ESSENTIAL DIMENSION OF PROJECTIVE ORTHOGONAL AND SYMPLECTIC GROUPS OF SMALL DEGREE

SANGHOON BAEK

Abstract. In this paper, we study the essential dimension of classes of central simple algebras with involutions of index less or equal to 4. Using structural theorems for simple algebras with involutions, we obtain the essential dimension of projective and symplectic groups of small degree.

1. Introduction

Let F be a field, A a central simple F-algebra, and (σ, f) a quadratic pair on A (see [5, 5.B]). A morphism of algebras with quadratic pair $\phi : (A, \sigma, f) \to (A', \sigma', f')$ is an F-algebra morphism $\phi : A \to A'$ such that $\sigma' \circ \phi = \phi \circ \sigma$ and $f \circ \phi = f'$. For any field extension K/F, we write $(A, \sigma, f)_K$ for $(A \otimes_F K, \sigma \otimes \mathrm{Id}_K, f_K)$, where $f_K : \mathrm{Sym}(A_K, \sigma_K) \to K$.

For $n \geq 2$, let D_n denote the category of central simple F-algebras of degree $2n$ with quadratic pair, where the morphisms are the F-algebra homomorphisms which preserve the quadratic pairs and let A_1^2 denote the category of quaternion algebras over an étale quadratic extension of F, where the morphisms are the F-algebra isomorphism. Then, there is an equivalence of groupoids:

$$D_2 \equiv A_1^2; \tag{1}$$

see [5, Theorem 15.7].

Moreover, if we consider the full subgroupoid ${}^1A_1^2$ of A_1^2 whose objects are F-algebras of the form $Q \times Q'$, where Q and Q' are quaternion algebras over F, and the full subgroupoid ${}^1D_n$ of D_n whose objects are central simple algebras over F with quadratic pair of trivial discriminant, then the equivalence in (1) specialize to the following equivalence of subgroupoids:

$${}^1D_2 \equiv {}^1A_1^2; \tag{2}$$

see [5, Corollary 15.12].

For $n \geq 1$, we denote by C_n be the category of central simple F-algebras of degree $2n$ with symplectic involution, where the morphisms are the F-algebra isomorphism which preserve the involutions.

By Galois cohomology, there are canonical bijections (see [5, §29.D,F])

$$D_n \longleftrightarrow H^1(F, \mathbf{PGO}_{2n}) \tag{3}$$

The work has been partially supported from the Fields Institute and from Zainoulline's NSERC Discovery grant 385795-2010.

and

$$C_n \longleftrightarrow H^1(F, \mathbf{PGSp}_{2n}). \tag{4}$$

Let $\mathcal{T} : \mathsf{Fields}/F \to \mathsf{Sets}$ be a functor from the category Fields/F of field extensions over F to the category Sets of sets and let p be a prime. We denote by $\mathrm{ed}(\mathcal{T})$ and $\mathrm{ed}_p(\mathcal{T})$ the essential dimension and essential p-dimension of $\mathcal{T}$, respectively. We refer to [4, Def. 1.2] and [8, Sec.1] for their definitions. Let G be an algebraic group over F. The *essential dimension* $\mathrm{ed}(G)$ (respectively, *essential p-dimension* $\mathrm{ed}_p(G)$) of G is defined to be $\mathrm{ed}(H^1(-, G))$ (respectively, $\mathrm{ed}_p(H^1(-, G))$), where $H^1(E, G)$ is the nonabelian cohomology set with respect to the finitely generated faithfully flat topology (equivalently, the set of isomorphism classes of G-torsors) over a field extension E of F.

A morphism $\mathcal{S} \to \mathcal{T}$ from Fields/F to Sets is called *p-surjective* if for any $E \in \mathsf{Fields}/F$ and any $\alpha \in \mathcal{T}(E)$, there is a finite field extension L/E of degree prime to p such that $\alpha_L \in \mathrm{Im}(\mathcal{S}(L) \to \mathcal{T}(L))$. A morphism of functors $\mathcal{S} \to \mathcal{T}$ from Fields/F to Sets is called *surjective* if for any $E \in \mathsf{Fields}/F$, $\mathcal{S}(E) \to \mathcal{T}(E)$ is surjective. Obviously, any surjective morphism is p-surjective for any prime p. Such a surjective morphism gives an upper bound for the essential (p)-dimension of $\mathcal{T}$ and a lower bound for the essential (p)-dimension of $\mathcal{S}$,

$$\mathrm{ed}(\mathcal{T}) \leq \mathrm{ed}(\mathcal{S}) \text{ and } \mathrm{ed}_p(\mathcal{T}) \leq \mathrm{ed}_p(\mathcal{S}); \tag{5}$$

see [4, Lemma 1.9] and [8, Proposition 1.3].

Example 1.1. Let $(M_2(F), \gamma) \in C_1$, where γ is the canonical involution on $M_2(F)$. As $(M_2(K), \gamma_K) \simeq (M_2(F), \gamma) \otimes_F K$ for any field extension K/F, we have $\mathrm{ed}((M_2(F), \gamma)) = 0$.

Assume that $\mathrm{char}(F) \neq 2$. The exact sequence

$$1 \to \boldsymbol{\mu}_2 \to \mathbf{Sp}_2 \to \mathbf{PGSp}_2 \to 1$$

induces the connecting morphism $\partial : H^1(-, \mathbf{PGSp}_2) \to \mathrm{Br}_2(-)$ which sends a pair (Q, γ) of a quaternion algebra with canonical involution to the Brauer class $[Q]$. As this morphism is nontrivial, by [4, Corollary 3.6] we have $\mathrm{ed}(\mathbf{PGSp}_2) \geq 2$ (or by Lemma 2.6). Consider the morphism $\mathbb{G}_m^2 \to C_1$ defined by $(x, y) \mapsto ((x, y), \gamma)$, where γ is the canonical involution. As this morphism is surjective, by (5) we have $\mathrm{ed}(C_1) \leq 2$, thus $\mathrm{ed}(\mathbf{PGSp}_2) = 2$. This can be recovered from the exceptional isomorphism $\mathbf{PGSp}_2 \simeq \mathbf{O}_3^+$.

Acknowledgements: I am grateful to A. Merkurjev for useful discussions. I am also grateful to J. P. Tignol and S. Garibaldi for helpful comments.

2. Essential dimension of projective orthogonal and symplectic groups associated to central simple algebras of index ≤ 4

First, we compute upper bounds for the essential dimension of certain classes of simple algebras with involutions of index less or equal to 2.

Let (Q,γ) be a pair of a quaternion over a field F and the canonical involution. For any field extension K/F and any integer $n \geq 3$, we write $\mathrm{QH}_n^+(K)$ (respectively, $\mathrm{QH}_n^-(K)$) for the set of isomorphism classes of $(M_n(Q),\sigma_h)$, where σ_h is the adjoint involution on $M_n(Q)$ with respect to a hermitian form (respectively, skew-hermitian form) h (with respect to γ). If n is odd, we write ${}^1\mathrm{QH}_n^-(K)$ for the set of isomorphism classes of $(M_n(Q),\sigma_h)$, where σ_h is the adjoint involution on $M_n(Q)$ with respect to a skew-hermitian form h with $\mathrm{disc}(\sigma_h)=1$.

Lemma 2.1. *Let F be a field and $n \geq 3$ any integer. Then*

(1) $\mathrm{ed}(\mathrm{QH}_n^+) \leq n+1$.

(2) $\mathrm{ed}(\mathrm{QH}_n^-) \leq 3n-3$ *if* $\mathrm{char}(F) \neq 2$.

(3) $\mathrm{ed}({}^1\mathrm{QH}_n^-) \leq 3n-4$ *if* $\mathrm{char}(F) \neq 2$.

Proof. (1) If h is a hermitian form, then $h = \langle t_1, t_2, \cdots, t_n\rangle$ for some $t_i \in F$. We consider the affine variety

$$X = \begin{cases} \mathbb{G}_m^2 \times \mathbb{A}_F^{n-1} & \text{if } \mathrm{char}(F) \neq 2, \\ \mathbb{G}_m \times \mathbb{A}_F^{n} & \text{if } \mathrm{char}(F) = 2, \end{cases}$$

and define a morphism $X(K) \to \mathrm{QH}_n^+(K)$ by

$$(a,b,t_1,\cdots,t_{n-1}) \mapsto \begin{cases} ((a,b)\otimes M_n(K), \sigma_{\langle 1,t_1,\cdots,t_{n-1}\rangle}) & \text{if } \mathrm{char}(F) \neq 2, \\ ([a,b)\otimes M_n(K), \sigma_{\langle 1,t_1,\cdots,t_{n-1}\rangle}), & \text{if } \mathrm{char}(F) = 2. \end{cases}$$

As a scalar multiplication does not change the adjoint involution, this morphism is surjective. Therefore, by (5), we have $\mathrm{ed}(\mathrm{QH}_n^+) \leq n+1$.

(2) From now we assume that $\mathrm{char}(F) \neq 2$. If h is a skew-hermitian form, then $h = \langle q_1, q_2, \cdots, q_n\rangle$ for some pure quaternions $q_i \in Q$. We consider the affine variety $Y = \mathbb{G}_m^2 \times \mathbb{A}^1 \times \mathbb{A}^{3(n-2)}$ with coordinates $(a,b,c,t_1,\cdots,t_{3n-6})$ and the conditions $ac^2+b \neq 0$, $at_{1+3(k-1)}^2 + bt_{2+3(k-1)}^2 - abt_{3k}^2 \neq 0$ for all $1 \leq k \leq n-2$. Define a morphism $\phi_K : Y(K) \to \mathrm{QH}_n^-(K)$ by

$$(a,b,c,t_1,\cdots,t_{3n-6}) \mapsto ((a,b)\otimes M_n(K), \sigma_h),$$

where $p = i$, $q = ci+j$, $r_k = t_{1+3(k-1)}i + t_{2+3(k-1)}j + t_{3k}ij$ for $1 \leq k \leq n-2$, and $h = \langle p, q, r_1, \cdots, r_{3n-2}\rangle$.

We show that Y is a classifying variety for QH_n^-. Suppose that we are given a quaternion K-algebra $Q = (a,b)$ and a skew hermitian form $h = \langle p, q, r_1, \cdots, r_{n-2}\rangle$ for some pure quaternions p, q, r_k. We can find a scalar $c \in K$ such that p and $q-cp$ anticommute, thus we have $Q \simeq (p^2, (q-cp)^2)$. For $1 \leq k \leq n-2$, let

$$r_k = t_{1+3(k-1)}p + t_{2+3(k-1)}(q-cp) + t_{3k}p(q-cp) \tag{6}$$

with $t_1, t_2, \cdots, t_{3n-6} \in K$. Then $(M_n(Q),\sigma_h) \simeq (M_n((p^2,(q-cp)^2)),\sigma_h)$ is the image of ϕ_K. Therefore, by (5), we have $\mathrm{ed}(\mathrm{QH}_n^-) \leq 3n-3$.

(3) Assume that $n = 2m + 1$ for $m \geq 1$. We consider the variety Y in (2) with an additional condition

$$-a(ac^2+b)\prod_{k=1}^{m-1} at_{1+3(k-1)}^2+bt_{2+3(k-1)}^2-abt_{3k}^2 = \prod_{k=m}^{n-2} at_{1+3(k-1)}^2+bt_{2+3(k-1)}^2-abt_{3k}^2.$$

We show that this variety with the same morphism ϕ_K in (2) is a classifying variety for ${}^1\mathrm{QH}_n^-$. Suppose that we are given a quaternion K-algebra $Q = (a, b)$ and a skew hermitian form $h = \langle p, q, r_1, \cdots, r_{n-2}\rangle$ for some pure quaternions p, q, r_k. We do the same procedure as in (1), so that we have $(M_n(Q), \sigma_h) \simeq (M_n((p^2, (q-cp)^2)), \sigma_h)$ and (6). As $\operatorname{disc}(\sigma_h) = 1$, there is a scalar $d \in K^\times$ such that

$$-p^2q^2r_1^2\cdots r_{m-1}^2 = (\frac{d}{r_m^2\cdots r_{n-2}^2})^2 r_m^2\cdots r_{n-2}^2.$$

We set $f = d/r_m^2\cdots r_{n-2}^2$. As a scalar multiplication does not change the adjoint involution, we can modify h by the scalar f. As $(p^2, (q-cp)^2) \simeq (f^2p^2, f^2(q-cp)^2)$, $(M_n(Q), \sigma_h)$ is the image of ϕ_K. Therefore, by (5), we have $\operatorname{ed}({}^1\mathrm{QH}_n^-) \leq 3n - 4$.

□

Remark 2.2. The main idea of the proof of the case where h is a skew-hermitian form is from Merkurjev's work on algebras of degree 4 in his private note.

Assume that n is odd. Then we have

$$\mathrm{QH}_n^+ = C_n, \mathrm{QH}_n^- = D_n, \text{ and } {}^1\mathrm{QH}_n^- = {}^1D_n.$$

Hence, by [5, Theorem 4.2] and the exceptional isomorphism $\mathbf{PGO}_6 \simeq \mathbf{PGU}_4$, we have

Corollary 2.3. *Assume that $n \geq 3$ is odd. Then*

1. $\operatorname{ed}(\mathbf{PGSp}_{2n}) \leq n + 1$.
2. $\operatorname{ed}(\mathbf{PGO}_{2n}) \leq 3n - 3$ *if* $\operatorname{char}(F) \neq 2$. *In particular,* $\operatorname{ed}(\mathbf{PGU}_4) \leq 6$.
3. $\operatorname{ed}(\mathbf{PGO}_{2n}^+) \leq 3n - 4$ *if* $\operatorname{char}(F) \neq 2$.

Remark 2.4. In fact, $\operatorname{ed}_2(\mathbf{PGSp}_{2n}) = \operatorname{ed}(\mathbf{PGSp}_{2n}) = n+1$ for $n \geq 3$ odd and $\operatorname{char}(F) \neq 2$. The lower bound was obtained by Chernousov and Serre in [6, Theorem 1] and the exact value was obtained by Macdonald in [7, Proposition 5.1].

Lemma 2.5. [2, Section 2.6] *Let F be a field of characteristic different from 2. Then*

$$\operatorname{ed}_2(\mathbf{PGL}_2^{\times n}) = \operatorname{ed}(\mathbf{PGL}_2^{\times n}) = 2n.$$

Proof. By [4, Lemma 1.11], we have

$$\operatorname{ed}_2(\mathbf{PGL}_2^{\times n}) \leq \operatorname{ed}(\mathbf{PGL}_2^{\times n}) \leq n \cdot \operatorname{ed}(\mathbf{PGL}_2) = 2n.$$

On the other hand, the natural morphism

$$H^1(-, \mathbf{PGL}_2^{\times n}) \to \mathit{Dec}_{2^n}(-) \tag{7}$$

is surjective, where $\mathsf{Dec}_{2^n}(K)$ is the set of all decomposable algebras of degree 2^n over a field extension K/F, hence, by (5), we have

$$2n = \mathrm{ed}_2(\mathsf{Dec}_{2^n}) \leq \mathrm{ed}_2(\mathbf{PGL}_2^{\times n}).$$

□

Lemma 2.6. *Let F be a field of characteristic different from 2. Then*

$$\mathrm{ed}_2(\mathbf{PGO}_{2^r}), \mathrm{ed}_2(\mathbf{PGSp}_{2^r}) \geq \begin{cases} 2 & \textit{if } r = 1, \\ 4 & \textit{if } r = 2, \\ (r-1)2^{r-1} & \textit{if } r \geq 3. \end{cases}$$

Proof. Consider the forgetful functors

$$H^1(-, \mathbf{PGO}_{2^r}) \to \mathsf{Alg}_{2^r,2} \tag{8}$$

and

$$H^1(-, \mathbf{PGSp}_{2^r}) \to \mathsf{Alg}_{2^r,2}, \tag{9}$$

where $\mathsf{Alg}_{2^r,2}(K)$ is the set of isomorphism classes of simple algebras of degree 2^n and exponent dividing 2 over a field extension K/F. These functors are surjective by a theorem of Albert.

It is well known that $\mathrm{ed}_2(\mathsf{Alg}_{2,2}) = 2$, $\mathrm{ed}_2(\mathsf{Alg}_{4,2}) = 4$. For $r \geq 3$, we have $\mathrm{ed}_2(\mathsf{Alg}_{2^r,2}) \geq (r-1)2^{r-1}$ by [3, Theorem]. Therefore, by (5), we have the above lower bound for $\mathrm{ed}_2(\mathbf{PGO}_{2^r})$ and $\mathrm{ed}_2(\mathbf{PGSp}_{2^r})$. □

The following Lemma 2.7(1) was proved by Rowen in [10, Theorem B] (see also [5, Proposition 16.16]) and Lemma 2.7(2) was proved by Serhir and Tignol in [11, Proposition]. We shall need the explicit forms of involutions on the decomposed quaternions as in (1):

Lemma 2.7. *Let F be a field of characteristic different from 2. Let (A, σ) be a central simple F-algebra of degree 4 with a symplectic involution σ.*

(1) *If A is a division algebra, then we have*

$$(A, \sigma) \simeq (Q, \sigma|_Q) \otimes (Q', \gamma),$$

where $\sigma|_Q$ is an orthogonal involution defined by $\sigma|_Q(x_0 + x_1 i + x_2 j + x_3 k) = x_0 + x_1 i + x_2 j - x_3 k$ with a quaternion basis $(1, i, j, k)$ for Q and γ is the canonical involution on a quaternion algebra Q'.

(2) *If A is not a division algebra, then we have*

$$(A, \sigma) \simeq (M_2(F), \mathrm{ad}_q) \otimes (Q', \gamma),$$

where q is a 2-dimensional quadratic form, ad_q is the adjoint involution on $M_2(F)$, and γ is the canonical involution on a quaternion algebra Q'.

Proof. (1) By [10, Proposition 5.3], we can choose a $i \in A\backslash F$ such that $\sigma(i) = i$ and $[F(i) : F] = 2$. Let ϕ be the nontrivial automorphism of $F(i)$ over F. By [10, Proposition 5.4], there is a $j \in A\backslash F$ such that $\sigma(j) = j$ and $ji = \phi(i)j$. Then i and j generate a quaternion algebra Q, $\sigma|_Q(i) = i$, $\sigma|_Q(j) = j$, and $\sigma|_Q(k) = -k$ with $k = ij$. Hence, $\sigma|_Q$ is an orthogonal involution on Q. By the double centralizer theorem, we have $A \simeq Q \otimes C_A(Q)$, where $C_A(Q)$ is the centralizer of $Q \subset A$ and is isomorphic to quaternion algebra Q' over F. By [5, Proposition 2.23], the restriction of σ on Q' is the canonical involution γ.

(2) See [11, Proposition]. $\square$

Proposition 2.8. *Let F be a field of characteristic different from 2. Then*

(1) $\mathrm{ed}_2(\mathbf{PGO}_4^+) = \mathrm{ed}(\mathbf{PGO}_4^+) = 4$.
(2) $\mathrm{ed}_2(\mathbf{PGO}_4) = \mathrm{ed}(\mathbf{PGO}_4) = 4$.
(3) $\mathrm{ed}_2(\mathbf{PGSp}_4) = \mathrm{ed}(\mathbf{PGSp}_4) = 4$.

Proof. (1) By the exceptional isomorphism (2), we have

$$\mathbf{PGO}_4^+ = \mathbf{PGL}_2 \times \mathbf{PGL}_2 .$$

The proof follows from Lemma 2.5 with $n = 2$.

(2) By Lemma 2.6, we have $\mathrm{ed}_2(\mathbf{PGO}_4) \geq 4$. For the opposite inequality, we consider the affine variety X defined in $\mathbb{A}^4_F$ with the coordinates (a, b, c, e) by $e(a^2 - b^2e)(c^2 - e) \neq 0$. Define a morphism $X \to A_1^2$ by

$$(a, b, c, e) \mapsto \begin{cases} (a + b\sqrt{e}, c + \sqrt{e}) & \text{if } F(\sqrt{e}) \text{ is a quadratic field extension,} \\ (a, b) \times (c, \sqrt{e}) & \text{otherwise,} \end{cases}$$

We show that X is a classifying variety for A_1^2. Let $Q = (a + b\sqrt{e}, c + d\sqrt{e})$ be a quaternion algebra over a quadratic extension $L = F(\sqrt{e})$. If $b = d = 0$, we can modify c by a norm of $L(\sqrt{a})/L$, hence we may assume that $d \neq 0$. Similarly, we can assume that $d = 1$, replacing e by ed^2. Thus, the morphism $X \to A_1^2$ is surjective. By (5), $\mathrm{ed}(A_1^2) \leq 4$. Hence, the opposite inequality $\mathrm{ed}(\mathbf{PGO}_4) \leq 4$ comes from the exceptional isomorphism (1) and the canonical bijection (3).

(3) By Lemma 2.6, we have $\mathrm{ed}_2(\mathbf{PGSp}_4) \geq 4$. For the opposite inequality, we define a morphism $\mathbb{G}_m^4 \to C_2$ by

$$(x, y, z, w) \mapsto \begin{cases} ((x, y), \sigma) \otimes ((z, w), \gamma) & \text{if } x \neq 1, \\ (M_2(F), \mathrm{ad}_q) \otimes ((z, w), \gamma) & \text{if } x = 1, \end{cases}$$

where σ is an involution defined by $\sigma(x_0+x_1i+x_2j+x_3k) = x_0+x_1i+x_2j-x_3k$ with a quaternion basis $(1, i, j, k)$ of the quaternion algebra (x, y), $q = \langle 1, y\rangle$ is a quadratic form, and γ is the canonical involution on the quaternion algebra (z, w). Note that multiplying any two dimensional quadratic form by a scalar does not change the adjoint involution. By Lemma 2.7, this morphism is surjective. Therefore, by (5), we have $\mathrm{ed}(C_2) \leq 4$, hence the result follows from the canonical bijection (4).

□

Remark 2.9.

(1) Assume that F is a field of characteristic 2. By [1, Corollary 2.2], we have $\mathrm{ed}_2(\mathit{Alg}_{4,2}) = \mathrm{ed}_2(\mathit{Dec}_4) \geq 3$. As the morphisms (7) and (8) are surjective, we get $\mathrm{ed}_2(\mathbf{PGO}_4^+) \geq 3$ and $\mathrm{ed}_2(\mathbf{PGO}_4) \geq 3$, respectively. On the other hand, the upper bounds in Proposition 2.8 (1) and (2) still hold, hence $3 \leq \mathrm{ed}(\mathbf{PGO}_4^+), \mathrm{ed}(\mathbf{PGO}_4) \leq 4$.

(2) As $\mathrm{ed}(\mathbf{O}_5^+) = 4$ by [9, Theorem 10.3], Proposition 2.8 (3) can be recovered from the exceptional isomorphism $\mathbf{PGSp}_4 \simeq \mathbf{O}_5^+$.

References

[1] S. Baek, *Essential dimension of simple algebras in positive characteristic*, C. R. Math. Acad. Sci. Paris **349** (2011) 375–378.

[2] S. Baek and A. Merkurjev, *Invariants of simple algebras*, Manuscripta Math. **129** (2009), no. 4, 409–421.

[3] S. Baek and A. Merkurjev, *Essential dimension of central simple algebras*, to appear in Acta Math.

[4] G. Berhuy and G. Favi, *Essential dimension: a functorial point of view* (*after A. Merkurjev*), Doc. Math. **8** (2003), 279–330 (electronic).

[5] M.-A. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol, *The book of involutions*, American Mathematical Society, Providence, RI, 1998, With a preface in French by J. Tits.

[6] V. Chernousov and J.-P. Serre, *Lower bounds for essential dimensions via orthogonal representations*, J. Algebra **305** (2006), no. 2, 1055–1070.

[7] M. L. MacDonald, *Cohomological invariants of odd degree Jordan algebras*, Math. Proc. Cambridge Philos. Soc. **145** (2008), no. 2, 295–303.

[8] A. S. Merkurjev, *Essential dimension*, Quadratic forms—algebra, arithmetic, and geometry, Contemp. Math., vol. 493, Amer. Math. Soc., Providence, RI, (2009), pp. 299–325.

[9] Z. Reichstein, *On the notion of essential dimension for algebraic groups*, Transform. Groups 5 (2000), no. 3, 265-304.

[10] L. Rowen, *Central simple algebras*, Israel J. Math. **29** (1978), no. 2-3, 285–301.

[11] A. Serhir and J.-P. Tignol, *The discriminant of a decomposable symplectic involution*, J. Algebra **273** (2004), no. 2, 601–607.

Department of Mathematics and Statistics, University of Ottawa, 585 King Edward, Ottawa, ON K1N6N5, Canada

E-mail address: sbaek@uottawa.ca