Admissibility of groups over function fields of p-adic curves

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Abstract

Let K be a field and G a finite group. The question of 'admissibility' of G over K was originally posed by Schacher, who gave partial results in the case $K = \mathbb{Q}$. In this paper, we give necessary conditions for admissibility of a finite group G over function fields of curves over complete discretely valued fields. Using this criterion, we give an example of a finite group which is not admissible over $\mathbb{Q}_p(t)$. We also prove a certain Hasse principle for division algebras over such fields.

Introduction

Let K be a field and G a finite group. We say that G is *admissible* over K if there exists a division ring D central over K and a maximal subfield L of D which is Galois over K with Galois group G. Schacher asked, given a field K, which finite groups are admissible over K and proved that if a finite group G is admissible over $\mathbb{Q}$, then every Sylow subgroup of G is metacyclic ([Sc], 4.1). This led to the conjecture that a finite group G is admissible over $\mathbb{Q}$ if and only if every Sylow subgroup of G is meta-cyclic. This conjecture has been proved for all solvable groups ([So]) and for certain non-solvable groups of small order ([CS], [FS], [Fe1], [Fe2], [Fe3]).

Recently Harbater, Hartman and Krashen ([HHK2], 4.5) gave a characterization of admissible groups over function fields of curves over complete discretely valued fields with algebraically closed residue fields. In this paper, we consider the function fields of curves over complete discretely valued fields without any assumptions on the residue fields and prove the following

Theorem 1. Let K be a complete discretely valued field with residue field k and $F = K(X)$ be the function field of a curve X over K. Let G be a finite group. Suppose that the order of G is coprime to char(k). If G is admissible over F then every Sylow subgroup P of G has a normal series $P \supseteq P_1 \supseteq P_2$ such that

(1) P/P_1 and P_2 are cyclic

(2) P_1/P_2 is admissible over a finite extension of the residue field of a discrete valuation of F.

A main ingredient for the proof of the above theorem is the following Hasse principle for central simple algebras, which has independent interest.

Theorem 2. Let K be a complete discretely valued field with residue field k and $F = K(X)$ be the function field of a curve X over K. Let A be a central simple algebra over F of degree $n = \ell^r$ for some prime ℓ and $r \geq 1$. Assume that ℓ is not equal to char(k) and K contains a primitive n^{th} root of unity. Then index(A) = index$(A \otimes F_v)$ for some discrete valuation v of F.

For the proof of the above theorem, we use the patching techniques of ([HHK1]). A similar Hasse principle is proved for quadratic forms over such fields in ([CTPS], 3.1). In ([HHK3], 9.12), it is proved that if a central simple algebra A over F (F as above), is split over F_ν for all discrete valuations on F, then A is split over F.

There are some examples of classes of finite groups which are admissible over the rational function fields. However there was no example, in the literature, of a finite group which is not admissible over $\mathbb{Q}_p(t)$. Using Theorem 1, we give an example of a finite group which is not admissible over $\mathbb{Q}_p(t)$. We also prove admissibility of a certain class of groups over $\mathbb{Q}_p(t)$ using patching techniques.

Theorem 3. Let K be a p-adic field and F the function field of a curve over F. Let G be a finite group with order coprime to $char(k)$. If every Sylow subgroup of G is a quotient of $\mathbb{Z}^4$, then G is admissible over F.

In [FSS], it was proved that every abelian group on three or less generators is admissible over $\mathbb{Q}(t)$. We conclude by showing that every abelian group of order n with four or less generators is admissible over $\mathbb{Q}(\zeta)(t)$, where ζ is a primitive n^{th} root of unity.

1. Some Preliminaries

In this section we recall a few basic definitions and facts about division algebras and patching techniques ([GS], [HH], [HHK1], [P], [S2], [Sc], [Sch],

[Ser]).

Let K be a field and $Br(K)$ be the Brauer group of central simple algebras over K. For an integer $n \geq 2$, let ${}_nBr(K)$ denote the n-torsion subgroup of $Br(K)$. If A and B are two central simple algebras over K, we write $A \simeq B$ if A and B are isomorphic as K-algebras and we write $A = B$ if they represent the same element in $Br(K)$. Let n be an integer coprime to char(K). Suppose E/K is a cyclic extension of degree n and σ a generator of $Gal(E/K)$. For $b \in K^*$, let $(E/K, \sigma, b)$ (or simply (E, σ, b)) be the K-algebra generated by E and y with $y^n = b$ and $\lambda y = y\sigma(\lambda)$ for all $\lambda \in E$. Then (E, σ, b) is a central simple algebra over K and represents an element in ${}_nBr(K)$. Suppose that K contains a primitive n^{th} root of unity. Then $E = K(\sqrt[n]{a})$ for some $a \in K^*$ and $\sigma(\sqrt[n]{a}) = \zeta\sqrt[n]{a}$ for a primitive n^{th} root of unity $\zeta \in K^*$. The algebra (E, σ, b) is generated by x, y with $x^n = a$, $y^n = b$ and $xy = \zeta yx$ and we denote the algebra (E, σ, b) by $(a, b)_n$.

Suppose that ν is a discrete valuation of K with residue field k. Let n be a natural number which is coprime to char(k). Then we have a *residue homomorphism* $\partial_\nu : {}_nBr(K) \to H^1(k, \mathbb{Z}/n\mathbb{Z})$, where $H^1(k, \mathbb{Z}/n\mathbb{Z})$ denotes the first Galois cohomology group. Suppose k contains a primitive n^{th} root of unity. By fixing a primitive n^{th} root of unity, we identify $H^1(k, \mathbb{Z}/n\mathbb{Z})$ with k^*/k^{*n}. With this identification we have $\partial_\nu((a, b)_n) = \overline{\frac{a^{\nu(b)}}{b^{\nu(a)}}} \in k^*/k^{*n}$, where for any $c \in K^*$ which is a unit at ν, $\overline{c}$ denotes its image in k^*. More generally, let E/K be a cyclic unramified inert extension of K and $\sigma \in Gal(E/K)$ be a generator. Let $\pi \in K^*$ be a parameter at ν. Then the residue of $(E/K, \sigma, \pi)$ is $(E_0/k, \sigma_0)$, where k is the residue field of K at ν, E_0 is the residue field of E at the unique extension of ν to E and σ_0 is the image of σ.

Let K be a field and n an integer. Then $H^1(K, \mathbb{Z}/n\mathbb{Z})$ classifies the equivalence classes of pairs (E, σ), where E is a cyclic Galois field extension of K of degree a divisor of n and σ a generator of the Galois group $G(E/K)$ of E/K. Let (E, σ) be a pair representing an element in $H^1(K, \mathbb{Z}/n\mathbb{Z})$. Let $m \geq 1$ be an integer. We now describe $m(E, \sigma)$ in the group $H^1(K, \mathbb{Z}/n\mathbb{Z})$. Let d be the greatest common divisor of m and $[E : K]$. Let $E(m)$ be the subfield of E fixed by $\sigma^{[E:K]/d}$ and $\sigma(m)$ be the restriction of $\sigma^{m/d}$ to $E(m)$. Then $m(E, \sigma) = (E(m), \sigma^{m/d})$. The order of (E, σ) in $H^1(K, \mathbb{Z}/n\mathbb{Z})$ is equal to $[E : K]$. Since the identity element of $H^1(K, \mathbb{Z}/n\mathbb{Z})$ is (K, id), we have $m(E, \sigma)$ is trivial in $H^1(K, \mathbb{Z}/n\mathbb{Z})$ if and only if $E(m) = K$ if and only if $[E : K]$ divides m.

Let K be a complete discretely valued field with residue field k. Let L/K be a finite extension of K. Since K is complete, the discrete valuation of K extends uniquely to a discrete valuation of L and L is complete with respect to this discrete valuation. Let n be a natural number which is coprime to the characteristic of k. Let L/K be a Galois extension of degree n. Let L_1 be the maximal unramified extension of K. Since n is coprime to char(k), the residue field L_0 of L is same as the residue field of L_1. Since L/K is Galois, L_1/K and L_0/k are also Galois and there is a natural isomorphism $Gal(L/K) \to Gal(L_0/k)$. Let E_0/k be a cyclic extension of degree n. Then there exists a unique (up to isomorphism) unramified cyclic extension E of K with residue field E_0 and a natural isomorphism $Gal(E/K) \to Gal(E_0/k)$. Let $(E_0, \sigma_0) \in H^1(k, \mathbb{Z}/n\mathbb{Z})$. Then we have a unique $(E, \sigma) \in H^1(K, \mathbb{Z}/n\mathbb{Z})$ with $[E : K] = [E_0 : k]$, E_0 as the residue field of E and the image of σ equal to σ_0; (E, σ) is called the *lift* of (E_0, σ_0).

Let $\mathcal{X}$ be a regular integral scheme with function field F. Let n be an integer which is a unit on $\mathcal{X}$. Let $f \in F$ and $P \in \mathcal{X}$ be a point. If f is regular at P, then we denote its image in the residue field $\kappa(P)$ at P by $f(P)$. Let $\mathcal{X}^1$ denote the set of codimension one points of $\mathcal{X}$. For each codimension one point x of $\mathcal{X}$, we have discrete valuation ν_x on F. Let $\kappa(x)$ denote the residue field at x. Since n is a unit on $\mathcal{X}$, n is coprime to char($\kappa(x)$) and we have the residue homomorphism $\partial_x : {}_nBr(F) \to H^1(\kappa(x), \mathbb{Z}/n\mathbb{Z})$. Let $\alpha \in {}_nBr(F)$. We say that α is *unramified* at x if $\partial_x(\alpha) = 0$. We say that α is *unramified* on $\mathcal{X}$ if it is unramified at every codimension one point of $\mathcal{X}$. Let A be a central simple algebra over F. We say tat A is *unramified* if its class in $Br(F)$ is unramified. If $\mathcal{X} = Spec(B)$, then we say that α is unramified on B if it is unramified on $\mathcal{X}$.

Let F be a field and A a central simple algebra over F. Then $A \simeq M_m(D)$ for some central division algebra D over F. The *degree* of A is defined as $\sqrt{dim_F A}$ and the *index* of A is defined as the degree of D. If L/K is a field extension, then index($A \otimes_F L$) divides index(A). Let B be an integral domain and F its field of fractions. Let $\mathcal{A}$ be an Azumaya algebra over B, then we define the *index* of $\mathcal{A}$ to be the index of $\mathcal{A} \otimes_B F$.

Let B be a regular integral domain of dimension at most 2 which is complete with respect to a prime ideal P. Assume that B/P is a regular integral domain of dimension at most 1 (for example P is a maximal ideal). Let $\kappa(P)$ be the field of fractions of B/P. Let A be a central simple algebra over the field of fractions F of B which is unramified on B. Then there exists an Azumaya algebra $\mathcal{A}$ over B such that $\mathcal{A} \otimes_B F \simeq A$ ([CTS], 6.13)). For an ideal I of B, we denote the algebra $\mathcal{A} \otimes_B B/I$ by $A(I)$.

Lemma 1.1 Let A, B, $\mathcal{A}$ and P be as above. Then index(A) = index ($\mathcal{A} \otimes_{B/P} \kappa(P)$).

Proof. Suppose $A \simeq M_n(D)$ for some division algebra D over F. Since A is unramified on B, D is also unramified on B. Since B is a regular domain of dimension 2, there exists an Azumaya algebra $\mathcal{D}$ over B such that $\mathcal{D} \otimes_B K \simeq D$ ([CTS], 6.13)). Since $Br(B) \to Br(F)$ is injective ([AG], 7.2), $\mathcal{A} = \mathcal{D}$ in $Br(B)$. In particular $\mathcal{A} \otimes_{B/P} \kappa(P) = \mathcal{D} \otimes_{B/P} \kappa(P)$ in $Br(\kappa(P))$. Hence $\mathrm{index}(A) = \mathrm{degree}(D) = \sqrt{\dim_F(D)} = \sqrt{\mathrm{rank}_B(\mathcal{D})} = \sqrt{\mathrm{rank}_{B/P}(\mathcal{D} \otimes_B B/P)} = \sqrt{\dim_{\kappa(P)}(\mathcal{D} \otimes_{B/P} \kappa(P))} \geq \mathrm{index}(\mathcal{A} \otimes_{B/P} \kappa(P))$.

Suppose $\mathcal{A} \otimes_{B/P} \kappa(P) \simeq M_m(D_0)$ for some central division algebra D_0 over $\kappa(P)$. Since B/P is regular integral domain of dimension 1, there exists an Azumaya algebra $\mathcal{D}_0$ over B/P such that $\mathcal{D}_0 \otimes_{B/P} \kappa(P) \simeq D_0$. Since B is P-adically complete, there exists an Azumaya algebra $\tilde{\mathcal{D}}_0$ over B such that $\tilde{\mathcal{D}}_0 \otimes_B B/P \simeq \mathcal{D}_0$ and $\mathcal{A} = \tilde{\mathcal{D}}_0$ in $Br(B)$ ([C], [KOS]). In particular $A = \tilde{\mathcal{D}}_0 \otimes_B F$ in $Br(F)$. Hence $\mathrm{index}(\mathcal{A} \otimes_{B/P} \kappa(P)) = \mathrm{degree}(D_0) = \sqrt{\dim_{\kappa(P)}(D_0)} = \sqrt{\mathrm{rank}_{B/P}(\mathcal{D}_0)} = \sqrt{\mathrm{rank}_B(\tilde{\mathcal{D}}_0)} = \sqrt{\dim_F(\tilde{\mathcal{D}}_0 \otimes F)} \geq \mathrm{index}(A)$. Thus $\mathrm{index}(A) = \mathrm{index}(\mathcal{A} \otimes_{B/P} \kappa(P))$. $\square$

Let K be a field and L a finite extension of K. Then L is called K-*adequate* if there is a division ring D central over K containing L as a maximal subfield. A finite group G is called K-*admissible* if there is a Galois extension L of K with G as the Galois group of L over K, and L is K-*adequate*.

A finite group G is called *metacyclic* if G has a normal subgroup H such that H is cyclic and G/H is cyclic.

2. A local-global principle for central simple algebras

Let K be a complete discretely valued field with residue field k. Let F be the function field of a curve over K. Let n be an integer which is coprime to the characteristic of k. Assume that K contains a primitive n^{th} root of unity. In this section we prove a certain Hasse principle for central simple algebras over F of index n. We begin with the following

Lemma 2.1. (cf. [FS], Proposition 1(3) and [JW], 5.15) Let R be a complete discrete valuated ring and K its field of fractions. Let A be a central simple algebra over K of index n which is unramified at R. Let E be an unramified cyclic extension of K of degree m and σ a generator of the Galois group of E/K. Let π be a parameter in R. Assume that mn is invertible in R. Then $\mathrm{index}(A \otimes (E, \sigma, \pi)) = \mathrm{index}(A \otimes E) \cdot [E : K]$.

The following two lemmas (2.2, 2.3) are well known.

Lemma 2.2. Let R be a regular ring of dimension 2 with field of fractions F. Let n be an integer which is a unit in R. Assume that F contains a primitive n^{th} root of unity. Suppose A is a central simple algebra over F which is unramified on R. Let $x \in R$ be a regular prime and ν be the discrete valuation on F given by x. Suppose that R is complete with respect to (x)-adic topology. Let $u \in R$ be a unit. Then $\text{index}(A \otimes F_\nu(\sqrt[n]{u})) = \text{index}(A \otimes F(\sqrt[n]{u}))$.

Proof. Let S be the integral closure of R in $F(\sqrt[n]{u})$. Since R is a regular ring and n, u are units in R, S is also a regular ring. Let x be a regular prime in R, then x is also a regular prime in S. Thus by replacing R by S, it is enough to show that $\text{index}(A \otimes F_\nu) = \text{index}(A)$.

Since R is a two-dimensional regular ring, there is an Azumaya algebra $\mathcal{A}$ over R with $\mathcal{A} \otimes F \simeq A$ ([CTS], 6.13)). Since R is complete with respect to (x)-adic topology, $\text{index}(A) = \text{index}(\mathcal{A} \otimes \kappa(x))$ (cf. 1.1), where $\kappa(x)$ is the field of fractions of $R/(x)$. Let R_ν be the ring of integers in F_ν. Since A is unramified on R, A is also unramified on R_ν. Since F_ν is complete, $\text{index}(A \otimes F_\nu) = \text{index}(\mathcal{A} \otimes \kappa(\nu))$ (cf. 1.1). Since $\kappa(\nu) = \kappa(x)$, $\text{index}(A) = \text{index}(A \otimes F_\nu)$. $\square$

Lemma 2.3. Let R be a complete regular local ring with field of fractions F and residue field k. Let n be an integer which is a unit in R. Let $u_1, \cdots, u_r \in R$ be units. Suppose $x \in R$ is a regular prime. Let ν be the discrete valuation on F given by x. Then $[F_\nu(\sqrt[n]{u_1}, \cdots, \sqrt[n]{u_r}) : F_\nu] = [F(\sqrt[n]{u_1}, \cdots, \sqrt[n]{u_r}) : F]$.

Proof. Since F_ν is a complete discretely valued field and n, u_1, $\cdots$, u_r are units in the ring of integers, $[F_\nu(\sqrt[n]{u_1}, \cdots, \sqrt[n]{u_r}) : F_\nu] = [\kappa(\nu)(\sqrt[n]{\overline{u}_1}, \cdots, \sqrt[n]{\overline{u}_r}) : \kappa(\nu)]$. Since $\kappa(\nu)$ is the field of fractions of the complete local ring $R/(x)$ and the the residue field of $R/(x)$ is k, we have $[\kappa(\nu)(\sqrt[n]{\overline{u}_1}, \cdots, \sqrt[n]{\overline{u}_r}) : \kappa(\nu)] = [k(\sqrt[n]{\overline{u_1}}, \cdots, \sqrt[n]{\overline{u_r}}) : k]$. Since R is complete and $u_1, \cdots, u_r$ are units, we also have $[F(\sqrt[n]{u_1}, \cdots, \sqrt[n]{u_r}) : F] = [k(\sqrt[n]{\overline{u}_1}, \cdots, \sqrt[n]{\overline{u}_r}) : k]$. Hence $[F_\nu(\sqrt[n]{u_1}, \cdots, \sqrt[n]{u_r}) : F_\nu] = [F(\sqrt[n]{u_1}, \cdots, \sqrt[n]{u_r}) : F]$. $\square$

Proposition 2.4. Let R be a 2-dimensional complete regular local ring with maximal ideal $m = (x, y)$. Let F be the field of fraction of R and k the residue field of R. Let n be an integer coprime to $\text{char}(k)$. Assume that F contains a primitive n^{th} root of unity. Let A be a central simple algebra over F of degree n. Suppose that A is unramified on R except possibly at x and

y. Then index$(A) =$ index$(A \otimes F_\nu)$ for the discrete valuation ν of F given by the prime ideals either (x) or (y) of R.

Proof. Suppose that A is unramified on R. Let ν be the discrete valuation of F given by x. Since A is unramified on R, by (2.2), we have index$(A \otimes F_\nu) =$ index(A).

Suppose that A is ramified on R only at the prime ideal (x). Let ν be the discrete valuation on F given by the prime ideal (x) of R. By ([S1]), we have $A = A' \otimes (u, x)$ for some unit u in R and a central simple algebra A' over F which is unramified on R. Since F_ν is complete and u is a unit at ν, there is a unique extension of ν to $F_\nu(\sqrt[n]{u})$ such that $F_\nu(\sqrt[n]{u})$ is complete with the residue field $\kappa(\nu)(\sqrt[n]{\overline{u}})$, where $\overline{u}$ is the image of u in $\kappa(\nu)$. Since A' is unramified on R, we have

$$\begin{array}{rcll} \text{index}(A \otimes F_\nu) & = & \text{index}(A' \otimes (u, x) \otimes F_\nu) & \\ & = & \text{index}(A' \otimes F_\nu(\sqrt[n]{u})) \cdot [F_\nu(\sqrt[n]{u}) : F_\nu] & \text{(by (2.1))} \\ & = & \text{index}(A' \otimes F(\sqrt[n]{u})) \cdot [F_\nu(\sqrt[n]{u}) : F_\nu] & \text{(by (2.2))} \\ & = & \text{index}(A' \otimes F(\sqrt[n]{u})) \cdot [F(\sqrt[n]{u}) : F] & \text{(by (2.3)).} \end{array}$$

Since $A = A' \otimes (u, x)$, the index of A divides index$(A' \otimes F(\sqrt[n]{u})) \cdot [F(\sqrt[n]{u}) : F] =$ index$(A \otimes F_\nu)$. Thus index$(A) =$ index$(A \otimes F_\nu)$.

Assume that A is ramified on R at both the primes (x) and (y). Then by ([S1]), either $A = A' \otimes (u_1, x) \otimes (u_2, y)$ or $A = A' \otimes (uy^r, x)$ where u_1, u_2, u are units in R, r coprime with n and A' unramified on R.

Suppose that $A = A' \otimes (u_1, x) \otimes (u_2, y)$ for some units $u_1, u_2 \in R$ and A' unramified on R. Let ν be the discrete valuation on F given by y. Then by (2.1), we have index$(A \otimes F_\nu) =$ index$(A' \otimes (u_1, x) \otimes F_\nu(\sqrt[n]{u_2})) \cdot [F_\nu(\sqrt[n]{u_2}) : F_\nu]$. By (2.2), we have $[F_\nu(\sqrt[n]{u_2}) : F_\nu] = [F(\sqrt[n]{u_2}) : F]$. We now compute index$(A' \otimes (u_1, x) \otimes F_\nu(\sqrt[n]{u_2}))$.

Since $A' \otimes (u_1, x)$ is unramified at ν and u_2 is a unit at ν, index$(A' \otimes (u_1, x) \otimes F_\nu(\sqrt[n]{u_2})) =$ index$(A' \otimes (u_1, x) \otimes \kappa(\nu)(\sqrt[n]{\overline{u_2}}))$ (cf. 1.1), where $\overline{u_2}$ is the image of u_2 in $\kappa(\nu)$. Since $\kappa(\nu)$ is the field of fractions of $R/(y)$, the image of x in $\kappa(\nu)$ gives a discrete valuation μ on the residue field $\kappa(\nu)$ with $\kappa(\mu) = k$ and $\kappa(\nu)$ is complete with respect to μ. Since $\overline{u_2}$ is a unit at μ, $\kappa(\nu)(\sqrt[n]{\overline{u_2}})$ is a complete discrete valuated field with $\overline{x}$ as parameter and residue field $k(\sqrt[n]{\overline{u_2}})$. Thus, we have

$$\begin{array}{l} \text{index}(A' \otimes (u_1, x) \otimes F_\nu(\sqrt[n]{u_2})) = \text{index}(A' \otimes (u_1, x) \otimes \kappa(\nu)(\sqrt[n]{\overline{u_2}}))\text{(by (1.1))} \\ = \text{index}(A' \otimes \kappa(\nu)(\sqrt[n]{\overline{u_2}}, \sqrt[n]{\overline{u_1}})) \cdot [\kappa(\nu)(\sqrt[n]{\overline{u_2}}, \sqrt[n]{\overline{u_1}}) : \kappa(\nu)(\sqrt[n]{\overline{u_2}})] \quad \text{(by (2.1))} \\ = \text{index}(A' \otimes F_\nu(\sqrt[n]{u_2}, \sqrt[n]{u_1})) \cdot [F_\nu(\sqrt[n]{u_2}, \sqrt[n]{u_1}) : F_\nu(\sqrt[n]{u_2})] \quad \text{(by (1.1))} \\ = \text{index}(A' \otimes F(\sqrt[n]{u_2}, \sqrt[n]{u_1})) \cdot [F(\sqrt[n]{u_2}, \sqrt[n]{u_1}) : F(\sqrt[n]{u_2})] \quad \text{(by (2.2), (2.3)).} \end{array}$$

Hence

$$\begin{aligned}&\mathrm{index}(A\otimes F_\nu)=\mathrm{index}(A'\otimes(u_1,x)\otimes F_\nu(\sqrt[n]{u_2}))\cdot[F_\nu(\sqrt[n]{u_2}):F_\nu](by\ (2.1))\\&=\mathrm{index}(A'\otimes F(\sqrt[n]{u_2},\sqrt[n]{u_1}))\cdot[F(\sqrt[n]{u_2},\sqrt[n]{u_1}):F(\sqrt[n]{u_2})]\cdot[F(\sqrt[n]{u_2}):F]\\&=\mathrm{index}(A'\otimes F(\sqrt[n]{u_2},\sqrt[n]{u_1}))\cdot[F(\sqrt[n]{u_2},\sqrt[n]{u_1}):F].\end{aligned}$$

On the other hand we have $\mathrm{index}(A)=\mathrm{index}(A'\otimes(u_1,x)\otimes(u_2,y))$ divides $\mathrm{index}(A'\otimes F(\sqrt[n]{u_1},\sqrt[n]{u_2}))\cdot[F(\sqrt[n]{u_1},\sqrt[n]{u_2}):F]$. In particular $\mathrm{index}(A\otimes F_\nu)=\mathrm{index}(A)$.

Assume that $A=A'\otimes(uy^r,x)$ for some unit $u\in R$, r coprime to n and A' unramified on R. Let ν be the discrete valuation on F given by the prime ideal (x) of R. By (2.1), we have $\mathrm{index}(A\otimes F_\nu)=\mathrm{index}(A'\otimes F_\nu(\sqrt[n]{uy^r}))\cdot[F_\nu(\sqrt[n]{uy^r}):F_\nu]$. Since y is a regular prime in R and r is coprime to n, it follows that $[F(\sqrt[n]{uy^r}):F]=[F_\nu(\sqrt[n]{uy^r}):F_\nu]=n$. Since F_ν is a complete discrete valuated field and A' is unramified, we have $\mathrm{index}(A'\otimes F_\nu(\sqrt[n]{uy^r}))=\mathrm{index}(A'\otimes\kappa(\nu)(\sqrt[n]{\overline{uy^r}}))$ (cf. 1.1). Since $\kappa(\nu)$ is a complete discrete valuated field with $\overline{y}$ as a parameter and residue field k, $\kappa(\nu)(\sqrt[n]{\overline{uy^r}})$ is also a complete discrete valuated field with residue field k. Since A' is unramified on R, we have $\mathrm{index}(A'\otimes\kappa(\nu)(\sqrt[n]{\overline{uy^r}}))=\mathrm{index}(A'\otimes k)$ (cf. (1.1)). Since R is complete, by (1.1), we have $\mathrm{index}(A')=\mathrm{index}(A'\otimes k)=\mathrm{index}(A'\otimes\kappa(\nu)(\sqrt[n]{\overline{uy^r}}))=\mathrm{index}(A'\otimes F_\nu(\sqrt[n]{uy^r}))$. Therefore, we have

$$\begin{aligned}\mathrm{index}(A\otimes F_\nu)&=\mathrm{index}(A'\otimes(uy^r,x)\otimes F_\nu)\\&=\mathrm{index}(A'\otimes F_\nu(\sqrt[n]{uy^r}))\cdot[F_\nu(\sqrt[n]{uy^r}):F_\nu]\\&=\mathrm{index}(A')\cdot n.\end{aligned}$$

Since $A=A'\otimes(uy^r,x)$, $\mathrm{index}(A)\leq\mathrm{index}(A')\cdot n=\mathrm{index}(A\otimes F_\nu)$. Thus $\mathrm{index}(A)=\mathrm{index}(A\otimes F_\nu)$. $\square$

Proposition 2.5. Let R be a 2-dimensional regular ring with field of fraction F and n an integer which is a unit in R. Assume that F contains a primitive n^{th} root of unity. Suppose that R is complete with respect to (s)-adic topology for some prime element $s\in R$ with $R/(s)$ a Dedekind domain. Let ν be the discrete valuation on F given by s. Let A be a central simple algebra over F of degree n which is unramified on R except at ν. Further assume that the residue of A at (s) is given by a unit a in $R/(s)$. Then $\mathrm{index}(A)=\mathrm{index}(A\otimes F_\nu)$.

Proof. Suppose that A is unramified on R. Since R is (s)-adically complete and $R/(s)$ is a regular domain of dimension 1 with field of fractions $\kappa(\nu)$, $\mathrm{index}(A)=\mathrm{index}(A\otimes_{R/(s)}\kappa(\nu))=\mathrm{index}(A\otimes F_\nu)$ (cf. (1.1)).

Suppose that A is ramified on R. Then by the assumption on A, A is ramified on R only at the prime ideal (s) of R and the residue of A at (s) is given by a unit a in $R/(s)$. Let $u \in R$ with image $a \in R/(s)$. Since R is (s)-adically complete and the image a of u modulo (s) is a unit, u is a unit in R. The cyclic algebra (u, s) is ramified on R only at (s) and the residue of (u, s) at (s) is (a). Since A is ramified only at the prime ideal (s) of R, and (a) is the residue of A at s, we have $A = A' \otimes (u, s)$ for some central simple algebra A' over F which is unramified on R. By (2.1), we have $\mathrm{index}(A \otimes F_\nu) = \mathrm{index}(A' \otimes F_\nu(\sqrt[n]{u})) \cdot [F_\nu(\sqrt[n]{u}) : F_\nu]$. Since R is (s)-adically complete and $R/(s)$ is integrally closed domain with u a unit, we have $[F(\sqrt[n]{u}) : F] = [F_\nu(\sqrt[n]{u}) : F_\nu]$. Since A' is unramified on R and R is (s)-adically complete, by (2.2), we have $\mathrm{index}(A' \otimes F_\nu(\sqrt[n]{u})) = \mathrm{index}(A' \otimes F(\sqrt[n]{u}))$. Hence we have $\mathrm{index}(A \otimes F_\nu) = \mathrm{index}(A' \otimes F(\sqrt[n]{u})) \cdot [F(\sqrt[n]{u}) : F]$. In particular $\mathrm{index}(A \otimes F_\nu) = \mathrm{index}(A)$. $\square$

Theorem 2.6. Let T be a complete discrete valuation ring with fraction field K and residue field k. Let X be a regular, projective, geometrically integral curve over K and $F = K(X)$ the function field of X. Let l be prime not equal to $\mathrm{char}(k)$ and A a central simple algebra over F of index $n = l^d$ for some $d \geq 1$. Assume that K contains a primitive n^{th} root of unity. Then $\mathrm{index}(A) = \mathrm{index}(A \otimes F_\nu)$ for some discrete valuation ν of F.

Proof. Let A be a central simple algebra over F. We choose a regular proper model $\mathcal{X}/T$ of X/K such that the support of the ramification divisor A and the components of the special fibre of $\mathcal{X}/T$ are a union of regular curves with normal crossings. Let $Y = \mathcal{X} \times_T k$ denote the special fibre.

For each irreducible curve C in the support of ramification divisor of A, let $a_C \in \kappa(C)^*$ be such that the residue of A at C is $(a_C) \in H^1(\kappa(C)^*, \mathbb{Z}/n\mathbb{Z})$.

Let S be a finite set of closed points of the special fibre containing all singular points of Y and all those points of irreducible curves C where a_C is not a unit.

For each $P \in S$, let F_P be the field of fractions of the completion $\hat{R}_P$ of the local ring R_P of $\mathcal{X}$ at P. Let t be a uniformizing parameter for T. For each irreducible component U of $Y \setminus S$, let R_U be the ring of elements in F which are regular on U. It is a regular ring. Let $\hat{R}_U$ be the completion of R_U at (t) and F_U the field of fractions of $\hat{R}_U$. As in ([CTPS], proof of 3.1), we choose S such that for every irreducible component U of $Y \setminus S$, $t = u.s^r$ for some integer $r \geq 1$, a unit $u \in R_U$ and $R_U/s = \hat{R}_U/s$ is a Dedekind domain with field of fractions $k(U)$. In particular, the t-adic completion $\hat{R}_U$ coincides with the s-adic completion of R_U.

Let $\mathcal{U}$ be the set of irreducible components of $X \setminus S$. By ([HHK1], 5.1), we have index(A) = lcm of index$(A \otimes F_\zeta)$, $\zeta \in \mathcal{U} \cup S$. Since index$(A \otimes F_\zeta)$ is a power of the prime l for all $\zeta \in \mathcal{P} \cup P$, we have index$(A) =$ index$(A \otimes F_\zeta)$ for some $\zeta \in \mathcal{U} \cup S$. In particular index$(A)$ equal to either index$(A \otimes F_U)$ for some irreducible components U of $Y \setminus S$ or index$(A \otimes F_P)$ for some $P \in S$.

Suppose that index$(A) =$ index$(A \otimes F_P)$ for some $P \in S$. The local ring $\hat{R}_P$ is a complete regular local ring of dimension 2 with maximal ideal (x, y) such that A is ramified on $\hat{R}_P$ at most at x and y. By (2.4), we have index$(A) =$ index$(A \otimes F_{P\nu})$ for the discrete valuation of F_P given by either (x) or (y). Since the restriction ν_0 of ν to F is non-trivial, ν_0 is a discrete valuation on F and index$(A) =$ index$(A \otimes F_{\nu_0})$.

Suppose that index$(A) =$ index$(A \otimes F_U)$ for some irreducible component U of $X \setminus S$. Then, by the choice of U, A is unramified on $\hat{R}_U$ except at (s) and the residue at (s) is given by a unit in $R_U/(s)$. Let ν be the discrete valuation on F_U given by (s). Since $\hat{R}_U$ is (s)-adically complete, by (2.5), index$(A) =$ index$(A \otimes F_{U\nu})$. Since the restriction of ν to F is also given by the ideal (s) in R_U, ν is non-trivial on F. In particular $F_\nu \subset F_{U\nu}$ and index$(A) =$ index$(A \otimes F_\nu)$. $\square$.

Remark 2.7. Let F and A be as in the above theorem. Then by (2.6), it follows that there exists a discrete valuation ν of F such that index$(A) =$ index$(A \otimes F_\nu)$. From the proof of (2.6) it follows that this discrete valuation comes from a codimension one point of a regular proper model of F. In particular, the residue field $\kappa(\nu)$ is either a finite extension of K or a function field of a curve over a finite extension of k.

3. Necessary conditions for Admissibility

In this section, we give a necessary condition for a finite group to be admissible over function fields of curves over complete discretely valued fields.

Let K be a complete discretely valued field with residue field k and π a parameter. Let D be a central division algebra over K of degree n and L a maximal subfield of D. Suppose that n is coprime to char(k). Let $(E_0, \sigma_0) \in H^1(k, \mathbb{Z}/n\mathbb{Z})$ be the residue of D and $(E, \sigma) \in H^1(K, \mathbb{Z}/n\mathbb{Z})$ the lift of (E_0, σ_0) (cf. §1). We fix an algebraic closure $\overline{K}$ of K and also an algebraic closure $\overline{k}$ of k. All the finite extensions of K and k are considered as subfields of $\overline{K}$ and $\overline{k}$ respectively. Let $F = L \cap E$. Let L_1 be the maximal unramified extension of F contained in L. Since E/K is unramified, $L \cap E = L_1 \cap E = F$.

Lemma 3.1. Let K, D, L, L_1, (E_0, σ_0), (E, σ), and F be as above. Then $D \otimes_K F \otimes_F (E/F, \sigma^{[F:K]}, \pi)^{op}$ is unramified at the discrete valuation of F.

Proof. Since the residue of D is (E_0, σ_0), $D \otimes (E, \sigma, \pi)^{op}$ is unramified at the discrete valuation of K. In particular, $D \otimes_K (E, \sigma, \pi) \otimes F$ is unramified at the discrete valuation of F. Since $F \subset E$, we have $(E, \sigma, \pi) \otimes_K F = (E/F, \sigma^{[F:K]}, \pi)$ in $Br(F)$. Thus $(D \otimes_K F) \otimes_F (E/F, \sigma^{[F:K]}, \pi)^{op}$ is unramified at the discrete valuation of F. $\square$

Lemma 3.2. Let K, D, L, L_1, (E_0, σ_0), (E, σ) and F be as above. Then $[E : F]$ divides $[L : L_1]$.

Proof. We have the following commutative diagram (cf., [S1], 10.4)

$$\begin{tikzcd} {}_nBr(K) \arrow[r] \arrow[d, "res"'] & H^1(k, \mathbb{Z}/n\mathbb{Z}) \arrow[d, "e.res"] \\ {}_nBr(L) \arrow[r] & H^1(L_0, \mathbb{Z}/n\mathbb{Z}) \end{tikzcd} ,$$

where L_0 is the residue field of L and e the ramification index of L/K. From the above commutative diagram, the residue of $D \otimes L$ is the restriction of $e(E_0, \sigma_0)$ to L_0. Since L is a maximal subfield of D, $D \otimes L$ is a split algebra. In particular, the residue of $D \otimes L$ is trivial. Hence the restriction of $e(E_0, \sigma_0)$ to L_0 is trivial and $[E : F]$ divides e (cf. §1). Since $[E : F] = [E_0 : F_0]$ and $e = [L : L_1]$, $[E : F]$ divides $[L : L_1]$. $\square$

Lemma 3.3. Let K, D, L, L_1, (E_0, σ_0), (E, σ) and F be as above. Then index$(D \otimes_K E) = [L_1 E : E]$.

Proof. Since $(E/F, \sigma^{[F:K]}, \pi)^{op} \otimes E$ is split, we have $D \otimes_K E = D \otimes_K F \otimes_F (E/F, \sigma^{[F:K]}, \pi)^{op} \otimes_F E$. By (3.1), $D \otimes_K F \otimes_F (E/F, \sigma^{[F:K]}, \pi)^{op}$ is unramified at the discrete valuation of F. Hence $D \otimes_K E$ is unramified at the discrete valuation of E. Since L is a maximal subfield of D, $D \otimes_K L$ is split. Since L/L_1 is totally ramified, LE/L_1E is totally ramified. In particular, the residue field of LE and L_1E are equal. Since $D \otimes_K E$ is unramified and $(D \otimes_K E) \otimes_E LE$ is plit, $(D \otimes_K E) \otimes_E L_1 E$ is split (cf. 1.1). In particular index$(D \otimes_K E) \leq [L_1 E : E]$.

Since $(D \otimes_K F) \otimes_F (E/F, \sigma^{[F:K]}, \pi)^{op}$ is unramified and π is a parameter in E, by (2.1), we have

$$\begin{aligned}
&\text{index}(D \otimes_K F) = \text{index}((D \otimes_K F) \otimes_F (E/F, \sigma^{[F:K]}, \pi)^{op} \otimes_F (E/F, \sigma^{[F:K]}, \pi)) \\
&= \text{index}((D \otimes_K F) \otimes_F (E/F, \sigma^{[F:K]}, \pi)^{op} \otimes_F E) \cdot [E : F] \\
&= \text{index}(D \otimes_K E) \cdot [E : F]
\end{aligned}$$

On the other hand, since $F \subset L$ and L is a maximal subfield of D, $\text{index}(D \otimes_K F) = [L : F]$. Hence $\text{index}(D \otimes_K E) \cdot [E : F] = [L : F] = [L : L_1] \cdot [L_1 : F]$. By (3.2), we have $[E : F]$ divides $[L : L_1]$. Hence $\text{index}(D \otimes_K E) \geq [L_1 : F] \geq [L_1E : E]$. Therefore $\text{index}(D \otimes_K E) = [L_1E : E]$. $\square$

Lemma 3.4. Let K be a complete discretely valued field with residue field k and P a p-group, p a prime. Suppose that p is coprime to $\text{char}(k)$. If P is admissible over K, then P has a normal series $P \supseteq P_1 \supseteq P_2$ such that

(1) P/P_1 and P_2 are cyclic

(2) P_1/P_2 is admissible over some finite extension of k.

Proof Suppose that P is admissible over K. Then there exists a Galois extension L/K and a division ring D central over K which contains L as maximal subfield such that $P = G(L/K)$. Let L_0 be the residue field of L. Let $\partial(D)=(E_0, \sigma_0) \in H^1(k, \mathbb{Z}/n\mathbb{Z})$ be the residue of D and $(E, \sigma) \in H^1(K, \mathbb{Z}/n\mathbb{Z})$ be the lift of (E_0, σ_0). Let L_1 be the maximal unramified extension of K contained in L. Since E is unramified extension of K, we have $L \cap E = L_1 \cap E$. Let $F = L \cap E$. Since E/K is cyclic, F/K is also cyclic.

Let P_1 be the Galois group of L/F. Since F/K is cyclic, P_1 is a normal subgroup of P and P/P_1 is cyclic.

Let P_2 be the Galois group of L/L_1. Then P_2 is a subgroup of P_1. Since L/L_1 is a totally ramified Galois extension of degree coprime to $\text{char}(k)$, P_2 is cyclic ([Se], Cor.2 and Cor.4 of Ch.IV, §2). Since L_1/F is a Galois extension (cf. §1), P_2 is a normal subgroup of P_1. The residue field F_0 of F is the same as the intersection of L_0 and E_0. We now show that P_1/P_2 is admissible over E_0.

Let D' be the division algebra with center E which is Brauer equivalent to $D \otimes_K E$. Since $D' \otimes_E L_1E = (D \otimes_K E) \otimes_E L_1E$ and $(D \otimes_K E) \otimes_E L_1E$ is split (cf. proof of (3.3)), $D' \otimes_E L_1E$ is also split. Since, by (3.3) $\text{degree}(D') = [L_1E : E]$, L_1E is a maximal subfield of D'. By (3.1), $D' = D \otimes_K E$ is unramified at the discrete valuation of E. Let $\overline{D'}$ be the image of D' over the residue field E_0 of E. Since E is complete, $\overline{D'}$ is central division algebra over E_0 and L_0E_0 is a maximal subfield of $\overline{D'}$. Hence $Gal(L_0E_0/E_0)$ is admissible over E_0. Since the residue field of L_1E is L_0E_0 and L_1E/E is an unramified Galois extension, we have $Gal(L_0E_0/E_0) \simeq Gal(L_1E/E)$. Since L_1/K and E/K are Galois and $F = L_1 \cap E$, we have $Gal(L_1E/E) \simeq Gal(L_1/F)$. Since $P_1/P_2 \simeq Gal(L_1/F) \simeq Gal(L_0E_0/E_0)$, P_1/P_2 is admissible over E_0. $\square$

The above lemma immediately gives the following

Proposition 3.5. Let K be a complete discretely valued field with residue filed k and G be a finite group. Suppose that the order of G is coprime to char(k). If G is admissible over K then every Sylow subgroup P of G has a normal sereis $P \supseteq P_1 \supseteq P_2$ such that

(1) P/P_1 and P_2 are cyclic

(2) P_1/P_2 is admissible over some finite extension of the residue field of K.

Proof. Let G be an admissible group over K. Then there is a field extension L/K and a division algebra D central over K containing L as a maximal subfield with $Gal(L/K) = G$. Let P be a Sylow subgroup of G. Let L^P be the fixed of P. Then L^P is a complete discretely valued field. Let D' be the commutant of L^P in D. Then D' is a central division algebra over L^P and $G(L/L^P) = P$ is admissible over L^P. Since L^P is also a complete discrete valued field, the result follows by (3.4). $\square$

Corollary 3.6. Let K be a complete discretely valued field with residue filed k either a local field or a global field. Let G be a finite group such that char(k) coprime to the order of G. If G is admissible over K then every Sylow subgroup P of G has a normal series $P \supseteq P_1 \supseteq P_2$ such that

(1) P/P_1 and P_2 are cyclic

(2) P_1/P_2 is metacyclic.

Proof. Every finite extension of the residue field is either a local field or a global field. The corollary follows from (3.5) and ([Sc]). $\square$

Theorem 3.7. Let K be a complete discretely valued field with residue field k. Let F be the function field of a curve over K. Let G be a finite group with order coprime to char(k). If G is admissible over F then every Sylow subgroup P of G has a normal series $P \supseteq P_1 \supseteq P_2$ such that

(1) P/P_1 and P_2 are cyclic

(2) P_1/P_2 is admissible over some finite extension of the residue field at a discrete valuation of F.

Proof First we reduce to a Sylow subgroup as in (3.5). Let G be an admissible group over F. Then there is a field extension L/F and a division algebra D central over F containing L as a maximal subfield with $Gal(L/F) = G$. Let P be a Sylow subgroup of G. Let L^P be the fixed of P. Then L^P is a complete discretely valued field. Let D' be the commutant of L^P in D. Then D' is a central division algebra over L^P and $G(L/L^P) = P$ is admissible over L^P. Since L^P is a finte extension of F, L^P is also a function field of a curve over a finite extension of K. Since the degree of D' is a power of prime and coprime to the char(k), by (2.6), there exists a discrete valuation ν of L^P such that $D' \otimes_{L^P} L^P_\nu$ is division. Since $L \otimes_{L^P} L^P_\nu$ is a maximal subfield of $D' \otimes_{L^P} L^P_\nu$ and $P = Gal(L \otimes_{L^P} L^P_\nu / L^P_\nu)$, the result follows from (3.5). $\square$

The following is immediate from (3.7) and (3.6).

Corollary 3.8. Let K be a local adic field and F the function field of a curve over K. Let n be a natural number which is coprime to the characteristic of the residue field of K and G a finite group of order n. If G is admissible over F then every Sylow subgroup P of G has a normal series $P \supseteq P_1 \supseteq P_2$ such that

(1) P/P_1 and P_2 are cyclic

(2) P_1/P_2 metacyclic.

The following is proved in ([HHK2], 4.5).

Corollary 3.9. Let K be a complete discretely valued field with residue field algebraically closed. Let F be the function field of a curve over K. Let G be a finite group of order n such that the characteristic of the residue field of K is coprime to n. If G is admissible over F then every Sylow subgroup P of G is metacyclic.

Proof. By (3.7), every Sylow subgroup P of G has a normal series $P \supseteq P_1 \supseteq P_2$ such that

(1) P/P_1 and P_2 are cyclic

(2) P_1/P_2 is admissible over some finite extension of the residue field of K.

Let k be the residue field of K. By (2.7) and the proof of (3.7), the residue field at the discrete valuation given in (2) is either a finite extension K or the function field of a curve over k. Since k is algebraically closed, these residue fields have cohomological dimension at most one and there are no non-trivial division algebras over such fields. Hence $P_1 = P_2$. $\square$

We now give an example of a finite group which is not $\mathbb{Q}_p(t)$-admissible.

Example 3.10. Let ℓ and p be two distinct primes. Let $P = (\mathbb{Z}/l\mathbb{Z})^5$. We claim that this group is not admissible over $\mathbb{Q}_p(t)$. Suppose that P is admissible over $\mathbb{Q}_p(t)$. Then by (3.8), there is a normal series $P \supseteq P_1 \supseteq P_2$ such that

(1)P/P_1 and P_2 are cyclic

(2)P_1/P_2 is metacyclic

Since P_2 and P/P_1 are cyclic their orders will be at most l. This implies that $|P_1/P_2| \geq l^3$. Since every element of P_1/P_2 has order at most l and $|P_1/P_2| \geq l^3$, P_1/P_2 cannot be metacyclic. $\square$

Remark 3.11. Let F be a field and p a prime. Suppose that for every finite extension E of F there is a set Ω_E of discrete valuations of E such that given a central division algebra D over E of degree a power of p, there exists a discrete valuation $\nu \in \Omega_E$ such that $D \otimes E_\nu$ is division. Let G be a finite group. The proof of (3.7) gives us the following: If G is admissible over F, then every p-Sylow subgroup of G has filtration as in (3.7).

Remark 3.12. Let G be a finite group satisfying the conditions of (3.5). Then every homomorphic image of G also satisfy the same conditions. However there is an example of a group G with a homomorphic image H such that G is admissible over the complet discrete valuation field $\mathbb{Q}((t))$ but H is not admissible ([FS]). Hence the conditions given in (3.5) for a group to be admissible are not sufficient.

4. A class of Admissible groups over $\mathbb{Q}_p(t)$

Let K be a discretely valued field with residue field k. Let F be the function field of a curve over K. Let n be an integer which is coprime to the characteristic of k. Suppose that K contains a primitive n^{th} root of unity. Then in ([HHK2], 4.4) it is proved that every finite group of order n with every Sylow subgroup product of at most two cyclic groups is admissible over F. They used the patching techniques to prove this result. In this section we prove a similar result for groups with every Sylow subgroup is a product of at most 4 cyclic groups, with an additional assumption on the residue field k. We begin with the following

Lemma 4.1. Let R be a regular local ring of dimension two with residue field k and field of fraction F. Let n_1 and n_2 be natural numbers which are coprime to the char(k). Assume that F contains a primitive $(n_1n_2)^{th}$ root of unity and there is an element in k^*/k^{*n_2} of order n_2. Then there is a central division algebra D over F of degree n_1n_2.

Proof. Let m be the maximal ideal of R. Since R is a regular local ring of dimension two, we have $m = (t, s)$. By the assumption on k, there is an element $\lambda_0 \in k^*$ such that its order in k^*/k^{*n_2} is n_2. Let $\lambda \in R$ which maps to λ_0. Let $a \in R$ be a unit with $a^{n_1} \neq 1$. Let ξ_1 be a primitive n_1^{th} root of unity and ξ_2 a primitive n_2^{th} root of unity.

Let

$$D_1 = (\frac{s}{s-t}, \frac{s-t^2}{s-a^{n_1}t^2})_{n_1}$$

and

$$D_2 = (\frac{s}{s-t^2}, \frac{s-\lambda t^2}{s-t^2})_{n_2}.$$

Let $D = D_1 \otimes_F D_2$. Then the degree of D is n_1n_2. We now show that D is a division algebra.

Let $S = R[x]/(s - t^2x)$. Then the field of fractions of S is isomorphic to F. We have

$$D_1 = (\frac{tx}{tx-1}, \frac{x-1}{x-a^{n_1}})_{n_1}$$

and

$$D_2 = (\frac{x}{x-1}, \frac{x-\lambda}{x-1})_{n_2}.$$

The ideal (t) of S is a prime ideal and gives a discrete valuation ν on F. Let F_ν be the completion of F at ν. To show that $D_1 \otimes_F D_2$ is a division algebra, it is enough to show that $D_1 \otimes_F D_2 \otimes_F F_\nu$ is a division algebra. Since $tx/tx - 1$ is a parameter at ν, by (2.1), we have

$$\mathrm{index}(D_1 \otimes_F D_2 \otimes_F F_\nu) = \mathrm{index}(D_2 \otimes_F F_\nu(\sqrt[n_1]{\frac{x-1}{x-a^{n_1}}})) \cdot [F_\nu(\sqrt[n_1]{\frac{x-1}{x-a^{n_1}}}) : F_\nu].$$

Since $\frac{x-1}{x-a^{n_1}}$ is a unit at ν and the residue field $\kappa(\nu)$ at ν is $k(x)$, by the assumption on a, we have $[F_\nu(\sqrt[n_1]{\frac{x-1}{x-a^{n_1}}}) : F_\nu] = n_1$. Since D_2 is unramified at ν, the index of $D_2 \otimes_F F_\nu(\sqrt[n_1]{\frac{x-1}{x-a^{n_1}}})$ is equal to the index of its image $(\frac{x}{x-1}, \frac{x-\lambda}{x-1})_{n_2}$ over the residue field $k(x)(\sqrt[n_1]{\frac{x-1}{x-a^{n_1}}})$. Let θ be the discrete valuation on $k(x)$ given by (x). Let υ be the extension of θ to $k(x)(\sqrt[n_1]{\frac{x-1}{x-a^{n_1}}})$.

Then the residue field of υ is k. The residue of $(\frac{x}{x-1}, \frac{x-\lambda}{x-1})_{n_2}$ at υ is the class of λ_0. Since the order of λ_0 in k^*/k^{*n_2} is n_2, the index of D_2 is n_2. Hence the index of $D_1 \otimes_F D_2 \otimes F_\nu$ is $n_1 n_2$. Since the degree of $D_1 \otimes_F D_2$ is $n_1 n_2$, $D_1 \otimes_F D_2$ is a division algebra. $\square$

Theorem 4.2. Let K be a discretely valued field with residue field k and F the function field of a curve over K. Let n be an integer which is coprime to the characteristic of k. Suppose that K contains a primitive n^{th} root of unity. Assume that for every finite extension L of k, there is an element in L^*/L^{*n} of order n. If G is a finite group of order n with every Sylow subgroup is a quotient of $\mathbb{Z}^4$, then G is admissible over F.

Proof. Let R be the ring of integers in K. Let $\mathcal{X}$ be a regular proper two dimensional scheme over R with function field F and the reduced special fibre is a union of regular curves with normal crossings. Let $p_1, \cdots, p_r$ be the prime factors of n. Let $Q_1, \cdots, Q_r$ be regular closed points on the special fibre of $\mathcal{X}$. Let R_{Q_i} be the regular local ring at Q_i, $\hat{R}_{Q_i}$ be the completion of R_{Q_i} at the maximal ideal and F_{Q_i} the field of fractions of $\hat{R}_{Q_i}$. Let $t_i \in R_{Q_i}$ be a prime defining the irreducible component of the special fibre of $\mathcal{X}$ containing Q_i. Let P_i be a p_i-Sylow subgroup of G. By ([HHK2], 4.2), it is enough to show that there exists a central division algebra D_i over F_{Q_i} and maximal subfield L_i of D_i with $Gal(L_i/F_{Q_i}) \simeq P_i$ and $L_i \otimes \hat{F}_{Q_i}$ a split algebra.

For a given i, $1 \leq i \leq r$, let $Q = Q_i$, $t = t_i$ and $P = P_i$. Since the residue field $\kappa(Q)$ of $\hat{R}_Q$ is a finite extension of k, by the assumption on k, there is an element in $\kappa(Q)^*/\kappa(Q)^{*n}$ of order m for any m dividing n. Since P is a quotient of $\mathbb{Z}^4$, $P \simeq C_{n_1} \times C_{n_2} \times C_{n_3} \times C_{n_4}$. Since $\hat{R}_Q$ is regular local ring of dimension 2 and t is a regular prime, we have $m_Q = (t, s)$. Let ξ_1 be a primitive ${n_1n_2}^{th}$ root of unity and ξ_2 a primitive ${n_3n_4}^{th}$ root of unity.

Let

$$D_1 = (\frac{s}{s-t}, \frac{s-t^2}{s-a^{n_1 n_1}t^2})_{n_1 n_2}$$

and

$$D_2 = (\frac{s}{s-t^2}, \frac{s-\lambda t^2}{s-t^2})_{n_3 n_4}.$$

for suitable a and λ as in (4.1). Let $D = D_1 \otimes_F D_2$. Then, by (4.1), D is a division algebra over F_Q.

In particular D_1 and D_2 are division algebras over F_Q. The cyclic algebra D_1 is generated by x_1 and y_1 with relations

$$x_1^{n_1 n_2} = \frac{s}{s-t}, \; y_1^{n_1 n_2} = \frac{s-t^2}{s-a^{n_1 n_2}t^2} \text{ and } x_1 y_1 = \xi_1 y_1 x_1.$$

Similarly D_2 is generated by x_2 and y_2 with relations

$$x_2^{n_3 n_4} = \frac{s}{s-t^2},\ y_2^{n_3 n_4} = \frac{s-\lambda t^2}{s-t^2} \text{ and } x_2 y_2 = \xi_2 y_2 x_2.$$

Let L_1 be the subalgebra of D_1 generated by $x_1^{n_1}$ and $y_1^{n_2}$. Let L_2 be the subalgebra of D_2 generated by $x_2^{n_3}$ and $y_2^{n_4}$. Then $L = L_1 \otimes L_2$ is a maximal subfield of $D_1 \otimes D_2$, $Gal(L/F) = C_{n_1} \times C_{n_2} \times C_{n_3} \times C_{n_4}$ and $L \otimes \hat{F}$ is a split algebra (cf., ([HHK2]) proof of 4.4,). $\square$

Corollary 4.3. Let K be a local field and k its residue field. Let F be the function field of a curve over K. Let n be an integer which is coprime to the characteristic of k. Suppose that K contains a primitive n^{th} root of unity. If G is a group of order n with every Sylow subgroup a quotient of $\mathbb{Z}^4$, then G is admissible over F.

Proof. Since k is a finite field, for any finite extension L of k and for any natural number n coprime to the characteristic of L, we have L^*/L^{*n} is cyclic group of order n. Hence k satisfies the condition of (4.2). $\square$.

Let K be a complete discretely valued field with residue field k. Let G be a finite abelian group of order n which is a product of at most four cyclic groups. Suppose that n is coprime to char(k) and the order of k^*/k^{*n} is at least n. If K contains a primitive n^{th} root of unity, then by (4.3), G is admissible over $K(t)$. We now prove that a similar result without the completeness assumption on K.

We begin with the following

Lemma 4.4. Let R be a discrete valuation ring and $\pi \in R$ a parameter. Let K be the field of fractions of R and k the residue filed of R. Let $F = K(t)$ be the rational function field in one variable over K. Let n be a natural number which is coprime to char(k). Assume that K contains a primitive n^{th} root of unity and there exists a $\lambda_0 \in k^*$ such that $[k(\sqrt[n]{\lambda_0}) : k] = n$. Then $(t, \pi - \lambda t)_n \otimes_F (t+1, \pi)_m$ is division over F for any $\lambda \in R$ which maps to λ_0.

Proof Since π is a parameter in R, the localisation $R[t]_{(\pi)}$ of $R[t]$ at the prime ideal (π) is a discrete valuation ring. Let ν be the discrete valuation on F given the discrete valuation ring $R[t]_{(\pi)}$ and F_ν be the completion of F at ν. Since the residue field at ν is $k(t)$, we have $[F_\nu(\sqrt[m]{t+1}) : F_\nu] = m$. Let w be the extension of ν to $F_\nu(\sqrt[m]{t+1})$. To show that $(t, \pi - \lambda t)_n \otimes_F (t+1, \pi)_m$ is division, it is enough to show that $(t, \pi - \lambda t)_n \otimes_F (t+1, \pi)_m \otimes_F F_\nu$ is division. Since F_ν is complete and $[F_\nu(\sqrt[m]{t+1}) : F_\nu] = m$, by (2.1), it is enough to show that $(t, \pi - \lambda t)_n \otimes F_\nu(\sqrt[m]{t+1})$ is division. Since t and $\pi - \lambda t$ are units at ν, the algebra $(t, \pi - \lambda t)_n \otimes F_\nu(\sqrt[m]{t+1})$ is unramified at w. Thus it is enough to show that its image $(t, -\lambda_0 t)_n = (t, \lambda_0)_n$ is division over the residue filed $k(t)(\sqrt[m]{t+1})$. Let γ be the discrete valuation on $k(t)$ given by t and $\tilde{\gamma}$ be the extension of γ to $k(t)(\sqrt[m]{t+1})$. Since $t+1$ is an m^{th} power in the completion of $k(t)$ at γ, the completion of $k(t)(\sqrt[m]{t+1})$ at $\tilde{\gamma}$ is $k((t))$. It is enough show that $(t, \lambda_0)_n$ is division over $k((t))$. Since $\lambda_0 \in k^*$ is an element of order n, $(t, \lambda_0)_n$ is division over $k((t))$. $\square$

Theorem 4.5. Let K be a field with a discrete valuation (not necessarily complete) and k its residue field. Let $G = \mathbb{Z}/l_1\mathbb{Z} \times \mathbb{Z}/l_2\mathbb{Z} \times \mathbb{Z}/l_3\mathbb{Z} \times \mathbb{Z}/l_4\mathbb{Z}$. Suppose that the $n = l_1 l_2 l_3 l_4$ is coprime to char(k), K contains a primitive n^{th} root of unity and there is a $\lambda_0 \in k$ such that $[k(\sqrt[l_1 l_2]{\lambda_0} : k] = l_1 l_2$. Then G is admissible over $K(t)$.

Proof. Let R be the ring of integers in K and $\lambda \in R$ mapping to λ_0 in k. Let $\pi \in R$ be a parameter. Let $D_1 = (t, \pi - \lambda t)_{l_1 l_2}$ and $D_2 = (t+1, \pi)_{l_3 l_4}$. Then, by (4.4), $D_1 \otimes D_2$ is a division algebra over $K(t)$. Let $L_1 = K(t)(y, z)$ where $y^{l_1} = t$, $z^{l_2} = \pi - \lambda t$. Then L_1 is a maximal subfield of D_1. Let $L_2 = K(t)(r, s)$ where $y^{l_3} = t+1$, $z^{l_4} = \pi$. Then L_2 is a maximal subfield of D_2. Therefore $L_1 \otimes_{K(t)} L_2$ is a maximal subfield of $D_1 \otimes D_2$. Since $Gal(L_1 \otimes_{K(t)} L_2 / K(t)) = G$, G is admissible over $K(t)$. $\square$

Corollary 4.6. Let p be a prim and l_1, l_2, l_3, l_4 be natural numbers which are coprime to p. Let $G = \mathbb{Z}/l_1\mathbb{Z} \times \mathbb{Z}/l_2\mathbb{Z} \times \mathbb{Z}/l_3\mathbb{Z} \times \mathbb{Z}/l_4\mathbb{Z}$. Let $n = l_1 l_2 l_3 l_4$ and ζ a primitive n^{th} root of unity. Then G is admisible over $\mathbb{Q}(\zeta)(t)$.

Remark 4.7. For p, l_1, l_2, l_3, l_4, G as in (4.6), G is admissible over $\mathbb{Q}_p(t)$ by (4.3). However (4.5) gives an explicite constriction of a division algebra over $\mathbb{Q}_p(t)$ and a maximal subfield with Galois group G.

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