(\mathfrak{S}) = \mathfrak{n}$ as desired.

We now establish the last assertions. Let n be the rank of $\mathfrak{S}$, so $\mathfrak{S} \simeq \mathbf{G}_{m,R}^n$ and the character lattice $X(\mathfrak{S})$ of $\mathfrak{S}$ is generated by the projections

$\pi_i : \mathbf{G}_{m,R}^n \to \mathbf{G}_{m,R}$. Since the kernel of the adjoint representation of $\mathfrak{G}$ is finite the sublattice of $X(\mathfrak{S})$ generated by $\lambda_1, \dots, \lambda_\ell$ has finite index; in particular every character π of $\mathfrak{S}$ can be written as a linear combination $\pi = a_1\lambda_1 + \cdots + a_\ell\lambda_\ell$ with rational coefficients $a_1, \dots, a_\ell$ and hence $d\pi = a_1 d\lambda_1 + \cdots + a_n d\lambda_\ell$. Similarly π can be written as $\pi = a_1\pi_1 + \cdots + a_n\pi_n$ with $a_1, \dots, a_n \in \mathbb{Z}$ and we then have $d\pi = a_1 d\pi_1 + \cdots + a_n d\pi_n$. It follows that

$$\begin{aligned} \mathfrak{m}(\mathfrak{S}) &= \{\, s \in \mathfrak{s} \ : \ d\lambda_i(s) \in k \ \forall i \,\} \\ &= \{\, s \in \mathfrak{s} \ : \ d\pi(s) \in k \ \forall \pi \in X(\mathfrak{S}) \,\} \\ &= \{\, s \in \mathfrak{s} \ : \ d\pi_i(s) \in k \ \forall i \,\}. \end{aligned}$$

The identification $\mathfrak{S} \simeq \mathbf{G}_{m,R}^n$ induces the identification $\mathfrak{s} \simeq \mathbf{G}_{a,R}^n$. The above equalities yield

$$\mathfrak{m}(\mathfrak{S}) \simeq \{\, (s_1, \dots, s_n) \ : \ s_i \in k \ \forall i \,\},$$

hence the last assertions follow immediately. $\square$

9.6. **Proposition.** *Let $\mathfrak{m}$ be an* AD *subalgebra of $\mathfrak{g}$. Then the submodule $R\mathfrak{m}$ is a direct summand of $\mathfrak{g}$.*

Proof. Let $M = \mathfrak{g}/R\mathfrak{m}$. Assume for a moment that M is a projective R–module. Then the exact sequence

$$0 \longrightarrow R\mathfrak{m} \longrightarrow \mathfrak{g} \longrightarrow M \longrightarrow 0$$

is split and the Proposition follows.

Thus it remains to show that M is a projective R–module or, equivalently, that for every prime ideal x of R the localized R_x–module M_x is free. Since localization is a left exact functor, and by Lemma 9.3 we have $(R\mathfrak{m})_x = R_x\mathfrak{m}$ the sequence

$$0 \longrightarrow R_x\mathfrak{m} \longrightarrow \mathfrak{g}_{R_x} \longrightarrow M_x \longrightarrow 0$$

is exact. By Lemma 9.3(3), the elements

$$v_1 \otimes 1, \dots, v_m \otimes 1 \in R_x\mathfrak{m} \subset \mathfrak{g} \otimes_R R_x = \mathfrak{g}_x$$

and the module $\mathfrak{g}_x$ satisfy the variation of Nakayama's lemma stated in [Lam] Cor 1.8. Hence $R_x\mathfrak{m}$ is a direct summand of $\mathfrak{g}$ and this implies that M_x is free. $\square$

9.7. **Proposition.** *Let $\mathfrak{m}$ be an AD subalgebra of $\mathfrak{g}$. Then $Z_{\mathfrak{G}}(\mathfrak{m})$ is an affine R–group whose geometric fibres are (connected) reductive groups.*

Proof. By Proposition 9.6 $R\mathfrak{m}$ is a direct summand of $\mathfrak{g}$. It follows from [DG] II §Prop.1.4 that $Z_{\mathfrak{G}}(R\mathfrak{m}) = Z_{\mathfrak{G}}(\mathfrak{m})$ is a closed subgroup of $\mathfrak{G}$. In particular, $Z_{\mathfrak{G}}(\mathfrak{m})$ is an affine scheme which is of finite type over $\mathrm{Spec}(R)$.

Let $x \in \mathrm{Spec}(R)$ be a point and let $k(\overline{x})$ be an algebraic closure of $k(x)$. Since the functor $Z_{\mathfrak{G}}(\mathfrak{m}) = Z_{\mathfrak{G}}(R\mathfrak{m})$ commutes with base change to verify the nature of its geometric fibers $Z_{\mathfrak{G}}(\mathfrak{m})(\overline{x})$ we may look at

$$Z_{\mathfrak{G}}(R\mathfrak{m}) \otimes_R k(\overline{x}) = Z_{\mathfrak{G}(\overline{x})}(k(\overline{x})\, \mathfrak{m}(\overline{x}))$$

where $\mathfrak{G}(\overline{x}) = \mathfrak{G} \otimes_R k(\overline{x})$ and $\mathfrak{m}(\overline{x})$ is the image of $\mathfrak{m}$ under $\mathfrak{g} \to \mathfrak{g} \otimes_R k(\overline{x})$. Thus we may assume without loss of generality that the ground ring is a field. By results of Steinberg ([St75, 3.3 and 3.8] and [St75, 0.2]) we conclude that $Z_{\mathfrak{G}}(\mathfrak{m})(\overline{x})$ is connected and reductive. □

9.8. **Flatness of $Z_{\mathfrak{G}}(\mathfrak{m})$.** Fix a split Cartan subalgebra $\mathbf{h}$ of $\mathbf{g}$. With respect to the adjoint representation $\mathrm{ad} : \mathbf{g} \to \mathrm{End}_k(\mathbf{g})$ we have the weight space decomposition

$$\mathbf{g} = \oplus_{\alpha \in \Sigma}\, \mathbf{g}_\alpha$$

where $\alpha : \mathbf{h} \to k$ is a linear function such that the corresponding eigenspace $\mathbf{g}_\alpha$ is non-zero. The kernel of the adjoint representation is trivial, $\dim \mathbf{g}_\alpha = 1$ if $\alpha \neq 0$ and $\mathbf{g}_0 = \mathbf{h}$.

9.9. **Lemma.** *Let $\mathbf{a} \subset \mathbf{h}$ be a subalgebra. Then:*

(1) *The centralizer $Z_{\mathbf{g}}(\mathbf{a})$ is a reductive Lie algebra whose centre is contained in $\mathbf{h}$.*

(2) *If $a \in \mathbf{a}$ is in generic position then $Z_{\mathbf{g}}(\mathbf{a}) = Z_{\mathbf{g}}(a)$.*

Proof. (1) The centralizer of $\mathbf{a}$ is generated by $\mathbf{h}$ and those $\mathbf{g}_\alpha$ for which $\alpha(x) = 0$ for every $x \in \mathbf{a}$. It is a well-known fact that this algebra is reductive.

(2) The inclusion $\subset$ is obvious. Conversely, the centralizer of a is generated by $\mathbf{h}$ and those $\mathbf{g}_\alpha$ for which $\alpha(a) = 0$. Since a is generic all such roots α also satisfy $\alpha(x) = 0$ for all $x \in \mathbf{a}$. □

9.10. **Lemma.** *Let $a_\alpha \in k,\ \alpha \in \Sigma$. Then there exists at most one element $h \in \mathbf{h}$ such that $\alpha(h) = a_\alpha$.*

Proof. Since the kernel of the adjoint representation of $\mathbf{g}$ is trivial the result follows. □

9.11. **Lemma.** *Let S be an object of k–alg. Let $v \in \mathbf{h} \otimes_k S$ be an ad k–diagonalizable element of $\mathbf{g} \otimes_k S$. If S is an integral domain then $v \in \mathbf{h}$.*

Proof. Let F be a field of quotients of S and view v as an element of $\mathbf{g} \otimes_k F$. The eigenvalues a_α of v with respect to the adjoint representation are $a_\alpha = \alpha(v)$. By assumption they all belong to k. Thus the nonhomogeneous linear system $\alpha(x) = a_\alpha,\ \alpha \in \Sigma$, has a solution over F, namely v. Since the coefficients of this system of equations are in k it also has a solution over k [see the proof of Lemma 9.3(2)]. By Lemma 9.10 such a solution is unique, hence $v \in \mathbf{h}$. □

9.12. **Remark.** The Lemma fails if S is not connected (hence not integral), but we do not know if connected instead of integral is the right hypothesis to make. Consider for example $\mathbf{g} = \mathfrak{sl}_2$ and $S = k \times k$. Let h, e, f be the standard generators of $\mathfrak{sl}_2$. Then $v = h \otimes (1,0)$ is an ad k-diagonalizable element of $\mathbf{h}_S$ which is not in $\mathbf{h} \simeq \mathbf{h} \otimes (1,1)$.

Fix an arbitrary element $h \in \mathbf{h}$. Recall that $\mathbf{G}$ acts on $\mathbf{g}$ by conjugation and it is known that the orbit $\mathcal{O}_h = \mathbf{G} \cdot h$ is a Zariski closed subset of $\mathbf{g}$ (because h is a semisimple element). Let $\mathbf{L} \subset \mathbf{G}$ be the isotropy subgroup of h in $\mathbf{G}$. It is known that $\mathbf{L}$ is a reductive subgroup and we have an exact sequence

$$1 \longrightarrow \mathbf{L} \longrightarrow \mathbf{G} \xrightarrow{\phi} \mathbf{G}/\mathbf{L} \longrightarrow 1$$

in the *fppf* topology of $\mathfrak{X} = \mathrm{Spec}(R)$.

The algebraic k–varieties $\mathcal{O}_h$ and $\mathbf{G}/\mathbf{L}$ have as distinguished points h and the coset $e = 1 \cdot \mathbf{L}$ respectively. The group $\mathbf{G}$ acts on both $\mathcal{O}_h$ and $\mathbf{G}/\mathbf{L}$ in a natural way and there exists a natural $\mathbf{G}$-equivariant isomorphism $\lambda : \mathcal{O}_h \simeq \mathbf{G}/\mathbf{L}$ which takes h into e (see [Bor] for details). Hence if R is an object in k–alg and $x \in \mathcal{O}_h(R)$, then x and h are conjugate by an element in $\mathbf{G}(R)$ if and only if $\lambda(x) \in \mathbf{G}(R) \cdot e$.

We now return to our simply connected semisimple R–group $\mathfrak{G}$ and its Lie algebra $\mathfrak{g}$

9.13. **Lemma.** *Let $\mathfrak{m}$ be an AD subalgebra of $\mathfrak{g}$. The affine scheme $Z_{\mathfrak{G}}(\mathfrak{m})$ is flat over* $\mathrm{Spec}(R)$*.*

Proof. That $Z_{\mathfrak{G}}(\mathfrak{m})$ is an affine scheme over R has already been established. Since flatness is a local property it will suffice to establish the result after we replace R by its localization at each element of $\mathfrak{X}$. Lemma 7.1 provides a finite étale connected cover $\widetilde{R}/R$ which splits $\mathfrak{G}$. By replacing R by $\widetilde{R}$ we reduce the problem to the split case. Summarizing, without loss of generality we may assume that $\mathfrak{G} = \mathbf{G} \times_k R$, $\mathfrak{g} = \mathbf{g} \otimes_k R := \mathbf{g}_R$ and R is a local domain.

As observed in Lemma 9.3 $\mathfrak{m}$ is contained in a split Cartan subalgebra $\mathcal{H}$ of $\mathbf{g} \otimes_k K := \mathbf{g}_K$. Fix a generic vector $v \in \mathfrak{m} \subset \mathbf{g}_K$. Let $\{ a_\alpha, \, \alpha \in \Sigma \}$ be the family of all eigenvalues of v with respect to the adjoint representation of $\mathbf{g}_K$. Clearly, $a_\alpha \in k$ for every $\alpha \in \Sigma$ (because $\mathfrak{m}$ is an AD subalgebra of $\mathbf{g}_R$).

9.14. **Sublemma.** *There exists a unique vector $h \in \mathbf{h}$ whose eigenvalues with respect to the adjoint representation are $\{ a_\alpha, \, \alpha \in \Sigma \}$. Moreover if v and h are viewed as elements of $\mathbf{g}_K$, then they are conjugate under $\mathbf{G}(K)$.*

Proof. Uniqueness follows from Lemma 9.10. As for existence, we note that $\mathcal{H}$ and $\mathbf{h}_K$ are conjugate over K, hence $\mathbf{h}_K$ clearly contains an element with the prescribed property. By Lemma 9.11 this element is contained in $\mathbf{h}$. The conjugacy assertion is by the very definition of h. □

We now come back to the $\mathbf{G}$-orbit $\mathcal{O}_h$ of h. We remind the reader that this is a closed subvariety of $\mathbf{g}$.

9.15. **Sublemma.** $v \in \mathcal{O}_h(R)$.

Proof. The element $v \in \mathbf{g}_R$ can be viewed as a morphism

$$\phi_v : \mathrm{Spec}\,(R) \to \mathbf{g}.$$

The image of the generic point $\mathrm{Spec}(K) \to \mathrm{Spec}(R) \to \mathbf{g}$ is contained in $\mathcal{O}_h$ for v and h are conjugate over K. Since $\mathcal{O}_h$ is a closed subvariety in $\mathfrak{g}$ and since $\mathrm{Spec}(R)$ is irreducible it follows that ϕ_v factors through the embedding $\mathcal{O}_h \hookrightarrow \mathfrak{g}$. □

To finish the proof of Lemma 9.13 we first consider the particular case when $\mathfrak{m}$ is contained in $\mathbf{h}$. Then $Z_{\mathfrak{G}}(\mathfrak{m})$ is obtained from the variety $Z_{\mathbf{G}}(\mathfrak{m})$ by the base change R/k so that flatness is clear.

In the general case, let $h \in \mathbf{h}$ be the element provided by Sublemma 9.14. By Sublemma 9.15 we have $v \in \mathcal{O}_h(R) = (\mathbf{G}/\mathbf{L})(R)$. Denote by R^{sh} the strict henselisation of the local ring R, that is the simply connected cover of R attached to a separable closure K_s of K as outlined in section 4.2. Since the map $p : \mathbf{G} \to \mathbf{G}/\mathbf{L}$ is smooth and surjective, Hensel's lemma [M1, §4] shows that $\mathbf{G}(R^{sh}) \to (\mathbf{G}/\mathbf{L})(R^{sh})$ is surjective. But R^{sh} is the inductive limit of the finite (connected) Galois covers of R, so there exists one such cover R' and a point $g' \in \mathbf{G}(R')$ such that $v = g'.h$. Up to replacing R by R' (which is a noetherian normal domain) we may assume that $v = h$.

We now recall that $Z_{\mathbf{g}_R}(h) = Z_{\mathbf{g}_R}(\mathfrak{m})$ since $h = v \in \mathfrak{m}$ is a generic vector. Since the center of $Z_{\mathbf{g}_R}(h)$ is contained in $\mathbf{h}_R$ and since $\mathfrak{m}$ is contained in the center of its centralizer we have $\mathfrak{m} \subset \mathbf{h}_R$. Applying Lemma 9.11 then shows that $\mathfrak{m} \subset \mathbf{h}$. Thus we have reduced the general case to the previous one. □

9.16. **Proposition.** *If $\mathfrak{m}$ is an AD subalgebra of $\mathfrak{g}$ then $Z_{\mathfrak{G}}(\mathfrak{m})$ is a reductive R–group.*

Proof. Since $Z_{\mathfrak{G}}(\mathfrak{m})$ is flat and also finitely presented over R the differential criteria for smoothness shows that $Z_{\mathfrak{G}}(\mathfrak{m})$ is in fact smooth over R because of Lemma 9.7. Thus $Z_{\mathfrak{G}}(\mathfrak{m})$ is affine and smooth over R with geometric fibers which are (connected) reductive groups in the usual sense (this last again by Lemma 9.7). By definition $Z_{\mathfrak{G}}(\mathfrak{m})$ is a reductive R–group. □

Proof of Theorem 9.1. (1) Let $\mathfrak{m}$ be a MAD subalgebra of $\mathfrak{g}$, and let $\mathfrak{S}$ denote the maximal split torus of the radical $\mathfrak{T}$ of the reductive R-group $Z_{\mathfrak{G}}(\mathfrak{m})$. By Remark 8.4 the Lie algebra of $\mathfrak{S}$ contains a unique maximal subalgebra $\mathfrak{m}(\mathfrak{S})$ which is an AD–subalgebra of $\mathfrak{g}$. By definition $\mathfrak{S} < \mathfrak{H} = Z_{\mathfrak{G}}(R\mathfrak{m})$. Let us as before denote $\mathrm{Lie}(\mathfrak{S})$ by $\mathfrak{s}$. Since $\mathfrak{s} \subset \mathfrak{S}(R[\varepsilon]))$ it follows that in $\mathfrak{g}$ we have $[\mathfrak{s}, R\mathfrak{m}] = 0$. In particular since $\mathfrak{m}\,(\mathfrak{S}) \subset \mathfrak{s}$ we have $[\,\mathfrak{m}(\mathfrak{S}), \mathfrak{m}\,] = 0$. But then by Remark 8.4 $\mathfrak{m} + \mathfrak{m}(\mathfrak{S})$ is an AD subalgebra of $\mathfrak{g}$. Since $\mathfrak{m}$ is a MAD we necessarily have $\mathfrak{m}(\mathfrak{S}) \subset \mathfrak{m}$ and now we are going to show that $\mathfrak{m}(\mathfrak{S}) = \mathfrak{m}$.

Recall that K denotes the quotient field of R. By Lemma 9.5 we have $\dim(\mathfrak{m}(\mathfrak{S})) = \mathrm{rank}\,(\mathfrak{S})$, so that to establish that $\mathfrak{m}(\mathfrak{S}) = \mathfrak{m}$ it will suffice to show that $\mathrm{rank}\,(\mathfrak{S}) \geq \dim_k(\mathfrak{m})$, or what is equivalent, that $\dim_K(\mathfrak{S}_K) \geq \dim_k(\mathfrak{m})$ where as usual $\mathfrak{S}_K = \mathfrak{S} \times_R K$.

We have $\mathfrak{H}_K = Z_{\mathfrak{G}_K}(R\mathfrak{m}) = Z_{\mathfrak{G}_K}(K\mathfrak{m})$, as can be seen from the fact that the computation of the centralizer commutes with base change. Since

$\mathfrak{S}$ is the maximal split torus of $\mathfrak{T}$ then $\mathfrak{S}_K$ is the maximal split torus of $\mathfrak{T}_K = \mathrm{rad}\,(\mathfrak{H}_K)$ by Lemma 7.4. We also have

$$\mathrm{Lie}\,(\mathfrak{H}_K) = \mathrm{Lie}\,(Z_{\mathfrak{G}_K}(R\mathfrak{m})) = \mathrm{Lie}\,(Z_{\mathfrak{G}_K}(K\mathfrak{m})) = Z_{\mathfrak{g}_K}(K\mathfrak{m}).$$

Since $K\mathfrak{m}$ is in the centre of $Z_{\mathfrak{g}_K}(K\mathfrak{m}) = \mathrm{Lie}\,(\mathfrak{H}_K)$ and the centre of $\mathrm{Lie}\,(\mathfrak{H}_K)$ coincides with $\mathrm{Lie}\,(\mathfrak{T}_K)$ we conclude that $K\mathfrak{m} \subset \mathrm{Lie}\,(\mathfrak{T}_K)$. On the other hand $K\mathfrak{m}$ is an AD subalgebra of $\mathfrak{g}_K$, so that by Lemma 8.7 $K\mathfrak{m} \subset \mathrm{Lie}\,(\mathfrak{S}_K)$. This shows that $\dim_K(K\mathfrak{m}) \leq \dim_K(\mathfrak{S}_K)$. But by Lemma 9.3(3) we have $\dim_k(\mathfrak{m}) = \dim_K(K\mathfrak{m})$. This completes the proof that $\mathfrak{m}(\mathfrak{S}) = \mathfrak{m}$.

Now it is easy to finish the proof that $\mathfrak{S}$ is a maximal split torus in $\mathfrak{G}$. Indeed, if $\mathfrak{S}$ is contained in a split torus $\mathfrak{S}'$ of larger rank then $\mathfrak{m}(\mathfrak{S}) \subset \mathfrak{m}(\mathfrak{S}')$ is a proper subalgebra which contradicts to the fact that $\mathfrak{m} = \mathfrak{m}(\mathfrak{S})$ is a MAD.

(2) Let $\mathfrak{S}$ be a maximal split torus of $\mathfrak{G}$, and let $\mathfrak{s} = \mathrm{Lie}\,(\mathfrak{S})$ be its Lie algebra. By Remark 8.4 $\mathfrak{s}$ contains a unique maximal subalgebra $\mathfrak{m}(\mathfrak{S}) = \mathfrak{m}$ which is an AD–subalgebra of $\mathfrak{g}$. This algebra, which was denoted by $\mathfrak{m}(\mathfrak{S})$ will for the remaining proof of (2) be denoted by $\mathfrak{m}$. We have by Lemma 9.5 that $R\mathfrak{m} = \mathrm{Lie}\,(\mathfrak{S})$. Thus, appealing to Proposition 5.6 and Lemma 9.3(1) we obtain

$$Z_{\mathfrak{G}}(\mathfrak{m}) = Z_{\mathfrak{G}}(R\mathfrak{m}) = Z_{\mathfrak{G}}(\mathfrak{s}) = Z_{\mathfrak{G}}(\mathfrak{S}).$$

We claim that $\mathfrak{m}$ is maximal. Assume otherwise. Then by Lemma 9.3(1) $\mathfrak{m}$ is properly included in a MAD subalgebra $\mathfrak{m}'$ of $\mathfrak{g}$. We have

$$\mathfrak{H}' := Z_{\mathfrak{G}}(R\,\mathfrak{m}') \subset \mathfrak{H} := Z_{\mathfrak{G}}(R\mathfrak{m}) = Z_{\mathfrak{G}}(\mathfrak{S}).$$

By Proposition 9.16 $\mathfrak{H}'$ and $\mathfrak{H}$ are reductive R–groups. Let $\mathfrak{T}'$ and $\mathfrak{T}$ be their radicals and let $\mathfrak{T}'_d$, $\mathfrak{T}_d$ be their maximal split tori. We have $\mathfrak{S} \subset \mathfrak{T} \subset \mathfrak{T}'$ and hence $\mathfrak{S} \subset \mathfrak{T}_d \subset \mathfrak{T}'_d$. But $\mathfrak{S}$ is a maximal split torus in $\mathfrak{G}$. Therefore $\mathfrak{S} = \mathfrak{T}'_d = \mathfrak{T}_d$ and this implies $\mathfrak{m} = \mathfrak{m}(\mathfrak{S}) = \mathfrak{m}(\mathfrak{T}_d) = \mathfrak{m}(\mathfrak{T}'_d)$. Recall that in part (1) we showed that $\mathfrak{m}(\mathfrak{T}'_d) = \mathfrak{m}'$ and thus $\mathfrak{m} = \mathfrak{m}'$ – a contradiction.

(3) If $\mathfrak{m}$ is a MAD subalgebra of $\mathfrak{g}$, the corresponding maximal split torus $\mathfrak{S}(\mathfrak{m})$ is the maximal split torus of the radical of $\mathfrak{H} = Z_{\mathfrak{G}}(R\mathfrak{m})$. The proof of (1) shows that the MAD subalgebra corresponding to $\mathfrak{S}(\mathfrak{m})$ is precisely $\mathfrak{m}$.

Conversely, if $\mathfrak{S}$ is a maximal split torus of $\mathfrak{G}$ then the maximal split torus corresponding to $\mathfrak{m}(\mathfrak{S})$ is the maximal split torus of the radical of the reductive group $Z_{\mathfrak{G}}(R\mathfrak{m}(\mathfrak{S})) = Z_{\mathfrak{G}}(\mathfrak{s}) = Z_{\mathfrak{G}}(\mathfrak{S})$ as explained in the proof of (1). Clearly $\mathfrak{S}$ is inside the radical of $Z_{\mathfrak{G}}(\mathfrak{S})$. Since $\mathfrak{S}$ is maximal split in $\mathfrak{G}$ it is maximal split in the radical of $Z_{\mathfrak{G}}(\mathfrak{S})$. Thus $\mathfrak{S} = \mathfrak{S}'$.

(4) Follows from the construction and functoriality in the definition of the adjoint action at the Lie algebra and group level. $\square$

10. A Cohomological obstruction to conjugacy

In this section R denotes a normal noetherian and integral domain and K its field of quotients.

Let $\mathfrak{G}$ be a reductive group scheme over R. We say that a maximal split torus $\mathfrak{S}$ of $\mathfrak{G}$ is *generically maximal split* if $\mathfrak{S}_K$ is a maximal split torus of $\mathfrak{G}_K$.

10.1. **Proposition.** *Let $\mathfrak{S}$ be a generically maximal split torus of $\mathfrak{G}$. If*

$$H^1_{Zar}\big(R, Z_{\mathfrak{G}}(\mathfrak{S})\big) = 1 \tag{10.1.1}$$

then all generically maximal split tori of $\mathfrak{G}$ are conjugate under $\mathfrak{G}(R)$.

Proof. We begin with a Lemma.

10.2. **Lemma.** *Let $\mathfrak{S}$ and $\mathfrak{S}'$ be generically maximal split tori of $\mathfrak{G}$. Then*

(i) *$\mathfrak{S}_{R_\mathfrak{p}}$ is a maximal split torus of $\mathfrak{G}_{R_\mathfrak{p}}$ for all $\mathfrak{p} \in \mathfrak{X} := \operatorname{Spec}(R)$.*

(ii) *The transporter $\tau_{\mathfrak{S},\mathfrak{S}'} = \mathbf{Trans}_{\mathfrak{G}}(\mathfrak{S}, \mathfrak{S}')$ is a (Zariski) locally trivial $N_{\mathfrak{G}}(\mathfrak{S})$–torsor over R.*

Proof. (i) If $\mathfrak{S}_{R_\mathfrak{p}}$ is not maximal split then neither is $\mathfrak{S}_K = \mathfrak{S}_{R_\mathfrak{p}} \otimes_{\operatorname{Spec}(R_\mathfrak{p})} \operatorname{Spec}(K)$.

(ii) By [SGA3, XI, 6.11 (a)], $\tau_{\mathfrak{S},\mathfrak{S}'}$ is a closed subscheme of $\mathfrak{G}$. It is clearly a right (formal) torsor under the affine R–group $N_{\mathfrak{G}}(\mathfrak{S})$. Since $\mathfrak{S}_{R_\mathfrak{p}}$ and $\mathfrak{S}'_{R_\mathfrak{p}}$ are maximal split tori of $\mathfrak{G}_{R_\mathfrak{p}}$ they are conjugate under $\mathfrak{G}(R_\mathfrak{p})$ by [SGA3, XXVI, 6.16]. Thus $\tau_{\mathfrak{S},\mathfrak{S}'}$ is an $N_{\mathfrak{G}}(\mathfrak{S})$–torsor which is locally trivial (i.e. there exists a Zariski open cover $\mathfrak{X} = \cup\, \mathfrak{X}_i$ such that $\tau_{\mathfrak{S},\mathfrak{S}'}(\mathfrak{X}_i) \neq \emptyset$). $\square$

Proof of the Proposition. Let $\mathfrak{S}'$ be a generically maximal split torus of $\mathfrak{G}$. The transporter $\tau_{\mathfrak{S},\mathfrak{S}'}$ yields according to Lemma 10.2 an element $\alpha \in H^1_{Zar}(R, N_{\mathfrak{G}}(\mathfrak{S}))$. Our aim is to show that α is trivial.

Consider the exact sequence (on $\mathfrak{X}_{ét}$) of R–groups

$$1 \longrightarrow Z_{\mathfrak{G}}(\mathfrak{S}) \longrightarrow N_{\mathfrak{G}}(\mathfrak{S}) \longrightarrow \mathfrak{W} \longrightarrow 1$$

with $\mathfrak{W} = N_{\mathfrak{G}}(\mathfrak{S})/Z_{\mathfrak{G}}(\mathfrak{S})$. Then $\mathfrak{W}$ is a finite étale group over R (see [SGA3, XI, 5.9]). By Lemma 7.7(2) the image of α in $H^1_{ét}(R, \mathfrak{W})$, which we know lies in $H^1_{Zar}(R, \mathfrak{W})$, is trivial. Thus we may assume $\alpha \in H^1_{ét}\big(R, Z_{\mathfrak{G}}(\mathfrak{S})\big)$. To finish the proof we need to show that

$$\alpha \in \operatorname{Im}\big[H^1_{Zar}\big(R, Z_{\mathfrak{G}}(\mathfrak{S})\big) \longrightarrow H^1_{ét}\big(R, Z_{\mathfrak{G}}(\mathfrak{S})\big)\big].$$

For this it suffices to show that the image $\alpha_\mathfrak{p}$ of α in

$$H^1_{ét}\big(R_\mathfrak{p}, Z_{\mathfrak{G}}(\mathfrak{S}) \times_R R_\mathfrak{p}\big) = H^1_{ét}\big(R_\mathfrak{p}, Z_{\mathfrak{G}_{R_\mathfrak{p}}}(\mathfrak{S}_{R_\mathfrak{p}})\big)$$

is trivial for all $\mathfrak{p} \in \mathfrak{X}$.

Since $\mathfrak{S}$ is generically maximal split, $\mathfrak{S}_{R_\mathfrak{p}}$ is a maximal split torus of $\mathfrak{G}_{R_\mathfrak{p}}$. Similarly for $\mathfrak{S}'_{R_\mathfrak{p}}$. Now by [SGA3, XXVI] $\mathfrak{S}_{R_\mathfrak{p}}$ and $\mathfrak{S}'_{R_\mathfrak{p}}$ are conjugate under $\mathfrak{G}_{R_\mathfrak{p}}(R_\mathfrak{p}) = \mathfrak{G}(R_\mathfrak{p})$, Thus the image of α under the composition of the natural maps

$$H^1_{ét}\big(R, N_{\mathfrak{G}}(\mathfrak{S})\big) \longrightarrow H^1_{ét}\big(R_P, N_{\mathfrak{G}_{R_\mathfrak{p}}}(\mathfrak{S}_{R_\mathfrak{p}})\big) \longrightarrow H^1_{ét}(R_\mathfrak{p}, \mathfrak{G}_{R_\mathfrak{p}}) \tag{10.2.1}$$

is trivial.

Let $\mathfrak{P}$ be a parabolic subgroup of $\mathfrak{G}_{R_\mathfrak{p}}$ containing $Z_{\mathfrak{G}_{R_\mathfrak{p}}}(\mathfrak{S}_{R_\mathfrak{p}})$ as a Levi subgroup (see Lemma 5.1). Then (*ibid.*) we have

$$H^1_{ét}(R_\mathfrak{p}, Z_{\mathfrak{G}_{R_\mathfrak{p}}}(\mathfrak{S}_{R_\mathfrak{p}})) \simeq H^1_{ét}(R_\mathfrak{p}, \mathfrak{P}) \hookrightarrow H^1_{ét}(R_\mathfrak{p}, \mathfrak{G}_{R_\mathfrak{p}}).$$

By (10.2.1) it now follows that $\alpha_\mathfrak{p}$ is trivial. □

11. A Counter-example to conjugacy for multiloop algebras

Let $\mathfrak{G}$ and $\mathfrak{g}$ be as in Theorem 9.1. We know that the conjugacy of two MADs in $\mathfrak{g}$ is equivalent to the conjugacy of the corresponding maximal split tori. The following example shows that in general maximal split tori are not necessarily conjugate.

Let D be the quaternion algebra over $R = R_2 = k[t_1^{\pm 1}, t_2^{\pm 1}]$ with generators T_1, T_2 and relations $T_1^2 = t_1$, $T_2^2 = t_2$ and $T_2T_1 = -T_1T_2$ and let $A = M_2(D)$. We may view A as the D-endomorphism algebra of the free right rank 2 module $V = D \oplus D$ over D. Let $\mathfrak{G} = \mathbf{SL}\,(1, A)$. This is a simple simply connected R-group of absolute type $\mathbf{SL}_{4,R}$. It contains a split torus $\mathfrak{S}$ whose R–points are matrices of the form

$$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$$

where $x \in R^\times$. It is well-known that this torus is maximal (one can for example see this by passing to F_2).

Consider now the D-linear map $f : V = D \oplus D \to D$ given by

$$(u, v) \to (1 + T_1)u - (1 + T_2)v.$$

Let $\mathcal{L}$ be its kernel. It is shown in [GP1] that this sequence is split and that $\mathcal{L}$ is a projective D–module of rank 1 which is not free. Since f is split, we have another decomposition $V \simeq \mathcal{L} \oplus D$. Let $\mathfrak{S}'$ be the split torus of $\mathfrak{G}$ whose R–points consist of linear transformations acting on the first summand $\mathcal{L}$ by multiplication $x \in R^\times$ and on the second summand by x^{-1}. As before, $\mathfrak{S}'$ is also a maximal split torus of $\mathfrak{G}$.

We claim that $\mathfrak{S}$ and $\mathfrak{S}'$ are not conjugate under $\mathfrak{G}(R)$. To see this we note that given $\mathfrak{S}$ we can restore the two summands in the decomposition $V = D \oplus D$ as eigenspaces of elements $\mathfrak{S}(R)$. Similarly, we can uniquely restore the two summands in the decomposition $V = \mathcal{L} \oplus D$ out of $\mathfrak{S}'$. Assuming now that $\mathfrak{S}$ and $\mathfrak{S}'$ are conjugate by an element in $\mathfrak{G}(R)$ we obtained immediately that the subspace $\mathcal{L}$ in V is isomorphic to one of the components of $V = D \oplus D$, in particular $\mathcal{L}$ is free – a contradiction.

12. The nullity one case

In this section we look in detail at the case $n = 1$, i.e. $R = k[t^{\pm 1}]$, where k is assumed to be *algebraically closed*. The twisted forms of $\mathbf{g} \otimes_k R$ are nothing but the derived algebra of the affine Kac-Moody Lie algebras modulo their centres [P2].

We maintain all of our previous notation, except for the fact that now we specify that $n = 1$.

12.1. **Lemma.** *Every maximal split torus of* $\mathfrak{G}$ *is generically maximal split.*

Proof. Let $\mathfrak{S}$ be a maximal split torus of our simply connected R–group $\mathfrak{G}$. We must show that $\mathfrak{S}_K$ is a maximal split torus of the algebraic K–group $\mathfrak{G}_K$.

We consider the reductive R-group $\mathfrak{H} = Z_{\mathfrak{G}}(\mathfrak{S})$, its derived (semisimple) group $\mathcal{D}(\mathfrak{H})$ which we denote by $\mathfrak{H}'$, and the radical $\mathrm{rad}(\mathfrak{H})$ of $\mathfrak{H}$.

Recall that $\mathrm{rad}(\mathfrak{H})$ is a central torus of $\mathfrak{H}$ and that we have an exact sequence of R–groups

$$1 \longrightarrow \boldsymbol{\mu} \longrightarrow \mathrm{rad}(\mathfrak{H}) \times_R \mathfrak{H}' \xrightarrow{m} \mathfrak{H} \longrightarrow 1$$

where m is the multiplication of $\mathfrak{H}$ and $\boldsymbol{\mu}$ is a finite group of multiplicative type.

The computation of centralizers, derived groups and radicals are compatible with base change,which gives the exact sequence of algebraic K-groups

$$1 \longrightarrow \boldsymbol{\mu}_K \longrightarrow \mathrm{rad}(\mathfrak{H}_K) \times_K \mathfrak{H}'_K \xrightarrow{m} \mathfrak{H}_K \longrightarrow 1$$

Since $\mathfrak{S}$ is central in $\mathfrak{H}$ it lies inside $\mathrm{rad}(\mathfrak{H})$, hence it is a maximal split torus of $\mathrm{rad}(\mathfrak{H})$. If $\mathfrak{S}_K$ is not a maximal split torus of $\mathfrak{G}_K$, there exists a split torus $\mathfrak{S}'$ of $\mathfrak{H}_K$ such that $\mathfrak{S}'$ is not a subgroup of $\mathrm{rad}(\mathfrak{H}_K)$. Thus if we set $(\mathfrak{S}' \cap \mathfrak{H}'_K)^\circ = \mathfrak{T}$ then $\mathfrak{T}$ is a non-trivial split torus of $\mathfrak{H}'_K$. Then $Z_{\mathfrak{H}'_K}(\mathfrak{T})$ is a Levi subgroup of a proper parabolic subgroup $\mathfrak{P}$ of $\mathfrak{H}'_K$. Let $\mathbf{t} = \mathrm{type}(\mathfrak{P})$ be the type of $\mathfrak{P}$. Let $\mathbf{Par_t}(\mathfrak{H}')$ be the R-scheme of parabolic subgroups of $\mathfrak{H}'$ of type $\mathbf{t}$. Then $\mathbf{Par_t}(\mathfrak{H}')(K) \neq \emptyset$. Since $\mathbf{Par_t}(\mathfrak{H}')$ is proper and R is regular of dimension 1, it follows that $\mathbf{Par_t}(\mathfrak{H}')(R) \neq \emptyset$. Let $\mathfrak{P}'$ be a parabolic subgroup of $\mathfrak{P}'$ of type $\mathbf{t}$. Then $\mathfrak{P}'$ is a proper subgroup, so that by Proposition 7.5 $\mathfrak{P}'$ contains a copy of $\mathbf{G}_{m,R}$. But then $m : \mathfrak{S} \times \mathbf{G}_{m,R} \to \mathfrak{H}$ yields a split torus of $\mathfrak{H}$ that properly contains $\mathfrak{S}$ (since the multiplication map has finite kernel), which contradicts the maximality of $\mathfrak{S}$. $\square$

12.2. **Theorem.** *In nullity one all MADs of* $\mathfrak{g}$ *are conjugate under the adjoint action of* $\mathfrak{G}(R)$.

Proof. In view of the last Lemma and Proposition 10.1 it will suffice to show that if $\mathfrak{S}$ is a maximal split torus of $\mathfrak{G}$, then $H^1_{Zar}(R, Z_{\mathfrak{G}}(\mathfrak{S})) = 1$.

Since $Z_{\mathfrak{G}}(\mathfrak{S})$ is a reductive R-group one in fact has a much stronger result, namely that $H^1_{ét}(R, Z_{\mathfrak{G}}(\mathfrak{S})) = 1$ (see [P2, Theorem 3.1]). $\square$

12.3. **Remark.** Let G be the "simply connected" Kac-Moody (abstract) group corresponding to $\mathfrak{g}$ (see [PK], and also [Kmr] and [MP] for details). We have the adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}_{k-\mathrm{Lie}}(\mathfrak{g})$. The celebrated Peterson-Kac conjugacy theorem [PK] for symmetrizable Kac-Moody (applied to the affine case) asserts that all MADs of $\mathfrak{g}$ are conjugate under the adjoint action of the group $\mathrm{Ad}(G)$ on $\mathfrak{g}$, while our result gives conjugacy under the image of $\mathfrak{G}(R)$, where the image is that of the adjoint representation $\mathrm{Ad} : \mathfrak{G} \to \mathrm{Aut}\,(\mathfrak{g})$ evaluated at R. In the untwisted case it is known that the two groups induce the same group of automorphisms of $\mathfrak{g}$ (see for example [Kmr]). The untwisted case appears to remain unstudied.

13. A density property for points of loop groups

In this section $\mathfrak{X} = \mathrm{Spec}(R_n)$. For a description of $\pi_1(\mathfrak{X}, a)$ see 4.3.

Let $\mathbf{G}$ be a linear algebraic k–group. Let $\eta \in Z^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(\overline{k})\big)$ be a loop cocycle and recall the decomposition $\eta = (\eta^{geo}, z)$ into geometric and arithmetic parts described in Lemma 6.6. Recall that we may view η^{geo} as a k–group homomorphism ${}_\infty\boldsymbol{\mu} \to {}_z\mathbf{G}$. We denote below by $({}_z\mathbf{G})^{\eta^{geo}}$ the centralizer k–subgroup of ${}_z\mathbf{G}$ with respect to the above group homomorphism.

13.1. **Remark.** By continuity there exists m and a Galois extension $\tilde{k}$ of k such that η factors through

$$\eta : \widetilde{\Gamma}_{n,m} \to \mathbf{G}(\tilde{k})$$

where

$$\widetilde{\Gamma}_{n,m} := \mathrm{Gal}(R_{n,m} \otimes_k \tilde{k}/R_n) = \boldsymbol{\mu}_m^n(\tilde{k}) \rtimes \mathrm{Gal}(\widetilde{k}/k)$$

where $m > 0$ and $\widetilde{k}/k$ is a finite Galois extension containing all m–roots of unity in $\overline{k}$. In this way η can be viewed as a Galois cocycle in $Z^1\big(\widetilde{\Gamma}_{n,m}, \mathbf{G}(R_{n,m} \otimes_k \tilde{k})\big)$. We can thus twist $\mathbf{G}_{R_n}$ by η. We call this procedure "reasoning at the finite level".

Recall that an abstract group M is *pro-solvable* if it admits a filtration

$$\cdots \subset M_{n+1} \subset M_n \subset \cdots \subset M_0 = M$$

by normal subgroups such that $\cap M_n = 1$ and M_n/M_{n+1} is abelian for all $n \geq 0$. If there exists a filtration such that M_n/M_{n+1} are k-vector spaces, we say that M is *pro-solvable in k-vector spaces.*

13.2. **Theorem.** *Let $\mathbf{G}$ be a linear algebraic k-group such that $\mathbf{G}^\circ$ is reductive. Let $\eta \in Z^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(\overline{k})\big)$ be a loop cocycle such that the twisted R_n–group $\mathfrak{H} = {}_\eta(\mathbf{G}_{R_n})$ is anisotropic. There exists a family of pro-solvable groups in k-vector spaces $(J_i)_{i=1,..,n}$ such that*

$$\mathfrak{H}(F_n) = J_n \rtimes J_{n-1} \rtimes \ldots \rtimes J_1 \rtimes ({}_z\mathbf{G})^{\eta^{geo}}(k) = (J_n \rtimes J_{n-1} \rtimes \ldots \rtimes J_1) \cdot \mathfrak{H}(R_n).$$

Proof. We may assume by twisting by z that z is trivial. We note that since $\mathfrak{H}$ is anisotropic the algebraic F_n–group $\mathfrak{H}_{F_n}$ is also anisotropic by [GP3] cor. 7.4.3. It is convenient to work at a finite level, namely with a cocycle

$$\eta : \widetilde{\Gamma}_{n,m} \to \mathbf{G}(\tilde{k})$$

as in Remark 13.1. We may also assume that $\tilde{k}$ splits a given chosen maximal torus of the k–group $\mathbf{G}^\circ$.

We proceed by induction on $n \geq 0$; the case $n = 0$ being obvious. We reason by means of a building argument and we view $\widetilde{F}_{n,m}$ and its subfield

$F_n = (\widetilde{F}_{n,m})^{\widetilde{\Gamma}_{n,m}}$ as local complete fields with the residue fields $\widetilde{F}_{n-1,m}$ and F_{n-1} respectively.

We consider the (enlarged) Bruhat-Tits building $\mathcal{B}_n = \mathcal{B}(\mathbf{G}_{\widetilde{F}_{n,m}})$ of the $\widetilde{F}_{n,m}$–group $\mathbf{G}_{\widetilde{F}_{n,m}}$ [Ti, §2.1]. It is equipped with a natural action of $\mathbf{G}(\widetilde{F}_{n,m}) \rtimes \widetilde{\Gamma}_{n,m}$. It is shown in [GP3] Theorem 7.9 that the building of $\mathfrak{H}_{F_n}$ inside $\mathcal{B}_n$ consists of a single point ϕ whose stabilizer is $\mathbf{G}\big(\widetilde{F}_{n-1,m}[[t_n^{\frac{1}{m}}]]\big)$. Since $\mathfrak{H}(F_n)$ stabilizes ϕ it follows that

$$(13.2.1) \qquad \mathfrak{H}(F_n) = \Big\{ g \in \mathbf{G}\big(\widetilde{F}_{n-1,m}[[t_n^{\frac{1}{m}}]]\big) \ \mid \ \eta(\sigma)\, \sigma(g) = g \ \ \forall \sigma \in \widetilde{\Gamma}_{n,m} \Big\}.$$

We next decompose $\boldsymbol{\mu}_m^n = \boldsymbol{\mu}_m^{n-1} \times \boldsymbol{\mu}_m$. Here the second component is a finite k-group of multiplicative type and it acts on $\mathbf{G}$ via η^{geo}. We let $\mathbf{G}_{n-1}$ denote the k–subgroup of $\mathbf{G}$ which is the centralizer of this action [DG, II 1.3.7]. The connected component of the identity of $\mathbf{G}_{n-1}$ is reductive according to [Ri]. Since the action of $\boldsymbol{\mu}_m^{n-1}$ on $\mathbf{G}$ given by η^{geo} commutes with that of $\boldsymbol{\mu}_m$ the k–group morphism $\eta^{geo} : \boldsymbol{\mu}_m^n \to \mathbf{G}$ factors through $\mathbf{G}_{n-1}$.

Denote by η^{geo}_{n-1} the restriction of η^{geo} to the k–subgroup $\boldsymbol{\mu}_m^{n-1}$ of $\boldsymbol{\mu}_m^n$. Set $\tilde{\Gamma}_{n-1,m} := \boldsymbol{\mu}_m^{n-1}(\tilde{k}) \rtimes \mathrm{Gal}(\tilde{k}/k)$ and consider the loop cocycle

$$\eta_{n-1} : \tilde{\Gamma}_{n-1,m} \to \mathbf{G}_{n-1}(\tilde{k})$$

attached to $(1, \eta^{geo}_{n-1})$. We define

$$\mathfrak{H}_{n-1,R_{n-1}} = {}_{\eta_{n-1}}(\mathbf{G}_{n-1,R_{n-1}}).$$

The crucial point for the induction argument is the fact that $\eta^{geo}_{n-1} : \boldsymbol{\mu}_m^{n-1} \to \mathbf{G}_{n-1}$ is anisotropic so that the twisted F_{n-1}–group ${}_{\eta_{n-1}}\mathbf{G}_{n-1}$ is anisotropic. This is established just as in [GP3, theo. 7.9]. We look now at the specialization map

$$sp_n : \mathfrak{H}(F_n) \hookrightarrow \mathbf{G}\big(\widetilde{F}_{n-1,m}[[t_n^{\frac{1}{m}}]]\big) \to \mathbf{G}(\widetilde{F}_{n-1,m}).$$

Let P be the parahoric subgroup of $\mathfrak{H}^\circ(F_n)$ attached to the point ϕ. Since the building of $\mathfrak{H}_{F_n}$ consists of the single point we have $P = \mathfrak{H}^\circ(F_n)$.

Recall that the notation P^* stands for the "pro-unipotent radical" of P as defined in §18.4 of the Appendix.

13.3. **Claim.** *We have* $P^* = \ker(sp_n)$ *and the image of* sp_n *is* $\mathfrak{H}_{n-1}(F_{n-1})$.

Because $\mathbf{G}$ is a k–group it is clear that the kernel of the specialization map $\mathbf{G}\big(\widetilde{F}_{n-1,m}[[t_n^{\frac{1}{m}}]]\big) \to \mathbf{G}(\widetilde{F}_{n-1,m})$ is contained in $\mathbf{G}^\circ\big(\widetilde{F}_{n-1,m}[[t_n^{\frac{1}{m}}]]\big)$. Since $(\mathfrak{H}/\mathfrak{H}^\circ)(F_n)$ injects into $(\mathfrak{H}/\mathfrak{H}^\circ)(\widetilde{F}_{n,m}) = (\mathbf{G}/\mathbf{G}^\circ)(\widetilde{F}_{n,m})$, the kernel of the specialization map sp_n is the same for $\mathfrak{H}^\circ(F_n)$ and $\mathfrak{H}(F_n)$. The parahoric subgroup of $\mathbf{G}^\circ(\widetilde{F}_{n,m})$ attached to the point ϕ is $Q = \mathbf{G}^\circ\Big(\widetilde{F}_{n-1,m}[[t_n^{\frac{1}{m}}]]\Big)$ and we have

$$Q^* = \ker\big(Q \to \mathbf{G}^\circ(\widetilde{F}_{n-1,m})\big)$$

by the very definition of Q^*. Hence $\ker(sp_n) = P \cap Q^* = P^*$ by Corollary 18.9 applied to the point ϕ.

The group $\mathfrak{H}_{n-1}(F_{n-1})$ is a subgroup of $\mathfrak{H}_n(F_n)$ which maps identically to itself by sp_n, so we have to verify that the specialization h_{n-1} of an element $h \in \mathfrak{H}(F_n)$ belongs to $\mathfrak{H}_{n-1}(F_{n-1})$. Specializing (13.2.1) at $t_n = 0$, we get

$$\eta(\gamma)\ {}^{\gamma}h_{n-1} = h_{n-1} \quad \forall \gamma \in \widetilde{\Gamma}_{n,m}. \tag{13.3.1}$$

We now apply the relation (13.3.1) to the generator τ_n of the Galois group $\mathrm{Gal}\big(\widetilde{F}_{n,m}/\widetilde{F}_{n-1,m}((t_n))\big)$; it yields

$$\eta(\tau_n)\, h_{n-1} = h_{n-1}, \tag{13.3.2}$$

where $\eta(\tau_n) \in \mathbf{G}(\widetilde{k})$, so that $h_{n-1} \in \mathbf{G}_{n-1}(\tilde{F}_{n-1,m})$. Furthermore, the equality (13.3.1) restricted to $\tilde{\Gamma}_{n-1,m}$ shows that $h_{n-1} \in \mathfrak{H}_{n-1}(F_{n-1})$. This establishes the Claim.

We can now finish the induction process. The group $\mathfrak{H}_{n-1}(F_{n-1})$ is a subgroup of $\mathfrak{H}(F_n)$, so

$$\mathfrak{H}(F_n) = J_n \rtimes \mathfrak{H}_{n-1}(F_{n-1})$$

where $J_n := \ker(sp_n)$ is the "pro-unipotent radical" and hence it is pro-solvable in k–spaces. By using the induction hypothesis, we have

$$\mathfrak{H}_{n-1}(F_{n-1}) = (J_{n-1} \rtimes \cdots \rtimes J_1) \rtimes \mathbf{G}_{n-1}^{\eta_{n-1}^{geo}}(k).$$

Since $\mathbf{G}_{n-1}^{\eta_{n-1}^{geo}} = \mathbf{G}^{\eta^{geo}}$, we conclude that

$$\mathfrak{H}(F_n) = (J_n \rtimes \cdots \rtimes J_1) \rtimes \mathbf{G}^{\eta^{geo}}(k)$$

as desired.

We have $\mathbf{G}^{\eta^{geo}}(k) \subset \mathfrak{H}(R_n)$, so we get the second identity as well. $\square$

14. Acyclicity, I

Let $\mathfrak{H}$ be a loop reductive group scheme. We will denote by $H^1_{toral}(R_n, \mathfrak{H})$ [resp. $H^1_{toral}(R_n, \mathfrak{H})_{irr}$] the subset of $H^1(R_n, \mathfrak{H})$ consisting of isomorphism classes of torsors $\mathfrak{E}$ such that the twisted R_n–group ${}_{\mathfrak{E}}\mathfrak{H}$ admits a maximal torus [resp. admits a maximal torus and is irreducible].

14.1. **Theorem.** *Let $\mathfrak{H}$ be a loop reductive group scheme. Then the natural map*

$$H^1_{toral}(R_n, \mathfrak{H})_{irr} \to H^1(F_n, \mathfrak{H}).$$

is injective.

Proof. By twisting, it is enough to show that for an irreducible loop reductive group $\mathfrak{H}$ the canonical map $H^1_{toral}(R_n, \mathfrak{H}) \to H^1(F_n, \mathfrak{H})$ has trivial kernel. Indeed reductive R_n–group schemes admitting a maximal torus are precisely the loop reductive groups [GP3, Theorem 6.1]. We now reason by succesive cases.

Case 1: *$\mathfrak{H}$ is adjoint and anisotropic.* We may view $\mathfrak{H}$ as a twisted form of a Chevalley group scheme $\mathbf{H}_{R_n}$ by a loop cocycle $\eta : \pi_1(R_n) \to \mathrm{Aut}(\mathbf{H})(\overline{k})$. We have the following commutative diagram of torsion bijections

$$\begin{tikzcd} H^1_{toral}\big(R_n, \mathbf{Aut}(\mathfrak{H})\big) \arrow[r] \arrow[d, "\tau_\eta"', "\simeq"] & H^1\big(F_n, \mathbf{Aut}(\mathfrak{H})\big) \arrow[d, "\tau_\eta"', "\simeq"] \\ H^1_{toral}\big(R_n, \mathbf{Aut}(\mathbf{H})\big) \arrow[r] & H^1\big(F_n, \mathbf{Aut}(\mathbf{H})\big). \end{tikzcd}$$

The vertical maps are bijective by [Gir, III 2.5.4] and Remark 6.9, while the bottom map is bijective by [GP3, theorem 8.1]. We thus have a bijection

$$\psi : H^1_{toral}\big(R_n, \mathbf{Aut}(\mathfrak{H})\big) \xrightarrow{\sim} H^1(F_n, \mathbf{Aut}(\mathfrak{H})).$$

The exact sequence $1 \to \mathfrak{H} \to \mathbf{Aut}(\mathfrak{H}) \to \mathbf{Out}(\mathfrak{H}) \to 1$ gives rise to the commutative diagram of exact sequence of pointed sets

$$\begin{tikzcd} \mathbf{Aut}(\mathfrak{H})(R_n) \arrow[r, "\delta"] \arrow[d] & \mathbf{Out}(\mathfrak{H})(R_n) \arrow[r, "\varphi"] \arrow[d, equal] & H^1_{\acute{e}t}(R_n, \mathfrak{H}) \arrow[r] \arrow[d] & H^1_{\acute{e}t}\big(R_n, \mathbf{Aut}(\mathfrak{H})\big) \arrow[d, "\psi"] \\ \mathbf{Aut}(\mathfrak{H})(F_n) \arrow[r, "\gamma"] & \mathbf{Out}(\mathfrak{H})(F_n) \arrow[r] & H^1(F_n, \mathfrak{H}) \arrow[r] & H^1\big(F_n, \mathbf{Aut}(\mathfrak{H})\big). \end{tikzcd}$$

Let $v \in H^1_{\acute{e}t}(R_n, \mathfrak{H})$ be a toral class mapping to $1 \in H^1(F_n, \mathfrak{H})$. Since ψ is bijective there exists $u \in \mathbf{Out}(\mathfrak{H})(R_n)$ such that $v = \varphi(u)$ and $u \in \mathrm{Im}\,\gamma$. Since $\mathbf{Out}(\mathfrak{H})(R_n)$ is a finite group, the Density Theorem 13.2 shows that $\mathbf{Aut}(\mathfrak{H})(R_n)$ and $\mathbf{Aut}(\mathfrak{H})(F_n)$ have the same image in $\mathbf{Out}(\mathfrak{H})(F_n)$. So $u \in \mathrm{Im}\,\delta$, which implies that $\gamma = 1 \in H^1_{\acute{e}t}(R_n, \mathfrak{H})$.

Case 2: $\mathfrak{H}$ is irreducible. Set $\mathfrak{Z} = Z(\mathfrak{H})$; it is an R_n–group of multiplicative type and we have an exact sequence of R_n–group schemes

$$1 \to \mathfrak{Z} \xrightarrow{i} \mathfrak{H} \to \mathfrak{H}_{ad} \to 1.$$

Here the adjoint group $\mathfrak{H}_{ad}$ is anisotropic since $\mathfrak{H}$ is irreducible. This exact sequence gives rise to the diagram

$$\begin{tikzcd} \mathfrak{H}_{ad}(R_n) \arrow[r, "\varphi_{R_n}"] \arrow[d] & H^1_{\acute{e}t}(R_n, \mathfrak{Z}) \arrow[r, "i_*"] \arrow[d, "\simeq"] & H^1_{\acute{e}t}(R_n, \mathfrak{H}) \arrow[r] \arrow[d] & H^1_{\acute{e}t}(R_n, \mathfrak{H}_{ad}) \arrow[d] \\ \mathfrak{H}_{ad}(F_n) \arrow[r, "\varphi_{F_n}"] & H^1(F_n, \mathfrak{Z}) \arrow[r] & H^1(F_n, \mathfrak{H}) \arrow[r] & H^1(F_n, \mathfrak{H}_{ad}). \end{tikzcd}$$

Note that the second vertical map is bijective by [GP2, Prop. 3.4.(3)] since $\mathfrak{Z}$ is of finite type ([SGA3, XII, §3]).

Let $v \in H^1_{\acute{e}t}(R_n, \mathfrak{H})$ be a toral class mapping to $1 \in H^1(F_n, \mathfrak{H})$. Taking into account the adjoint anisotropic case, a diagram chase provides an element $u \in H^1_{\acute{e}t}(R_n, \mathfrak{Z})$ such that $v = i_*(u)$ and u belongs to the image of the characteristic map φ_{F_n}. Since $H^1_{\acute{e}t}(R_n, \mathfrak{Z})$ is an abelian torsion group, the Density Theorem 13.2 shows that $\mathfrak{H}_{ad}(F_n)$ and $\mathfrak{H}_{ad}(R_n)$ have the same image in $H^1_{\acute{e}t}(R_n, \mathfrak{Z})$. So u belongs to the image of φ_{R_n}, and this implies that $v = i_*(u) = 1 \in H^1_{\acute{e}t}(R_n, \mathfrak{H})$ as desired. $\square$

15. Conjugacy of certain parabolic subgroup schemes and maximal split tori

15.1. **Theorem.** *Let $\mathfrak{H}$ be a loop reductive group scheme over R_n. Then there exists a unique $\mathfrak{H}(R_n)$-conjugacy class of*

(a) Couples $(\mathfrak{L}, \mathfrak{P})$ where $\mathfrak{P}$ is a minimal parabolic R_n–subgroup scheme of $\mathfrak{H}$ and $\mathfrak{L}$ is a Levi subgroup of $\mathfrak{P}$ such that $\mathfrak{L}$ is a loop reductive group scheme.

(b) Maximal split subtorus $\mathfrak{S}$ of $\mathfrak{H}$ such that $Z_{\mathfrak{H}}(\mathfrak{S})$ is a loop reductive group scheme.

15.2. **Remark.** The counter-example in §11 shows that the assumption that $\mathfrak{L}$ and $Z_{\mathfrak{H}}(\mathfrak{S})$ be loop reductive group schemes is not superflous.

15.3. **Lemma.** *The proof of Theorem 15.1.a) reduces to the semisimple simply connected case.*

Proof. Denote by $\mathfrak{H}^{sc}$ the simply connected covering of the derived group scheme of $\mathfrak{H}$, and denote by $\mathfrak{E}$ the radical torus of $\mathfrak{H}$. We assume that Theorem 15.1.a) holds for $\mathfrak{H}^{sc}$. There is a canonical central isogeny [H, §1.2]

$$1 \to \mathfrak{C} \to \mathfrak{H}^{sc} \times \mathfrak{E} \xrightarrow{f} \mathfrak{H} \to 1.$$

Let $(\mathfrak{L}, \mathfrak{P})$ where $\mathfrak{P}$ is a parabolic subgroup of $\mathfrak{H}$ containing a Levi subgroup $\mathfrak{L}$. Then

$$f^{-1}(\mathfrak{P}) = \mathfrak{P}^{sc} \times \mathfrak{E}, f^{-1}(\mathfrak{L}) = \mathfrak{L}^{sc} \times \mathfrak{E}$$

where $\mathfrak{P}^{sc}$ is a minimal parabolic subgroup of the R_n–group $\mathfrak{H}^{sc}$ and $\mathfrak{L}^{sc}$ is a Levi subgroup of $\mathfrak{P}^{sc}$. Similarly for $(\mathfrak{L}', \mathfrak{P}')$. Conversely, from a couple $(\mathfrak{M}, \mathfrak{Q})$ for $\mathfrak{H}^{sc}$, we can define a couple $\big((\mathfrak{M} \times \mathfrak{E})/\mathfrak{C}, (\mathfrak{Q} \times \mathfrak{E})/\mathfrak{C}\big)$ for $\mathfrak{H}$. By [GP3, cor. 6.3], loop group schemes are exactly those carrying a maximal torus. Since this last property is insensitive to central extensions [SGA3, XII.4.7], the correspondence described above exchanges loop objects $\mathfrak{L}$ with loop objects $\mathfrak{L}^{sc}$. Also it exchanges minimal parabolics of $\mathfrak{H}$ with minimal parabolics of $\mathfrak{H}^{sc}$.

Existence: By the simply connected case, we know that there exists a couple $(\mathfrak{M}, \mathfrak{Q})$ for $\mathfrak{H}^{sc}$ such that $\mathfrak{Q}$ is a minimal parabolic subgroup of $\mathfrak{H}^{sc}$ and $\mathfrak{M}$ is a Levi subgroup which is loop reductive. Then $(\mathfrak{L}, \mathfrak{P}) := \big((\mathfrak{M} \times \mathfrak{E})/\mathfrak{C}, (\mathfrak{Q} \times \mathfrak{E})/\mathfrak{C}\big)$ is as desired for $\mathfrak{H}$.

Conjugacy: We are given another couple $(\mathfrak{L}', \mathfrak{P}')$. Since $(\mathfrak{L}')^{sc}$ is loop reductive, the simply connected case yields that $(\mathfrak{L}^{sc}, \mathfrak{P}^{sc})$ and $(\mathfrak{L}'^{sc}, \mathfrak{P}'^{sc})$ are $\mathfrak{H}^{sc}(R_n)$–conjugate. By applying f, we conclude that $(\mathfrak{L}, \mathfrak{P})$ and $(\mathfrak{L}', \mathfrak{P}')$ are $\mathfrak{H}(R_n)$–conjugate. □

Proof of Theorem 15.1

15.4. **Existence.** (a) Lemma 15.3 enables to assume that $\mathfrak{H}$ is semisimple simply connected. There exists a split semisimple simply connected k–group $\mathbf{H}$ (namely the Chevalley k–form of $\mathfrak{H}$) and a loop cocycle $\eta : \pi_1(R_n) \to \mathbf{Aut}(\mathbf{H})(\overline{k})$ such that $\mathfrak{H} = {}_\eta(\mathbf{H}_{R_n})$. Let $(\mathbf{T}, \mathbf{B})$ be a Killing couple of $\mathbf{H}$ and $\Pi \subset \Delta(\mathbf{H}, \mathbf{T})$ is the base of the finite root system associated to $(\mathbf{T}, \mathbf{B})$. We denote by $\mathbf{H}_{ad}$ the adjoint group of $\mathbf{H}$ and by $(\mathbf{T}_{ad}, \mathbf{B}_{ad})$ the corresponding Killing couple. We have $\mathbf{Aut}(\mathbf{H}) = \mathrm{Aut}(\mathbf{H_{ad}})$.

For each $I \subset \Delta$, we have the standard parabolic subgroup $\mathbf{P}_I$ of $\mathbf{H}$ and its Levi subgroup $\mathbf{L}_I$, as well as $\mathbf{P}_{I,ad}$ and $\mathbf{L}_{I,ad}$ for $\mathbf{H}_{ad}$.

Let I be the subset of uncircled vertices in the Witt-Tits diagram of $\mathfrak{H}_{F_n}$. The version of the "Witt-Tits decomposition" given in [GP3, Cor. 8.4] applied to $\mathrm{Aut}(\mathbf{H}_{ad})$ shows that

$$[\eta] \in \mathrm{Im}\Big(H^1_{loop}\big(R_n, \mathbf{Aut}(\mathbf{H}_{ad}, \mathbf{P}_{I,ad}, \mathbf{L}_{I,ad})\big)_{irr} \to H^1_{loop}\big(R_n, \mathbf{Aut}(\mathbf{H}_{ad})\big)\Big).$$

Thus we may assume that η has values in $\mathbf{Aut}(\mathbf{H}, \mathbf{P}_I, \mathbf{L}_I)(\overline{k}) = \mathbf{Aut}(\mathbf{H}_{ad}, \mathbf{P}_{I,ad}, \mathbf{L}_{I,ad})(\overline{k})$. The twisted R_n–group schemes $\mathfrak{P} = {}_\eta(\mathbf{P}_I)$ and $\mathfrak{L} = {}_\eta(\mathbf{L}_I)$ are as desired since $\mathfrak{P}_{F_n}$ is a minimal F_n–parabolic subgroup of $\mathfrak{H}_{F_n}$ by the definition of the Witt-Tits index.

(b) Let $\mathfrak{S}$ be the maximal split subtorus of the radical of $\mathfrak{L}$, with $\mathfrak{L}$ as in (a) By Proposition 7.3 we have $Z_{\mathfrak{H}}(\mathfrak{S}) = \mathfrak{L}$. This implies that $Z_{\mathfrak{H}}(\mathfrak{S})$ is a loop reductive group. Now $\mathfrak{S}$ is a maximal split torus of $\mathfrak{H}$ (hence has both required properties) because it is so over F_n. To see this note that by our construction the semisimple part of the centralizer of $\mathfrak{S}$ is anisotropic over F_n. So if $\mathfrak{S} \subset \mathfrak{S}'$ is a proper inclusion over F_n, then $\mathfrak{S}'$ sits inside the radical, say $\mathfrak{T}$, of the centralizer of $\mathfrak{S}$. But by Lemma 7.4, the torus $\mathfrak{S}$ is still maximal split in $\mathfrak{T}$ over K_n and hence over F_n because $\mathfrak{T}$ is split over a Galois extension $\widetilde{R}_{n,m}/R_n$ for some integer m.

15.5. **Conjugacy.** Let $(\mathfrak{L}, \mathfrak{P})$ be a couple as prescribed by Theorem 15.1. Recall that $\mathfrak{S}$ is the maximal split subtorus of the radical of $\mathfrak{L}$.

(a) Let $(\mathfrak{M}, \mathfrak{Q})$ be another couple (satisfying the same conditions). Consider the R_n–scheme $\mathfrak{Y} = \mathfrak{H}/\mathfrak{P}$ of parabolic subgroups of type $\mathbf{t}(\mathfrak{P})$ [SGA3, XXVI]. The exact sequence of étale R_n–sheaves $1 \to \mathfrak{P} \to \mathfrak{H} \xrightarrow{f} \mathfrak{Y} \to 1$ induces exact sequences of pointed sets [Gir, III.3.2.2]

$$\begin{array}{ccccccc}
\mathfrak{H}(R_n) & \xrightarrow{\psi} & \mathfrak{Y}(R_n) & \xrightarrow{\varphi} & H^1_{\acute{e}t}(R_n, \mathfrak{P}) & \longrightarrow & H^1_{\acute{e}t}(R_n, \mathfrak{H}) \\
 & & & & \Big\uparrow{\simeq} & & \\
 & & & & H^1_{\acute{e}t}(R_n, \mathfrak{L}) & &
\end{array}$$

(note that a natural mapping $H^1_{ét}(R_n, \mathfrak{L}) \to H^1_{ét}(R_n, \mathfrak{P})$ is a bijection by [SGA3, XXVI, 3.2]) and by base change

$$\begin{array}{ccccccc} \mathfrak{H}(F_n) & \xrightarrow{\psi_{F_n}} & \mathfrak{Y}(F_n) & \xrightarrow{\varphi_{F_n}} & H^1(F_n, \mathfrak{P}) & \longrightarrow & H^1(F_n, \mathfrak{H}) \\ & & & & \uparrow \simeq & & \\ & & & & H^1(F_n, \mathfrak{L}) & & \end{array}$$

The R_n–parabolic subgroup $\mathfrak{Q}$ defines a point $y \in \mathfrak{Y}(R_n)$.

15.6. **Claim.** $\varphi(y) \in H^1_{toral}(R_n, \mathfrak{L}) \simeq H^1_{toral}(R_n, \mathfrak{P})$.

Indeed $\varphi(y)$ is the class of the $\mathfrak{P}$–torsor $\mathfrak{E} := f^{-1}(y)$. We can assume without loss of generality that $\mathfrak{E}$ is obtained from an $\mathfrak{L}$–torsor $\mathfrak{F}$. Then $\mathfrak{Q}$ is isomorphic[13] to the twist ${}_{\mathfrak{F}}\mathfrak{P}$, and ${}_{\mathfrak{F}}\mathfrak{L}$ is a Levi subgroup of the R_n–group ${}_{\mathfrak{F}}\mathfrak{P}$. Since Levi subgroups of ${}_{\mathfrak{F}}\mathfrak{P}$ are conjugate under $R_u({}_{\mathfrak{F}}\mathfrak{P})(R_n)$ [SGA3, XXVI, 1.8], it follows that ${}_{\mathfrak{F}}\mathfrak{L}$ is R_n–isomorphic to $\mathfrak{M}$. The group scheme ${}_{\mathfrak{F}}\mathfrak{L}$ carries then a maximal torus and the claim is proved.

On the other hand, we have already observed before in §15.4 that $\mathfrak{P}_{F_n}$ and $\mathfrak{Q}_{F_n}$ are minimal parabolic subgroups of $\mathfrak{H}_{F_n}$, hence they are conjugate under $\mathfrak{H}(F_n)$. In other words, y viewed as an element of $\mathfrak{Y}(F_n)$ is in the image of ψ_{F_n}, hence $\varphi_{F_n}(y) = 1$. It follows that $\varphi(y)$ belongs to the kernel of

$$H^1_{toral}(R_n, \mathfrak{L})_{irr} \to H^1(F_n, \mathfrak{L})$$

which is trivial by Theorem 14.1. Thus $y \in \operatorname{Im} \psi$, i.e. $\mathfrak{P}$ and $\mathfrak{Q}$ are $\mathfrak{H}(R_n)$–conjugate and so are the couples $(\mathfrak{L}, \mathfrak{P})$ and $(\mathfrak{M}, \mathfrak{Q})$.

(b) Let $\mathfrak{S}'$ be a maximal split subtorus of $\mathfrak{H}$ such that its centralizer $\mathfrak{L}' = Z_{\mathfrak{H}}(\mathfrak{S}')$ is a loop reductive group scheme. By Lemma 5.1, $Z_{\mathfrak{H}}(\mathfrak{S}')$ is a Levi subgroup of a parabolic subgroup of $\mathfrak{P}'$ of $\mathfrak{H}$. By Proposition 5.4.c, $\mathfrak{P}'$ is a minimal parabolic subgroup of $\mathfrak{H}$. By (1), the couple $(\mathfrak{L}', \mathfrak{P}')$ is conjugate under $\mathfrak{H}(R_n)$ to $(\mathfrak{L}, \mathfrak{P})$. We may thus assume that $\mathfrak{L} = \mathfrak{L}'$, i.e. $Z_{\mathfrak{H}}(\mathfrak{S}) = Z_{\mathfrak{H}}(\mathfrak{S}')$. It follows $\mathfrak{S}'$ is a central split subtorus of $\mathfrak{L}$, hence that $\mathfrak{S}' \subset \mathfrak{S}$. But $\mathfrak{S}'$ is a maximal split subtorus of $\mathfrak{H}$, so we conclude that $\mathfrak{S} = \mathfrak{S}'$ as desired.

We record further properties of the couples considered in the Theorem.

15.7. **Corollary.** *Let $(\mathfrak{P}, \mathfrak{L})$ be as in Theorem 15.1.*

(1) If $\mathfrak{S}$ is the maximal split central subtorus of $\mathfrak{L}$, $\mathfrak{S}$ is maximal in $\mathfrak{H}$ and $\mathfrak{L} = Z_{\mathfrak{H}}(\mathfrak{S})$.

(2) $\mathfrak{P}_{F_n}$ is a minimal parabolic subgroup of $\mathfrak{H}_{F_n}$.

(3) $Z_{\mathfrak{H}}(\mathfrak{S})_{F_n}$ is irreducible,

(4) $(Z_{\mathfrak{H}}(\mathfrak{S})/\mathfrak{S})_{F_n}$ is anisotropic.

(5) $\mathfrak{S}_{F_n}$ is maximal split in $\mathfrak{H}_{F_n}$.

[13] Surprisingly enough, this compatibility is not in Giraud's book. A proof can be found in [De, lemme 4.2.33 page 175].

Proof. (1) This is established in the proof in §15.4.b).

(2) Since $\mathfrak{H}$ is a loop reductive, this follows from [GP3, cor. 7.4.(1)].

(3) The R_n–group $Z_{\mathfrak{H}}(\mathfrak{S})$ is irreducible by 5.4.a and it is also loop reductive. The statement follows again from [GP3, cor. 7.4.(1)].

(4) The R_n–group $Z_{\mathfrak{H}}(\mathfrak{S})/\mathfrak{S}$ is anisotropic by 5.4.c. The statement follows then from [GP3, cor. 7.4.(3)].

(5) The key observation is that $\mathfrak{S}$ is a maximal torus of $\mathfrak{H}$ hence equal to the maximal central split subtorus of $Z_{\mathfrak{H}}(\mathfrak{S})$. Proposition 5.4.a applied to the anisotropic group $(Z_{\mathfrak{H}}(\mathfrak{S})/\mathfrak{S})_{F_n}$ shows that $\mathfrak{S}_{F_n}$ is a maximal R_n–torus of $\mathfrak{H}_{F_n}$.

□

16. Applications to infinite dimensional Lie Theory

This section should be considered as an outline of the main application of our conjugacy theorem to infinite dimensional Lie theory. A detailed version of these results, as well as others, will be made available in a forthcoming paper intended for specialists in the area.

***

Throughout this section we assume that k is algebraically closed of characteristic zero. $\mathbf{G}$ will denote a simple simply connected Chevalley group over k, and $\mathbf{g}$ its Lie algebra. We fix integers $n \geq 0$, $m > 0$ and an n–tuple $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ of commuting elements of $\mathrm{Aut}_k(\mathbf{g})$ satisfying $\sigma_i^m = 1$. Let

$$R = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \ \textit{and} \ \widetilde{R} = k[t_1^{\pm \frac{1}{m}}, \dots, t_n^{\pm \frac{1}{m}}].$$

Recall that $\widetilde{R}/R$ is Galois and that we can identify $\mathrm{Gal}(\widetilde{R}/R)$ with $(\mathbb{Z}/m\mathbb{Z})^n$ via our choice of compatible roots of unity. Recall also from the Introduction the multiloop algebra based on $\mathbf{g}$ corresponding to $\boldsymbol{\sigma}$, is

$$L(\mathbf{g}, \boldsymbol{\sigma}) = \bigoplus_{(i_1, \dots, i_n) \in \mathbb{Z}^n} \mathbf{g}_{i_1 \dots i_n} \otimes t_1^{\frac{i_1}{m}} \dots t_n^{\frac{i_n}{m}} \subset \mathbf{g} \otimes_k \widetilde{R}$$

It is a twisted form of the R–Lie algebra $\mathbf{g} \otimes_k R$ which is split by $\widetilde{R}$:

$$L(\mathbf{g}, \boldsymbol{\sigma}) \otimes_R \widetilde{R} \simeq \mathbf{g} \otimes_k \widetilde{R} \simeq (\mathbf{g} \otimes_k R) \otimes_R \widetilde{R}.$$

The $\widetilde{R}/R$ form $L(\mathbf{g}, \boldsymbol{\sigma})$ is given by a natural loop cocycle (denoted by z in [GP1, §5])

$$\eta = \eta(\boldsymbol{\sigma}) \in Z^1\big(\Gamma, \mathbf{Aut}(\mathbf{g})(k)\big) \subset Z^1\big(\Gamma, \mathbf{Aut}(\mathbf{g})(\widetilde{R})\big).$$

Since $\mathbf{Aut}(\mathbf{g}) \simeq \mathbf{Aut}(\mathbf{G})$ we can also consider by means of η the twisted R–group $\mathfrak{G} = {}_\eta\mathbf{G}_R$. As before we denote the Lie algebra of $\mathfrak{G}$ by $\mathfrak{g}$. It is well known, and in any case easy to verify (see for example the proof of [GP1, prop 4.10]) that the determination of Lie algebras commutes with the twisting process. Thus

$$\mathfrak{g} \simeq L(\mathbf{g}, \boldsymbol{\sigma})$$

16.1. **Borel-Mostow MADS.** By a Theorem of Borel and Mostow [BM] there exists a Cartan subalgebra $\mathbf{h}$ of $\mathbf{g}$ that is stable under the action of $\boldsymbol{\sigma}$ (by which we mean that each of the σ_i stabilizes $\mathbf{h}$). By restricting $\boldsymbol{\sigma}$ to $\mathbf{h}$ we can consider the loop algebra based on $\mathbf{h}$ with respect to $\boldsymbol{\sigma}$,

$$L(\mathbf{h},\boldsymbol{\sigma}) = \bigoplus_{(i_1,\dots,i_n)\in\mathbb{Z}^n} \mathbf{h}_{i_1\dots i_n} \otimes t_1^{\frac{i_1}{m}} \dots t_n^{\frac{i_n}{m}} \subset \mathbf{h} \otimes_k \widetilde{R}$$

Let $\mathbf{T}$ be the maximal torus of $\mathbf{G}$ corresponding to $\mathbf{h}$. Denote by $\mathbf{T}^{\boldsymbol{\sigma}}$ (resp. $\mathbf{h}^{\boldsymbol{\sigma}}$) the fixed point subgroup of $\mathbf{T}$ (resp. subalgebra of $\mathbf{h}$) under $\boldsymbol{\sigma}$, i.e the elements of $\mathbf{T}$ (resp. $\mathbf{h}$) that are fixed by each of the σ_i. Since the torus $\mathbf{T}$ is also $\boldsymbol{\sigma}$–stable and, just as above, we can consider its twisted form $\mathfrak{T} = {}_\eta\mathbf{T}_R$ and corresponding Lie algebra $\mathfrak{h} = {}_\eta\mathbf{h}_R$. The same formalism already mentioned yields that

$$\mathfrak{h} \simeq L(\mathbf{h},\boldsymbol{\sigma}).$$

It is not difficult to see that $\mathfrak{h}$ is a direct summand of $\mathfrak{g}$ as an R–module whose geometric fibers $\mathfrak{h}(\overline{x})$ are Cartan subalgebras of $\mathfrak{g}(\overline{x})$ for all $x \in \mathfrak{X} = \operatorname{Spec}(R)$. Thus $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ in the sense of [SGA3].

Let $\mathfrak{T}_d$ be the maximal split torus of $\mathfrak{T}$. It is easy to see that

$$\mathfrak{T}_d \simeq \mathbf{T}^{\boldsymbol{\sigma}}_R = {}_\eta(\mathbf{T}^{\boldsymbol{\sigma}}_R) \subset \mathfrak{G} = {}_\eta\mathbf{G}_R.$$

According to Remark 8.4 its Lie algebra $\mathfrak{t}_d$ contains a unique maximal subalgebra $\mathfrak{m}$ which is an AD–subalgebra of $\mathfrak{g}$. The description of this algebra is quite simple:

$$\mathfrak{m} = \mathbf{h}_{0,\dots,0} \otimes_k 1 \subset L(\mathbf{g},\boldsymbol{\sigma}) \simeq \mathfrak{g}.$$

Note that $\mathfrak{m}$ is precisely $\mathbf{h}^{\boldsymbol{\sigma}} \otimes_k 1$.

By Theorem 9.1 $\mathfrak{m}$ is a MAD if and only if $\mathfrak{T}_d$ is a maximal split torus of $\mathfrak{G}$, in which case $\mathfrak{m} = \mathfrak{m}(\mathfrak{T}_d)$. We will call MADs of a multiloop algebra which are of this form *Borel Mostow* MADs of $\mathfrak{g}$.

Let $\Delta_{\boldsymbol{\sigma}} = \Delta_{\boldsymbol{\sigma}}(\mathbf{g},\mathbf{h})$ be the subset of $\Delta(\mathbf{g},\mathbf{h})$ consisting of those roots that vanish on $\mathbf{h}^{\boldsymbol{\sigma}} = \mathbf{h}_{0,\dots,0}$. We have

$$Z_{\mathbf{g}}(\mathbf{h}^{\boldsymbol{\sigma}}) = \mathbf{h} \bigoplus \bigoplus_{\alpha\in\Delta_{\boldsymbol{\sigma}}} \mathbf{g}^{\alpha}.$$

Because $\mathfrak{m} \subset \mathbf{g}_{0,\dots,0}$ it is straighforward to verify that $Z_{\mathfrak{g}}(\mathfrak{m})$ is precisely the multiloop algebra $L\big(Z_{\mathbf{g}}(\mathbf{h}^{\boldsymbol{\sigma}}),\boldsymbol{\sigma}\big)$. Note that $Z_{\mathbf{g}}(\mathbf{h}^{\boldsymbol{\sigma}})$ is the Lie algebra of the reductive k–group $\mathbf{H} := Z_{\mathbf{G}}(\mathbf{h}^{\boldsymbol{\sigma}}) = Z_{\mathbf{G}}(\mathbf{T}^{\boldsymbol{\sigma}})$ and hence by twisting we conclude that $Z_{\mathfrak{g}}(\mathfrak{m})$ is the Lie algebra of

$$Z_{\mathfrak{G}}(\mathfrak{T}_d) = Z_{\mathfrak{G}}(\mathbf{T}^{\boldsymbol{\sigma}}_R) \simeq {}_\eta\mathbf{H}_R.$$

16.2. **Proposition.** *(1) $Z_{\mathfrak{G}}(\mathfrak{T}_d)$ is a loop reductive group.*

(2) $\mathfrak{m}$ is a MAD if and only if the dimension of $\mathbf{h}_{0,\dots,0}$ is maximal among the Cartan subalgebras of $\mathbf{g}$ normalised by $\boldsymbol{\sigma}$. In particular, Borel-Mostow MADs exist.

Proof. (1) We have explained above that $Z_{\mathfrak{G}}(\mathfrak{T}_d) \simeq {}_{\eta}\mathbf{H}_R$. This last group is loop reductive by definition since η is a loop cocycle.
(2) We know that all MADs in $\mathfrak{g}$ and all maximal split tori in $\mathfrak{G}$ have the same dimension, say r. Since $\mathfrak{m}$ is an AD–subalgebra we have $\dim_k \mathfrak{m} = \dim_k(\mathbf{h}_{0,\dots,0}) \leq r$ and hence $\mathfrak{m}$ is a MAD if and only if $\dim_k(\mathbf{h}_{0,\dots,0}) = r$. It is then enough to show that there exists a Borel-Mostow AD of rank r, that is we need to find a Cartan subalgebra $\mathbf{h}'$ of $\mathbf{g}$ normalized by $\boldsymbol{\sigma}$ such that $\dim_k(\mathbf{h}'_{0,\dots,0}) = r$.

Up to conjugating by an element of $\mathbf{G}(k)$, we can assume that $\boldsymbol{\sigma}$ normalises in an irreducible way a standard parabolic group $\mathbf{P}_I$ [GP1, 3.4] and also the standard Levi subgroup $\mathbf{L}_I$ by complete reductivity [Mt]. Then the twist ${}_z(\mathbf{P}_I)_R$ is still minimal parabolic of $\mathfrak{G}$ over the field F_n. It follows that if $\mathbf{S}$ is the torus consisting of the fixed point subgroup of the radical of $\mathbf{L}_I$ under $\boldsymbol{\sigma}$ then $\mathbf{S}_R \hookrightarrow {}_z(\mathbf{L}_R) \subset {}_z\mathbf{G}_R$ is the maximal split torus in the radical of ${}_z(\mathbf{L}_I)_R$ and hence a maximal split torus in $\mathfrak{G}$; in particular $\dim_k \mathbf{S} = r$.

Let $\mathfrak{s}$ be the Lie algebra of $\mathbf{S}$. We have $\dim_k \mathfrak{s} = \dim_k \mathbf{S} = r$ and by our construction $\boldsymbol{\sigma}$ acts trivially on $\mathfrak{s}$. The reductive subalgebra $Z_{\mathbf{g}}(\mathfrak{s})$ is stable under $\boldsymbol{\sigma}$, so the application of Borel-Mostow's theorem provides a Cartan subalgebra $\mathbf{h}'$ of $Z_{\mathbf{g}}(\mathfrak{s})$ stable under $\boldsymbol{\sigma}$.

Now we have a Lie algebras $\mathfrak{s}$ over k of dimension r which is contained in $\mathbf{h}'$ and is inside of $\mathfrak{s}' = (\mathbf{h}')^{\sigma}$. If $\mathfrak{s}'$ is strictly larger than $\mathfrak{s}$ then it gives rise to a MAD of dimension $> r$, which is impossible. Thus $\mathfrak{s}$ is the subalgebra of $\mathbf{h}'$ consisting of the fixed point under $\boldsymbol{\sigma}$. $\square$

According to our Conjugacy Theorem all Borel-Mostow MADs of a multiloop algebra are conjugate under $\mathfrak{G}(R)$. There is a very important class of multiloop algebras, the so called Lie tori, where Borel-Mostow MADs play a crucial role. We now turn our attention to them.[14]

16.3. **Applications to EALAs.** As in [AABGP] and [N1] it will be convenient for us to work with root systems that contain 0. So by a *finite irreducible root system* we will mean a finite subset Δ of a finite dimensional vector space V over k such that $0 \in \Delta$ and $\Delta^{\times} := \Delta \setminus \{0\}$ is a finite irreducible root system in V in the usual sense; see [Bbk1, chap. VI, §1, déf. 1]. Note that $\Delta^{\times}$ is not assumed to be reduced.

We will use the following notation for the root system Δ. Let

$$Q = Q(\Delta) := \Sigma_{\alpha \in \Delta} \mathbb{Z}\alpha \subset V$$

be the *root lattice* of Δ. Let $\langle \ , \ \rangle : V \times V^* \to k$ denote the natural pairing of V with V^*. If $\alpha \in \Delta^{\times}$, $\alpha^{\vee}$ will denote the *coroot* of α in V^*; that is, $\alpha^{\vee}$ is the unique element of V^* such that $\langle \alpha, \alpha^{\vee} \rangle = 2$ and the map $r_{\alpha} : \beta \to \beta - \langle \beta, \alpha^{\vee} \rangle \alpha$ stabilizes Δ.

[14]Lie tori were introduced by Y. Yoshii [Y1, Y2] and further studied by E. Neher in [N1, N2]. The terminology is consistent with that of tori in the theory of non-associative algebras, e.g. Jordan tori. But in the presence of algebraic groups, where tori are well defined objects, the terminology is a bit unfortunate.

Let

$$\Delta_{ind}^{\times} := \{\alpha \in \Delta^{\times} : \frac{1}{2}\alpha \notin \Delta\}$$

denote the set of *indivisible* roots in Δ, and let

$$\Delta_{ind} := \Delta_{ind}^{\times} \cup \{0\}.$$

Lie tori are Lie algebras that are graded by groups of the form $Q \times \Lambda$ and satisfy some additional axioms that we are going to recall. For our purpose we may assume that $\Lambda \simeq \mathbb{Z}^n$ for some $n \geq 0$ called the *nullity* of the Lie torus.

Before proceeding with the definition and basic properties of Lie tori let us recall some basic notation on $Q \times \Lambda$-graded algebras.

Let $\mathcal{L} = \oplus_{(\alpha,\lambda)\in Q\times\Lambda} \mathcal{L}_{\alpha}^{\lambda}$ be a $Q \times \Lambda$-graded algebra. Then $\mathcal{L} = \oplus_{\lambda\in\Lambda}\mathcal{L}^{\lambda}$ is Λ-graded and $\mathcal{L} = \oplus_{\alpha\in Q}\mathcal{L}_{\alpha}$ is Q-graded algebra, with

$$\mathcal{L}^{\lambda} = \oplus_{\alpha\in Q}\mathcal{L}_{\alpha}^{\lambda} \text{ for } \lambda \in \Lambda \quad \text{and} \quad \mathcal{L}_{\alpha} = \oplus_{\lambda\in\Lambda}\mathcal{L}_{\alpha}^{\lambda} \text{ for } \alpha \in Q,$$

and we have $\mathcal{L}_{\alpha}^{\lambda} = \mathcal{L}_{\alpha} \cap \mathcal{L}^{\lambda}$. Conversely if $\mathcal{L}$ has a Q-grading and a Λ-grading that are compatible (which means that each $\mathcal{L}_{\alpha}$ is a Λ-graded subspace of $\mathcal{L}$ or equivalently that each $\mathcal{L}^{\lambda}$ is a Q-graded subspace of $\mathcal{L}$), then $\mathcal{L}$ is $Q \times \Lambda$-graded with $\mathcal{L}_{\alpha}^{\lambda} = \mathcal{L}_{\alpha} \cap \mathcal{L}^{\lambda}$. Hence, a $Q \times \Lambda$-graded algebra $\mathcal{L}$ has three different associated support sets, namely the $Q \times \Lambda$-support, the Q-support and the Λ-support denoted respectively by $\mathrm{supp}_{Q\times\Lambda}(\mathcal{L})$, $\mathrm{supp}_{Q}(\mathcal{L})$ and $\mathrm{supp}_{\Lambda}(\mathcal{L})$. Of course a degree belongs to one of these supports if the corresponding homogeneous subspace is non-zero.

We now recall the standard definition of a Lie torus. Afterwards we will adapt it to our purposes.

16.4. **Definition.** *A* Lie Λ-torus of (relative) type Δ *is a $Q \times \Lambda$-graded Lie algebra $\mathcal{L}$ over k which (with the notation as above) satisfies:*

(LT1) $\mathrm{supp}_{Q}(\mathcal{L}) = \Delta$.

(LT2)

(i) $(\Delta_{ind}^{\times}, 0) \subset \mathrm{supp}_{Q\times\Lambda}(\mathcal{L})$.

(ii) *If $(\alpha, \lambda) \in \mathrm{supp}_{Q\times\Lambda}(\mathcal{L})$ and $\alpha \neq 0$, then there exist elements $e_{\alpha}^{\lambda} \in \mathcal{L}_{\alpha}^{\lambda}$ and $f_{\alpha}^{\lambda} \in \mathcal{L}_{-\alpha}^{-\lambda}$ such that*

$$\mathcal{L}_{\alpha}^{\lambda} = k e_{\alpha}^{\lambda}, \quad \mathcal{L}_{-\alpha}^{-\lambda} = k f_{\alpha}^{\lambda},$$

and

$$[[e_{\alpha}^{\lambda}, f_{\alpha}^{\lambda}], x_{\beta}] = \langle \beta, \alpha^{\vee} \rangle x_{\beta} \tag{16.4.1}$$

for $x_{\beta} \in \mathcal{L}_{\beta}$, $\beta \in Q$.

(LT3) *$\mathcal{L}$ is generated as an algebra by the spaces $\mathcal{L}_{\alpha}$, $\alpha \in \Delta^{\times}$.*

(LT4) $\langle \mathrm{supp}_{\Lambda}(\mathcal{L}) \rangle = \Lambda$.

Q is the root grading group *and we call the Q-grading of $\mathcal{L}$ the root grading. Similarly, Λ is the* external grading group *and we call the Λ-grading of $\mathcal{L}$ the external grading.*

16.5. **Definition.** *Let $\mathcal{L}$ be a Lie Λ-torus of type Δ. Following* [N1]*, we define the* root grading pair *for $\mathcal{L}$ to be the pair* $(\mathbf{s}, \mathbf{t})$ *of subalgebras of $\mathcal{L}$ where* $\mathbf{s}$ *is the subalgebra of $\mathcal{L}$ generated by $\{\mathcal{L}_\alpha^0\}_{\alpha\in\Delta^\times}$ and $\mathbf{t} = \sum_{\alpha\in\Delta^\times}[\mathcal{L}_\alpha^0, \mathcal{L}_{-\alpha}^0]$. Notice that $\mathbf{t}$ is a subalgebra of $\mathbf{s}$, $\mathbf{s} \subseteq \mathcal{L}^\circ$ and $\mathbf{t} \subseteq \mathcal{L}_0^0$. We will see below in* (16.6.2) *that the subalgebra $\mathbf{t}$ completely determines the root grading of $\mathcal{L}$.*

The next proposition summarizes the basic properties of the root grading pair of a Lie torus. Parts (i)–(v) were announced in [N1, §3]. See [ABFP2] for details.

16.6. **Proposition.** *Let $\mathcal{L}$ be a* centreless *Lie Λ-torus of type Δ with root grading pair* $(\mathbf{s}, \mathbf{t})$*.*

(i) *If $\alpha \in \Delta_{ind}^\times$, then $\Lambda_{2\alpha}^0 = 0$.*

(ii) $\mathbf{s}$ *is a finite dimensional split simple Lie algebra and $\mathbf{t}$ is a split Cartan subalgebra of* $\mathbf{s}$*.*

(iii) *There is a unique linear isomorphism $\alpha \to \tilde{\alpha}$ of V onto $\mathbf{t}^*$ such that $\widetilde{\Delta_{ind}} = \Delta(\mathbf{s}, \mathbf{t})$ and*

$$[e_\alpha^0, f_\alpha^0] = \tilde{\alpha}^\vee \tag{16.6.1}$$

for $\alpha \in \Delta_{ind}^\times$. Here $\tilde{\alpha}^\vee \in (\mathbf{t}^*)^* \simeq \mathbf{t}$.

(iv) *If $\alpha \in Q$ then*

$$\mathcal{L}_\alpha = \{x \in \mathcal{L} : [h, x] = \tilde{\alpha}(h)x \text{ for } h \in \mathbf{t}\}. \tag{16.6.2}$$

(v) *If $(\alpha, \lambda) \in \mathrm{supp}_{Q\times\Lambda}(\mathcal{L})$ and $\alpha \in \Delta^\times$ then*

$$[e_\alpha^\lambda, f_\alpha^\lambda] = \tilde{\alpha}^\vee \tag{16.6.3}$$

(vi) *We have $\mathcal{L}^0 = \mathbf{s}$ and $\mathcal{L}_0^0 = \mathbf{t}$.* □

16.7. **Remark.** We see that a Lie torus is really two things, more precisely a pair $\mathcal{L} = (L, D)$ consisting of a k–Lie algebra L together with an "external root data" D comprised of a finite irreducible root system Δ and a $Q \times \Lambda$–grading on L satisfying certain axioms. This is reminiscent of the concept of "epinglage" in the finite dimensional case.

The concept of isomorphism also deserves some comments. On the one hand we have isomorphisms of Lie tori as Lie algebras. On the other we have an obvious concept of isomorphism of what we've called external root data, and we can define isomorphism that preserve this information (they are called bi-graded isomorphism in [ABFP2]). This is the strongest form of isomorphism, for it preserves the structure of the full root system. There is an intermediate concept, that of isotopy. The interested reader can look up §2 of [ABFP2] for details.

For our purpose we want to observe that given an automorphism θ of L we can construct a new data, call it D_θ, in an obvious way so that $\mathcal{L}_\theta = (L, D_\theta)$ is a Lie torus. Δ, Q, Λ do not change, nor do the supports. As the reader may have guessed, all we do is to set $(\mathcal{L}_\theta)^\lambda_\alpha = \theta(\mathcal{L}^\lambda_\alpha)$, and use the elements $\theta(e^\lambda_\alpha)$ and $\theta(f^\lambda_\alpha)$ to satisfy axiom LT2(ii).

By (vi) of the last Proposition we see that the grading pair of $\mathcal{L}_\theta$ is $\big(\theta(\mathbf{s}), \theta(\mathbf{t})\big)$. Clearly the isomorphism $\theta : \mathcal{L} \to \mathcal{L}_\theta$ preserves the external root data.

The relevance of centreless Lie tori is that they sit at the "bottom" of every EALA (see [AABGP], [N1] and [N2]). A good example is provided by the affine Kac-Moody Lie algebras. They are of the form (see [Kac])

$$\mathcal{E} = \mathcal{L} \oplus kc \oplus kd$$

where $\mathcal{L}$ is a loop algebra of the form $L(\mathbf{g}, \pi)$ for some (unique) $\mathbf{g}$ and some (unique up to conjugacy) diagram automorphism π of $\mathbf{g}$. The element c is central and d is a degree derivation for a natural grading of $\mathcal{L}$. If $\mathbf{h}$ is the standard Chevalley split Cartan subalgebra of $\mathbf{g}$, then $\mathcal{H} = \mathbf{h}^\pi + kc + kd$ plays the role of the Cartan subalgebra for $\mathcal{E}$.

The infinite dimensional Lie algebra $\mathcal{E}$ admits an invariant non-degenerate bilinear form whose restriction to $\mathcal{H}$ is non-degenerate. With respect to $\mathcal{H}$ our algebra $\mathcal{E}$ admits a root space decomposition. The roots are of two types: anisotropic (real) or isotropic (imaginary). This terminology comes from transferring the form to $\mathcal{H}^*$ and computing the "length" of the roots.

The *core* of $\mathcal{E}$ is the subalgebra generated by all the anisotropic roots. The correct way to recover $\mathcal{L}$ inside $\mathcal{E}$ is as its core (which is $\mathcal{L} \oplus kc$) modulo its centre (which is kc).[15]

It is known that EALAs of nullity 1 are precisely the affine Kac-Moody Lie algebras [ABGP]. Neher has shown that this "pattern" of realizations holds in all nullities. Loosely speaking an EALA is always of the form

$$\mathcal{E} = \mathcal{L} \oplus \mathcal{C} \oplus \mathcal{D}$$

where $\mathcal{L}$ is a Lie torus, $\mathcal{C}$ is central and $\mathcal{D}$ is a space of derivations. Given $\mathcal{L}$, the recipes for all possibles $\mathcal{C}$ is central and $\mathcal{D}$ are completely understood. Thus many questions about EALAs (e.g. their classification) come down to analogous questions about Lie tori.

With the above in mind as motivation, we now return to our paper. The definition of Lie tori has as a central ingredient the relative type Δ, and the double grading by $Q \times \Lambda$ with its corresponding root grading pair $(\mathbf{s}, \mathbf{t})$. On the other hand $\mathcal{L}$ is a Lie algebra, and it is natural to ask (and essential for the classification) whether Δ and the root grading pairs are *invariants* of $\mathcal{L}$. In other words.

Question: Let $\mathcal{L}$ be a centreless Lie Λ-torus of type Δ with root grading pair $(\mathbf{s}, \mathbf{t})$. Assume that $\mathcal{L}$ admits the structure of a centreless Lie Λ'-torus of type Δ' with root grading pair $(\mathbf{s}', \mathbf{t}')$. Is $\Lambda \simeq \Lambda'$, $\Delta \simeq \Delta'$, $\mathbf{s} \simeq \mathbf{s}'$ and $\mathbf{t} \simeq \mathbf{t}'$?

[15] In nullity one the core coincides with the derived algebra, but this is not necessarilty true in higher nullities.

We shall see as an application of our Conjugacy Theorem that the answer to this question is affirmative whenever $\mathcal{L}$ is finitely generated as a module over its centroid, which we henceforth assume.[16]

That $\Lambda \simeq \Lambda'$ is an assertion about the nullity being an invariant of $\mathcal{L}$. It is well know that this is true. For example, the nullity is the transcendence degree of the field of quotients of the centroid of $\mathcal{L}$ [ABP2.5, cor.6.4].

To address the remaining questions we turn to the realization theorem of [ABFP2]. The finiteness assumption and the realization theorem tells us that the Lie tori are multiloop algebras.[17] This is the key that allows us to bring conjugacy into the picture.

The data (Δ, Λ) leads to a Lie algebra isomorphism $\phi : \mathcal{L} \to L(\mathbf{g}, \boldsymbol{\sigma})$. Similarly we have $\phi' : \mathcal{L} \to L(\mathbf{g}', \boldsymbol{\sigma}')$. By descent considerations it is clear that $\mathbf{g} \simeq \mathbf{g}'$: This is the Invariance of the Absolute Type. See for example [GP1, theo. 4.9] or [ABP2.5, theo 8.16.]. The explicit nature of the Realization Theorem shows that ϕ maps $(\mathbf{s}, \mathbf{t})$ to $(\mathbf{g}^{\boldsymbol{\sigma}}, \mathbf{h}_{\mathbf{t}})$ where $\mathbf{h}_{\mathbf{t}}$ is a Cartan subalgebra of the *simple* Lie algebra $\mathbf{g}^{\boldsymbol{\sigma}}$. Furthermore $\mathbf{h}_{\mathbf{t}}$ is a Borel-Mostow MAD $\mathbf{h}^{\boldsymbol{\sigma}}$ of $\mathbf{g}^{\boldsymbol{\sigma}}$. Similar considerations apply to (Δ', Λ').

We now use $\phi \circ \phi'^{-1}$ to put two Λ Lie tori structures of relative type Δ and Δ' on $L = L(\mathbf{g}, \boldsymbol{\sigma})$ (see Remark 16.7). The original structure has root grading pair $(\mathbf{s}, \mathbf{t})$ where $\mathbf{s} = \mathbf{g}^{\boldsymbol{\sigma}}$ and $\mathbf{t}$ is a Borel-Mostow MAD. About the second pair $(\mathbf{s}'', \mathbf{t}'')$ what we do know is $\mathbf{s}' \simeq \mathbf{s}''$, $\mathbf{t}' \simeq \mathbf{t}''$ and that $Z_{\mathfrak{G}}(\mathbf{t}'')$ is a loop reductive group. By the Conjugacy Theorem the exists an element $\theta \in \mathfrak{G}(R)$ such that $\theta(\mathbf{t}') = \mathbf{t}$. This leads to yet another Lie Λ torus of type Δ' on L with grading pairs $(\mathbf{s}''', \mathbf{t})$ with $\mathbf{s}''' \simeq \mathbf{s}''$. The root system of $(\mathbf{s}''', \mathbf{t})$ is of type Δ' while that of $(\mathbf{s}, \mathbf{t})$ is of type Δ. Since $\mathbf{t}$ determines the Q–grading we conclude that $\Delta \simeq \Delta'$, hence that $\mathbf{s} \simeq \mathbf{s}'$. This completes the proof that our Question has an affirmative answer.

As a consequence we see that the (relative) type Δ is an invariant of a Lie torus. The spirit of this result should be interpreted as the analogue that on $\mathbf{g}$ we cannot choose two different Cartan subalgebras that will lead to root systems of different type. More generally, it is the analogue of the fact that the relative type of a finite dimensional simple Lie algebra (in characteristic 0) or of a simple algebraic group is an invariant of the algebra or group in question.

The invariance of the relative type was established in [Als] by using strictly methods from EALA theory. Allison also showed that under the assumption that conjugacy (as established in this paper) holds, any isotopy between Lie tori necessarily preserves the external root data information. This is a very important result for the theory of EALAs for, together with conjugacy, it yields a very precise description of the group of automorphisms of Lie tori.

---

[16] The exceptions occur only on type A and they are completely classified.

[17] The converse is false in nullity > 1.

17. Acyclicity, II

17.1. **Theorem.** *Let $\mathfrak{H}$ be a loop reductive group scheme over R_n. Then the natural map*

$$H^1_{toral}(R_n, \mathfrak{H}) \to H^1(F_n, \mathfrak{H}).$$

is bijective.

17.2. **Remark.** The theorem generalizes (in characteristic 0) our main result in [CGP]. Indeed, in that paper we showed that if $n = 1$ and $\mathbf{G}$ is a reductive group over an arbitrary field k of good characteristic then $H^1_{ét}(R_1, \mathbf{G}) \to H^1(F_1, \mathbf{G})$ is bijective and that every $\mathbf{G}$-torsor is toral. The Theorem also generalizes the Acyclicity result of [GP3], which is used in the present proof and covers the case when $\mathfrak{H}$ is "constant".

The proof of the theorem is based on the following statement which generalizes the Density Theorem 13.2 to the case of arbitrary loop reductive group schemes, not necessary anisotropic.

17.3. **Theorem.** *Let $\mathbf{H}$ be a linear algebraic k–group whose connect component of the identity is reductive. Let $\eta : \pi_1(R_n) \to \mathbf{H}(\overline{k})$ be a loop cocycle and consider the loop reductive R_n–groups $\mathfrak{H} = {}_\eta\mathbf{H}_{R_n}$ and $\mathfrak{H}^\circ = {}_\eta\mathbf{H}^\circ{}_{R_n}$. Let $(\mathfrak{P}, \mathfrak{L})$ be a couple given by Theorem 15.1 for $\mathfrak{H}^\circ$. Then there exists a normal subgroup J of $\mathfrak{L}(F_n)$ which is a quotient of a group admitting a composition serie whose quotients are pro-solvable groups in k–vector spaces such that*

$$\mathfrak{H}(F_n) = \Big\langle \, \mathfrak{H}(R_n), \; J, \; \mathfrak{H}(F_n)^+ \Big\rangle$$

where $\mathfrak{H}(F_n)^+$ stands for the normal subgroup of $\mathfrak{H}(F_n)$ generated by one parameter additive F_n–subgroups.

17.4. **Remark.** If $\mathfrak{H}$ is semisimple simply connected, isotropic and F_n-simple, we know that $\mathfrak{H}(F_n)/\mathfrak{H}(F_n)^+ \cong \mathfrak{H}(F_n)/R$ [G, 7.2] so that the group $\mathfrak{H}(F_n)/\mathfrak{H}(F_n)^+$ has finite exponent (*ibid*, 7.6). In this case, the decomposition reads $\mathfrak{H}(F_n) = \Big\langle \, \mathfrak{H}(R_n), \, \mathfrak{H}(F_n)^+ \Big\rangle$.

Proof. Case (1)*: $\mathfrak{H}$ is a torus $\mathfrak{T}$.* We leave it to the reader to reason by induction on n to establish the case of a split torus $\mathbf{T} = \mathbf{G}_m^n$ (the case $n = 1$ follows from the identity $F_1^\times = R_1^\times \,.\, \ker(k[[t_1]]^\times \to k^\times)$. Since all finite connected étale coverings of R_n are also Laurent polynomial rings over field extensions of k [GP3, Lemma 2.8] and the statement is stable under products, the theorem also holds for induced tori.

Let $\mathfrak{T}$ be an arbitrary torus. Since $\mathfrak{T}$ is isotrivial, it is a quotient of an induced torus $\mathfrak{E}$. We have then an exact sequence

$$1 \to \mathfrak{S} \xrightarrow{i} \mathfrak{E} \xrightarrow{f} \mathfrak{T} \to 1$$

of multiplicative R_n–group schemes. It gives rise to a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \mathfrak{S}(R_n) \arrow[r, "i_{R_n}"] \arrow[d] & \mathfrak{E}(R_n) \arrow[r, "f_{R_n}"] \arrow[d] & \mathfrak{T}(R_n) \arrow[r, "\varphi_{R_n}"] \arrow[d] & H^1_{\acute{e}t}(R_n, \mathfrak{S}) \arrow[r] \arrow[d, "\simeq"'] & 1 \\ 1 \arrow[r] & \mathfrak{S}(F_n) \arrow[r, "i_{F_n}"] & \mathfrak{E}(F_n) \arrow[r, "f_{F_n}"] & \mathfrak{T}(F_n) \arrow[r, "\varphi_{F_n}"] & H^1(F_n, \mathfrak{S}) \arrow[r] & 1 \end{tikzcd}$$

with exact rows. Note that the right vertical map is an isomorphism by [GP2, Prop. 3.4] and that surjectivity on the right horizontal maps is due to the fact $H^1_{\acute{e}t}(R_n, \mathfrak{E}) = H^1(F_n, \mathfrak{E}) = 1$. By diagram chasing we see that

$$\mathfrak{T}(R_n)/f_{F_n}\big(\mathfrak{E}(R_n)\big) \xrightarrow{\sim} \mathfrak{T}(F_n)/f_{R_n}\big(\mathfrak{E}(F_n)\big).$$

Therefore the case of the induced torus $\mathfrak{E}$ provides a suitable group J such that $\mathfrak{T}(F_n) = \mathfrak{T}(R_n) \cdot f_{F_n}(J)$.

Case (2)*: $\mathfrak{H} = \mathfrak{L}$ is irreducible.* Let $\mathfrak{C}$ be the radical torus of $\mathfrak{L}$. We have an exact sequence [SGA3, XXI, 6.2.4]

$$1 \longrightarrow \boldsymbol{\mu} \xrightarrow{\ i\ } \mathcal{D}\mathfrak{L} \times_{R_n} \mathfrak{C} \xrightarrow{\ f\ } \mathfrak{L} \longrightarrow 1.$$

It gives rise to a commutative diagram of exact sequences of pointed sets

$$\begin{tikzcd} (\mathcal{D}\mathfrak{L} \times \mathfrak{C})(R_n) \arrow[r, "f_{R_n}"] \arrow[d] & \mathfrak{L}(R_n) \arrow[r, "\varphi_{R_n}"] \arrow[d] & H^1_{\acute{e}t}(R_n, \boldsymbol{\mu}) \arrow[r, "i_{*,R_n}"] \arrow[d, "\simeq"'] & H^1_{loop}(R_n, \mathcal{D}\mathfrak{L} \times \mathfrak{C}) \arrow[d] \\ (\mathcal{D}\mathfrak{L} \times \mathfrak{C})(F_n) \arrow[r, "f_{F_n}"] & \mathfrak{L}(F_n) \arrow[r, "\varphi_{F_n}"] & H^1(F_n, \boldsymbol{\mu}) \arrow[r, "i_{*,F_n}"] & H^1(F_n, \mathcal{D}\mathfrak{L} \times \mathfrak{C}). \end{tikzcd}$$

Note that the image of the map $H^1_{\acute{e}t}(R_n, \boldsymbol{\mu}) \to H^1_{\acute{e}t}(R_n, \mathcal{D}\mathfrak{L})$ is contained in $H^1_{toral}(R_n, \mathcal{D}\mathfrak{L})_{irr}$. So taking into consideration Theorem 14.1 (applied to the irreducible loop reductive group scheme $\mathcal{D}\mathfrak{L}$ and chasing the above diagram we see that

$$\mathfrak{L}(R_n)/f_{R_n}\big((\mathcal{D}\mathfrak{L})(R_n) \times \mathfrak{C}(R_n)\big) \xrightarrow{\sim} \mathfrak{L}(F_n)/f_{F_n}\big((\mathcal{D}\mathfrak{L})(F_n) \times \mathfrak{C}(F_n)\big).$$

The case of $\mathcal{D}\mathfrak{L}$ done in Proposition 13.2 together with the case of the torus $\mathfrak{C}$ provide a suitable normal group J such that $\mathfrak{L}(F_n) = \mathfrak{L}(R_n) \cdot J$.

Case (3). $\mathfrak{H} = \mathfrak{H}^\circ$. Since $\mathfrak{H}$ is loop reductive by assumption it suffices to observe that $\mathfrak{H}(F_n)$ is generated by $\mathfrak{L}(F_n)$ and $\mathfrak{H}^+(F_n)$ [BT73, 6.11].

Case (4). For the general case it remains to show that for an arbitrary element $g \in \mathfrak{H}(F_n)$ the coset $g\mathfrak{H}^\circ(F_n)$ contains at least one R_n–point of $\mathfrak{H}$.

Let $\mathfrak{S}$ be the maximal split torus of the radical of $\mathfrak{L}$. The torus $g\mathfrak{S}_{F_n}g^{-1} \subset \mathfrak{H}^\circ_{F_n}$ is maximal split, hence there is $g_1 \in \mathfrak{H}^\circ(F_n)$ such that $g\mathfrak{S}_{F_n}g^{-1} = g_1\mathfrak{S}_{F_n}g_1^{-1}$. Thus replacing g by $g_1^{-1}g$ if necessary, we may assume that $g\mathfrak{S}_{F_n}g^{-1} = \mathfrak{S}_{F_n}$. Then we also have $g(\mathfrak{L}_{F_n})g^{-1} = \mathfrak{L}_{F_n}$, so that $g \in N_{\mathfrak{H}}(\mathfrak{L})(F_n)$.

The torus $\mathfrak{S}$ is clearly normal in $N_{\mathfrak{H}}(\mathfrak{L})$. Hence we have an exact sequence

$$1 \longrightarrow \mathfrak{S} \longrightarrow N_{\mathfrak{H}}(\mathfrak{L}) \longrightarrow \mathfrak{H}' := N_{\mathfrak{H}}(\mathfrak{L})/\mathfrak{S} \longrightarrow 1.$$

Note that since $H^1_{\acute{e}t}(R_n, \mathfrak{S}) = 1$, the natural maps $N_{\mathfrak{H}}(\mathfrak{L})(R_n) \to \mathfrak{H}'(R_n)$ and $N_{\mathfrak{H}}(\mathfrak{L})(F_n) \to \mathfrak{H}'(F_n)$ are surjective. Furthermore, $\mathfrak{H}'$ satisfies all conditions of Theorem 13.2, so that the required fact follows immediately from that theorem applied to $\mathfrak{H}'$ and from the surjectivity of the above maps. $\square$

We can proceed to the proof of Theorem 17.1 which is very similar to that of Theorem 14.1.

Proof. Injectivity: By twisting, it is enough to show that the natural map $H^1_{toral}(R_n, \mathfrak{H}) \to H^1(F_n, \mathfrak{H})$ has trivial kernel.
We first assume that $\mathfrak{H}$ is adjoint. We may view $\mathfrak{H}$ as the twisted form of a Chevalley group scheme $\mathbf{H}_{R_n}$ by a loop cocycle $\eta : \pi_1(R_n) \to \mathrm{Aut}\big(\mathbf{H}(\overline{k})\big)$. The same reasoning given in Case 1 of the proof of Theorem 14.1 shows that we have a natural bijection

$$H^1_{toral}\big(R_n, \mathbf{Aut}(\mathfrak{H})\big) \xrightarrow{\sim} H^1\big(F_n, \mathbf{Aut}(\mathfrak{H})\big). \tag{17.4.1}$$

The exact sequence

$$1 \to \mathfrak{H} \to \mathbf{Aut}(\mathfrak{H}) \to \mathbf{Out}(\mathfrak{H}) \to 1$$

gives rise to a commutative diagram of exact sequence of pointed sets

$$\begin{array}{ccccccc}
\mathbf{Aut}(\mathfrak{H})(R_n) & \xrightarrow{\gamma} & \mathbf{Out}(\mathfrak{H})(R_n) & \xrightarrow{\varphi} & H^1_{\acute{e}t}(R_n, \mathfrak{H}) & \longrightarrow & H^1_{\acute{e}t}(R_n, \mathbf{Aut}(\mathfrak{H})) \\
\downarrow & & || & & \downarrow & & \downarrow \\
\mathbf{Aut}(\mathfrak{H})(F_n) & \xrightarrow{\psi} & \mathbf{Out}(\mathfrak{H})(F_n) & \longrightarrow & H^1(F_n, \mathfrak{H}) & \longrightarrow & H^1(F_n, \mathbf{Aut}(\mathfrak{H})).
\end{array}$$

Let $v \in H^1_{\acute{e}t}(R_n, \mathfrak{H})$ be a toral class mapping to $1 \in H^1(F_n, \mathfrak{H})$. In view of bijection (17.4.1) there exists $u \in \mathrm{Out}(\mathfrak{H})(R_n)$ such that $v = \varphi(u)$ and u belongs to the image of ψ. Since $\mathrm{Out}(\mathfrak{H})(R_n)$ is a finite group, the Density Theorem 17.3 shows that $\mathrm{Aut}(\mathfrak{H})(R_n)$ and $\mathrm{Aut}(\mathfrak{H})(F_n)$ have same image in $\mathrm{Out}(\mathfrak{H})(F_n)$. So u belongs to the image of γ, hence $v = 1 \in H^1_{\acute{e}t}(R_n, \mathfrak{H})$.
Let now $\mathfrak{H}$ be an arbitrary reductive group. Set $\mathfrak{C} = Z(\mathfrak{H})$. This is an R_n–group of multiplicative type and we have an exact (central) sequence of R_n-group schemes

$$1 \to \mathfrak{C} \xrightarrow{i} \mathfrak{H} \to \mathfrak{H}_{ad} \to 1.$$

This exact sequence gives rise to the diagram of exact sequence of pointed sets

$$\begin{array}{ccccccccc}
\mathfrak{H}_{ad}(R_n) & \xrightarrow{\varphi_{R_n}} & H^1_{\acute{e}t}(R_n, \mathfrak{C}) & \xrightarrow{i_*} & H^1_{\acute{e}t}(R_n, \mathfrak{H}) & \longrightarrow & H^1_{\acute{e}t}(R_n, \mathfrak{H}_{ad}) & \xrightarrow{\Delta} & H^2_{\acute{e}t}(R_n, \mathfrak{C}) \\
\downarrow & & \downarrow \simeq & & \downarrow & & \downarrow & & \downarrow \simeq \\
\mathfrak{H}_{ad}(F_n) & \xrightarrow{\varphi_{F_n}} & H^1(F_n, \mathfrak{C}) & \longrightarrow & H^1(F_n, \mathfrak{H}) & \longrightarrow & H^1(F_n, \mathfrak{H}_{ad}) & \xrightarrow{\Delta_{F_n}} & H^2_{\acute{e}t}(F_n, \mathfrak{C}).
\end{array}$$

The isomorphisms $H^i_{\acute{e}t}(R_n, \mathfrak{C}) \cong H^i(F_n, \mathfrak{C})$ comes from [GP2, prop. 3.4.(3)] for $i = 1, 2$.

Let $v \in H^1_{\acute{e}t}(R_n, \mathfrak{H})$ be a toral class mapping to $1 \in H^1(F_n, \mathfrak{H})$. Taking into account the adjoint case, a diagram chase provides $u \in H^1_{\acute{e}t}(R_n, \mathfrak{C})$ such

that $v = i_*(u)$ and u_{F_n} belongs to the image of the characteristic map φ_{F_n}. Since $H^1_{ét}(R_n, \mathfrak{C})$ is an abelian torsion group, the Density Theorem 17.3 shows that $\mathfrak{H}_{ad}(F_n)$ and $\mathfrak{H}_{ad}(R_n)$ have the same images in $H^1_{ét}(R_n, \mathfrak{C})$. So u belongs to the image of φ_{R_n}. Hence $v = i_*(u) = 1 \in H^1_{ét}(R_n, \mathfrak{H})$.
Surjectivity: Follows by a simple chasing in the diagrams above. □

Question. Assume that $\mathfrak{H}$ is loop semisimple simply connected, isotropic and F_n-simple, Let $\mathfrak{H}(R_n)^+ \subset \mathfrak{H}(R_n)$ be the (normal) subgroup generated by the $R_u(\mathfrak{P})(R_n)$ where $\mathfrak{P}$ runs over the set of parabolic subgroups of $\mathfrak{H}$ considered in Theorem 15.1. Is the map

$$\mathfrak{H}(R_n)/\mathfrak{H}(R_n)^+ \to \mathfrak{H}(F_n)/\mathfrak{H}(F_n)^+$$

an isomorphism? Note that the map is surjective by Remark 17.4. The question is then all about the injectivity of the map in question.

18. Appendix: Greenberg functors, Bruhat-Tits theory and pro-unipotent radicals

We are given a complete discrete valuation field K of valuation ring $O = O_K$ and of the perfect residue field $k = O/\pi O$. Here $\pi \in O$ is a uniformizer. In the inequal characteristic case denote by e_0 the absolute ramification index of O, i.e. $p = u\pi^{e_0}$ for a unit $u \in O$ where $p = \mathrm{char}(k)$; in the equal characteristic case, put $e_0 = 1$. We denote by O^{sh} the strict henselization of O, or in other words, its maximal unramified extension.

18.1. **Greenberg functor.** We recall here basic facts, see the references [Gb], [M2, §III.4], [BLR], [B].

Assume first that we are in the unequal characteristic case, that is K is of characteristic 0 and k is of characteristic $p > 0$.

For each k-algebra Λ and $r \geq 0$, we denote by $W_r(\Lambda)$ the group of Witt vectors of length r and by $W(\Lambda) = \varprojlim W_r(\Lambda)$ the ring of Witt vectors (see [Se2, §II.6]). There exists a unique ring homomorphism $W(k) \to O$ commuting with the projection on $k = W_0(k)$ (*ibid*, II.5).

Let $\mathfrak{S}$ be an affine $W(k)$-scheme. Recall that for each $r \geq 0$, the functor k–*alg* $\to$ *Sets* given by

$$\Lambda \to \mathfrak{S}(W_r(\Lambda))$$

is representable by an affine k–scheme $\mathrm{Green}_r(\mathfrak{S})$. The projective limit

$$\mathrm{Green}\,(\mathfrak{S}) := \varprojlim_r \mathrm{Green}_r(\mathfrak{S})$$

is a scheme which satisfies $\mathrm{Green}(\mathfrak{S})(\Lambda) = \mathfrak{S}(W(\Lambda))$. If $\mathfrak{X}$ is an affine O–scheme, we deal also with the relative versions of the Greenberg functor

$$\underline{G}_r(\mathfrak{X}) := \mathrm{Green}_r(\prod_{O/W(k)} \mathfrak{X}), \quad \underline{G}(X) := \mathrm{Green}\,(\prod_{O/W(k)} \mathfrak{X}).$$

We have $\underline{G}_r(\mathfrak{X})(k) = \mathfrak{X}(O/p^r O)$ and $\underline{G}(\mathfrak{X})(k) = \mathfrak{X}(O)$. We have $\underline{G}(\mathrm{Spec}(O)) = \mathrm{Spec}(k)$; if $\mathfrak{X}$ is a O–group scheme, then $\underline{G}(\mathfrak{X})$ and the $\underline{G}_r(\mathfrak{X})$ carry a natural k-group structure [B, 4.1].

18.2. **Lemma.** *Let L/K be a finite extension, O_L the valuation ring of L and l/k the corresponding residue extension. Let $\mathfrak{Y}/O_L$ be an affine scheme. Let $\underline{H}/l$ be the relative Greenberg functor of $\mathfrak{Y}$ with respect to $W(l)$. Then we have natural isomorphisms of k-schemes (for all $r \geq 1$)*

$$\underline{G}_r\Big(\prod_{O_L/O} \mathfrak{Y} \Big) \simeq \prod_{l/k} \underline{H}_r(\mathfrak{Y}), \quad \underline{G}\Big(\prod_{O_L/O} \mathfrak{Y} \Big) \simeq \prod_{l/k} \underline{H}(\mathfrak{Y}).$$

In particular if $k = l$ then we have $\underline{G}_r\Big(\prod\limits_{O_L/O} \mathfrak{Y} \Big) = \underline{H}_r(\mathfrak{Y})$ and $\underline{G}\Big(\prod\limits_{O_L/O} \mathfrak{Y} \Big) \simeq \underline{H}(\mathfrak{Y})$.

Proof. We have a commutative square

$$\begin{array}{ccc} O & \longrightarrow & O_L \\ \uparrow & & \uparrow \\ W(k) & \longrightarrow & W(l). \end{array}$$

So by the functorial properties of the Weil restriction, we have

$$\prod_{O/W(k)} \prod_{O_L/O} \mathfrak{Y} = \prod_{O_L/W(k)} \mathfrak{Y} = \prod_{W(l)/W(k)} \prod_{O_L/W(l)} \mathfrak{Y}. \tag{18.2.1}$$

Let Λ be a k–algebra. Using (18.2.1) and the definitions of the Greenberg functors, we have

$$\begin{aligned} \underline{G}_r\Big(\prod_{O_L/O} \mathfrak{Y} \Big)(\Lambda) &= \mathrm{Green}_r\Big(\prod_{O_L/W(k)} \mathfrak{Y} \Big)(\Lambda) \\ &= \Big(\prod_{W(l)/W(k)} \prod_{O_L/W(l)} \mathfrak{Y} \Big)(W_r(\Lambda)) \\ &= \Big(\prod_{O_L/W(l)} \mathfrak{Y} \Big)\big(W(l) \otimes_{W(k)} W_r(\Lambda)\big). \end{aligned}$$

Since $W_r(\Lambda)$ is a $W_r(k)$-module, we have

$$W(l) \otimes_{W(k)} W_r(\Lambda) = W_r(l) \otimes_{W_r(k)} W_r(\Lambda) = W_r(\Lambda \otimes_k l)$$

by [I, 1.5.7]. Hence

$$\underline{G}_r\Big(\prod_{O_L/O} \mathfrak{Y} \Big)(\Lambda) = \Big(\prod_{O_L/W(l)} \mathfrak{Y} \Big)(W_r(\Lambda \otimes_k l)) = R_{l/k}(\underline{H}_r)(\Lambda)$$

as desired. By passing to the limit, we get the second identity. $\square$

18.3. **Lemma.** (1) *Let $\mathfrak{X}/O$ be an affine scheme of finite type such that $\mathfrak{X}_K = \emptyset$. Then $\underline{G}(\mathfrak{X}) = \emptyset$.*

(2) *Let $\mathfrak{N}/O$ be an affine group scheme of finite type such that $\mathfrak{N}_K = \mathrm{Spec}\,(K)$. Then $\underline{G}(\mathfrak{N}) = \underline{G}(\mathrm{Spec}\,(O)) = \mathrm{Spec}(k)$.*

Proof. (1) We have $\mathfrak{X} = \mathrm{Spec}\,(A)$ where A is an $O/\pi^d O$–algebra of finite type for d large enough Put $r_0 = d\,e_0$. Then $p^{r_0} A = 0$. For a k–algebra Λ we have by definition

$$\underline{G}(\mathfrak{X})(\Lambda) = \mathrm{Hom}_O(A, W(\Lambda) \otimes_{W(k)} O).$$

But $W(\Lambda) \otimes_{W(k)} O$ is p–torsion free, so $\underline{G}(\mathfrak{X}) = \emptyset$.
(2) We have $N = \mathrm{Spec}(B)$ and we have the decomposition $B = O \oplus I$ where I is the kernel of the co-unit of the corresponding Hopf algebra. The O-module I is an ideal of B which is an $O/\pi^d O$-algebra of finite type. The same reasoning as above shows that

$$\begin{aligned} \underline{G}(\mathfrak{N})(\Lambda) &= \mathrm{Hom}_O(B, W(\Lambda) \otimes_{W(k)} O) \\ &= \mathrm{Hom}_O(O, W(\Lambda) \otimes_{W(k)} O) \\ &= \underline{G}(\mathrm{Spec}(O))(\Lambda). \end{aligned}$$

Thus $\underline{G}(\mathfrak{N}) = \underline{G}(\mathrm{Spec}(O))$ which is nothing but that $\mathrm{Spec}(k)$ as reminded above. $\square$

Secondly, assume that k and K have the same characteristic (0 or $p > 0$) and we still assume that k is perfect. Then k embeds in O (in an unique way, [EGA4, 21.5.3]) and for an O-scheme $\mathfrak{X}$ the functors

$$\underline{G}(\mathfrak{X}) := \prod_{O|k} \mathfrak{X} \quad \text{and} \quad \underline{G}_r(\mathfrak{X}) := \prod_{O/\pi^r O \;|\; k} (\mathfrak{X} \times_O O/\pi^r O)$$

play the desired role [BLR, §9.6] and allow us to write

$$\mathfrak{X}(O) = \varprojlim_r \mathfrak{X}(O/\pi^r O) \; = \varprojlim_r \underline{G}_r(\mathfrak{X})(k)$$

where the $\underline{G}_r(\mathfrak{X})$ are k-schemes (by Weil restriction [BLR, §7.6]). The two lemmas are true as well.

18.4. **Congruence filtration.** Let G be a reductive K–group and denote by $\mathcal{B} = \mathcal{B}(G, K)$ its (extended) Bruhat-Tits building. Let x be a point of $\mathcal{B}$ and denote by P_x the parahoric subgroup

$$P_x = \Big\{ g \in G(K) \;\mid\; g(x) = x \Big\}.$$

Denote by $\mathfrak{P}_x$ the canonical smooth group scheme over O defined by Bruhat-Tits [BT2, §5.1] with generic fiber G and such that $\mathfrak{P}_x(O) = P_x$ or, more precisely,

$$\mathfrak{P}_x(O^{sh}) = \Big\{ g \in G(K^{sh}) \;\mid\; g(x) = x \Big\}$$

where x is viewed as an element in $\mathcal{B}(G, K^{sh})$ via the canonical mapping $\mathcal{B}(G,K) \hookrightarrow \mathcal{B}(G, K^{sh})$. Since $\mathfrak{P}_x$ is smooth we have

$$\mathfrak{P}_x(O) = \varprojlim_{n\geq 1} \mathfrak{P}_x(O/\pi^n O)$$

and the transition maps $\mathfrak{P}_x(O/\pi^{n+1}O) \to \mathfrak{P}_x(O/\pi^n O)$ are surjective with kernel $\mathrm{Lie}(\mathfrak{P}_x) \otimes_O k$ ([M2, III.4.3])

The application of the relative Greenberg functor to the smooth affine group scheme $\mathfrak{P}_x$ defines a projective system of affine k-groups $\mathbf{P}_{x,n}$ $(n \geq 1)$ such that

$$\mathbf{P}_{x,n}(k) = \mathfrak{P}_x(O/\pi^{ne_0}O).$$

The $\mathbf{P}_{x,n}$ are smooth according to [B, Lemme 4.1.1]. The kernel $\mathbf{P}_{x,n+1/n}$ of the transition maps $\mathbf{P}_{x,n+1} \to \mathbf{P}_{x,n}$ are k-unipotent abelian groups which are successive extensions of the vector group of $\mathrm{Lie}(\mathfrak{P}_x) \otimes_O k$ (*ibid.* or [M2, III.4.3]).

For each $n \geq 1$, we denote by $\mathbf{R}_{n,x} := R_u(\mathbf{P}_{x,n})$ the unipotent radical of $\mathbf{P}_{x,n}$; since k is perfect, it is defined over k and split [DG, IV.2.3.9]. The quotient $\mathbf{M}_x$ of $\mathbf{P}_{x,n}$ by $\mathbf{R}_{x,n}$ is independent of n. It is nothing but the quotient of the special fiber of $\mathfrak{P}_x$ by its k-unipotent radical R_x. The k-group $\mathbf{M}_x^\circ$ is reductive according to [BT2, 4.6.12].

We consider the "maximal pro-unipotent normal subgroup"

$$P_x^* := \ker\Big(\mathfrak{P}_x(O) \to \mathbf{M}_x(k)\Big)$$

which is of analytic nature. Denote by

$$\mathbf{P}_x/k := \varprojlim_{n\geq 1} \mathbf{P}_{x,n}$$

and by $\mathbf{P}_x^*/k = \ker\big(\mathbf{P}_x \to \mathbf{M}_x\big)$. By construction we have $P_x^* = \mathbf{P}_x^*(k)$.

18.5. **Lemma.** *For each $n \geq 1$, there is a short exact sequence of affine k–groups*

$$1 \to \ker(\mathbf{P}_x \to \mathbf{P}_{x,n}) \to \mathbf{P}_x^* \to \mathbf{R}_{x,n} \to 1.$$

Proof. Apply the snake lemma to the commutative diagram of k–groups

$$\begin{array}{ccccccc}
1 \longrightarrow & \mathbf{P}_x^* & \longrightarrow & \mathbf{P}_x & \longrightarrow & \mathbf{M}_x \to 1 \\
& \downarrow & & \downarrow & & || \\
1 \longrightarrow & \mathbf{R}_{x,n} & \longrightarrow & \mathbf{P}_{x,n} & \longrightarrow & \mathbf{M}_x \to 1.
\end{array}$$

□

18.6. **Lemma.** *The k-group $\mathbf{P}_x^*$ is the unique maximal split pro-unipotent closed normal subgroup of the pro-algebraic affine k-group $\mathbf{P}_x$.*

Proof. Since

$$\ker(\mathbf{P}_x \to \mathbf{P}_{x,1}) = \varprojlim_n \ker(\mathbf{P}_{x,n} \to \mathbf{P}_{x,1})$$

is pro-unipotent, the above exact sequence shows that $\mathbf{P}_x^*$ is pro-unipotent. Let $\mathbf{U}_x$ be a pro-unipotent normal closed subgroup of $\mathbf{P}_x$. The image of $\mathbf{U}_x$ by the map $\mathbf{P}_x \to \mathbf{M}_x$ is a normal unipotent connected k-subgroup. Since $\mathbf{M}_x^\circ$ is reductive, its image is trivial. Therefore $\mathbf{U}_x \subset \mathbf{P}_x^*$ which completes the proof. □

18.7. **Behaviour under a Galois extension.** Just as does the whole theory, the construction of P_x^* has a very nice behaviour with respect to unramified extensions of K. The behaviour under a given tamely ramified finite Galois field extension L/K is subtle. Since such an extension is a tower of an unramified extension and a totally ramified one, we may concentrate on the case when L/K is totally (tamely) ramified. Then L/K is cyclic of degree e invertible in $k = \overline{K} = \overline{L}$. The Galois group $\Gamma = \mathrm{Gal}(L/K)$ acts on the building $\mathcal{B}(G, L)$. The Bruhat-Tits-Rousseau theorem ([Ro, §5], see also [Pr]) states that the natural map

$$j : \mathcal{B}(G, K) \to \mathcal{B}(G, L)$$

induces a bijection $\mathcal{B}(G, K) \xrightarrow{\sim} \mathcal{B}(G, L)^\Gamma$. For $z \in \mathcal{B}(G, L)$, we denote by Q_z the parahoric subgroup of $G(L)$ and by $\mathfrak{Q}_z$ the canonical group scheme over O_L attached to the point z.

For $\sigma \in \Gamma$, we have $\sigma(Q_z) = Q_{\sigma(z)}$. Hence for the canonical group schemes over O_L attached to z and $\sigma(z)$, there is a natural cartesian square

$$\begin{tikzcd} \mathfrak{Q}_{\sigma(z)} \arrow[r, "f_{\sigma,z}"] \arrow[d] & \mathfrak{Q}_z \arrow[d] \\ \mathrm{Spec}(O_L) \arrow[r, "(\sigma^{-1})^*"] & \mathrm{Spec}(O_L). \end{tikzcd}$$

Put $y = j(x) \in \mathcal{B}(G, L)^\Gamma$. and we have then an O-action of Γ on the scheme $\mathfrak{Q}_y$. We note that

$$P_x = G(K) \cap Q_y \quad = \quad G(L)^\Gamma \cap Q_y = Q_y^\Gamma. \tag{18.7.1}$$

As above we consider the groups $\mathbf{Q}_{y,n}$ and their projective limit $\mathbf{Q}_y$. Since k is the residue field of O_L, all $\mathbf{Q}_{y,n}$ and $\mathbf{Q}_y$ are k-groups. The action of Γ on $\mathfrak{Q}_y$ induces its action on $\mathbf{Q}_{y,n}$, hence on $\mathbf{M}_y$ where $\mathbf{M}_y$ stands for the reductive k–group attached to y, and on their projective limit $\mathbf{Q}_y$. By Lemma 18.6, $\mathbf{Q}_y^*$ is a characteristic k-subgroup of $\mathbf{Q}_y$, hence Γ also acts on the pro-algebraic k-group $\mathbf{Q}_y^*$. Our goal is to prove the following fact:

18.8. **Proposition.** *There is a natural closed embedding $\mathbf{P}_x \to \mathbf{Q}_y$ and we have*

$$\mathbf{P}_x^* = \mathbf{P}_x \cap \mathbf{Q}_y^*.$$

This gives rise to an isomorphism $\mathbf{M}_x \xrightarrow{\sim} \mathbf{M}_y^\Gamma$.

By taking k–points we get the following wished compatibility, namely.

18.9. **Corollary.** *We have*

$$P_x^* \xrightarrow{\sim} P_x \cap Q_y^*.$$

Consider the Weil restriction $\mathfrak{J}_x := \Pi_{O_L/O}\big(\mathfrak{Q}_y\big)$ and recall it is a smooth O-scheme [Yu, §2.5]. Let $\mathfrak{N}$ be the kernel of the natural map $\mathfrak{P}_x \to \mathfrak{J}_x$, its generic fiber is trivial. As above, applying the Greenberg functors to the O-schemes $\mathfrak{J}_x$ and $\mathfrak{N}$ we get k-groups $\mathbf{J}_{x,n},\ \mathbf{J}_x$ and $\mathbf{N}_n,\ \mathbf{N}$.

Since the Greenberg functor is left exact, we get an exact sequence

$$1 \to \mathbf{N} \to \mathbf{P}_x \to \mathbf{J}_x.$$

Since $\mathfrak{N}_K = 1$, we have $\mathbf{N} = 1$ according to Lemma 18.3 (2). Hence we may view $\mathbf{P}_x$ as a closed subgroup of $\mathbf{J}_x$. But according to Lemma 18.2, $\mathbf{J}_{x,n}$ is nothing but $\mathbf{Q}_{y,n}$. This implies $\mathbf{J}_x$ is isomorphic in a natural way to $\mathbf{Q}_y$. Thus we have constructed a natural closed embedding $\mathbf{P}_x \to \mathbf{Q}_y$.

Define the k-subgroups $\mathbf{Q}_y^\Gamma := \varprojlim_n \mathbf{Q}_{y,n}^\Gamma$ and $(\mathbf{Q}_y^*)^\Gamma = \mathbf{Q}_y^\Gamma \cap \mathbf{Q}_y^*$ of $\mathbf{Q}_y$ and $\mathbf{Q}_y^*$ respectively.

18.10. **Lemma.** (1) *If k'/k is a finite extension of fields, the projective system $\big(\mathbf{Q}_{y,n}^\Gamma(k')\big)_{n\geq 1}$ has surjective transitions maps. Therefore the projective system of k-groups $\big(\mathbf{Q}_{y,n}^\Gamma\big)_{n\geq 1}$ has surjective transitions maps.*

(2) *If k'/k is a field finite extension, we have an exact sequence*

$$1 \to (\mathbf{Q}_y^*)^\Gamma(k') \to \mathbf{Q}_y^\Gamma(k') \to \mathbf{M}_y^\Gamma(k') \to 1;$$

hence the sequence of the pro-algebraic k-groups

$$1 \to (\mathbf{Q}_y^*)^\Gamma \to \mathbf{Q}_y^\Gamma \to \mathbf{M}_y^\Gamma \to 1$$

is also exact.

(3) *The algebraic k–group $\mathbf{M}_y^\Gamma$ is smooth and its connected component of the identity is reductive.*

Proof. (1) Since Bruhat-Tits theory is insensitive to finite unramified extensions, we may assume that $k = k'$ without loss of generality. Since $\mathbf{Q}_{y,n+1/n}$ is a k-split unipotent group, we have an exact sequence

$$1 \to \mathbf{Q}_{y,n+1/n}(k) \to \mathbf{Q}_{y,n+1}(k) \to \mathbf{Q}_{y,n}(k) \to 1.$$

It gives rise to the exact sequence of pointed sets

$$1 \to \mathbf{Q}_{y,n+1/n}(k)^\Gamma \to \mathbf{Q}_{y,n+1}(k)^\Gamma \to \mathbf{Q}_{y,n}(k)^\Gamma \to H^1\big(\Gamma, \mathbf{Q}_{y,n+1/n}(k)\big).$$

Since $\mathbf{Q}_{y,n+1/n}(k)$ admits a characteristic central composition serie in k-vector spaces and the order of Γ is invertible in k, the right hand side is trivial. A fortiori, the system $(\mathbf{Q}_{y,n}^\Gamma)$ of k–groups is surjective (because $\mathbf{Q}_{y,n}^\Gamma(k) = \mathbf{Q}_{y,n}(k)^\Gamma$).

(2) By part (1), the map $\mathbf{Q}_y^\Gamma(k) \to (\mathbf{Q}_{y,1})^\Gamma(k)$ is surjective. The same argument as in (1) shows that $(\mathbf{Q}_{y,1})^\Gamma(k) \to \mathbf{M}_y^\Gamma(k)$ is also surjective. By

taking the composition of these maps we conclude the map $\mathbf{Q}_y^\Gamma(k) \to \mathbf{M}_y^\Gamma(k)$ is surjective whence the desired exactness of both sequences.

(3) The group Γ may be viewed as a finite abelian constant group scheme whose order is invertible in k. Hence Γ is also a (smooth) k-group of multiplicative type. Since $\mathbf{M}_y$ is affine and smooth, Grothendieck's theorem of smoothness of centralizers [SGA3, XI, 5.3] shows that $\mathbf{M}_y^\Gamma$ is smooth. Its connected component of the identity is reductive by a result of Richardson [Ri, prop. 10.1.5]. □

We can now proceed to the proof of Proposition 18.8.

Proof. We have to show that our closed embedding $\mathbf{P}_x \to \mathbf{Q}_y$ which we constructed above induces an isomorphism $\mathbf{P}_x^* \xrightarrow{\sim} \mathbf{P}_x \cap \mathbf{Q}_y^*$. Since $\mathbf{P}_x \cap \mathbf{Q}_y^*$ is a normal closed split pro-unipotent subgroup of $\mathbf{P}_x$ it is contained in $\mathbf{P}_x^*$. Hence it remains only to show that $\mathbf{P}_x^* \subset \mathbf{Q}_y^*$.

We now recall from (18.7.1) that $P_x = Q_y^\Gamma$ and $\mathbf{Q}_y^\Gamma(k) = \mathbf{Q}_y(k)^\Gamma = Q_y^\Gamma$. By Lemma 18.10, $\mathbf{Q}_y^\Gamma(k)$ projects onto $\mathbf{M}_y^\Gamma(k)$, so the composite map

$$P_x = \mathbf{P}_x(k) \to \mathbf{Q}_y^\Gamma(k) \to \mathbf{M}_y^\Gamma(k)$$

is surjective. Since this is true for all finite extensions of k, the homomorphism of k-algebraic groups $\mathbf{P}_x \to \mathbf{M}_y^\Gamma$ is surjective. But $(\mathbf{M}_y^\Gamma)^\circ$ is reductive, hence this map is trivial on the pro-unipotent radical $\mathbf{P}_x^*$. We get then a surjective map $\mathbf{M}_x \to \mathbf{M}_y^\Gamma$ and also a homomorphism $\mathbf{P}_x^* \to (\mathbf{Q}_y^*)^\Gamma \subset \mathbf{Q}_y^*$ as required. □

References

[Als] B. Allison, *Some isomorphism invariants of Lie tori*, Jour. Lie Theory (to appear).

[AABGP] B. Allison, S. Azam, S. Berman, Y. Gao and A. Pianzola, *Extended affine Lie algebras and their root systems*, Mem. Amer. Math. Soc. **126** #603 (1997).

[ABFP2] B. Allison, S. Berman, J. Faulkner, A. Pianzola, *Multiloop realization of extended affine Lie algebras and Lie tori*, Trans. Amer. Math. Soc. **361** (2009), 4807–4842.

[ABGP] B. Allison, S. Berman, Y. Gao and A. Pianzola, *A characterization of affine Kac-Moody Lie algebras*, Comm. Math. Phys. 185 (1997), 671688.

[ABP2.5] B. Allison, S. Berman and A. Pianzola, *Iterated loop algebras*, Pacific J. Math. **227** (2006), 1–42.

[B] L. Bégueri, *Dualité sur un corps local à corps résiduel algébriquement clos*, Mémoires de la SMF **4** (1980), 1-121.

[BLR] S. Bosch, W. Lütkebohmert, M. Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete **21** (1990), Springer-Verlag.

[Bor] A. Borel, *Linear Algebraic Groups (Second enlarged edition)*, Graduate text in Mathematics **126** (1991), Springer.

[BM] A. Borel and G.D Mostow, *On semi-simple automorphisms of Lie algebras*, Ann. Math. **61** (1955), 389-405.

[Bbk] N. Bourbaki, *Commutative algebra*,

[Bbk1] N. Bourbaki, *Groupes et algèbres de Lie*, Ch. 4,5 et 6, Masson. .

[BT65] A. Borel and J. Tits, *Groupes réductifs*, Inst. Hautes Études Sci. Publ. Math. **27** (1965), 55–150.

[BT1] F. Bruhat, J. Tits, *Groupes réductifs sur un corps local. I. Données radicielles valuées*, Inst. Hautes Etudes Sci. Publ. Math. **41** (1972), 5–251.

[BT73] A. Borel and J. Tits, *Homomorphismes "abstraits" de groupes algébriques simples*, Annals of Mathematics **97** (1973), 499–571.

[BT2] F. Bruhat, J. Tits, *Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée*, Inst. Hautes Etudes Sci. Publ. Math. **60** (1984), 197–376.

[BT3] F. Bruhat, J. Tits, *Groupes réductifs sur un corps local. III. Comléments et applications à la cohomologie galoisienne*, J. Fac. Sci. Univ. Tokyo **34** (1987), 671-698.

[CGP] V. Chernousov, P. Gille, A. Pianzola, *Torsors over the punctured affine line*, preprint (2008).

[CTO] J.–L. Colliot–Thélène, M. Ojanguren, *Espaces principaux homogènes localement triviaux*, I.H.É.S. Publ. Math. **75** (1992), 97–122.

[De] C. Demarche, *Méthodes cohomlogiques pour l'étude des points rationnels sur les espaces homogènes*, thèse (Orsay, 2009), author's URL.

[DG] M. Demazure, P. Gabriel, *Groupes algébriques*, North-Holland (1970).

[EGA4] A. Grothendieck (avec la collaboration de J. Dieudonné), *Eléments de Géométrie Algébrique IV*, Inst. Hautes Etudes Sci. Publ. Math. no 20, 24, 28 and 32 (1964 - 1967).

[G] P. Gille, Le problème de Kneser-Tits, exposé Bourbaki n0 983, Astérisque **326** (2009), 39-81.

[GP1] Gille, P. and Pianzola, A. *Galois cohomology and forms of algebras over Laurent polynomial rings*, Math. Annalen **338** (2007) 497-543.

[GP2] P. Gille and A. Pianzola, *Isotriviality and étale cohomology of Laurent polynomial rings*, Jour. Pure Applied Algebra, **212** 780–800 (2008).

[GP3] P. Gille and A. Pianzola, *Torsors, Reductive group Schemes and Extended Affine Lie Algebras*, preprint 125pp. (2011). ArXiv:1109.3405v1

[Gir] J. Giraud, *Cohomologie non-abélienne*, Springer (1970).

[Gb] M. Greenberg, *Schemata over local rings*, Annals of Math. **73** (1961), 624-648.

[Gr] A. Grothendieck, *Techniques de descente et théorèmes d'existence en géométrie algébrique. IV. Les schémas de Hilbert*, Séminaire Bourbaki, Vol. 6, Exp. No. 221, 249–276, Soc. Math. France, Paris, 1995.

[EGA IV] A. Grothendieck (avec la collaboration de J. Dieudonné), *Eléments de Géométrie Algébrique IV*, Publications mathématiques de l'I.H.É.S. no 20, 24, 28 and 32 (1964 - 1967).

[H] G. Harder, *Halbeinfache Gruppenschemata über vollständigen Kurven*, Invent. Math. **6** (1968), 107–149.

[Hu] J. Humphreys, Linear algebraic groups, Springer-Verlag, 1975.

[I] L. Illusie, *Complexe de de Rham-Witt et cohomologie cristalline*, Annales Scientifiques de l'Ecole normale supérieure **12** (1979), 501-561.

[Kac] V. Kac, *Infinite dimensional Lie algebras*, third edition, Cambridge University Press, Cambridge, 1990.

[Kmr] S. Kumar, *Kac-moody groups, their flag varieties and representation theory*, Springer Verlag (2002)

[Lam] T.-Y. Lam, *Serre's problem on projective modules*, second edition (2007), Springer.

[LS] S. Lang and J.-P. Serre, *Sur les revêtements non ramifiés des variétés algébriques*, American Journal of Mathematics **79** (1957), 319-330.

[M1] J. S. Milne, *Étale Cohomology*, Princeton University Press.

[M2] J. S. Milne, *Arithmetic duality theorem*, second edition (2004).

[MP] R.V. Moody and A. Pianzola, *Lie algebras with triangular decomposition*, John Wiley, New York, 1995.

[Mt] G. D. Mostow, *Fully reducible subgroups of algebraic groups*, Amer. J. Math. **78** (1956), 200–221.

[N1] E. Neher, *Lie tori*, C.R. Math. Acad. Sci. Soc. R. Can., **26** (2004), pp. 84–89.

[N2] E. Neher, *Extended affine Lie algebras*, C.R. Math. Acad. Sci. Soc. R. Can., **26** (2004), pp. 90–96.

[P1] A. Pianzola, *Locally trivial principal homogeneous spaces and conjugacy theorems for Lie algebras*, J. Algebra **275** (2004), no. 2, 600–614.

[P2] A. Pianzola, *Vanishing of H^1 for Dedekind rings and applications to loop algebras*, C. R. Acad. Sci. Paris, Ser. I **340** (2005), 633-638.

[PK] D.H. Peterson and V. Kac, *Infinite flag varieties and conjugacy theorems*, Proc. Natl. Acad. Sci. USA **80** (1983), 1778–1782.

[Pr] G. Prasad, *Galois-fixed points in the Bruhat-Tits building of a reductive group*, Bull. Soc. Math. France **129**, (2001), 169–174.

[Ra] M. Raynaud, *Anneaux locaux henséliens*, Lecture Notes in Math. 169. Springer (1971).

[Ri] R. W. Richardson, *On orbits of algebraic groups and Lie groups*, Bull. Austral. Math. Soc. **25** (1982), 1–28.

[Ro] G. Rousseau, *Immeubles des groupes réductifs sur les coprs locaux*, Thèse, Université de Paris-Sud (1977).

[SGA1] *Séminaire de Géométrie algébrique de l'I.H.E.S., Revêtements étales et groupe fondamental, dirigé par A. Grothendieck*, Lecture Notes in Math. 224. Springer (1971).

[SGA3] *Séminaire de Géométrie algébrique de l'I.H.E.S., 1963-1964, schémas en groupes, dirigé par M. Demazure et A. Grothendieck*, Lecture Notes in Math. 151-153. Springer (1970).

[Se1] J.-P. Serre, *Galois Cohomology*, Springer, 1997.

[Se2] J.-P. Serre, *Local fields*, Springer-Verlag, New York, 1979.

[St75] R. Steiberg, Torsion in reductive groups, Advances in Mathematics **15** (1975), 63–92.

[Ti] J. Tits, *Reductive groups over local fields*, Proceedings of the Corvallis conference on L- functions etc., Proc. Symp. Pure Math. **33** (1979), part 1, 29-69.

[Y1] Y. Yoshii, *Lie tori—A simple characterization of extended affine Lie algebras*, Publ. Res. Inst. Math. Sci. **42** (2006), 739–762.

[Y2] Y. Yoshii, *Root systems extended by an abelian group and their Lie algebras*, J. Lie Theory **14** (2004), pp. 371–374.

[Yu] J.-K. Yu, *Smooth models associated to concave functions in Bruhat-Tits theory*, preprint (2002).

Department of Mathematics, University of Alberta, Edmonton, Alberta T6G 2G1, Canada
E-mail address: chernous@math.ualberta.ca

UMR 8553 du CNRS, École Normale Supérieure, 45 rue d'Ulm, 75005 Paris, France.
E-mail address: gille@ens.fr

Department of Mathematics, University of Alberta, Edmonton, Alberta T6G 2G1, Canada.
Centro de Altos Estudios en Ciencia Exactas, Avenida de Mayo 866, (1084) Buenos Aires, Argentina.
E-mail address: a.pianzola@math.ualberta.ca