

VERSALITY OF ALGEBRAIC GROUP ACTIONS AND RATIONAL POINTS ON TWISTED VARIETIES

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ABSTRACT. We formalize and study several competing notions of versality for an action of a linear algebraic group on an algebraic variety X . Our main result is that these notions of versality are equivalent to various statements concerning rational points on twisted forms of X (existence of rational points, existence of a dense set of rational points, etc.) We give applications of this equivalence in both directions, to study versality of group actions and rational points on algebraic varieties. We obtain similar results on p -versality for a prime integer p . An appendix, by J.-P. Serre, puts the notion of versality in a historical perspective.

1. INTRODUCTION

Let k be a base field and G be a linear algebraic group defined over k . We say that a G -action on an irreducible k -variety X is

- *weakly versal*, if for every field K/k , with K infinite, and every G -torsor $T \rightarrow \text{Spec}(K)$ there is a G -equivariant k -morphism $f: T \rightarrow X$, and
- *versal*, if every G -invariant dense open subvariety of X is weakly versal.

Note that here we view T as a k -scheme; it will not be of finite type in general. The advantage of the second notion over the first is that it only depends on X up to (a G -equivariant) birational isomorphism. In the case where $X \rightarrow B$ is a G -torsor over some irreducible base space B , our definition of versality is identical to [GMS03, Definition 5.1]. Versal or “generic” objects naturally arise in many parts of algebra and algebraic geometry, such as the theory of central simple algebras [Pro67, Ami72, Sal99], Galois theory [JLY02], and the study of algebraic surfaces [Dun09, Tok06]. For a historical perspective we refer the reader to the appendix.

Our main result, Theorem 1.1 below, relates versality of X to the existence of K -points on certain K -forms of X , for field extensions K/k . To state it,

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we need the following additional definitions, which will be used throughout the paper. We will say that a G -action on X is

- *very versal*, if there exists a linear representation $G \rightarrow \mathrm{GL}(V)$ and a G -equivariant dominant rational map $V \dashrightarrow X$,
- *birationally linear*, if there exists a linear representation $G \rightarrow \mathrm{GL}(V)$ and a G -equivariant birational isomorphism between V and X ,
- *stably birationally linear*, if there exists a linear representation $G \rightarrow \mathrm{GL}(W)$ such that $X \times W$ is birationally linear.

If K/k is a field extension, with K infinite, and $\pi: T \rightarrow \mathrm{Spec}(K)$ is a G -torsor, we will refer to (T, K) as a *twisting pair* (see Definition 4.1) and write ${}^T X$ for the twist of X_K by T (see Section 3).

Theorem 1.1. *Let G be a linear algebraic group defined over k . A G -action on an irreducible quasiprojective k -variety X is*

- (a) *weakly versal if and only if, for every twisting pair (T, K) , ${}^T X(K) \neq \emptyset$,*
- (b) *versal if and only if, for every twisting pair (T, K) , K -points are dense in ${}^T X$,*
- (c) *very versal if and only if, for every twisting pair (T, K) , ${}^T X$ is K -unirational,*
- (d) *stably birationally linear if and only if, for every twisting pair (T, K) , ${}^T X$ is stably K -rational.*

Theorem 1.1 tells us, in particular, that for a G -variety X , none of the implications

$$(1.1) \quad \text{stably birationally linear} \implies \text{very versal} \implies \text{versal} \implies \text{weakly versal}$$

can be reversed in general, even if $G = \{1\}$. On the other hand, in many natural examples X is geometrically unirational, i.e., X becomes unirational over the algebraic closure \bar{k} . In this situation the twisted K -variety ${}^T X$ is geometrically unirational for every twisting pair (T, K) . For a smooth geometrically unirational variety Y defined over an infinite field K , it is not known whether or not the following properties are equivalent: (i) Y is K -unirational, (ii) K -points are dense in Y , and (iii) Y has a K -point; see [Kol02, Question 1.3]. It is thus conceivable that if X is smooth and geometrically unirational then the second and third implications in (1.1) may, indeed, be reversed. This explains why part (b) of Theorem 1.1, which takes the most delicate arguments to prove, is never truly used in the specific examples in this paper, i.e., why versal varieties in these examples turn out to be very versal.

The rest of this paper is structured as follows. Section 2 is devoted to notation and preliminaries, and Section 3 to a discussion of the twisting operation. In Sections 4 and 5 we prove Theorem 1.1. In Section 6 we show that every K -form of the moduli space $\overline{M}_{0,n}$ of stable curves of genus 0 with $n \geq 5$ marked points is unirational over K . In Section 7 we use Theorem 1.1 in the other direction to give a versality criterion for the action

of a closed subgroup $G \subset A$ on a homogeneous space A/B . In Section 8 we define and study the related notions of p -versality, where p is a prime integer. We show that p -versality is related to 0-cycles on twisted varieties (rather than points) and that for smooth varieties, weak p -versality is equivalent to p -versality. Sections 9 and 10 feature versality criteria for group actions on projective spaces and low degree hypersurfaces. As an application, we show that a recent conjecture of I. Dolgachev on the Cremona dimension is incompatible with a long-standing conjecture of J. W. S. Cassels and P. Swinnerton-Dyer about rational points on cubic hypersurfaces.

2. NOTATION AND PRELIMINARIES

Let k be a field; we will denote the algebraic closure of k by \bar{k} .

A k -variety X is a reduced, quasiprojective scheme of finite type over k , not necessarily irreducible. A *morphism of k -varieties* is a morphism of schemes respecting the structure morphism to k .

An *algebraic group over k* is a smooth affine group scheme of finite type over k . An *action of G on X* will always be a morphic action, i.e., a morphism $\sigma : G \times X \rightarrow X$ satisfying the standard conditions [MFK94]. We will sometimes refer to X with an action of G as a G -variety.

Given a k -variety X and a field extension K/k , the symbol X_K denotes the K -variety $X \otimes_{\text{Spec}(k)} \text{Spec}(K)$. A *k -form of X* is a k -variety X' such that $X_{\bar{k}} \simeq X'_{\bar{k}}$.

A *right* (resp. *left*) G -torsor over Y is a morphism $\psi : X \rightarrow Y$ of k -schemes such that G acts on X on the right (resp. left), $\psi : X \rightarrow Y$ is flat, and the map $G \times_Y X \rightarrow X \times_Y X$ defined via $(g, x) \mapsto (x, x \cdot g)$ (resp. $(g, x) \mapsto (x, g \cdot x)$) is an isomorphism. The set of G -torsors over a field K is in bijection with the Galois cohomology set $H^1(K, G)$.

A k -variety is *rational* if it is k -birationally equivalent to \mathbb{A}^n , for some positive integer n . A k -variety X is *unirational* if there exists a dominant rational k -map $\mathbb{A}^n \dashrightarrow X$.

A G -action on X is *generically free* if there exists a dense G -invariant open subvariety $X_0 \subset X$ such that the scheme-theoretic stabilizer of every point $x \in X_0$ is trivial. This is equivalent to the existence of a dense G -invariant open subvariety U of X which is the total space of a G -torsor $\pi : U \rightarrow B$; see [BF03, Theorem 4.7]. If B is irreducible, we say that X is a *primitive G -variety*.

Given a generically free primitive G -variety X one obtains a G -torsor over $\text{Spec}(k(B))$ by pullback of the generic point of B . Conversely, given a G -torsor over a finitely generated field extension K/k , one can recover a birational equivalence class of generically free primitive G -varieties for which $k(B) = K$. Indeed, the G -torsor is an affine K -variety and thus can be defined over some finitely generated k -subalgebra of K .

The following remark is an immediate consequence of this correspondence.

Remark 2.1. A G -action on an irreducible k -variety X is weakly versal if and only if, for every generically free primitive G -variety Y defined over k , with $k(Y)^G$ infinite, there exists a G -equivariant k -rational map $Y \dashrightarrow X$.

Proposition 2.2. *Let X be an irreducible G -variety defined over k . If G has a fixed k -point $x \in X(k)$ then X is weakly versal.*

Proof. For every field K/k and every G -torsor $T \rightarrow \mathrm{Spec}(K)$, the constant map $T \rightarrow X$, sending all of T to x , is G -equivariant. \square

Remark 2.3. Let G be an algebraic k -group. Then there exists a generically free linear representation $G \rightarrow \mathrm{GL}(V)$; see [BF03, Remark 4.12]. Moreover, adding a copy of the trivial representation if necessary, we can choose V so that $k(V)^G$ is an infinite field.

Proposition 2.4. *If X is a versal irreducible G -variety then X is geometrically irreducible.*

Proof. It suffices to show that X_{k^s} is irreducible, where k^s denotes the separable closure of k ; see [Har77, Exercise 2.3.15(a)]. Let X_1, \dots, X_n denote the irreducible components of X_{k^s} . We want to show that $n = 1$. We will argue by contradiction. Assume $n \geq 2$. Since X is irreducible over k , the absolute Galois group $\mathrm{Gal}(k)$ permutes X_1, \dots, X_n transitively. Thus $Y := X_1 \cap \dots \cap X_n$ is a proper closed G -invariant k -subvariety of X .

Let V be a generically free linear G -representation with $k(V)^G$ infinite. By Remark 2.1 there exists a G -equivariant rational k -map $f: V \dashrightarrow X \setminus Y$. Since V is geometrically irreducible, the image of f is contained in one of the components X_i . Since $\mathrm{Gal}(k)$ transitively permutes the components, it is also contained in X_2, \dots, X_n and thus in Y , a contradiction. \square

3. TWISTING

Let G/k be an algebraic group, X/k be a G -variety, and $T \rightarrow \mathrm{Spec}(k)$ be a right G -torsor. The diagonal action of G on $T \times X$ makes $T \times X$ into the total space of a G -torsor $T \times X \rightarrow B$. The base space B of this torsor is unique (it is the geometric quotient of $T \times X$ by G); it is usually called *the twist* of X by T and denoted by ${}^T X$. This construction relies on our standing assumption that X is quasiprojective; for details, see [Flo08, Section 2] or [CTKPR11, Section 2].

Note that there is no natural G -action on ${}^T X$; we lose the G -action in the course of constructing ${}^T X$. However, ${}^T X$ carries a natural action of the twisted group ${}^T G$; see Propositions 3.5 and 3.6 below.

If T is split over k , it is easy to see that ${}^T X$ is k -isomorphic to X . Hence, ${}^T X$ is a k -form of X , i.e., X and ${}^T X$ become isomorphic over any splitting field L/k of T . Combining this observation with Hilbert's Theorem 90 ([Ser79, Proposition X.1.3]), we obtain the following.

Lemma 3.1. *Let V be a linear representation $G \rightarrow \mathrm{GL}(V)$ viewed as a G -variety and $T \rightarrow \mathrm{Spec}(k)$ be a G -torsor. Then ${}^T V$ is k -isomorphic to V . In particular, ${}^T V(k)$ is dense in ${}^T V$. \square*

It is well-known that quasiprojectivity is a geometric property; i.e., if a k -variety is quasiprojective over \bar{k} then it is quasiprojective over k . Thus, twisting is performed entirely within the category of quasiprojective varieties.

Twisting is functorial in the following sense: a G -equivariant morphism $f: X \rightarrow Y$ (respectively, a rational map $f: X \dashrightarrow Y$) of G -varieties gives rise to a K -morphism ${}^T f: {}^T X \rightarrow {}^T Y$ (respectively, a K -rational map ${}^T f: {}^T X \dashrightarrow {}^T Y$). For details, see [Flo08, Lemma 2.2] (where only rational maps are considered, but the construction of ${}^T f$ is even more straightforward if f is regular).

The following Proposition amplifies [CTKPR11, Lemma 3.4].

Proposition 3.2. *Let k be a field, G be a k -group and $T \rightarrow \text{Spec}(k)$ be a G -torsor. Denote by \mathbf{Var} the category of k -varieties and by $G\text{-}\mathbf{Var}$ the category of k -varieties with a G -action. Morphisms in the latter category are G -equivariant k -maps.*

Let $\mathcal{L}_T: \mathbf{Var} \rightarrow G\text{-}\mathbf{Var}$ be the functor $T \times -$ which takes a k -variety Y to $T \times Y$ (viewed as a G -variety, with G acting trivially on Y). Let $\mathcal{R}_T: G\text{-}\mathbf{Var} \rightarrow \mathbf{Var}$ be the twisting functor $({}^T -)$ described above.

Then the functors $(\mathcal{L}_T, \mathcal{R}_T)$ form an adjoint pair. In other words, for any $Y \in \mathbf{Var}$ and $X \in G\text{-}\mathbf{Var}$, we have an isomorphism

$$(3.1) \quad \text{Hom}_{G\text{-}\mathbf{Var}}(T \times Y, X) \simeq \text{Hom}_{\mathbf{Var}}(Y, {}^T X)$$

which is functorial in both X and Y .

Proof. The isomorphism is easiest to see by considering the intermediate set $\mathcal{F}(Y, X)$ consisting of G -equivariant morphisms $\gamma: T \times Y \rightarrow T \times X$ such that the following diagram commutes:

$$\begin{array}{ccc} T \times Y & \xrightarrow{\gamma} & T \times X \\ & \searrow \text{pr}_1 & \downarrow \text{pr}_1 \\ & & T \end{array}$$

where the vertical maps are projections.

Given $\gamma \in \mathcal{F}(Y, X)$, we obtain $\alpha = \text{pr}_2 \circ \gamma$ in $\text{Hom}_{G\text{-}\mathbf{Var}}(T \times Y, X)$. Mapping $\alpha \in \text{Hom}_{G\text{-}\mathbf{Var}}(T \times Y, X)$ to $\gamma = \text{pr}_1 \times \alpha$ is an inverse. Given $\gamma \in \mathcal{F}(Y, X)$, we obtain $\beta \in \text{Hom}_{\mathbf{Var}}(Y, {}^T X)$ by taking quotients. All of these operations are clearly functorial.

It remains to reconstruct $\gamma \in \mathcal{F}(Y, X)$ given $\beta: Y \rightarrow {}^T X$. Pulling back by the torsor $T \times X \rightarrow {}^T X$ we obtain a G -torsor $\pi: Y' \rightarrow Y$:

$$\begin{array}{ccc} Y' & \xrightarrow{f} & T \times X \\ \downarrow \text{G-torsor} & & \downarrow \text{G-torsor} \\ Y & \xrightarrow{\beta} & {}^T X. \end{array}$$

The G -equivariant map $\phi = (\text{pr}_1 \circ f) \times \pi$ is a morphism of torsors $\phi : Y' \rightarrow T \times Y$. By a standard result on torsors, this means ϕ is an isomorphism.

Thus, we have a G -equivariant morphism $\gamma' : T \times Y \rightarrow T \times X$ which lifts β . However, since pullbacks are only defined up to isomorphism, the projections $T \times Y \rightarrow T$ and $T \times X \rightarrow T$ do not necessarily commute with γ' . Nevertheless, there exists a unique $g \in G$ such that $g \circ \gamma'$ is in $\mathcal{F}(Y, X)$. This is the desired γ . The construction is easily seen to be functorial. \square

Corollary 3.3. *Let X be a G -variety, and $T \rightarrow \text{Spec}(k)$ be a G -torsor. Let L/k be a splitting field of T , let s be a point in $T(L)$, and let $t_s : ({}^T X)_L \rightarrow X_L$ be the L -isomorphism such that $s \times t_s$ is a section of $\pi_L : T_L \times X_L \rightarrow ({}^T X)_L$. If $\alpha : T \rightarrow X$ and $\beta : \text{Spec}(k) \rightarrow {}^T X$ are corresponding maps under the adjunction of Proposition 3.2 (with $Y = \text{Spec}(k)$) then $\alpha_L(s) = t_s(\beta_L)$ in $X(L)$.*

Proof. By definition, α and β fit into the following commutative diagram of G -equivariant k -morphisms:

$$\begin{array}{ccc} T & \xrightarrow{\text{id} \times \alpha} & T \times X \\ \downarrow & & \downarrow \\ \text{Spec}(k) & \xrightarrow{\beta} & {}^T X . \end{array}$$

The vertical maps are G -torsors; we split them by base-changing from k to L . By the definition of t_s the resulting diagram

$$\begin{array}{ccc} T_L & \xrightarrow{\text{id} \times \alpha_L} & T_L \times X_L \\ \downarrow & & \downarrow \\ \text{Spec}(L) & \xrightarrow{\beta_L} & ({}^T X)_L . \end{array} \begin{array}{l} s \\ \left(\begin{array}{c} \downarrow \\ \downarrow \end{array} \right) s \times t_s \end{array}$$

is commutative. Tracing from the lower left corner to the upper right, we see that $s \times \alpha_L(s) = s \times t_s(\beta_L)$ as morphisms $\text{Spec}(L) \rightarrow T_L \times X_L$. Composing these morphisms with the projection $T_L \times X_L \rightarrow X_L$, we see that $\alpha_L(s) = t_s(\beta_L)$ as maps $\text{Spec}(L) \rightarrow X$. \square

Corollary 3.4. *Let X and Y be G -varieties defined over k , and let $T \rightarrow \text{Spec}(k)$ be a G -torsor. Then*

- (a) ${}^T(X \times Y)$ is canonically isomorphic to ${}^T X \times {}^T Y$.
- (b) Let $f : X \rightarrow Y$ be a G -equivariant closed (resp. open) immersion. Then ${}^T f : {}^T X \rightarrow {}^T Y$ is also a closed (resp. open) immersion.
- (c) If $f : X \dashrightarrow Y$ is a G -equivariant dominant rational map then the induced rational map ${}^T f : {}^T X \dashrightarrow {}^T Y$ is also dominant.

Proof. (a) follows from the fact that the twisting functor is a right adjoint and, thus, is left exact. To prove (b) and (c) note that by [EGA IV, Proposition 2.7.1] the properties of being a closed or open immersion and of being dominant are geometric. In other words, for the purpose of checking that ${}^T f$

has these properties, we may pass to any field extension L/k . In particular, we may replace k by a splitting field of T and thus assume without loss of generality that $T \rightarrow \text{Spec}(k)$ is split. In this case ${}^T X$, ${}^T Y$ and ${}^T f$ become X , Y , and f , respectively, and the assertions of parts (b) and (c) become obvious. \square

Proposition 3.5. (cf. [CTKPR11, Lemma 3.5]) *Suppose H and G are algebraic k -groups, and G acts on H by group automorphisms. Let $T \rightarrow \text{Spec}(k)$ be a G -torsor. Then ${}^T H$ is a k -form of the algebraic group H . In particular, ${}^T H$ is an affine algebraic k -group.*

Proof. The commutative diagrams defining the group scheme structure on H are all G -equivariant. Applying the twisting functor to these diagrams, and using Corollary 3.4(a), we see that ${}^T H$ is an algebraic group. If L/k is a splitting field of T then clearly $({}^T H)_L$ and H_L are isomorphic. The assertion that H is affine follows by descent; see [EGA IV, Proposition 2.7.1(xiii)]. \square

Proposition 3.6. *Let $T \rightarrow \text{Spec}(k)$ be a G -torsor and let ${}^T G$ denote the twist by T of the conjugation action of G on itself. For every G -variety X , the G -action on X induces a ${}^T G$ -action on ${}^T X$. Moreover, for every G -equivariant morphism f the morphism ${}^T f$ is ${}^T G$ equivariant. In other words, the twisting functor factors through the category of ${}^T G$ -varieties.*

Proof. The action map $G \times X \rightarrow X$ and associated commutative diagrams are all G -equivariant. As in the proof of Proposition 3.5, we obtain an action map ${}^T G \times {}^T X \rightarrow {}^T X$ and commutative diagrams which show that ${}^T f$ is ${}^T G$ equivariant. \square

4. PROOF OF THEOREM 1.1(A) AND (B)

We will use repeatedly the fact that twisting commutes with base field extension. Given a k -variety X , a field extension K/k , and a G -torsor $T \rightarrow \text{Spec}(K)$, we will use the shorthand notation ${}^T X$ to denote ${}^T(X_K)$.

For brevity, we use the following terminology throughout the paper.

Definition 4.1. Let k be a field and G be an algebraic k -group. By a G -twisting pair (T, K) we shall mean a choice of a field extension K/k , with K infinite, and a G -torsor $T \rightarrow \text{Spec}(K)$. In situations where the choice of G is clear from the context and there is no risk of ambiguity, we will simply refer to (T, K) as a *twisting pair*.

Proof of Theorem 1.1(a). Let (T, K) be any twisting pair. Setting $Y = \text{Spec}(K)$ in Proposition 3.2, we see that the K -points of ${}^T X$ are in a natural 1 – 1 correspondence with G -equivariant maps $T \rightarrow X$. \square

The proof of Theorem 1.1(b) is considerably more delicate. Before we proceed with the details, we would like to explain a new obstacle our argument will have to overcome.

Given a G -action on an irreducible X , and a G -twisting pair (T, K) , let us say that X is (T, K) -weakly versal if there exists a morphism $T \rightarrow X$

defined over k . The G -action on X is, by definition, weakly versal if it is (T, K) -weakly versal for every twisting pair (T, K) . Note that our proof of Theorem 1.1(a) establishes the following stronger statement:

Choose a G -twisting pair (T, K) . Then an irreducible G -variety X is (T, K) -weakly versal if and only if ${}^T X$ has a K -point.

Similarly, given a G -twisting pair (T, K) , we will say that an irreducible G -variety X is (T, K) -versal if every dense G -invariant open subvariety of X is (T, K) -weakly versal. One is thus naturally led to try to prove Theorem 1.1(b) by showing that for any given G -twisting pair (T, K) , X is (T, K) -versal if and only if K -points are dense in ${}^T X$. The following example shows that this stronger version of Theorem 1.1(b) fails.

Example 4.2. Let $k = \mathbb{C}$ and let X be a smooth irreducible projective complex curve of genus $g \geq 2$, whose automorphism group $G := \text{Aut}(X)$ is non-trivial. By Hurwitz's theorem, G is finite.

Let $\pi: X \rightarrow X/G$ be the quotient map, let $K := k(X)^G = k(X/G)$, and let $T \rightarrow \text{Spec}(K)$ be the G -torsor obtained by pulling back π via the generic point $\text{Spec}(K) \rightarrow X/G$. Then the G -action on X is (T, K) -versal, since the identity map $X \rightarrow X$ restricts to a G -equivariant morphism $T \rightarrow U$ for any G -invariant open subset $U \subset X$.

On the other hand, we claim that the K -curve ${}^T X$ has only finitely many K -points, and hence, K -points cannot be dense in ${}^T X$. Indeed, arguing as in the proof of Theorem 1.1(a), we see that K -points of ${}^T X$ are in a natural bijective correspondence with G -equivariant k -morphisms $T \rightarrow X$ or equivalently, with G -equivariant rational maps $X \dashrightarrow X$, or equivalently (since X is a smooth complete curve) with G -equivariant morphisms $X \rightarrow X$. The latter can be of two types: (i) dominant and (ii) constant (i.e., the image is a single point of X). It thus suffices to show that there are only finitely many morphisms $X \rightarrow X$ of each type.

(i) Since $g \geq 2$, the Hurwitz formula tells us that any dominant morphism $X \rightarrow X$ is, in fact, an automorphism of X . As we mentioned above, X has only finitely many automorphisms.

(ii) If the image of f is a point of X , this point has to be fixed by G , and X has only finitely many G -fixed points. This completes the proof of the claim. \square

The above example demonstrates that, given a twisting pair (T, K) , we cannot hope to deduce the density of K -points in ${}^T X$ from the fact that X is (T, K) -versal. We will deduce the density of K -points in ${}^T X$, for every twisting pair (T, K) , from the fact that X is (S, F) -versal, where S and F are as follows.

Definition 4.3. For the rest of this section and in Section 5:

- V will denote a generically free linear representation of G ,
- F will denote the field $k(V)^G$. We will choose V so that F is infinite (see Remark 2.3).

- V_0 will denote a dense open G -invariant subvariety of V which is the total space of a G -torsor $U \rightarrow B$.
- $S \rightarrow \text{Spec}(F)$ will denote the G -torsor obtained by pulling back $V_0 \rightarrow B$ via the generic point $\eta: \text{Spec}(F) \rightarrow B$.

We now proceed with the proof of Theorem 1.1(b).

Lemma 4.4. *Let X/k be a geometrically irreducible G -variety, and suppose X is (T, K) -versal for some twisting pair (T, K) . Then, for any field extension L/k and for any proper closed L -subvariety $Y \subsetneq X_L$ (not necessarily G -invariant), there exists a G -invariant k -morphism $\psi: T \rightarrow X$ such that the image of $\psi_L: T \times_k \text{Spec}(L) \rightarrow X_L$ is not contained in Y .*

Proof. First assume $L = k$. If Y is G -invariant, the lemma follows from the definition of (T, K) -versality. If Y is not G -invariant, we proceed as follows. Let Z be the closure of the union $\bigcup \text{im}(\psi)$, where $\psi: T \rightarrow X$ varies over all G -equivariant k -morphisms whose image is contained in Y . Since each ψ is G -equivariant, the closure of each $\text{im}(\psi)$ is G -invariant, as is the closure of their union. In other words, the subvariety Z is G -invariant. Note that $Z \subseteq Y \subsetneq X$. Since X is (T, K) -versal, there is a map $\psi: T \rightarrow X$ whose image is in the complement of Z . By the construction of Z , the image of any such map is not contained in Y . This completes the proof of the lemma in the case where $k = L$.

Now assume L/k is arbitrary. Let $X = U_1 \cup \dots \cup U_m$ be an open affine cover of X defined over k . (We do not assume that the U_i are G -invariant.) The defining equations of Y in each U_i involve only a finite number of elements of L . Let R be the k -subalgebra of L generated by all these elements. Then Y is, in fact, defined over $\text{Spec}(R)$. In other words, there exists a closed k -subvariety $Y_0 \subset X_R = X \times_k \text{Spec}(R)$ such that $Y = Y_0 \times_R L$.

Since $Y \neq X_L$, clearly $Y_0 \neq X_R$. Let $\pi: X_R = X \times_k \text{Spec}(R) \rightarrow X$ be the natural projection and

$$C := \{x \in X \mid \pi^{-1}(x) \subset Y_0\}.$$

Then C is a closed subvariety of X defined over k and $C \neq X$ (because $Y_0 \neq X_R$). As we showed above, there is a G -equivariant k -morphism $\psi: T \rightarrow X$ whose image is not contained in C . Then the image of $\psi \times_k \text{Spec}(R): T \times \text{Spec}(R) \rightarrow X_R$ is not contained in Y_0 and thus, the image of $\psi_L: T \times \text{Spec}(L) \rightarrow X_L$ is not contained in Y . \square

Corollary 4.5. *Let X be a geometrically irreducible k -variety, and let L/k be a field extension. Note that there is a natural inclusion of sets $X(k) \hookrightarrow X_L(L)$ by pulling back $\text{Spec}(k) \rightarrow X$ by $\text{Spec}(L) \rightarrow \text{Spec}(k)$. Then $X(k)$ is dense in X if and only if $X(k)$ is dense in X_L .*

Proof. If L is finite then the result is immediate: in this case $X(k)$ is dense in X if and only if $X(k)$ is dense in $X(L)$ if and only if X is a point. Thus we may assume that L is infinite. The morphism $X_L \rightarrow X$ is dominant, so one

direction is obvious. The other implication is a special case of Lemma 4.4 with $K = L$, $G = \{1\}$ and $T = \text{Spec}(K)$. \square

Lemma 4.6. *Let X/k be a geometrically irreducible G -variety, let (T, K) be a twisting pair, and let L/K be a field extension which splits T . Fix an L -point $s \in T(L)$. Then the following are equivalent:*

- (a) $({}^T X)(K)$ is dense in ${}^T X$,
- (b) the set of points $f(s)$, where f varies over all G -equivariant k -morphisms $f: T \rightarrow X$, is dense in X_L .

Note that condition (b) is considerably stronger than the condition that the union of $f(T)$ is dense in X_L , which came up in Lemma 4.4. This discrepancy is precisely the source of the difficulty we encountered in Example 4.2.

Proof. Since X is geometrically irreducible, so is ${}^T X$. By Corollary 4.5, condition (a) is equivalent to

- (c) $({}^T X)(K)$ is dense in $({}^T X)_L$.

Let t_s be an L -isomorphism between $({}^T X)_L$ and X_L , chosen so that

$$(s, t_s): ({}^T X)_L \rightarrow T_L \times X_L$$

is a section (defined over L) of the G -torsor $T \times X \rightarrow {}^T X$, as in the statement of Corollary 3.3. Then (c) is equivalent to

- (d) the set of L -points of the form $t_s(q)$, where q varies over $({}^T X)(K)$, is dense in X_L .

By Corollary 3.3, (d) is equivalent to (b). \square

Proof of Theorem 1.1(b). \Leftarrow (cf. [FF08, Proposition 1.12]): Assume K -points are dense in ${}^T X$ for every twisting pair (T, K) . We want to show that every dense G -invariant open subset $U \subset X$ is weakly versal. By Theorem 1.1(a) it suffices to show that ${}^T U$ contains a K -point for every twisting pair (T, K) , as above. This follows from the fact that ${}^T U$ is a dense open subset of ${}^T X$; see Corollary 3.4(b).

\Rightarrow : Assume X is versal. Then X is geometrically irreducible; see Proposition 2.4. Fix a twisting pair (T, K) . We want to show K -points are dense in ${}^T X$. Let L be a splitting field for T , and let s be a point in $T(L)$. By Lemma 4.6, it suffices to show that for every closed subset $Y \subsetneq X_L$ defined over L , there exists a G -equivariant k -morphism $f: T \rightarrow X$ such that $f(s) \notin Y$.

As explained above, we cannot construct f directly using only the fact that X is (T, K) -versal. We will instead construct f in two steps, as a composition of a G -equivariant k -morphism $f_1: T \rightarrow V$ and a G -equivariant rational map $f_2: V \dashrightarrow X$. Here V , S and F are as in Definition 4.3.

Let us begin by constructing f_2 . By Lemma 4.4, there exists a G -equivariant k -morphism $\psi: S \rightarrow X$ such that the image of ψ_L is not contained in Y . Equivalently, there exists a G -equivariant rational k -map

$f_2: V \dashrightarrow X$ such that the image of $(f_2)_L$ is not contained in Y . (Note that our construction of f_2 makes use of the fact that X is (S, F) -versal, not just (T, K) -versal.)

Now let U be an open subset of V_L such that $(f_2)_L$ is regular on U and $(f_2)_L(U) \cap Y = \emptyset$. Replacing X by V simplifies matters considerably, because we know that ${}^T V(K)$ is dense in ${}^T V \simeq V_K$; see Lemma 3.1. Thus, by Lemma 4.6, there exists a G -equivariant k -morphism $f_1: T \rightarrow V$ such that $f_1(s) \in U$. In particular, there exists a component of T whose image under f_1 intersects the domain of definition of f_2 . Since T is a torsor over a field K , and f_2 is G -equivariant, the composition $f_2 f_1$ will be a regular G -equivariant morphism $T \rightarrow X$. Moreover, since $f_1(s) \in U$, we have $f_2 f_1(s) \notin Y$, as desired. This completes the proof of Theorem 1.1(b). \square

5. PROOF OF THEOREM 1.1(C) AND (D)

Let

$$\begin{array}{ccccc} S & \longrightarrow & V_0 & \hookrightarrow & V \\ \downarrow & & \downarrow & & \\ \mathrm{Spec}(F) & \xrightarrow{\eta} & B & & \end{array}$$

be as in Definition 4.3.

Lemma 5.1. *Let X, Y be geometrically irreducible k -varieties and suppose X has a G -action. If there exists a dominant rational (resp. birational) F -map $f: Y_F \dashrightarrow {}^S X$ then there exists a G -equivariant dominant rational (resp. birational) k -map $V \times Y \dashrightarrow V \times X$.*

Proof. Choose an open F -subvariety $Z \subset Y_F$ such that $f|_Z: Z \rightarrow {}^S X$ is regular. By Proposition 3.2, f gives rise to a commutative diagram

$$\begin{array}{ccc} S \times_F Z & \longrightarrow & S \times_F X_F \\ \downarrow & & \downarrow \\ Z & \xrightarrow{f|_Z} & {}^S X_F \\ & \searrow & \downarrow \\ & & \mathrm{Spec}(F), \end{array}$$

where the vertical maps in the square are G -torsors. By a well known property of torsors, since the map $Y_F \dashrightarrow {}^S X$ is dominant (resp. birational), so is the top horizontal map.

Note that Z may not descend to a variety over k ; however, after replacing B by a dense open subset, we may assume that the open immersion $Z \subset Y_F$ descends to B , i.e., there exists a k -variety Z' such that the pull-back

diagram

$$\begin{array}{ccc}
 & & Y_F \longrightarrow Y_B \\
 & \nearrow & \nearrow \\
 Z & \longrightarrow & Z' \\
 \downarrow & & \downarrow \\
 \text{Spec}(F) & \xrightarrow{\eta} & B
 \end{array}$$

commutes and $Z' \subset Y_B$ is an open immersion; see [EGA IV, Proposition 2.7.1(x)].

By the naturality of the fiber product operation, the G -equivariant F -map

$$S \times_F Z \rightarrow S \times_F X_F$$

is G -equivariantly isomorphic to an F -map

$$(V_0 \times_B Z')_F \rightarrow (V_0 \times_B X_B)_F ;$$

see [EGA I, Corollaire 3.3.10]. Shrinking B once more, we obtain a dominant (resp. birational) G -equivariant map $V_0 \times_B Z' \rightarrow V_0 \times_B X_B$ such that the commutative triangle on the left is the pull-back of the commutative triangle on the right to the generic point η :

$$\begin{array}{ccc}
 S \times_F Z \longrightarrow S \times_F X_F & & V_0 \times_B Z' \longrightarrow V_0 \times_B X_B \\
 \searrow & \downarrow & \searrow \\
 & \text{Spec}(F) & \xrightarrow{\eta} B
 \end{array}$$

Since $Z' \hookrightarrow Y_B$ is an open immersion, we obtain a dominant (resp. birational) G -equivariant rational map

$$(5.1) \quad V_0 \times_B Y_B \dashrightarrow V_0 \times_B X_B$$

We now note that $V_0 \times_B Y_B \simeq V_0 \times_k Y$ and $V_0 \times_B X_B \simeq V_0 \times_k X$, where \simeq denotes G -equivariant isomorphism over k . Thus (5.1) gives us a dominant (resp. birational) G -equivariant rational k -map $V_0 \times_k Y \dashrightarrow V_0 \times_k X$ or equivalently, a dominant (resp. birational) G -equivariant rational k -map $V \times_k Y \dashrightarrow V \times_k X$, as desired. \square

Proof of Theorem 1.1(c). \implies : Suppose X is very versal, i.e., there exists a dominant G -equivariant map $f: W \dashrightarrow X$. Then for any twisting pair (T, K) , the K -rational map ${}^T f: {}^T W \dashrightarrow {}^T X$ is dominant; see Corollary 3.4. By Lemma 3.1, ${}^T W \simeq_K W_K$. Thus ${}^T X$ is K -unirational.

\impliedby : By assumption, there exists a dominant rational map $\mathbb{A}_F^n \dashrightarrow {}^S X$ for some integer n . By Lemma 5.1, we obtain a dominant rational G -equivariant k -map $V \times \mathbb{A}_k^n \dashrightarrow V \times X$, where G acts trivially on \mathbb{A}_k^n . Composing this map with the projection $V \times X \rightarrow X$, we see that X is very versal. \square

Proof of Theorem 1.1(d). \implies : Suppose the G -action on X is stably birationally linear, i.e., there exists a G -equivariant birational isomorphism $\phi: X \times W_1 \xrightarrow{\sim} W_0$ for some linear representations $G \rightarrow \mathrm{GL}(W_1)$ and $G \rightarrow \mathrm{GL}(W_0)$ defined over k . Twisting ϕ by a twisting pair (T, K) , we obtain a birational isomorphism

$${}^T\phi: ({}^T X) \times_K ({}^T W_1) \xrightarrow{\sim} {}^T W_0.$$

Since ${}^T W_1$ and ${}^T W_0$ are affine spaces over K (cf. Lemma 3.1), this tells us that ${}^T X$ is stably rational over K .

\impliedby : By assumption, there is a birational isomorphism $\mathbb{A}_F^n \xrightarrow{\sim} {}^S X \times \mathbb{A}_F^m$ defined over F . Now note that ${}^S X \times \mathbb{A}_F^m \simeq {}^S(X \times \mathbb{A}_k^m)$, where G acts trivially on \mathbb{A}_k^m ; cf. Corollary 3.4(a). By Lemma 5.1, we obtain a G -equivariant birational isomorphism $V \times \mathbb{A}_k^n \xrightarrow{\sim} V \times X \times \mathbb{A}_k^m$, defined over k . Here G acts trivially on both \mathbb{A}_k^n and \mathbb{A}_k^m . This shows that the G -action on X is stably birationally linear. \square

Example 5.2. (cf. [CTKPR11, Proposition 3.3]) Let H be a connected algebraic group. If $\mathrm{Char}(k) > 0$, assume that H is reductive. Then every action of an algebraic group G on H by group automorphisms is very versal.

Proof. Let (T, K) be a twisting pair. By Proposition 3.5, ${}^T H$ carries the structure of an affine algebraic group over K . By Chevalley's theorem [Bor91, Theorem 18.2(ii)] ${}^T H$ is unirational over K . The desired conclusion now follows from Theorem 1.1(c). \square

6. FORMS OF $\overline{M}_{0,n}$

Proposition 6.1. *Let k be a base field of characteristic 0 and $\overline{M}_{0,n}$ be the moduli space of stable curves of genus 0 with $n \geq 5$ marked points, defined over k . Suppose K/k is a field extension. Then every K -form of $\overline{M}_{0,n}$ is K -unirational.*

It is well known that $\overline{M}_{0,5}$ is a Del Pezzo surface of degree 5 and that every Del Pezzo surface of degree 5 over a field K/k is a K -form of $\overline{M}_{0,5}$. Thus, for $n = 5$, we recover the theorem of H. P. F. Swinnerton-Dyer [SD72]; about the existence of rational points on such surfaces (in characteristic 0 only). For alternative proofs of Swinnerton-Dyer's theorem, see [SB92] and [Sko93].

Proof. The natural action of S_n on $X = M_{0,n}$ permuting the n points on \mathbb{P}^1 extends to $\overline{M}_{0,n}$. Our proof relies on a recent theorem of A. Bruno and M. Mella [BM10] which says that S_n is, in fact, the full automorphism group of $\overline{M}_{0,n}$. (In [BM10] the base field is assumed to be the field of complex numbers. However, using the Lefschetz principle one easily deduces that the automorphism group of $\overline{M}_{0,n}$ is S_n over any base field k of characteristic 0.) Consequently, every K -form of $\overline{M}_{0,n}$, over a field extension K/k is isomorphic to ${}^T X$ for some S_n -torsor $T \rightarrow \mathrm{Spec}(K)$.

By Theorem 1.1(c) it suffices to show that the S_n -action on $\overline{M_{0,n}}$ is very versal. To do this, consider dominant S_n -equivariant maps

$$(\mathbb{A}^2)^n \dashrightarrow (\mathbb{P}^1)^n \dashrightarrow \overline{M_{0,n}}.$$

Here the first map is the n th power of the natural projection $\mathbb{A}^2 \setminus \{(0,0)\} \rightarrow \mathbb{P}^1$, and the second map takes an n -tuple of distinct points on \mathbb{P}^1 to its class in $M_{0,n}$. The symmetric group S_n acts on the $2n$ -dimensional affine space $(\mathbb{A}^2)^n$ linearly, by permuting the n factors of \mathbb{A}^2 . \square

7. HOMOGENEOUS SPACES

Proposition 7.1. *Let A be a (not necessarily connected) algebraic group. If $\text{Char}(k) > 0$, assume that A is reductive. Suppose G and B are closed subgroups of A , and $X := A/B$ is geometrically irreducible. Consider X as a G -variety. The following are equivalent:*

- (a) X is very versal,
- (b) X is versal,
- (c) X is weakly versal,
- (d) the image of the natural map $H^1(K, G) \rightarrow H^1(K, A)$ is contained in the image of the natural map $H^1(K, B) \rightarrow H^1(K, A)$ for every field extension K/k where K is infinite.

Proof. Let (T, K) be a twisting pair. In view of Proposition 3.6, ${}^T X$ is a homogeneous space for the twisted group ${}^T A$.

By Theorem 1.1 it suffices to show that the following conditions are equivalent:

- (i) ${}^T X$ has a K -point,
- (ii) There exists a dominant ${}^T A$ -equivariant map $f: {}^T A \rightarrow {}^T X$ defined over K ,
- (iii) ${}^T X$ is K -unirational,
- (iv) K -points are dense in ${}^T X$,
- (v) The class of T lies in the image of the natural map $H^1(K, B) \rightarrow H^1(K, A)$.

(i) \implies (ii) By Proposition 3.6 ${}^T A$ acts on ${}^T X$. If $p \in X(K)$, we can define f to be the orbit map $f(g) = g \cdot p$. Passing to a splitting field of T , we see that f is dominant.

(ii) \implies (iii) is an immediate consequence of Chevalley's theorem [Bor91, Theorem 18.2(ii)], which asserts that, under our assumption on A , ${}^T A$ is unirational.

The implications (iii) \implies (iv) \implies (i) are obvious.

(ii) \iff (v) is proved [Ser02, Proposition I.5.37]; see also [Spr66, Proposition 1.11]. \square

Example 7.2. We record several interesting special cases of Proposition 7.1 when A is connected.

(a) Suppose $B = \{1\}$. Then the translation action of a subgroup G on A is versal if and only if the natural map $H^1(K, G) \rightarrow H^1(K, A)$ is trivial for every field extension K/k , where K is infinite. The same is true whenever B is a special group, i.e., whenever $H^1(K, B)$ is trivial for every K/k .

(b) Setting $B = G$ yields the following: For any closed subgroup $G \subset A$, the translation action of G on A/G is versal.

(c) If B is the normalizer of a maximal torus in A , we see that the translation action of G on A/B is versal for any $G \subset A$. This is because the natural map $H^1(K, B) \rightarrow H^1(K, A)$ is surjective for every field extension K/k ; see [Ser02, III.4.3, Lemma 6] if K is perfect and [CGR08, Corollary 5.3] otherwise.

8. p -VERSALITY

Throughout this section, p is a prime number.

Definition 8.1. Let G/k be an algebraic group and let X/k be an irreducible G -variety. We say that X is

- *weakly p -versal* if for every twisting pair (T, K) , there exists a field extension L/K of degree prime to p and a G -equivariant k -morphism $T_L \rightarrow X$.
- *p -versal* if every G -invariant dense open subset $U \subset X$ is weakly p -versal (cf. [Mer09, Section 2.2]).

Recall that a field L is called *p -closed* if the degree of every finite field extension of L is a power of p . For every field K , there exists an algebraic extension $K^{(p)}/K$, such that $K^{(p)}$ is p -closed and, for every finite subextension $K \subset K' \subset L$, the degree $[K' : K]$ is prime to p . The field $K^{(p)}$ satisfying these conditions is unique up to K -isomorphism; it usually called the *p -closure* of K and is denoted by $K^{(p)}$. For details, see [EKM08, Proposition 101.16].

Lemma 8.2. *Let X be a geometrically irreducible G -variety defined over k . Then the following conditions are equivalent:*

- (a) X is weakly p -versal,
- (b) for every twisting pair (T, K) , ${}^T X$ has a point whose degree over K is prime to p ,
- (c) for every twisting pair (T, K) , ${}^T X$ has a 0-cycle whose degree is prime to p ,
- (d) for every twisting pair (T, K) , the variety $({}^T X)_{K^{(p)}}$ has a 0-cycle of degree 1,
- (e) for every twisting pair (T, K) , the variety ${}^T X$ has a $K^{(p)}$ -point.

Proof. (a) \iff (b): By Proposition 3.2, the existence of an L -point of ${}^T X$ is equivalent to the existence of a G -equivariant k -morphism $T_L \rightarrow X$.

(b) \implies (c) is obvious.

(c) \implies (d): Suppose $Z \subset {}^T X$ is a 0-cycle of degree d , where d is prime to p . Since the degree of every point of $({}^T X)_{K^{(p)}}$ is a power of p , there

exists a 0-cycle $Z' \subset ({}^T X)_{K^{(p)}}$ whose degree is a power of p . A desired 0-cycle of degree 1 on $({}^T X)_{K^{(p)}}$ can then be constructed as an integer linear combination of Z and Z' .

(d) \implies (e): (cf. [Ful85, Example 13.1]) This is immediate from the fact that the degree of every closed point on $({}^T X)_{K^{(p)}}$ is a power of p .

(e) \implies (b): Every $K^{(p)}$ -point of ${}^T X$ descends to a finitely generated subextension $K \subset L \subset K^{(p)}$. The field L is then a finite extension of K whose degree is prime to p . \square

Theorem 8.3. *Let G be an algebraic k -group acting on a smooth geometrically irreducible k -variety X . Then X is p -versal if and only if X is weakly p -versal.*

Proof. We will assume that X is weakly p -versal and prove that X is p -versal; the other direction is obvious.

Let $U \subset X$ be a G -invariant dense open subvariety. We want to show that U is weakly versal. Let (T, K) be a twisting pair. By Lemma 8.2, it suffices to prove that if ${}^T X$ has a 0-cycle whose degree is prime to p , then so does ${}^T U$. Since ${}^T U$ is a dense open subvariety of ${}^T X$ (see Corollary 3.4(b)), this is a special case of Chow's Moving Lemma [R70]. \square

Corollary 8.4. (a) *Let X/k be a geometrically irreducible generically smooth G -variety.*

(a) *Assume that G has a closed subgroup H whose index is finite and prime to p . Then the G -action on X is p -versal if and only if the restricted H -action is p -versal.*

(b) *Suppose there exists a smooth k -point $x \in X(k)$ such that the orbit $G \cdot x$ is finite and $\deg([G \cdot x])$ is prime to p . Then the G -action on X is p -versal.*

Proof. After replacing X by its smooth locus, we may assume that X is smooth.

(a) From the proof of [MR09, Lemma 4.1], for any field K/k , the map $H^1(K, H) \rightarrow H^1(K, G)$ is p -surjective. That is, for any $\alpha \in H^1(K, G)$ there exists a finite extension L/K of degree prime to p such that α_L lies in the image of the natural map $H^1(L, H) \rightarrow H^1(L, G)$. If K is p -closed, then $[L : K]$ is a power of p , so $L = K$, and the map $H^1(K, H) \rightarrow H^1(K, G)$ is surjective. In other words, for any H -torsor $T \rightarrow \text{Spec}(K)$, there exists a G -torsor $T' \rightarrow \text{Spec}(K)$ such that ${}^T X$ and ${}^{T'} X$ become isomorphic over $K^{(p)}$. In particular, ${}^T X$ has a $K^{(p)}$ -point if and only if ${}^{T'} X$ has a $K^{(p)}$ -point. Lemma 8.2 now tells us that the G -action on X is weakly p -versal if and only if the H -action is weakly p -versal. By Theorem 8.3, the same is true if “weakly p -versal” is replaced by “ p -versal”.

(b) Let H be the stabilizer of x in G . Then x is fixed by H , and the index $[G : H] = \deg([G \cdot x])$ is finite and prime to p . By Proposition 2.2, the H -action on X is weakly versal. By Theorem 8.3, the H -action on X is p -versal. By part (a) the G -action on X is p -versal as well. \square

We also note the following immediate consequence of Theorem 8.3 and Lemma 8.2, in the spirit of Theorem 1.1.

Corollary 8.5. *A G -action on a smooth geometrically irreducible variety X is p -versal for every prime p if and only if, for every twisting pair (T, K) , ${}^T X$ has a 0-cycle of degree 1. \square*

Every versal G -variety is clearly p -versal for every prime p . However, the converse is not true in general, even if $G = \{1\}$; after all, there exist k -varieties with 0-cycles of degree 1 but no k -points. On the other hand, no counterexample is known for the following weaker statement:

Conjecture 8.6 (cf. [Dun09]). *Let G be a finite constant group, X be a G -variety and G_p be a Sylow p -subgroup of G . If X is G_p -versal for every prime p , then X is G -versal.*

Note that the key assumption here is that X is versal and not just p -versal as a G_p -variety.

Remark 8.7. It is natural to define a G -variety X to be “very p -versal” if there exists a linear representation V , and a diagram of dominant rational G -equivariant maps of the form

$$\begin{array}{ccc} V' & & \\ | & \searrow & \\ | & & \\ \downarrow & & \\ V & & X, \end{array}$$

where the degree of $V' \dashrightarrow V$ is prime to p . (Note that V' is not required to be a vector space.) Under mild assumptions on X this notion also turns out to be equivalent to p -versality.

9. PROJECTIVE REPRESENTATIONS

Let G be a finite subgroup of PGL_n defined over k and G' be the preimage of G in GL_n . The diagram

$$(9.1) \quad 1 \longrightarrow \mathbb{G}_m \longrightarrow G' \longrightarrow G \longrightarrow 1$$

gives rise to the connecting morphism $\partial_K : H^1(K, G) \rightarrow H^2(K, \mathbb{G}_m)$ for every field K/k .

Proposition 9.1. (cf. [Dun09, Corollary 3.4]) *Let G be a finite subgroup of PGL_n defined over k . Then the following conditions are equivalent:*

- (a) *The G -action on \mathbb{P}^{n-1} is stably birationally linear,*
- (b) *the G -action on \mathbb{P}^{n-1} is very versal,*
- (c) *the G -action on \mathbb{P}^{n-1} is versal,*
- (d) *the G -action on \mathbb{P}^{n-1} is weakly versal,*
- (e) *the G -action on \mathbb{P}^{n-1} is p -versal for every prime p ,*
- (f) *$\partial_K = 0$ for every K/k .*
- (g) *G lifts to a subgroup of GL_n , i.e., the exact sequence (9.1) splits.*

Proof. Let (T, K) be a G -twisting pair. Then $X = {}^T(\mathbb{P}^{n-1})$ is a Brauer-Severi variety over K whose class is $\partial_K([T])$, where $[T]$ is the class of T in $H^1(K, G)$. It is well known that a Brauer-Severi variety X over K is K -rational if and only if X has a zero cycle of degree 1 if and only if the class of X in $H^2(K, \mathbb{G}_m)$ is trivial. This shows that condition (a) - (f) are all equivalent.

(g) \implies (b). If G lifts to GL_n then the natural projection map $\mathbb{A}^n \setminus \{0\} \rightarrow \mathbb{P}^{n-1}$ is dominant and G -equivariant, and (b) follows.

(f) \implies (g). By [KM, Theorem 4.4 and Remark 4.5], (f) implies

$$\gcd_{\rho} \dim(\rho) = 1,$$

as ρ ranges over representations $G' \rightarrow \mathrm{GL}(V)$ such that $\rho(t) = tI_V$ for every $t \in \mathbb{G}_m$. Here I_V is the identity map on V . Thus there exist representations ρ_1, \dots, ρ_m of G and integers d_1, \dots, d_m such that the multiplicative character $\chi = \det(\rho_1)^{d_1} \dots \det(\rho_m)^{d_m} : G' \rightarrow \mathbb{G}_m$ has the property that $\chi(t) = t$ and hence splits the sequence (9.1) ($\mathrm{Ker}(\chi)$ is a complement of \mathbb{G}_m in G'). \square

10. GROUP ACTIONS ON QUADRIC AND CUBIC HYPERSURFACES

Lemma 10.1. *Let V be a finite-dimensional k -vector space and $G \rightarrow \mathrm{GL}(V)$ be a linear representation. Then*

- (a) *for any twisting pair (T, K) , ${}^T\mathbb{P}(V)$ is K -isomorphic to $\mathbb{P}(V)_K$.*
- (b) *Suppose X be a closed G -invariant subvariety of $\mathbb{P}(V)$. Then the inclusion $\iota : X \hookrightarrow \mathbb{P}(V)$ induces a closed embedding ${}^T\iota : {}^TX \hookrightarrow \mathbb{P}(V)_K$ with the same Hilbert polynomial as X .*

Proof. (a) By Lemma 3.1, ${}^TV \simeq V_K$. The (T, K) -twist of the natural projection $V \dashrightarrow \mathbb{P}(V)$, is thus a dominant rational map $V_K \dashrightarrow {}^T\mathbb{P}(V)$. Consequently, the Brauer-Severi variety ${}^T\mathbb{P}(V)$ has a K -point, and part (a) follows.

(b) Since the embeddings ${}^T\iota : {}^TX \rightarrow \mathbb{P}(V)_K$ and $\iota : X \rightarrow \mathbb{P}(V)$ become projectively equivalent over the algebraic closure \bar{K} , they have the same Hilbert polynomial. \square

Theorem 10.2. *Let G be an algebraic group over k , $G \rightarrow \mathrm{GL}(V)$ be a finite-dimensional k -representation, and $X \subset \mathbb{P}(V)$ be an irreducible, quadratic G -invariant hypersurface. The following are equivalent:*

- (a) *X is stably birationally linear,*
- (b) *X is very versal,*
- (c) *X is versal,*
- (d) *X is weakly versal.*
- (e) *X is 2-versal.*

Assume further that G is finite, and G_2 is a Sylow 2-subgroup of G . Then conditions (a) - (e) are equivalent to

- (f) *X is versal for the action of G_2 .*

Proof. Let (T, K) be a twisting pair. By Lemma 10.1 $Q := {}^T X$ is an irreducible quadratic hypersurface in \mathbb{P}_K^n , defined over K . The equivalence of conditions (a)–(d) now follows from the following well-known property of irreducible quadric hypersurfaces $Q \subset \mathbb{P}(V)_K$:

$$(10.1) \quad \text{if } Q \text{ has a } K\text{-point then } X \text{ is } K\text{-rational.}$$

The equivalence of (a) and (e) is an immediate consequence of Springer’s theorem: if Q has an L -point for some odd degree extension L/K then Q has a K -point.

If G is a finite group then (f) \implies (e) by Corollary 8.4(a). On the other hand, (b) \implies X is very versal as a G_2 -variety \implies (f). \square

If we replace a quadric hypersurface by a cubic hypersurface of dimension ≥ 2 then property (10.1) in the above proof remains true, provided that “rational” is replaced by “unirational”, and Springer’s Theorem becomes an open conjecture. The precise statements are as follows.

Theorem 10.3. ([Kol02]) *Let $X \subset \mathbb{P}_k^n$ be a smooth cubic hypersurface where $n \geq 3$. If X has a k -point then X is k -unirational.*

Conjecture 10.4 (J. W. S. Cassels, P. Swinnerton-Dyer; see [Cor76]). *Suppose $X \subset \mathbb{P}_k^n$ is a cubic hypersurface. If X has a 0-cycle of degree prime to 3, then X has a k -point.*

The argument we used to prove Theorem 10.2 now yields the following analogous statement for cubic hypersurfaces.

Theorem 10.5. *Let G be an algebraic k -group, $G \rightarrow \mathrm{GL}(V)$ be a finite-dimensional k -representation and $X \subset \mathbb{P}(V)$ be a smooth G -invariant cubic hypersurface. Assume $\dim(V) \geq 4$. Then following are equivalent:*

- (a) X is very versal,
- (b) X is versal,
- (c) X is weakly versal,

Now suppose G is finite, G_3 is a Sylow 3-subgroup of G , and Conjecture 10.4 holds. Then (a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e), where

- (d) X is 3-versal, and
- (e) X is versal for the action of G_3 . \square

Corollary 10.6. *Suppose an algebraic group G acts on a smooth cubic hypersurface X as in Theorem 10.5. If G fixes a k -point $x \in X(k)$ then X is G -versal.*

Proof. By Proposition 2.2 the G -action on X is weakly versal. Theorem 10.5 now tells us that this action is versal. \square

We now recall the definitions of two important numerical invariants of a finite group G . The *essential dimension*, $\mathrm{ed}(G)$ of G is the minimal dimension of a versal G -variety with a faithful G -action; see [BR97]. The *Cremona*

dimension, $\text{Crdim}(G)$ is the minimal integer n such that G embeds into the Cremona group $\text{Cr}(n)$ of birational automorphisms of the affine space \mathbb{A}^n .

For the rest of this section we will assume that the base field k is the field \mathbb{C} of complex numbers.

Conjecture 10.7. (*I. Dolgachev, unpublished*) $\text{ed}(G) \geq \text{Crdim}(G)$ for every finite group G .

Proposition 10.8. (a) Conjecture 8.6 implies $\text{ed}(\text{PSL}_2(\mathbb{F}_{11})) \leq 3$.

(b) Conjecture 10.4 implies $\text{ed}(\text{PSL}_2(\mathbb{F}_{11})) \leq 3$.

(c) Conjecture 10.7 implies $\text{ed}(\text{PSL}_2(\mathbb{F}_{11})) \geq 4$.

Proof. Consider the Klein cubic, i.e., the smooth cubic threefold $X \subset \mathbb{P}^4$ cut out by

$$x_0^2x_1 + x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_0 = 0.$$

The automorphism group of X is $G = \text{PSL}_2(\mathbb{F}_{11})$. The action of this group on X is induced by a linear representation $G \rightarrow \text{GL}_5$ [Adl78].

It is shown in [Bea11] that X has a G_p -fixed point x_p for any p -Sylow subgroup G_p of G . Hence, by Corollary 10.6, the G_p -action on X is versal for every prime p . Now

(a) Conjecture 8.6 implies that X is G -versal. Thus $\text{ed}(G) \leq \dim(X) \leq 3$.

(b) Conjecture 10.4 also implies that the G -action on X is versal; see Theorem 10.5. Consequently, $\text{ed}(G) \leq 3$, as in part (a).

(c) From [Pro09, Remark 1.6] we see that there are no rational complex threefolds with a faithful action of $\text{PSL}_2(\mathbb{F}_{11})$. (In particular, the Klein cubic threefold X is not rational.) Thus $\text{Crdim}(\text{PSL}_2(\mathbb{F}_{11})) \geq 4$, and part (c) follows. \square

We conclude that Conjectures 10.4 and 10.7 are incompatible; they cannot both be true. Same for Conjectures 8.6 and 10.7.

Remark 10.9. It is easy to show that $3 \leq \text{ed}(\text{PSL}_2(\mathbb{F}_{11})) \leq 4$. Thus the three inequalities in the statement of Proposition 10.8 can be replaced by equalities. If we knew whether $\text{ed}(\text{PSL}_2(\mathbb{F}_{11}))$ is 3 or 4, we would be able to complete the classification of finite simple groups of essential dimension 3 over \mathbb{C} . For details, see [Bea11].

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APPENDIX: LETTER FROM J.-P. SERRE TO Z. REICHSTEIN

Paris, June 10, 2010

Dear Reichstein,

About “versal” :

There was first the notion of a “universal object” , a notion which appeared in several branches of mathematics around 1930-1950; there is even a section of Bourbaki’s *Théorie des Ensembles* (chap.IV, §2) on the general properties of this notion. An especially interesting case being the universal G -principal homogeneous space (now “ G -torsor”); the case of $G = \mathrm{GL}(n)$ was basically due to Chern. Such spaces ($E_G \rightarrow B_G$ was the standard notation) were very useful to topologists; see e.g. Borel’s thesis.

In the definition of “universal”, there is a uniqueness property (up to homotopy, sometimes) which is required. There are many interesting cases where it does not hold (e.g. deformations of complex manifolds, à la Kodaira-Spencer); people called them “almost universal” (or quasi , or semi . . .). I do not know exactly when somebody had the amusing idea to call them “versal”, by deleting the “uni” which suggests uniqueness. I seem to remember that it was Douady who did this (he enjoyed playing with words); the date should be close to 1966, but I have not looked into his publications, and I cannot give you a precise reference.¹

That this idea applied to Galois cohomology was obvious from the beginning, both to people with a topologist background (such as Rost or myself), and to algebraists trying to parametrize equations (they rather used the word “generic”, which I find a bit confusing). But I don’t think^(*) the word “versal” got into print [in this context] before my UCLA lectures of 2001 (do you know an earlier reference?)², even though I had used it in some College lectures around 1990 (especially those on “negligible cohomology”, which were never written down).

Note that the definition in UCLA has a rather non standard restriction: it asks for a density property which may seem artificial (but it is essential in Duncan’s work!).

Best wishes,
J-P.Serre

(*) I have asked Google about “versal torsor”, but all the references there seem to be post 2001.

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¹The earliest reference I have been able to find is [Dou60], page 2-04. Z.R.

²The term *versal* is used in [BR97], Section 7. The *versal polynomials* defined there give rise to versal G -torsors, in the sense of [GMS03, Section I.5], where G is a finite group. Z.R.

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