AROUND RATIONALITY OF CYCLES

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Abstract. In this article we prove certain results comparing rationality of algebraic cycles over the function field of a quadric and over the base field. Those results have already been proved by Alexander Vishik in the case of characteristic 0, which allowed him to work with algebraic cobordism theory. Our proofs use the modulo 2 Steenrod operations in the Chow theory and work in any characteristic $\neq 2$.

In characteristic 0, the results of this note (Theorem 1.1 and 2.4 and Proposition 2.1) have been proved several years ago by Alexander Vishik in [3] (exact references are given right before each statement) with the help of the algebraic cobordism theory and especially *symmetric operations* of [4]. In fact, putting aside characteristic, the original versions are stronger. Indeed, an exponent 2 element appears in our conclusions, while the use of symmetric operations in the algebraic cobordism theory allows to obtain results without an exponent 2 element (see Remark on page 370 of [3]).

In a way, most results of this note are generalizations of some results proved by Nikita Karpenko in [2]. Our proofs are, to a great extent, inspired by the proofs of [2].

In our proofs, the base field is allowed to be of any characteristic different from 2 because the Landweber-Novikov operations used in [3, Remark after proof of Theorem 3.1] are replaced here by the Steenrod operations on the modulo 2 Chow groups.

We refer to [3] and [2] for an introduction into the subject. Notation is introduced in the beginning of Section 1.

1. Main Result

Let F be a field of characteristic $\neq 2$, Q a smooth projective quadric over F of dimension $n \geq 0$, Y a smooth quasi-projective F-variety (a *variety* is a separated scheme of finite type over a field).

The function field $F(Q)$ is defined if $n \geq 1$ or if Q is anisotropic. In the case of $n = 0$ and isotropic Q we have $Q = SpecF \coprod SpecF$ and we set $F(Q) := F$.

We write $CH(Y)$ for the integral Chow group of Y (see [1, Chapter X]) and we write $Ch(Y)$ for $CH(Y)$ modulo 2. We write $\overline{Y} := Y_{\overline{F}}$ where $\overline{F}$ is an algebraic closure of F. Let X be a geometrically integral variety over F. An element $\overline{y}$ of $Ch(\overline{Y})$ (or of $CH(\overline{Y})$) is $F(X)$-*rational* if its image $\overline{y}_{\overline{F}(X)}$ under $Ch(\overline{Y}) \to Ch(Y_{\overline{F}(X)})$ (resp. $CH(\overline{Y}) \to CH(Y_{\overline{F}(X)})$) is in the image of $Ch(Y_{F(X)}) \to Ch(Y_{\overline{F}(X)})$ (resp. $CH(Y_{F(X)}) \to CH(Y_{\overline{F}(X)})$). Finally, an element $\overline{y}$ of $Ch(\overline{Y})$ (or of $CH(\overline{Y})$) is called *rational* if it is in the image of $Ch(Y) \to Ch(\overline{Y})$ (resp. $CH(Y) \to CH(\overline{Y})$).

In a way, the following result is a generalization of [3, Theorem 3.1(1)]. Indeed, the use of the Steenrod operations on the modulo 2 Chow groups allows to obtain a valid result in any characteristic different from 2. Nevertheless, an exponent 2 element appears in our conclusion while it is not the case in [3, Theorem 3.1(1)]. In addition, this result is also a generalization of [2, Theorem 2.1] in the sense that it allows a larger codimension for the considered cycle.

Theorem 1.1. *Assume that $m < n/2+j$. Let $\overline{y}$ be an $F(Q)$-rational element of $Ch^m(\overline{Y})$. Then $S^j(\overline{y})$ is the sum of a rational element and the class modulo 2 of an integral element of exponent 2.*

Proof. We assume that $m \geq 0$ in the proof. We also assume that $j \leq m$ (otherwise we get $S^j(\overline{y}) = 0$). The element $\overline{y}$ being $F(Q)$-rational, there exists $y \in Ch^m(Y_{F(Q)})$ mapped to $\overline{y}_{\overline{F(Q)}}$ under the homomorphism

$$Ch^m(Y_{F(Q)}) \to Ch^m(Y_{\overline{F(Q)}}).$$

Let us fix an element $x \in Ch^m(Q \times Y)$ mapped to $y \mod 2$ under the surjection

$$Ch^m(Q \times Y) \twoheadrightarrow Ch^m(Y_{F(Q)}).$$

Since over $\overline{F}$ the variety Q becomes cellular, the image $\overline{x} \in Ch^m(\overline{Q} \times \overline{Y})$ of x decomposes as

$$\overline{x} = h^0 \times y^m + \cdots + h^{[\frac{n}{2}]} \times y^{m-[\frac{n}{2}]} + l_{[\frac{n}{2}]} \times z^{m+[\frac{n}{2}]-n} + \cdots + l_{[\frac{n}{2}]-j} \times z^{m+[\frac{n}{2}]-j-n}$$

with some $y^i \in Ch^i(\overline{Y})$ and some $z^i \in Ch^i(\overline{Y})$, where $y^m = \overline{y}$, and where $h^i \in Ch^i(\overline{Q})$ is the ith power of the hyperplane section class while $l_i \in Ch_i(\overline{Q})$ is the class of an i-dimensional subspace of $\mathbb{P}(W)$, where W is a maximal totally isotropic subspace associated with the quadric $\overline{Q}$ (see [1, §68]).

For every $i = 0, ..., m$, let s^i be the image in $CH^{m+i}(\overline{Q} \times \overline{Y})$ of an element in $CH^{m+i}(Q \times Y)$ representing $S^i(x) \in Ch^{m+i}(Q \times Y)$. We also set $s^i := 0$ for $i > m$ as well as for $i < 0$.

The integer n can be uniquely written in the form $n = 2^t - 1 + s$, where t is a non-negative integer and $0 \leq s < 2^t$. Let us denote $2^t - 1$ as d. Since $d \leq n$, we can fix a smooth subquadric P of Q of dimension d; we write in for the imbedding

$$(P \hookrightarrow Q) \times id_Y : P \times Y \hookrightarrow Q \times Y.$$

Lemma 1.2. *For any integer r, one has*

$$S^r pr_* in^* x = \sum_{i=0}^{r} pr_*(c_i(-T_P) \cdot in^* S^{r-i}(x)) \quad \text{in } Ch^{r+m-d}(Y)$$

(where T_P is the tangent bundle of P, c_i are the Chern classes, et pr is the projection $P \times Y \to Y$).

Proof. The morphism $pr : P \times Y \to Y$ is a smooth projective morphism between smooth schemes. Thus, for any integer r, we have by [1, Proposition 61.10],

$$S^r \circ pr_* = \sum_{i=0}^{r} pr_*(c_i(-T_{pr}) \cdot S^{r-i})$$

where T_{pr} is the relative tangent bundle of pr over $P \times Y$. Furthermore, since pr is the projection $P \times Y \to Y$, one has $T_{pr} = T_P$. Hence, we get

$$S^r pr_* in^* x = \sum_{i=0}^{r} pr_*(c_i(-T_P) \cdot S^{r-i}(in^* x)).$$

Finally, since $in : P \times Y \hookrightarrow Q \times Y$ is a morphism between smooth schemes, the Steenrod operations of cohomological type commute with in^* (see [1, Theorem 61.9]), we are done. □

We apply Lemma 1.2 taking $r = d + j$. Since $pr_* in^* x \in Ch^{m-d}(Y)$ and $m - d < d + j$ (indeed, $m - d < n/2 + j - d$ by assumption, and $n/2 < 2d$ thanks to our choice of d), we have $S^{d+j} pr_* in^* x = 0$.

Hence, we have by Lemma 1.2,

$$\sum_{i=0}^{d+j} pr_*(c_i(-T_P) \cdot in^* S^{d+j-i}(x)) = 0 \quad \text{ in } \; Ch^{m+j}(Y).$$

In addition, for any $i = 0, ..., d$, by [1, Lemma 78.1] we have $c_i(-T_P) = \binom{-d-2}{i} \cdot h^i$, where $h^i \in Ch^i(P)$ is the ith power of the hyperplane section class, and where the binomial coefficient is considered modulo 2. Furthermore, for any $i = 0, ..., d$, the binomial coefficient $\binom{-d-2}{i} = \binom{d+i+1}{i}$ is odd (because d is a power of 2 minus 1, cf. [1, Lemma 78.6]). Moreover, for $i > d$, we have $c_i(-T_P) = 0$ in $CH^i(P)$ by definition of Chern classes. Thus, we get

$$\sum_{i=0}^{d} pr_*(h^i \cdot in^* S^{d+j-i}(x)) = 0 \quad \text{ in } \; Ch^{m+j}(Y).$$

Therefore, the element

$$\sum_{i=0}^{d} pr_*(h^i \cdot in^* s^{d+j-i}) \in CH^{m+j}(\overline{Y})$$

is twice a rational element.

Furthermore, for any $i = 0, ..., d$, we have

$$pr_*(h^i \cdot in^* s^{d+j-i}) = pr_*(in_*(h^i \cdot in^* s^{d+j-i}))$$

(the first pr is the projection $P \times Y \to Y$ while the second pr is the projection $Q \times Y \to Y$). Since in is a proper morphism between smooth schemes, we have by [1, Proposition 56.9],

$$in_* h^i \cdot in^* s^{d+j-i} = in_* h^i \cdot s^{d+j-i} = h^{n-d+i} \cdot s^{d+j-i}$$

and we finally get

$$pr_*(h^i \cdot in^* s^{d+j-i}) = pr_*(h^{n-d+i} \cdot s^{d+j-i}).$$

Hence, we get that the element

$$\sum_{i=0}^{d} pr_*(h^{n-d+i} \cdot s^{d+j-i}) \in CH^{m+j}(\overline{Y})$$

is twice a rational element.

We would like to compute the sum obtained modulo 4. Since $s^{d+j-i}=0$ if $d+j-i>m$, the ith summand is 0 for any $i<d+j-m$ ($(j-m)\leq 0$ by assumption). Otherwise – if $i\geq d+j-m$ – the factor h^{n-d+i} is divisible by 2 (indeed, we have $h^{n-d+i}=2l_{d-i}$ because $n-d+i\geq n+j-m>n/2$) and in order to compute the ith summand modulo 4 it suffices to compute s^{d+j-i} modulo 2, that is, to compute $S^{d+j-i}(\overline{x})$.

We recall that

$$\overline{x}=h^0\times y^m+\cdots+h^{[\frac{n}{2}]}\times y^{m-[\frac{n}{2}]}+l_{[\frac{n}{2}]}\times z^{m+[\frac{n}{2}]-n}+\cdots+l_{[\frac{n}{2}]-j}\times z^{m+[\frac{n}{2}]-j-n}.$$

Therefore, we have

$$S^{d+j-i}(\overline{x})=\sum_{k=0}^{[\frac{n}{2}]}S^{d+j-i}(h^k\times y^{m-k})+\sum_{k=0}^{j}S^{d+j-i}(l_{[\frac{n}{2}]-k}\times z^{m+[\frac{n}{2}]-k-n}).$$

And we set

$$A_i:=\sum_{k=0}^{[\frac{n}{2}]}S^{d+j-i}(h^k\times y^{m-k})\ \ \textit{and}\ \ B_i:=\sum_{k=0}^{j}S^{d+j-i}(l_{[\frac{n}{2}]-k}\times z^{m+[\frac{n}{2}]-k-n}).$$

For any $k=0,...,[\frac{n}{2}]$, we have by [1, Theorem 61.14],

$$S^{d+j-i}(h^k\times y^{m-k})=\sum_{l=0}^{d+j-i}S^{d+j-i-l}(h^k)\times S^l(y^{m-k}).$$

Moreover, for any $l=0,...,d+j-i$, we have by [1, Corollary 78.5],

$$S^{d+j-i-l}(h^k)=\binom{k}{d+j-i-l}h^{d+j+k-i-l}.$$

Thus, choosing an integral representative $\varepsilon_{k,l}\in CH^{m-k+l}(\overline{Y})$ of $S^l(y^{m-k})$ (we choose $\varepsilon_{k,l}=0$ if $l>m-k$), we get that the element

$$\sum_{k=0}^{[\frac{n}{2}]}\sum_{l=0}^{d+j-i}\binom{k}{d+j-i-l}(h^{d+j+k-i-l}\times\varepsilon_{k,l})\in CH^{m+d+j-i}(\overline{Q}\times\overline{Y})$$

is an integral representative of A_i.

Therefore, for any $i\geq d+j-m$, choosing an integral representative $\tilde{B}_i$ of B_i, there exists $\gamma_i\in CH^{m+d+j-i}(\overline{Q}\times\overline{Y})$ such that

$$s^{d+j-i}=\sum_{k=0}^{[\frac{n}{2}]}\sum_{l=0}^{d+j-i}\binom{k}{d+j-i-l}(h^{d+j+k-i-l}\times\varepsilon_{k,l})+\tilde{B}_i+2\gamma_i.$$

Hence, according to the multiplication rules in the ring $CH(\overline{Q})$ described in [1, Proposition 68.1], for any $i\geq d+j-m$, we have

$$h^{n-d+i}\cdot s^{d+j-i}=2\sum_{k=0}^{[\frac{n}{2}]}\sum_{l=0}^{d+j-i}\binom{k}{d+j-i-l}(l_{l-j-k}\times\varepsilon_{k,l})+h^{n-d+i}\cdot\tilde{B}_i+4l_{d-i}\cdot\gamma_i.$$

If $k \leq d-i$, one has $j+k \leq d+j-i$, and for any $0 \leq l \leq d+j-i$, we have by dimensional reasons,

$$pr_*(l_{l-j-k} \times \varepsilon_{k,l}) = \begin{cases} \varepsilon_{k,l} & \text{if } l = j+k \\ 0 & \text{otherwise.} \end{cases}$$

Otherwise $k > d-i$, and $pr_*(l_{l-j-k} \times \varepsilon_{k,l}) = 0$ for any $0 \leq l \leq d+j-i$. Moreover, for $k > d-i$, one has $j+k > j+d-i \geq m > m-k$, therefore $\varepsilon_{k,j+k} = 0$.

Thus we deduce the identity

$$pr_*\left(2\sum_{k=0}^{[\frac{n}{2}]}\sum_{l=0}^{d+j-i}\binom{k}{d+j-i-l}(l_{l-j-k} \times \varepsilon_{k,l})\right) = 2\sum_{k=0}^{[\frac{n}{2}]}\binom{k}{d-i-k}\varepsilon_{k,j+k}.$$

Then,

$$\sum_{i=d+j-m}^{d} pr_*\left(2\sum_{k=0}^{[\frac{n}{2}]}\sum_{l=0}^{d+j-i}\binom{k}{d+j-i-l}(l_{l-j-k} \times \varepsilon_{k,l})\right)$$
$$= 2\sum_{i=d+j-m}^{d}\sum_{k=0}^{[\frac{n}{2}]}\binom{k}{d-i-k}\varepsilon_{k,j+k}.$$

In the latest expression, for every $k = 0, ..., [\frac{n}{2}]$, the total coefficient near $\varepsilon_{k,j+k}$ is

$$2\sum_{i=d+j-m}^{d}\binom{k}{d-i-k} = 2\sum_{i=d-2k}^{d-k}\binom{k}{d-i-k} = 2\sum_{s=0}^{k}\binom{k}{s} = 2^{k+1},$$

which is divisible by 4 for $k \geq 1$.

Therefore, the cycle $\sum_{i=d+j-m}^{d} pr_*(h^{n-d+i} \cdot s^{d+j-i}) \in CH^{m+j}(\overline{Y})$ is congruent modulo 4 to

$$2\varepsilon_{0,j} + \sum_{i=d+j-m}^{d} pr_*(h^{n-d+i} \cdot \tilde{B}_i).$$

Thus, the cycle $2\varepsilon_{0,j} + \sum_{i=d+j-m}^{d} pr_*(h^{n-d+i} \cdot \tilde{B}_i)$ is congruent modulo 4 to twice a rational element.

Finally, the following lemma will lead to the conclusion.

Lemma 1.3. *For any $d+j-m \leq i \leq d$, one can choose an integral representative $\tilde{B}_i$ of B_i so that*

$$pr_*(h^{n-d+i} \cdot \tilde{B}_i) = 0.$$

Proof. We recall that $B_i := \sum_{k=0}^{j} S^{d+j-i}(l_{[\frac{n}{2}]-k} \times z^{m+[\frac{n}{2}]-k-n})$. For any $k = 0, ..., j$, we have by [1, Theorem 61.14],

$$S^{d+j-i}(l_{[\frac{n}{2}]-k} \times z^{m+[\frac{n}{2}]-k-n}) = \sum_{l=0}^{d+j-i} S^{d+j-i-l}(l_{[\frac{n}{2}]-k}) \times S^{l}(z^{m+[\frac{n}{2}]-k-n}).$$

And for any $l = 0, ..., d+j-i$, we have by [1, Corollary 78.5],

$$S^{d+j-i-l}(l_{[\frac{n}{2}]-k}) = \binom{n+1-[\frac{n}{2}]+k}{d+j-i-l} l_{[\frac{n}{2}]-k-d-j+i+l}.$$

Thus, choosing an integral representative $\delta_{k,l} \in CH^{m-k+l}(\overline{Y})$ of $S^l(z^{m+[\frac{n}{2}]-k-n})$ (we choose $\delta_{k,l} = 0$ if $l > m + [\frac{n}{2}] - k - n$), we get that the element

$$\sum_{k=0}^{j} \sum_{l=0}^{d+j-i} \binom{n+1-[\frac{n}{2}]+k}{d+j-i-l} (l_{[\frac{n}{2}]-k-d-j+i+l} \times \delta_{k,l}) \in CH^{m+d+j-i}(\overline{Q} \times \overline{Y})$$

is an integral representative of B_i. Let us note it $\tilde{B}_i$.

Hence, we have

$$h^{n-d+i} \cdot \tilde{B}_i = \sum_{k=0}^{j} \sum_{l=0}^{d+j-i} \binom{n+1-[\frac{n}{2}]+k}{d+j-i-l} (l_{[\frac{n}{2}]-k-n-j+l} \times \delta_{k,l}).$$

Moreover, we have

$$pr_*(l_{[\frac{n}{2}]-k-n-j+l} \times \delta_{k,l}) \neq 0 \Longrightarrow l = j+k+n-[\frac{n}{2}].$$

Furthermore, for any $0 \leq k \leq j$, we have $d+j-i \leq m < j + \frac{n}{2} \leq j+n-[\frac{n}{2}] \leq j+k+n-[\frac{n}{2}]$. Thus, for any $0 \leq l \leq d+j-i$ and for any $0 \leq k \leq j$, we have $pr_*(l_{[\frac{n}{2}]-k-n-j+l} \times \delta_{k,l}) = 0$. It follows that $pr_*(h^{n-d+i} \cdot \tilde{B}_i) = 0$ and we are done. $\square$

We deduce from Lemma 1.3 that the cycle $2\varepsilon_{0,j} \in CH^{m+j}(\overline{Y})$ is congruent modulo 4 to twice a rational cycle. Therefore, there exist a cycle $\gamma \in CH^{m+j}(\overline{Y})$ and a rational cycle $\alpha \in CH^{m+j}(\overline{Y})$ so that

$$2\varepsilon_{0,j} = 2\alpha + 4\gamma,$$

hence, there exists an exponent 2 element $\delta \in CH^{m+j}(\overline{Y})$ so that

$$\varepsilon_{0,j} = \alpha + 2\gamma + \delta.$$

Finally, since $\varepsilon_{0,j}$ is an integral representative of $S^j(\overline{y})$, we get that $S^j(\overline{y})$ is the sum of a rational element and the class modulo 2 of an integral element of exponent 2. We are done with the proof of Theorem 1.1. $\square$

2. Other results

In this section we continue to use notation introduced in the beginning of Section 1. In the same way as before, the following proposition is a generalization of [3, Proposition 3.3(2)] (although, putting aside characteristic, our proposition is still weaker than the original version in the sense that an exponent 2 element appears in the conclusion).

Proposition 2.1. *Let $x \in Ch^m(Q \times Y)$ be some element, and $y^i, z^i \in Ch^i(\overline{Y})$ be the coordinates of $\overline{x}$ as in the beginning of proof of Theorem 1.1. Assume that $m = [\frac{n+1}{2}] + j$ and that $n \geq 1$. Then $S^j(y^m) + y^m \cdot z^j$ differs from a rational element by the class of an exponent 2 element of $CH^{m+j}(\overline{Y})$.*

Proof. The image $\overline{x} \in Ch^m(\overline{Q} \times \overline{Y})$ of x decomposes as

$$\overline{x} = h^0 \times y^m + \cdots + h^{[\frac{n}{2}]} \times y^{m-[\frac{n}{2}]} + l_{[\frac{n}{2}]} \times z^{m+[\frac{n}{2}]-n} + \cdots + l_{[\frac{n}{2}]-j-1} \times z^{m+[\frac{n}{2}]-j-n}.$$

Let $\mathbf{x} \in CH^m(Q \times Y)$ be an integral representative of x. The image $\overline{\mathbf{x}} \in CH^m(\overline{Q} \times \overline{Y})$ decomposes as

$$\overline{\mathbf{x}} = h^0 \times \mathbf{y}^m + \cdots + h^{[\frac{n}{2}]} \times \mathbf{y}^{m-[\frac{n}{2}]} + l_{[\frac{n}{2}]} \times \mathbf{z}^{m+[\frac{n}{2}]-n} + \cdots + l_{[\frac{n}{2}]-j-1} \times \mathbf{z}^{m+[\frac{n}{2}]-j-n}$$

where the elements $\mathbf{y}^i \in CH^i(\overline{Y})$ (resp. $\mathbf{z}^i \in CH^i(\overline{Y})$) are some integral representatives of the elements y^i (resp. z^i).

For every $i = 0, ..., m-1$, let s^i be the image in $CH^{m+i}(\overline{Q} \times \overline{Y})$ of an element in $CH^{m+i}(Q \times Y)$ representing $S^i(x) \in Ch^{m+i}(Q \times Y)$. We also set $s^i := 0$ for $i > m$ as well as for $i < 0$. Finally, we set $s^0 := \overline{\mathbf{x}}$ and $s^m := (s^0)^2$. Therefore, for any integer i, s^i is the image in $CH^{m+i}(\overline{Q} \times \overline{Y})$ of an integral representative of $S^i(x)$.

The integer n can be uniquely written in the form $n = 2^t - 1 + s$, where t is a non-negative integer and $0 \leq s < 2^t$. Let us denote $2^t - 1$ as d.

We would like to use again Lemma 1.2 to get that the sum

$$\sum_{i=d+j-m}^{d} pr_*(h^{n-d+i} \cdot s^{d+j-i}) \in CH^{m+j}(\overline{Y})$$

is twice a rational element. To do this, it suffices to check that $m - d < d + j$. Then the same reasoning as the one used during the proof of Theorem 1.1 gives us the desired result.

We have $m - d = [\frac{n+1}{2}] + j - d = d + j + ([\frac{n+1}{2}] - 2d)$, and since our choice of d and the assumption $n \geq 1$, one can easily check that $2d > [\frac{n+1}{2}]$. Thus we do get that the sum

$$\sum_{i=d+j-m}^{d} pr_*(h^{n-d+i} \cdot s^{d+j-i}) \in CH^{m+j}(\overline{Y})$$

is twice a rational element. We would like to compute that sum modulo 4.

For any $i \geq d + j - m$, the factor s^{d+j-i} present in the ith summand is congruent modulo 2 to $S^{d+j-i}(\overline{x})$, which is represented by $\tilde{A}_i + \tilde{B}_i$, where

$$\tilde{A}_i := \sum_{k=0}^{[\frac{n}{2}]} \sum_{l=0}^{d+j-i} \binom{k}{d+j-i-l} (h^{d+j+k-i-l} \times \varepsilon_{k,l})$$

and

$$\tilde{B}_i := \sum_{k=0}^{j} \sum_{l=0}^{d+j-i} \binom{n+1-[\frac{n}{2}]+k}{d+j-i-l} (l_{[\frac{n}{2}]-k-d-j+i+l} \times \delta_{k,l})$$

where $\varepsilon_{k,l} \in CH^{m-k+l}(\overline{Y})$ (resp. $\delta_{k,l} \in CH^{m-k+l}(\overline{Y})$) is an integral representative of $S^l(y^{m-k})$ (resp. of $S^l(z^{m+[\frac{n}{2}]-k-n})$), and we choose $\varepsilon_{k,l} = 0$ if $l > m - k$ (resp. $\delta_{k,l} = 0$ if $l > m + [\frac{n}{2}] - k - n$). Finally, in the case of even $m - j$, we choose $\varepsilon_{\frac{m-j}{2}, \frac{m+j}{2}} = (\mathbf{y}^{\frac{m+j}{2}})^2$.

Furthermore, for any $i \geq d+j-m$, we have

$$h^{n-d+i} \cdot \tilde{B}_i = \sum_{k=0}^{j} \sum_{l=0}^{d+j-i} \binom{n+1-[\frac{n}{2}]+k}{d+j-i-l} (l_{[\frac{n}{2}]-k-n-j+l} \times \delta_{k,l}).$$

And we have

$$pr_*(l_{[\frac{n}{2}]-k-n-j+l} \times \delta_{k,l}) \neq 0 \Longrightarrow l = j+k+n-[\frac{n}{2}].$$

On the one hand, for any $i > d+j-m$, we have $d+j-i < m = n-[\frac{n}{2}]+j \leq j+k+n-[\frac{n}{2}]$. Hence, for any $0 \leq l \leq d+j-i$ and for any $0 \leq k \leq j$, we have $pr_*(l_{[\frac{n}{2}]-k-n-j+l} \times \delta_{k,l}) = 0$. Then, for any $i > d+j-m$, we get that $pr_*(h^{n-d+i} \cdot \tilde{B}_i) = 0$.

On the other hand, for $i = d+j-m$, we have $d+j-i = j+n-[n/2]$ and

$$l = j+k+n-[\frac{n}{2}] \Longleftrightarrow k = 0 \ \ and \ \ l = d+j-i.$$

Thus, we have

$$pr_*(h^{n+j-m} \cdot \tilde{B}_{d+j-m}) = \delta_{0,m}.$$

Since $m > m+[n/2]-n$, we get that $\delta_{0,m} = 0$.

Therefore, for any $i \geq d+j-m$, we have

$$pr_*(h^{n-d+i} \cdot \tilde{B}_i) = 0.$$

Then, for any $i > d+j-m$, the cycle h^{n-d+i} is divisible by 2. Hence, according to the multiplication rules in the ring $CH(\overline{Q})$ described in [1, Proposition 68.1] and by doing the same computations as those done during the proof of Theorem 1.1, for any $i > d+j-m$, we get the congruence

$$pr_*(h^{n-d+i} \cdot s^{d+j-i}) \equiv 2 \sum_{k=0}^{[\frac{n}{2}]} \binom{k}{d-i-k} \varepsilon_{k,j+k} \ \ (mod\ 4).$$

Moreover, since $d-i-k \leq k$ if and only if $k \leq [\frac{m-j}{2}]$, for any $i > d+j-m$, we have the congruence

$$pr_*(h^{n-d+i} \cdot s^{d+j-i}) \equiv 2 \sum_{k=0}^{[\frac{m-j}{2}]} \binom{k}{d-i-k} \varepsilon_{k,j+k} \ \ (mod\ 4). \tag{1}$$

Now, we would like to study the $(d+j-m)$th summand, that is to say the cycle $pr_*(h^{n+j-m} \cdot s^m)$ modulo 4. That is the purpose of the following lemma.

Lemma 2.2. *One has*

$$pr_*(h^{n+j-m} \cdot s^m) \equiv \begin{cases} 2\varepsilon_{\frac{m-j}{2},\frac{m+j}{2}} + 2\boldsymbol{y}^m \cdot \boldsymbol{z}^j \ \ (mod\ 4) & if\ m-j\ is\ even \\ 2\boldsymbol{y}^m \cdot \boldsymbol{z}^j \ \ (mod\ 4) & if\ m-j\ is\ odd. \end{cases}$$

Proof. We recall that $s^m = (\overline{\mathbf{x}})^2$. Thus, we have

$$h^{n+j-m} \cdot s^m = h^{n+j-m} \cdot (A+B+C)$$

where

$$A := \sum_{0 \leq i,l \leq [\frac{n}{2}]} h^{i+l} \times (\mathbf{y}^{m-i} \cdot \mathbf{y}^{m-l}),$$

$$B := \sum_{0 \leq i,l \leq j} (l_{[\frac{n}{2}]-i} \cdot l_{[\frac{n}{2}]-l}) \times (\mathbf{z}^{j-i} \cdot \mathbf{z}^{j-l})$$

and

$$C := 2 \sum_{i=0}^{[\frac{n}{2}]} h^i \times \mathbf{y}^{m-i} \cdot \sum_{l=0}^{j} l_{[\frac{n}{2}]-l} \times \mathbf{z}^{j-l}.$$

First of all, we have

$$h^{n+j-m} \cdot A = \sum_{0 \leq i,l \leq [\frac{n}{2}]} h^{n+j-m+i+l} \times (\mathbf{y}^{m-i} \cdot \mathbf{y}^{m-l}).$$

Now we have $m = [\frac{n+1}{2}] + j$, so $n+j-m+i+l = [\frac{n}{2}] + i + l$. Thus, if $i \geq 1$ or $l \geq 1$, we have $n+j-m+i+l > [\frac{n}{2}]$, and in this case we have $h^{n+j-m+i+l} = 2l_{m-i-l-j}$. Therefore, the cycle $h^{n+j-m} \cdot A$ is equal to

$$h^{n+j-m} \times (\mathbf{y}^m)^2 + 4 \sum_{1 \leq i,l \leq [\frac{n}{2}]} l_{m-i-l-j} \times (\mathbf{y}^{m-i} \cdot \mathbf{y}^{m-l}) + 2 \sum_{i=1}^{[\frac{n}{2}]} l_{m-j-2i} \times (\mathbf{y}^{m-i})^2.$$

Then, since $n \geq 1$, we have $n+j-m \neq n$. It follows that $pr_*(h^{n+j-m} \times (\mathbf{y}^m)^2) = 0$.

Furthermore, we have

$$pr_*(\sum_{i=1}^{[\frac{n}{2}]} l_{m-j-2i} \times (\mathbf{y}^{m-i})^2) = \begin{cases} (\mathbf{y}^{\frac{m+j}{2}})^2 & \text{if } m-j \text{ is even} \\ 0 & \text{if } m-j \text{ is odd.} \end{cases}$$

Therefore, $pr_*(h^{n+j-m} \cdot A)$ is congruent modulo 4 to $2\varepsilon_{\frac{m-j}{2},\frac{m+j}{2}}$ if $m-j$ is even, and to 0 if $m-j$ is odd.

Then, by dimensional reasons, we have $l_{[\frac{n}{2}]-i} \cdot l_{[\frac{n}{2}]-l} = 0$ if $i \geq 1$ or if $l \geq 1$. Hence, we have $B = (l_{[\frac{n}{2}]} \cdot l_{[\frac{n}{2}]}) \times (\mathbf{z}^j)^2$. It follows that

$$h^{n+j-m} \cdot B = (l_0 \cdot l_{[\frac{n}{2}]}) \times (\mathbf{z}^j)^2$$

and $l_0 \cdot l_{[\frac{n}{2}]} = 0$ by dimensional reasons. Therefore, we get that $h^{n+j-m} \cdot B = 0$.

Finally, we have

$$h^{n+j-m} \cdot C = 2 \sum_{i=0}^{[\frac{n}{2}]} h^{n+j-m+i} \times \mathbf{y}^{m-i} \cdot \sum_{l=0}^{j} l_{[\frac{n}{2}]-l} \times \mathbf{z}^{j-l}.$$

Now for any $i \geq 1$, we have $n+j-m+i > [\frac{n}{2}]$, and in this case the cycle $h^{n+j-m+i}$ is divisible by 2. Thus, the element $h^{n+j-m} \cdot C$ is congruent modulo 4 to

$$2\sum_{l=0}^{j}(h^{[\frac{n}{2}]} \cdot l_{[\frac{n}{2}]-l}) \times (\mathbf{y}^m \cdot \mathbf{z}^{j-l}),$$

and, by dimensional reasons, in the latest sum, each summand is 0 except the one corresponding to $l=0$. Therefore, the cycle $h^{n+j-m} \cdot C$ is congruent modulo 4 to $2l_0 \times (\mathbf{y}^m \cdot \mathbf{z}^j)$. It follows that $pr_*(h^{n+j-m} \cdot C)$ is congruent modulo 4 to $2\mathbf{y}^m \cdot \mathbf{z}^j$. We are done. □

By the congruence (1) and Lemma 2.2, we deduce that the cycle

$$\sum_{i=d+j-m}^{d} pr_*(h^{n-d+i} \cdot s^{d+j-i})$$

is congruent modulo 4 to

$$2\sum_{i=d+j-m}^{d}\sum_{k=0}^{[\frac{m-j}{2}]}\binom{k}{d-i-k}\varepsilon_{k,j+k} + 2\mathbf{y}^m \cdot \mathbf{z}^j.$$

It follows that the cycle

$$2\sum_{i=d+j-m}^{d}\sum_{k=0}^{[\frac{m-j}{2}]}\binom{k}{d-i-k}\varepsilon_{k,j+k} + 2\mathbf{y}^m \cdot \mathbf{z}^j$$

is congruent modulo 4 to twice a rational element $\alpha \in CH^{m+j}(\overline{Y})$. Then, we finish as in the proof of Theorem 1.1. For every $k = 0, ..., [(m-j)/2]$, the total coefficient near $\epsilon_{k,j+k}$ is 2^{k+1}, which is divisible by 4 for $k \geq 1$. Therefore, there exists a cycle $\gamma \in CH^{m+j}(\overline{Y})$ such that

$$2\varepsilon_{0,j} + 2\mathbf{y}^m \cdot \mathbf{z}^j = 2\alpha + 4\gamma,$$

hence, there exists an exponent 2 element $\delta \in CH^{m+j}(\overline{Y})$ so that

$$\varepsilon_{0,j} + \mathbf{y}^m \cdot \mathbf{z}^j = \alpha + 2\gamma + \delta.$$

Finally, since $\varepsilon_{0,j}$ is an integral representative of $S^j(y^m)$ and $\mathbf{y}^m$ (resp. $\mathbf{z}^j$) is an integral representative of y^m (resp. of z^j), we get that $S^j(y^m) + y^m \cdot z^j$ differs from a rational element by the class of an exponent 2 element of $CH^{m+j}(\overline{Y})$. We are done with the proof of Proposition 2.1. □

Remark 2.3. In the case of $j=0$, and if we make the extra assumption that the image of x under the composition

$$Ch^m(Q \times Y) \to Ch^m(Q_{F(Y)}) \to Ch^m(Q_{\overline{F}(Y)}) \to Ch^m(\overline{Q})$$

(the last passage is given by the inverse of the change of field isomorphism) is rational, then we get the stronger result that the cycle y^m differs from a rational element by the class of an exponent 2 element of $CH^m(\overline{Y})$. That is the object of [2, Proposition 4.1].

Finally, the following theorem is a consequence of Proposition 2.1. In the same way as before, it is a generalization of [3, Theorem 3.1(2)]. For a variety X, we write r for the restriction map $Ch^*(X) \to Ch^*(\overline{X})$.

Theorem 2.4. *Assume that $m = [\frac{n+1}{2}] + j$ and $n \geq 1$. Let $\overline{y}$ be an $F(Q)$-rational element of $Ch^m(\overline{Y})$. Then there exists a rational element $z \in Ch^j(\overline{Y})$ such that $S^j(\overline{y}) + \overline{y} \cdot z$ is the sum of a rational element and the class modulo 2 of an integral element of exponent 2.*

Proof. The element $\overline{y}$ being $F(Q)$-rational, there exists $x \in Ch^m(Q \times Y)$ mapped to $\overline{y}_{\overline{F(Q)}}$ under the composition

$$Ch^m(Q \times Y) \to Ch^m(Y_{F(Q)}) \to Ch^m(Y_{\overline{F(Q)}}).$$

Moreover, the image $\overline{x} \in Ch^m(\overline{Q} \times \overline{Y})$ of x decomposes as

$$\overline{x} = h^0 \times y^m + \cdots + h^{[\frac{n}{2}]} \times y^{m-[\frac{n}{2}]} + l_{[\frac{n}{2}]} \times z^{m+[\frac{n}{2}]-n} + \cdots + l_{[\frac{n}{2}]-j-1} \times z^{m+[\frac{n}{2}]-j-n}$$

with some $y^i \in Ch^i(\overline{Y})$, and some $z^i \in Ch^i(\overline{Y})$, and where $y^m = \overline{y}$.

Thus, by Proposition 2.1, the cycle $S^j(\overline{y}) + \overline{y} \cdot z^j$ is the sum of a rational element and the class an element of exponent 2.

Finally, we have by [1, Proposition 49.20],

$$r \circ (pr)_*(x \cdot h^{[\frac{n}{2}]}) = pr_*(\overline{x} \cdot h^{[\frac{n}{2}]}) = z^j.$$

We are done with the proof of Theorem 2.4. □

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