SIGNATURES OF HERMITIAN FORMS AND THE KNEBUSCH TRACE FORMULA

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Abstract. Signatures of quadratic forms have been generalized to hermitian forms over algebras with involution. In the literature this is done via Morita theory, which causes sign ambiguities in certain cases. In this paper, a hermitian version of the Knebusch Trace Formula is established and used as a main tool to resolve these ambiguities.

1. Introduction

In this paper we study signatures of hermitian forms over central simple algebras with involution of any kind, defined over formally real fields. These generalize the classical signatures of quadratic forms.

Following [3] we do this via extension to real closures and Morita equivalence. This leads to the notion of M-signature of hermitian forms in Section 3.2. We study its properties, make a detailed analysis of the impact of choosing different real closures and different Morita equivalences and show in particular that sign changes can occur. This motivates the search for a more intrinsic notion of signature, where such sign changes do not occur.

In Section 3.3 we define such a signature, the H-signature, which only depends on the choice of a tuple of hermitian forms, mimicking the fact that in quadratic form theory the form $\langle 1\rangle$ always has positive signature. The H-signature generalizes the definition of signature in [3] and is in particular well-defined when the involution becomes hyperbolic after scalar extension to a real closure of the base field, addressing an issue with the definition proposed in [3].

Our main tool is a generalization of the Knebusch Trace Formula to M-signatures of hermitian forms, which we establish in Section 5. In Section 7 we show that the total H-signature of a hermitian form is a continuous map and in Section 8 we prove the Knebusch Trace Formula for H-signatures.

2. Preliminaries

2.1. **Algebras with Involution.** The general reference for this section is [11, Chapter I].

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Let F be a field of characteristic different from 2. An *F-algebra with involution* is a pair (A, σ) consisting of a finite-dimensional simple F-algebra A with centre a field $K = Z(A)$ and an F-linear involution $\sigma : A \to A$. Either σ is of *the first kind*, in which case $K = F$ and $\sigma|_K = \mathrm{id}_K$, or σ is of *the second kind*, in which case $K = F(\sqrt{d})$ is a quadratic field extension of F and $\sigma|_K$ is the nontrivial element in the Galois group of K/F.

Consider the F-subspaces

$$\mathrm{Sym}(A, \sigma) = \{a \in A \mid \sigma(a) = a\} \text{ and } \mathrm{Skew}(A, \sigma) = \{a \in A \mid \sigma(a) = -a\}$$

of A. Then $A = \mathrm{Sym}(A, \sigma) \oplus \mathrm{Skew}(A, \sigma)$. Assume that σ is of the first kind. Then $\dim_F(A) = m^2$ for some positive integer m. Furthermore, σ is either *orthogonal* (or, *of type* $+1$) if $\dim_F \mathrm{Sym}(A, \sigma) = m(m + 1)/2$, or *symplectic* (or, *of type* -1) if $\dim_F \mathrm{Sym}(A, \sigma) = m(m - 1)/2$. If σ is of the second kind, then $\dim_F(A) = 2m^2$ for some positive integer m and $\dim_F \mathrm{Sym}(A, \sigma) = \dim_F \mathrm{Skew}(A, \sigma) = m^2$. Involutions of the second kind are also called *unitary*.

Let σ and τ be two involutions on A that have the same restriction to K. By the Skolem-Noether theorem they differ by an inner automorphism:

$$\tau = \mathrm{Int}(u) \circ \sigma$$

for some $u \in A^\times$, uniquely determined up to a factor in $F^\times$, such that $\sigma(u) = u$ if σ and τ are both orthogonal, both symplectic or both unitary and $\sigma(u) = -u$ if one of σ, τ is orthogonal and the other symplectic. Here $\mathrm{Int}(u)(x) := uxu^{-1}$ for $x \in A$.

2.2. **ε-Hermitian Spaces and Forms.** The general references for this section are [10, Chapter I] and [20, Chapter 7], both for rings with involution. Treatments of the central simple and division cases can also be found in [6] and [14], respectively.

Let (A, σ) be an F-algebra with involution. Let $\varepsilon \in \{-1, 1\}$. An *$\varepsilon$-hermitian space* over (A, σ) is a pair (M, h), where M is a finitely generated right A-module (which is automatically projective since A is semisimple) and $h : M \times M \longrightarrow A$ is a sesquilinear form such that $h(y, x) = \varepsilon\sigma(h(x, y))$ for all $x, y \in M$. We call (M, h) a *hermitian space* when $\varepsilon = 1$ and a *skew-hermitian space* when $\varepsilon = -1$. If (A, σ) is a field equipped with the identity map, we say (*skew-*) *symmetric bilinear* space instead of (skew-) hermitian space.

Consider the left A-module $M^* = \mathrm{Hom}_A(M, A)$ as a right A-module via the involution σ. The form h induces an A-linear map $h^* : M \to M^*$, $x \mapsto h(x, \cdot)$. We call (M, h) *nonsingular* if h^* is an isomorphism. All spaces occurring in this paper are assumed to be nonsingular. We often simply write h instead of (M, h) and speak of a *form* instead of a space.

If $A = D$ is a division algebra (so that $M \simeq D^n$ for some integer n) such that $(D, \sigma, \varepsilon) \neq (F, \mathrm{id}_F, -1)$, then h can be diagonalized: there exist invertible elements $a_1, \dots, a_n \in \mathrm{Sym}(D, \sigma)$ such that, after a change of basis,

$$h(x, y) = \sum_{i=1}^{n} \sigma(x_i) a_i y_i, \ \forall x, y \in D^n.$$

In this case we use the shorthand notation

$$h = \langle a_1, \dots, a_n \rangle_\sigma,$$

which resembles the usual notation for diagonal quadratic forms. If A is not a division algebra we can certainly consider diagonal hermitian forms defined on free A-modules of finite rank, but some hermitian forms over (A, σ) may not be diagonalizable.

Witt cancellation and Witt decomposition hold for ε-hermitian spaces over (A, σ). Let $W_\varepsilon(A, \sigma)$ denote the Witt group of Witt classes of ε-hermitian spaces over (A, σ). When $\varepsilon = 1$ we drop the subscript and simply write $W(A, \sigma)$. We denote the usual Witt ring of F by $W(F)$. We find it convenient to identify forms over (A, σ) with their classes in $W_\varepsilon(A, \sigma)$.

Lemma 2.1. *Let (A, σ) be an F-algebra with involution.*

(*i*) *If σ is of the first kind, then $W_{-1}(A, \sigma) \simeq W(A, \tau)$ for some involution τ of opposite type to σ.*

(*ii*) *If σ is of the second kind, then $W_{-1}(A, \sigma) \simeq W(A, \sigma)$.*

Proof. (*i*) Let $u \in A^\times$ be such that $\sigma(u) = -u$ and define $\tau := \mathrm{Int}(u) \circ \sigma$. Let h be a skew-hermitian form over (A, σ). Then uh is a hermitian form over (A, τ). The one-to-one correspondence $h \mapsto uh$ respects isometries, orthogonal sums and hyperbolicity and so induces the indicated isomorphism.

(*ii*) Let $Z(A) = K = F(\sqrt{d})$ for some $d \in F^\times$. Then $\sigma(\sqrt{d}) = -\sqrt{d}$. The one-to-one correspondence $h \mapsto \sqrt{d}h$ induces the indicated isomorphism. ■

2.3. **Adjoint Involutions.** The general reference for this section is [11, 4.A].

Let (A, σ) be an F-algebra with involution. Let (M, h) be an ε-hermitian space over (A, σ). The algebra $\mathrm{End}_A(M)$ is again central simple over F since M is finitely generated [11, 1.10]. The involution ad_h on $\mathrm{End}_A(M)$, defined by

$$h(x, f(y)) = h(\mathrm{ad}_h(f)(x), y), \ \forall x, y \in M, \forall f \in \mathrm{End}_A(M)$$

is called the *adjoint involution* of h. The involutions σ and ad_h are of the same kind and $\sigma(\alpha) = \mathrm{ad}_h(\alpha)$ for all $\alpha \in K$. In case ad_h and σ are of the first kind we also have

$$\mathrm{type}(\mathrm{ad}_h) = \varepsilon \ \mathrm{type}(\sigma).$$

Furthermore, every involution on $\mathrm{End}_A(M)$ is of the form ad_h for some ε-hermitian form h over (A, σ) and the correspondence between ad_h and h is unique up to a multiplicative factor in $F^\times$ in the sense that $\mathrm{ad}_h = \mathrm{ad}_{\lambda h}$ for every $\lambda \in F^\times$.

Let (A, σ) be an F-algebra with involution. By a theorem of Wedderburn there exists a division algebra D (unique up to isomorphism) with centre $Z(A)$ and a finite-dimensional right D-vector space V such that $A \simeq \mathrm{End}_D(V)$. Thus $A \simeq M_m(D)$ for some positive integer m. Furthermore there exists an involution ϑ on D of the same kind as σ and an ε_0-hermitian form φ_0 over (D, ϑ) with $\varepsilon_0 \in \{-1, 1\}$ such that (A, σ) and $(\mathrm{End}_D(V), \mathrm{ad}_{\varphi_0})$ are isomorphic as algebras with involution. In matrix form ad_{φ_0} is described as follows:

$$\mathrm{ad}_{\varphi_0}(X) = \Phi_0 \vartheta^t(X) \Phi_0^{-1}, \ \forall X \in M_m(D),$$

where $\vartheta^t(X) := (\vartheta(x_{ij}))^t$ for $X = (x_{ij})$ and $\Phi_0 \in \mathrm{GL}_m(D)$ is the Gram matrix of φ_0. Thus $\vartheta^t(\Phi_0) = \varepsilon_0\Phi_0$.

2.4. **Hermitian Morita Theory.** We refer to [2, §1], [5], [6, Chapters 2–3], [10, Chapter I, §9], or [13] for more details.

Let (M, h) be an ε-hermitian space over (A, σ). One can show that the algebras with involution $(\mathrm{End}_A(M), \mathrm{ad}_h)$ and (A, σ) are Morita equivalent: for every $\mu \in \{-1, 1\}$ there is an equivalence between the categories $\mathscr{H}_\mu(\mathrm{End}_A(M), \mathrm{ad}_h)$ and $\mathscr{H}_{\varepsilon\mu}(A, \sigma)$ of non-singular μ-hermitian forms over $(\mathrm{End}_A(M), \mathrm{ad}_h)$ and non-singular $\varepsilon\mu$-hermitian forms over (A, σ), respectively (where the morphisms are given by isometry), cf. [10, Chapter I, Theorem 9.3.5]. This equivalence respects isometries, orthogonal sums and hyperbolic forms. It induces a group isomorphism

$$W_\mu(\mathrm{End}_A(M), \mathrm{ad}_h) \simeq W_{\varepsilon\mu}(A, \sigma).$$

The Morita equivalence and the isomorphism are not canonical.

The algebras with involution (A, σ) and (D, ϑ) are also Morita equivalent. An example of such a Morita equivalence is obtained by composing the following three non-canonical equivalences of categories, the last two of which we will call *scaling* and *collapsing*. For computational purposes we describe them in matrix form. We follow the approach of [17]:

$$\mathscr{H}_\varepsilon(A, \sigma) \longrightarrow \mathscr{H}_\varepsilon(M_m(D), \mathrm{ad}_{\varphi_0}) \xrightarrow{\text{scaling}} \mathscr{H}_{\varepsilon_0\varepsilon}(M_m(D), \vartheta^t) \xrightarrow{\text{collapsing}} \mathscr{H}_{\varepsilon_0\varepsilon}(D, \vartheta). \quad (1)$$

Scaling: Let (M, h) be an ε-hermitian space over $(M_m(D), \mathrm{ad}_{\varphi_0})$. Scaling is given by

$$(M, h) \longmapsto (M, \Phi_0^{-1}h).$$

Note that Φ_0^{-1} is only determined up to a scalar factor in $F^\times$ since $\mathrm{ad}_{\varphi_0} = \mathrm{ad}_{\lambda\varphi_0}$ for any $\lambda \in F^\times$ and that replacing Φ_0 by $\lambda\Phi_0$ results in a different Morita equivalence.

Collapsing: Recall that $M_m(D) \simeq \mathrm{End}_D(D^m)$ and that we always have $M \simeq (D^m)^k \simeq M_{k,m}(D)$ for some integer k. Let $h : M \times M \longrightarrow M_m(D)$ be an $\varepsilon_0\varepsilon$-hermitian form with respect to ϑ^t. Then

$$h(x, y) = \vartheta^t(x)By, \ \forall x, y \in M_{k,m}(D),$$

where $B \in M_k(D)$ satisfies $\vartheta^t(B) = \varepsilon_0\varepsilon B$, so that B determines an $\varepsilon_0\varepsilon$-hermitian form b over (D, ϑ). Collapsing is then given by

$$(M, h) \longmapsto (D^k, b).$$

3. Signatures of Hermitian Forms

3.1. **Signatures of Forms: the Real Closed Case.** Let R be a real closed field, $C = R(\sqrt{-1})$ (which is algebraically closed) and $H = (-1, -1)_R$ Hamilton's quaternion division algebra over R. We recall the definitions of signature for the various types of forms (all assumed to be nonsingular) over R, C, $R \times R$ and H. We will use them in the definition of the M-signature of a hermitian form over (A, σ) in Section 3.2.

(a) Let b be a symmetric bilinear (or quadratic) form over R. Then $b \simeq \langle \alpha_1, \dots, \alpha_n \rangle$ for some $n \in \mathbb{N}$ and $\alpha_i \in \{-1, 1\}$. We let

$$\operatorname{sign} b := \sum_{i=1}^{n} \alpha_i.$$

By Sylvester's Law of Inertia, $\operatorname{sign} b$ is well-defined.

(b) Let b be a skew-symmetric form over R. Then b is hyperbolic and we let

$$\operatorname{sign} b := 0.$$

(c) Let h be a hermitian form over $(C, -)$, where $\overline{\sqrt{-1}} = -\sqrt{-1}$. Then $h \simeq \langle \alpha_1, \dots, \alpha_n \rangle_-$ for some $n \in \mathbb{N}$ and $\alpha_i \in \{-1, 1\}$. By a theorem of Jacobson [7], h is up to isometry uniquely determined by the symmetric bilinear form $b_h := 2 \times \langle \alpha_1, \dots, \alpha_n \rangle$ defined over R. We let

$$\operatorname{sign} h := \frac{1}{2} \operatorname{sign} b_h = \operatorname{sign}\langle \alpha_1, \dots, \alpha_n \rangle.$$

(d) Let h be a hermitian form over $(R \times R, \widehat{\ })$, where $\widehat{(x, y)} = (y, x)$ is the exchange involution. Then h is a torsion form [19, p. 43] and we let

$$\operatorname{sign} h := 0.$$

(e) Let h be a hermitian form over $(H, -)$, where $-$ denotes quaternion conjugation. Then $h \simeq \langle \alpha_1, \dots, \alpha_n \rangle_-$ for some $n \in \mathbb{N}$ and $\alpha_i \in \{-1, 1\}$. By a theorem of Jacobson [7], h is up to isometry uniquely determined by the symmetric bilinear form $b_h := 4 \times \langle \alpha_1, \dots, \alpha_n \rangle$ defined over R. We let

$$\operatorname{sign} h := \frac{1}{4} \operatorname{sign} b_h = \operatorname{sign}\langle \alpha_1, \dots, \alpha_n \rangle.$$

(f) Let h be a skew-hermitian form over $(H, -)$, where $-$ denotes quaternion conjugation. Then h is a torsion form [20, Chapter 10, Theorem 3.7] and we let

$$\operatorname{sign} h := 0.$$

Remark 3.1. The signature maps defined in (a), (c) and (e) above give rise to unique group isomorphisms

$$W(R) \simeq \mathbb{Z}, \ W(C, -) \simeq \mathbb{Z}, \text{ and } W(H, -) \simeq \mathbb{Z}$$

such that $\operatorname{sign}\langle 1 \rangle = 1$, $\operatorname{sign}\langle 1 \rangle_- = 1$ and $\operatorname{sign}\langle 1 \rangle_- = 1$, respectively. In addition, we have the group isomorphisms

$$W_{-1}(R, \mathrm{id}_R) = 0, \ W_{-1}(R \times R, \widehat{\ }) \simeq W_{-1}(H, -) \simeq \mathbb{Z}/2\mathbb{Z}.$$

See also [8, Chapter I, 10.5] and [19, p. 43].

3.2. **The M-Signature of a Hermitian Form.** Our approach in this section is inspired by [3, §3.3,§3.4]. We only consider hermitian forms over (A,σ), cf. Lemma 2.1.

Let F be a formally real field and let (A,σ) be an F-algebra with involution. Consider an ordering $P \in X_F$, the space of orderings of F. By a *real closure of F at P* we mean a field embedding $\iota : F \to K$, where K is real closed, $\iota(P) \subseteq K^2$ and K is algebraic over $\iota(F)$.

Let h be a hermitian form over (A,σ). Choose a real closure $\iota : F \to F_P$ of F at P, and use it to extend scalars from F to F_P:

$$W(A,\sigma) \longrightarrow W(A \otimes_F F_P, \sigma \otimes \mathrm{id}),\ h \longmapsto h \otimes F_P := (\mathrm{id}_A \otimes \iota)^*(h),$$

where the tensor product is along ι. The extended algebra with involution $(A \otimes_F F_P, \sigma \otimes \mathrm{id})$ is Morita equivalent to an F_P-algebra with involution (D_P, ϑ_P), chosen as follows:

(i) If σ is of the first kind, D_P is equal to one of F_P or $H_P := (-1,-1)_{F_P}$. Furthermore, we may choose $(D_P,\vartheta_P) = (F_P, \mathrm{id}_{F_P})$ in the first case and $(D_P,\vartheta_P) = (H_P, -)$ in the second case by Morita theory (scaling).

(ii) If σ is of the second kind, recall that $Z(A) = K = F(\sqrt{d})$. Now if $d <_P 0$, then D_P is equal to $F_P(\sqrt{-1})$, whereas if $d >_P 0$, then D_P is equal to $F_P \times F_P$ and $A \otimes_F F_P$ is a direct product of two simple algebras. Furthermore, we may choose $(D_P,\vartheta_P) = (F_P(\sqrt{-1}), -)$ in the first case and $(D_P,\vartheta_P) = (F_P \times F_P, \widehat{\ })$ in the second case, again by Morita theory (scaling).

Note that ϑ_P is of the same kind as σ in each case.

The extended involution $\sigma \otimes \mathrm{id}_{F_P}$ is adjoint to an ε_P-hermitian form φ_P over (D_P,ϑ_P) where $\varepsilon_P = -1$ if one of σ and ϑ_P is orthogonal and the other is symplectic, whereas $\varepsilon_P = 1$ if σ and ϑ_P are of the same type, i.e. both orthogonal, symplectic or unitary.

Now choose any Morita equivalence

$$\mathscr{M} : \mathscr{H}(A \otimes_F F_P, \sigma \otimes \mathrm{id}) \longrightarrow \mathscr{H}_{\varepsilon_P}(D_P, \vartheta_P) \tag{2}$$

with $(D_P,\vartheta_P) \in \{(F_P, \mathrm{id}_{F_P}), (H_P, -), (F_P(\sqrt{-1}), -), (F_P \times F_P, \widehat{\ })\}$, which exists by the analysis above. This Morita equivalence induces an isomorphism, which we again denote by $\mathscr{M}$,

$$\mathscr{M} : W(A \otimes_F F_P, \sigma \otimes \mathrm{id}) \xrightarrow{\sim} W_{\varepsilon_P}(D_P, \vartheta_P). \tag{3}$$

Definition 3.2. Let $P \in X_F$. Fix a real closure $\iota : F \to F_P$ of F at P and a Morita equivalence $\mathscr{M}$ as above. Define the *M-signature of h at* $(\iota, \mathscr{M})$, denoted $\mathrm{sign}_\iota^{\mathscr{M}} h$, as follows:

$$\mathrm{sign}_\iota^{\mathscr{M}} h := \mathrm{sign}\, \mathscr{M}(h \otimes F_P),$$

where $\mathrm{sign}\, \mathscr{M}(h \otimes F_P)$ can be computed as shown in Section 3.1.

This definition relies on two choices: firstly the choice of the real closure $\iota : F \to F_P$ of F at P and secondly the choice of the Morita equivalence $\mathscr{M}$. Note that there is no canonical choice for $\mathscr{M}$. We now study the dependence of the M-signature on the choice of ι and $\mathscr{M}$.

Let $\iota_1 : F \to L_1$ and $\iota_2 : F \to L_2$ be two real closures of F at P, and let (D_1, ϑ_1) and ε_1 play the role of (D_P, ϑ_P) and ε_P, respectively, obtained above when ι is replaced by ι_1. Let

$$\mathscr{M}_1 : \mathscr{H}(A \otimes_F L_1, \sigma \otimes \mathrm{id}) \longrightarrow \mathscr{H}_{\varepsilon_1}(D_1, \vartheta_1)$$

be a fixed Morita equivalence. By the Artin-Schreier theorem [20, Chapter 3,Theorem 2.1] there is a unique isomorphism $\rho : L_1 \to L_2$ such that $\rho \circ \iota_1 = \iota_2$. It extends to an isomorphism $\mathrm{id} \otimes \rho : (A \otimes_F L_1, \sigma \otimes \mathrm{id}) \to (A \otimes_F L_2, \sigma \otimes \mathrm{id})$. The isomorphism ρ also extends canonically to $D_1 \in \{L_1, (-1,-1)_{L_1}, L_1(\sqrt{-1}), L_1 \times L_1\}$. Consider the L_2-algebra with involution $(D_2, \vartheta_2) := (\rho(D_1), \rho \circ \vartheta_1 \circ \rho^{-1})$. We define $\rho(\mathscr{M}_1)$ to be the Morita equivalence from $(A \otimes_F L_2, \sigma \otimes \mathrm{id})$ to (D_2, ϑ_2), described by the following diagram:

$$\begin{array}{ccc}
\mathscr{H}(A \otimes_F L_1, \sigma \otimes \mathrm{id}) & \xrightarrow{\ \mathscr{M}_1\ } & \mathscr{H}_{\varepsilon_1}(D_1, \vartheta_1) \\
\Big\downarrow{\scriptstyle (\mathrm{id}\otimes\rho)^*} & & \Big\downarrow{\scriptstyle \rho^*} \\
\mathscr{H}(A \otimes_F L_2, \sigma \otimes \mathrm{id}) & \xrightarrow{\ \rho(\mathscr{M}_1)\ } & \mathscr{H}_{\varepsilon_1}(D_2, \vartheta_2)
\end{array}$$

Proposition 3.3 (Change of Real Closure)**.** *With notation as above we have for every* $h \in W(A, \sigma)$*,*

$$\operatorname{sign} \mathscr{M}_1(h \otimes L_1) = \operatorname{sign} \rho(\mathscr{M}_1)(h \otimes L_2),$$

in other words

$$\operatorname{sign}_{\iota_1}^{\mathscr{M}_1} h = \operatorname{sign}_{\iota_2}^{\rho(\mathscr{M}_1)} h.$$

Proof. The statement is trivially true when $\varepsilon_1 = -1$, by cases (b), (d) and (f) in Section 3.1, so we may assume that $\varepsilon_1 = 1$. Consider the diagram

$$\begin{array}{ccccccc}
 & & W(A \otimes_F L_1, \sigma \otimes \mathrm{id}) & \xrightarrow[\sim]{\ \mathscr{M}_1\ } & W(D_1, \vartheta_1) & & \\
 & {\scriptstyle (\mathrm{id}\otimes\iota_1)^*}\nearrow & \Big\downarrow{\scriptstyle (\mathrm{id}\otimes\rho)^*} & & \Big\downarrow{\scriptstyle \rho^*} & \searrow{\scriptstyle \mathrm{sign}} & \\
W(A, \sigma) & & & & & & \mathbb{Z} \\
 & {\scriptstyle (\mathrm{id}\otimes\iota_2)^*}\searrow & & & & \nearrow{\scriptstyle \mathrm{sign}} & \\
 & & W(A \otimes_F L_2, \sigma \otimes \mathrm{id}) & \xrightarrow[\sim]{\ \rho(\mathscr{M}_1)\ } & W(D_2, \vartheta_2) & &
\end{array}$$

The left triangle commutes by the definition of ρ. The square commutes by the definition of $\rho(\mathscr{M}_1)$. The right triangle commutes since $\rho^*(\langle 1 \rangle_{\vartheta_1}) = \langle 1 \rangle_{\vartheta_2}$ and by Remark 3.1. The statement follows. ∎

Proposition 3.4 (Change of Morita Equivalence)**.** *Let* $\mathscr{M}_1$ *and* $\mathscr{M}_2$ *be two different Morita equivalences as in* (2)*. Then there exists* $\delta \in \{-1, 1\}$ *such that for every* $h \in W(A, \sigma)$*,*

$$\operatorname{sign}_{\iota}^{\mathscr{M}_1} h = \delta \operatorname{sign}_{\iota}^{\mathscr{M}_2} h.$$

Proof. Note that $\varepsilon_1 = \varepsilon_2$. The statement is trivially true when $\varepsilon_1 = -1$, by cases (b), (d) and (f) in Section 3.1, so we may assume that $\varepsilon_1 = 1$. The two different Morita equivalences give rise to two different group isomorphisms

$$m_i : W(A \otimes_F F_P, \sigma \otimes \mathrm{id}) \xrightarrow{\sim} \mathbb{Z} \qquad (i = 1, 2),$$

by (3) and Remark 3.1. The map $m_1 \circ m_2^{-1}$ is an automorphism of $\mathbb{Z}$ and is therefore equal to $\mathrm{id}_{\mathbb{Z}}$ or $-\mathrm{id}_{\mathbb{Z}}$. ∎

Propositions 3.3 and 3.4 immediately imply

Corollary 3.5. *Let $\iota_1 : F \to L_1$ and $\iota_2 : F \to L_2$ be two real closures of F at P and let $\mathscr{M}_1$ and $\mathscr{M}_2$ be two different Morita equivalences as in* (2). *Then there exists $\delta \in \{-1, 1\}$ such that for every $h \in W(A, \sigma)$,*

$$\mathrm{sign}_{\iota_1}^{\mathscr{M}_1} h = \delta \, \mathrm{sign}_{\iota_2}^{\mathscr{M}_2} h.$$

The following result easily follows from the properties of Morita equivalence:

Proposition 3.6.

(i) *Let h be a hyperbolic form over (A, σ), then*

$$\mathrm{sign}_{\iota}^{\mathscr{M}} h = 0.$$

(ii) *Let h_1 and h_2 be hermitian forms over (A, σ), then*

$$\mathrm{sign}_{\iota}^{\mathscr{M}} (h_1 \perp h_2) = \mathrm{sign}_{\iota}^{\mathscr{M}} h_1 + \mathrm{sign}_{\iota}^{\mathscr{M}} h_2.$$

(iii) *The M-signature at $(\iota, \mathscr{M})$, $\mathrm{sign}_{\iota}^{\mathscr{M}}$, induces a homomorphism of additive groups*

$$W(A, \sigma) \longrightarrow \mathbb{Z}.$$

(iv) *Let h be a hermitian form over (A, σ) and q a quadratic form over F, then*

$$\mathrm{sign}_{\iota}^{\mathscr{M}} (q \otimes h) = \mathrm{sign}_P q \cdot \mathrm{sign}_{\iota}^{\mathscr{M}} h,$$

where $\mathrm{sign}_P q$ denotes the usual signature of the quadratic form q at P.

Definition 3.7. Let h be a hermitian form over (A, σ). From Definition 3.2 and Section 3.1 it follows that $\mathrm{sign}_{\iota}^{\mathscr{M}} h$ is automatically zero whenever P belongs to the following subset of X_F, which we call set of *nil-orderings*:

$$\mathrm{Nil}[A, \sigma] := \begin{cases} \{P \in X_F \mid D_P = H_P\} & \text{if } \sigma \text{ is orthogonal} \\ \{P \in X_F \mid D_P = F_P\} & \text{if } \sigma \text{ is symplectic} \\ \{P \in X_F \mid D_P = F_P \times F_P\} & \text{if } \sigma \text{ is unitary} \end{cases},$$

where the square brackets indicate that $\mathrm{Nil}[A, \sigma]$ depends only on the Brauer class of A and the type of σ.

3.3. **The H-Signature of a Hermitian Form.** It follows from Corollary 3.5 that $\operatorname{sign}_\iota^{\mathscr{M}}$ is uniquely defined up to a choice of sign. We can arbitrarily choose the sign of the signature of a form at each ordering P. See for instance Remark 3.13 for a way to change sign using Morita equivalence (scaling).

A more intrinsic definition is therefore desirable, in particular when considering the total signature map of a hermitian form $X_F \to \mathbb{Z}$ since such arbitrary changes of sign would prevent it from being continuous. We are thus led to define a signature that is independent of the choice of ι and $\mathscr{M}$.

Lemma 3.8. *Let $P \in X_F \setminus \mathrm{Nil}[A,\sigma]$. Let $\iota_1 : F \to L_1$ and $\iota_2 : F \to L_2$ be two real closures of F at P and let $\mathscr{M}_1$ and $\mathscr{M}_2$ be two different Morita equivalences as in* (2). *Let $h_0 \in W(A,\sigma)$ be such that $\operatorname{sign}_{\iota_1}^{\mathscr{M}_1} h_0 \neq 0$ and let $\delta_k \in \{-1,1\}$ be the sign of $\operatorname{sign}_{\iota_k}^{\mathscr{M}_k} h_0$ for $k = 1,2$. Then*

$$\delta_1 \operatorname{sign}_{\iota_1}^{\mathscr{M}_1} h = \delta_2 \operatorname{sign}_{\iota_2}^{\mathscr{M}_2} h,$$

for all $h \in W(A,\sigma)$.

Proof. Let $\delta \in \{-1,1\}$ be as in Corollary 3.5. Then, we have for all $h \in W(A,\sigma)$ that $\operatorname{sign}_{\iota_2}^{\mathscr{M}_2} h = \delta \operatorname{sign}_{\iota_1}^{\mathscr{M}_1} h$ and in particular that $\operatorname{sign}_{\iota_2}^{\mathscr{M}_2} h_0 = \delta \operatorname{sign}_{\iota_1}^{\mathscr{M}_1} h_0$. It follows that $\delta_1 = \delta\delta_2$. Thus $\delta_1 \operatorname{sign}_{\iota_1}^{\mathscr{M}_1} h = \delta\delta_2 \operatorname{sign}_{\iota_1}^{\mathscr{M}_1} h = \delta_2 \operatorname{sign}_{\iota_2}^{\mathscr{M}_2} h$. ∎

We will show in Theorem 6.4 that there exists a finite tuple $H = (h_1,\ldots,h_s)$ of diagonal hermitian forms of rank one over (A,σ) such that for every $P \in X_F \setminus \mathrm{Nil}[A,\sigma]$ there exists $h_0 \in H$ such that $\operatorname{sign}_\iota^{\mathscr{M}} h_0 \neq 0$.

Definition 3.9. Let $h \in W(A,\sigma)$ and let $P \in X_F$. We define the *H-signature of h at P* as follows: If $P \in \mathrm{Nil}[A,\sigma]$, define $\operatorname{sign}_P^H h := 0$. If $P \notin \mathrm{Nil}[A,\sigma]$, let $i \in \{1,\ldots,s\}$ be the least integer such that $\operatorname{sign}_\iota^{\mathscr{M}} h_i \neq 0$ (for any ι and $\mathscr{M}$, cf. Corollary 3.5), let $\delta_{\iota,\mathscr{M}} \in \{-1,1\}$ be the sign of $\operatorname{sign}_\iota^{\mathscr{M}} h_i$ and define

$$\operatorname{sign}_P^H h := \delta_{\iota,\mathscr{M}} \operatorname{sign}_\iota^{\mathscr{M}} h.$$

Lemma 3.8 shows that this definition is independent of the choice of ι and $\mathscr{M}$ (but it does depend on the choice of H).

A choice of Morita equivalence which is convenient for computations of signatures is given by (1) with (A,σ) replaced by $(A \otimes_F F_P, \sigma \otimes \mathrm{id})$. We denote this Morita equivalence by $\mathscr{N}$ and now describe the induced isomorphisms of Witt groups:

$$\begin{array}{ccccccc} W(A\otimes_F F_P,\sigma\otimes\mathrm{id}) & \xrightarrow{\xi_P^*} & W(M_m(D_P),\mathrm{ad}_{\varphi_P}) & \xrightarrow{\text{scaling}} & W(M_m(D_P),\vartheta_P{}^t) & \xrightarrow{\text{collapsing}} & W(D_P,\vartheta_P) \\ h\otimes F_P & \longmapsto & \xi_P^*(h\otimes F_P) & \longmapsto & \Phi_P^{-1}\xi_P^*(h\otimes F_P) & \longmapsto & \mathscr{N}(h\otimes F_P), \end{array} \tag{4}$$

where h is a hermitian form over (A,σ), $P \in X_F \setminus \mathrm{Nil}[A,\sigma]$ (so that $\varepsilon_P = 1$), ξ_P^* is the group isomorphism induced by some fixed isomorphism

$$\xi_P : (A \otimes_F F_P, \sigma\otimes\mathrm{id}) \xrightarrow{\sim} (M_m(D_P), \mathrm{ad}_{\varphi_P}),$$

and Φ_P is the Gram matrix of the form φ_P. Observe that $\operatorname{sign}\varphi_P$ can be computed as in Section 3.1.

Lemma 3.10. *Let $P \in X_F \setminus \mathrm{Nil}[A,\sigma]$, let $\iota : F \to F_P$ be a real closure of F and let $\mathscr{N}$ and φ_P be as above. Then* $\operatorname{sign}_\iota^{\mathscr{N}} \langle 1\rangle_\sigma = \operatorname{sign}\varphi_P$.

Proof. We extend scalars from F to F_P via ι, $\langle 1\rangle_\sigma \longmapsto \langle 1\rangle_\sigma \otimes F_P = \langle 1\otimes 1\rangle_{\sigma\otimes\mathrm{id}}$ and push $\langle 1\otimes 1\rangle_{\sigma\otimes\mathrm{id}}$ through the sequence (4),

$$\langle 1\otimes 1\rangle_{\sigma\otimes\mathrm{id}} \longmapsto \xi_P^*(\langle 1\otimes 1\rangle_{\sigma\otimes\mathrm{id}}) = \langle \xi_P(1\otimes 1)\rangle_{\mathrm{ad}_{\varphi_P}} = \langle I_m\rangle_{\mathrm{ad}_{\varphi_P}} \longmapsto \Phi_P^{-1}\langle I_m\rangle_{\mathrm{ad}_{\varphi_P}} = \langle \Phi_P^{-1}\rangle_{\vartheta_P{}^t}.$$

(Note that $\xi_P(1\otimes 1) = I_m$, the $m\times m$-identity matrix in $M_m(D_P)$ since ξ_P is an algebra homomorphism.) By collapsing, the matrix Φ_P^{-1} now corresponds to a quadratic form over F_P, a hermitian form over $(F_P(\sqrt{-1}), -)$ or a hermitian form over $(H_P, -)$. In either case Φ_P^{-1} is congruent to Φ_P. Thus $\operatorname{sign}_\iota^{\mathscr{N}} \langle 1\rangle_\sigma = \operatorname{sign}\varphi_P$. ∎

Remark 3.11. It follows from Lemma 3.10 that the signature defined in [3, §3.3, §3.4] is actually sign_P^H with $H = (\langle 1\rangle_\sigma)$. It is now clear that this signature cannot be computed when $\operatorname{sign}_\iota^{\mathscr{N}} \langle 1\rangle_\sigma = 0$, i.e. when $\sigma\otimes\mathrm{id}_{F_P} \simeq \mathrm{ad}_{\varphi_P}$ is hyperbolic. In contrast, if we take $H = (h_1,\dots,h_s)$, as described before Definition 3.9 we are able to compute the signature in all cases. Note that we may choose $h_1 = \langle 1\rangle_\sigma$, so that Definition 3.9 generalizes the definition of signature in [3, §3.3, §3.4].

Example 3.12. Let $(A,\sigma) = (M_4(\mathbb{R}), \mathrm{ad}_\varphi)$, where $\varphi = \langle 1,-1,1,-1\rangle$. Then $\operatorname{sign}\varphi = 0$. Consider the rank one hermitian forms

$$h = \left\langle\!\left(\begin{smallmatrix}1&&&\\&1&&\\&&-1&\\&&&1\end{smallmatrix}\right)\!\right\rangle_\sigma,\ h_1 = \left\langle\!\left(\begin{smallmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{smallmatrix}\right)\!\right\rangle_\sigma,\ \text{and } h_2 = \left\langle\!\left(\begin{smallmatrix}1&&&\\&-1&&\\&&1&\\&&&-1\end{smallmatrix}\right)\!\right\rangle_\sigma$$

over (A,σ). Then $\operatorname{sign}^{\mathscr{N}} h = -2$, $\operatorname{sign}^{\mathscr{N}} h_1 = 0$ and $\operatorname{sign}^{\mathscr{N}} h_2 = 4$, where we suppressed the index ι since $\mathbb{R}$ is real closed. Let $H_1 = (h_1)$ and $H = (h_1, h_2)$, then $\operatorname{sign}^{H_1} h$ is not defined, whereas $\operatorname{sign}^H h = -2$. Observe that taking $H = (h_1, -h_2)$ instead would result in $\operatorname{sign}^H h = 2$.

Remark 3.13. Let $a \in A^\times$ be such that $\sigma(a) = \varepsilon a$ with $\varepsilon \in \{-1, 1\}$. The Morita equivalence *scaling by a*,

$$\mathscr{H}(A,\sigma) \longrightarrow \mathscr{H}_\varepsilon(A, \mathrm{Int}(a)\circ\sigma),\ h \longmapsto ah$$

induces an isomorphism

$$\zeta_a : W(A,\sigma) \longrightarrow W_\varepsilon(A, \mathrm{Int}(a)\circ\sigma),\ h \longmapsto ah.$$

It is clear that

$$\operatorname{sign}_\iota^{\mathscr{M}} h = \operatorname{sign}_\iota^{\mathscr{M}\circ(\zeta_a^{-1}\otimes\mathrm{id})} ah.$$

Consider the special case where $a \in F^\times$. Thus $\varepsilon = 1$ and $\mathrm{Int}(a)\circ\sigma = \sigma$. Assume that $a <_P 0$. Then

$$\operatorname{sign}_\iota^{\mathscr{M}} \zeta_a(h) = \operatorname{sign}\mathscr{M}(ah\otimes F_P) = \operatorname{sign}\mathscr{M}(-h\otimes F_P) = -\operatorname{sign}_\iota^{\mathscr{M}} h,$$

where the last equality follows from Proposition 3.6(*iii*). The same computation shows that $\mathrm{sign}_P^H \zeta_a(h) = -\mathrm{sign}_P^H h$ for any choice of H. Thus, scaling by a changes the sign of the signature, which is contrary to what is claimed in [3, p. 662].

Remark 3.14. For any choice of H, P and ι as in Definition 3.9, there exists a Morita equivalence $\mathscr{M}'$ such that $\mathrm{sign}_P^H h = \mathrm{sign}_\iota^{\mathscr{M}'} h$ for any $h \in W(A,\sigma)$ (i.e. such that $\mathrm{sign}_\iota^{\mathscr{M}'} h_i > 0$ with h_i as in Definition 3.9). Indeed, for $\mathscr{M}$ as in Definition 3.9, it suffices to take $\mathscr{M}' = \delta_{\iota,\mathscr{M}}\mathscr{M}$.

It remains to be shown that a tuple H as described just before Definition 3.9 does exist. In order to reach this conclusion we first need to develop more theory in Sections 4 and 5.

4. Signatures of Involutions

Let (A,σ) be an F-algebra with involution. Consider the *involution trace form*

$$T_\sigma : A \times A \longrightarrow K, (x,y) \longmapsto \mathrm{Trd}_A(\sigma(x)y),$$

where Trd_A denotes the reduced trace of A. If σ is of the first kind, T_σ is a symmetric bilinear form over $K = F$. If σ is of the second kind, T_σ is a hermitian form over $(K, \sigma|_K)$. Let $P \in X_F$. The *signature of the involution σ at P* is defined by

$$\mathrm{sign}_P \sigma := \sqrt{\mathrm{sign}_P T_\sigma}$$

and is a nonnegative integer, since $\mathrm{sign}_P T_\sigma$ is always a square; cf. Lewis and Tignol [15] for involutions of the first kind and Quéguiner [18] for involutions of the second kind. We call the involution σ *positive at P* if $\mathrm{sign}_P \sigma \neq 0$.

Example 4.1.

(*i*) Let $(A,\sigma) = (M_n(F), t)$, where t denotes transposition. Then $T_\sigma \simeq n^2 \times \langle 1 \rangle$. Hence $\mathrm{sign}_P \sigma = n$ for all $P \in X_F$.

(*ii*) Let $(A,\sigma) = ((a,b)_F, -)$, where $-$ denotes quaternion conjugation. Then $T_\sigma \simeq \langle 2 \rangle \otimes \langle 1, -a, -b, ab \rangle$. Hence $\mathrm{sign}_P \sigma = 2$ for all $P \in X_F$ such that $a <_P 0, b <_P 0$ and $\mathrm{sign}_P = 0$ for all other $P \in X_F$.

(*iii*) Let $(A,\sigma) = (F(\sqrt{d}), -)$, where $-$ denotes conjugation. Then $T_\sigma \simeq \langle 1 \rangle_\sigma$. We have $\mathrm{sign}_P \langle 1 \rangle_\sigma = \frac{1}{2}\mathrm{sign}_P \langle 1, -d \rangle$, cf. [20, Chapter 10, Examples 1.6(iii)]. Hence $\mathrm{sign}_P \sigma = 0$ for all $P \in X_F$ such that $d >_P 0$ and $\mathrm{sign}_P \sigma = 1$ for all $P \in X_F$ such that $d <_P 0$.

Remark 4.2. Let (A,σ) and (B,τ) be two F-algebras with involution.

(*i*) Consider the tensor product $(A \otimes_F B, \sigma \otimes \tau)$. Then $T_{\sigma\otimes\tau} \simeq T_\sigma \otimes T_\tau$ and so $\mathrm{sign}_P(\sigma \otimes \tau) = (\mathrm{sign}_P \sigma)(\mathrm{sign}_P \tau)$ for all $P \in X_F$.

(*ii*) If $(A,\sigma) \simeq (B,\tau)$, then $T_\sigma \simeq T_\tau$ so that $\mathrm{sign}_P \sigma = \mathrm{sign}_P \tau$ for all $P \in X_F$.

Remark 4.3. Pfister's local-global principle holds for algebras with involution (A,σ) and also for hermitian forms h over such algebras, [16].

Remark 4.4. The map $\operatorname{sign}\sigma$ is continuous from X_F (equipped with the Harrison topology, see [12, Chapter VIII 6] for a definition) to $\mathbb{Z}$ (equipped with the discrete topology). Indeed: define the map $\sqrt{}$ on $\mathbb{Z}$ by setting $\sqrt{k} = -1$ if k is not a square in $\mathbb{Z}$. Since $\mathbb{Z}$ is equipped with the discrete topology, this map is continuous. Since T_σ is a symmetric bilinear form or a hermitian form over $(K, \sigma|_K)$, the map $\operatorname{sign} T_\sigma$ is continuous from X_F to $\mathbb{Z}$ (by [12, VIII, Proposition 6.6] and [20, Chapter 10, Example 1.6(iii)]). Thus, by composition, $\operatorname{sign}\sigma = \sqrt{\operatorname{sign} T_\sigma}$ is continuous from X_F to $\mathbb{Z}$.

Lemma 4.5. *Let $P \in X_F$. If $P \in \operatorname{Nil}[A, \sigma]$, then*

$$\operatorname{sign}_P \sigma = \operatorname{sign}\varphi_P = 0.$$

Otherwise,

$$\operatorname{sign}_P \sigma = \lambda_P\, |\operatorname{sign}\varphi_P|,$$

where $\lambda_P = 1$ if $(D_P, \vartheta_P) = (F_P, \mathrm{id}_{F_P})$ or $(D_P, \vartheta_P) = (F_P(\sqrt{-1}), -)$ and $\lambda_P = 2$ if $(D_P, \vartheta_P) = (H_P, -)$.

Proof. This is a reformulation of [15, Theorem 1] and part of its proof for involutions of the first kind and [18, Proposition 3] for involutions of the second kind. ∎

Lemma 4.6. *Let (M, h) be a hermitian space over (A, σ), let $P \in X_F$, let $\iota : F \to F_P$ be a real closure of F at P and let $\mathscr{M}$ be a Morita equivalence as in* (2)*. If $P \in \operatorname{Nil}[A, \sigma]$, then*

$$\operatorname{sign}_P \mathrm{ad}_h = \operatorname{sign}_\iota^{\mathscr{M}} h = 0.$$

Otherwise,

$$\operatorname{sign}_P \mathrm{ad}_h = \lambda_P\, |\operatorname{sign}_\iota^{\mathscr{M}} h|,$$

with λ_P as defined in Lemma 4.5. In particular,

$$\operatorname{sign}_\iota^{\mathscr{M}} h = 0 \Leftrightarrow \operatorname{sign}_P \mathrm{ad}_h = 0.$$

Proof. Assume first that $P \in \operatorname{Nil}[A, \sigma]$. Then $\operatorname{sign}_\iota^{\mathscr{M}} h = 0$. Consider the adjoint involution ad_h on $\operatorname{End}_A(M)$. Since h is hermitian, σ and ad_h are of the same type. Furthermore, A and $\operatorname{End}_A(M)$ are Brauer equivalent by [11, 1.10]. Thus $\operatorname{Nil}[A, \sigma] = \operatorname{Nil}[\operatorname{End}_A(M), \mathrm{ad}_h]$. By Lemma 4.5 we conclude that $\operatorname{sign}_P \mathrm{ad}_h = 0$.

Next, assume that $P \in X_F \setminus \operatorname{Nil}[A, \sigma]$. Without loss of generality we may replace F by F_P. Consider a Morita equivalence

$$\mathscr{M}' : \mathscr{H}(A, \sigma) \longrightarrow \mathscr{H}(D, \vartheta)$$

with $(D, \vartheta) = (F, \mathrm{id})$, $(D, \vartheta) = (H, -)$ or $(D, \vartheta) = (F(\sqrt{-1}), -)$. Let (N, b) be the hermitian space over (D, ϑ) corresponding to (M, h) under $\mathscr{M}'$. Then $\operatorname{sign}^{\mathscr{M}'} h = \operatorname{sign} b$. By [2, Remark 1.4.2] we have $(\operatorname{End}_A(M), \mathrm{ad}_h) \simeq (\operatorname{End}_D(N), \mathrm{ad}_b)$ so that $\operatorname{sign}\mathrm{ad}_h = \operatorname{sign}\mathrm{ad}_b$. By [15, Theorem 1] and [18, Proposition 3] we have $\operatorname{sign}\mathrm{ad}_b = \lambda\, |\operatorname{sign} b|$ with $\lambda = 1$ if $(D, \vartheta) = (F, \mathrm{id})$ or $(D, \vartheta) = (F(\sqrt{-1}), -)$ and $\lambda = 2$ if $(D, \vartheta) = (H, -)$. We conclude that $\operatorname{sign}\mathrm{ad}_h = \lambda\, |\operatorname{sign}^{\mathscr{M}'} h| = \lambda\, |\operatorname{sign}^{\mathscr{M}} h|$, where the last equality follows from Corollary 3.5. ∎

5. The Knebusch Trace Formula for M-Signatures

We start with two preliminary sections in order not to overload the proof of Theorem 5.1 below.

5.1. Hermitian Forms over a Product of Rings with Involution.

Let

$$(A, \sigma) = (A_1, \sigma_1) \times \cdots \times (A_t, \sigma_t),$$

where $A, A_1, \dots, A_t$ are rings and $\sigma, \sigma_1, \dots, \sigma_t$ are involutions. We write an element $a \in A$ indiscriminately as $(a_1, \dots, a_t)$ or $a_1 + \cdots + a_t$ with $a_i \in A_i$ for $i = 1, \dots, t$. Writing $1_A = (e_1, \dots, e_t)$, the elements $e_1, \dots, e_t$ are central idempotents, and the coordinates of $a \in A$ are given by

$$A \longrightarrow A_1 \times \cdots \times A_t, \ a \longmapsto (ae_1, \dots, ae_t).$$

Note that $e_i e_j = 0$ whenever $i \neq j$. We assume that $\sigma(1) = 1$ and thus $\sigma(e_i) = e_i$ for $i = 1, \dots, t$.

Let M be an A-module and let $h : M \times M \to A$ be a hermitian form over (A, σ). Following [9, Proof of Lemma 1.9] we can write

$$M = \coprod_{i=1}^{t} Me_i, \ m = (me_1, \dots, me_t),$$

where $\coprod_{i=1}^t Me_i$ is the A-module with set of elements $\prod_{i=1}^t Me_i$, whose sum is defined coordinate by coordinate and whose product is defined by $(m_1e_1, \dots, m_te_t)a = (m_1e_1a_1, \dots, m_te_ta_t)$ for $m_1, \dots, m_t \in M$ and $a \in A$.

Define $h_i = h|_{Me_i}$. Then $h_i(xe_i, ye_i) = \sigma(e_i)h(x,y)e_i = h(x,y)e_i^2 = h(x,y)e_i$ and $h_i : Me_i \times Me_i \to A_i$ is a hermitian form over (A_i, σ_i). We also have

$$\begin{aligned} h(xe_1 + \cdots + xe_t, ye_1 + \cdots + ye_t) &= \sum_{i,j=1}^{t} h(xe_i, ye_j)e_ie_j \\ &= \sum_{i=1}^{t} h(xe_i, ye_i)e_i \end{aligned}$$

which proves that $h = h_1 \perp \ldots \perp h_t$.

5.2. Algebraic Extensions and Real Closures.

We essentially follow [20, Chapter 3, Lemma 2.6, Lemma 2.7, Theorem 4.4].

Let $P \in X_F$ and let F_P denote a real closure of F at P. Let L be a finite extension of F. Writing $L = F[X]/(R)$ for some $R \in F[X]$ and $R = R_1 \cdots R_t$ as a product of pairwise distinct irreducibles in $F_P[X]$ with $\deg R_1 = \cdots = \deg R_r = 1$ and $\deg R_{r+1} = \cdots = \deg R_t = 2$, we obtain canonical F_P-isomorphisms $L \otimes_F F_P \simeq F_P[X]/(R_1 \cdots R_t)$ and

$$L \otimes_F F_P \xrightarrow{\omega} F_1 \times \cdots \times F_t, \tag{5}$$

where $F_i = F_P[X]/(R_i)$ is a real closed field for $1 \leq i \leq r$ and is algebraically closed for $r+1 \leq i \leq t$. We write $1 = (e_1, \dots, e_t)$ in $F_1 \times \cdots \times F_t$ and define $\omega_i(x) = \omega(x)e_i$ for $x \in L \otimes_F F_P$, the projection of $\omega(x)$ on its i-th coordinate.

Let $\iota_0 : F_P \to L \otimes_F F_P$ be the canonical inclusion. Then $\omega_i \circ \iota_0 : F_P \to F_i$ is an isomorphism of fields and of F_P-modules for $1 \leq i \leq r$. In particular, for $1 \leq i \leq r$, F_i is naturally an F_P-module of dimension one, and it is easily seen that $\mathrm{Tr}_{F_i/F_P} = (\omega_i \circ \iota_0)^{-1}$. It follows that Tr_{F_i/F_P} is an isomorphism of fields.

Let $\iota_1 : L \to L \otimes_F F_P$ be the canonical inclusion. Then $\omega_i \circ \iota_1 : L \to F_i$ ($i = 1, \dots, r$) denote the r different embeddings of ordered fields corresponding to the orderings Q_i on L that extend P. In other words, if $\{Q_1, \dots, Q_r\}$ are the different extensions of P to L, then for every $1 \leq i \leq r$, the map

$$L \xrightarrow{\iota_1} L \otimes_F F_P \xrightarrow{\omega_i} F_i$$

is a real closure of L at Q_i. Since Tr_{F_i/F_P} is an isomorphism of fields, it follows that the map

$$L \xrightarrow{\iota_1} L \otimes_F F_P \xrightarrow{\omega_i} F_i \xrightarrow{\mathrm{Tr}_{F_i/F_P}} F_P$$

is also a real closure of L at Q_i.

5.3. **The Knebusch Trace Formula.** Let (A, σ) be an F-algebra with involution. Let L/F be a finite extension. The trace $\mathrm{Tr}_{L/F} : L \to F$ induces an A-linear homomorphism

$$\mathrm{Tr}_{A \otimes_F L} = \mathrm{id}_A \otimes \mathrm{Tr}_{L/F} : A \otimes_F L \longrightarrow A$$

which induces a group homomorphism (transfer map)

$$\mathrm{Tr}^*_{A \otimes_F L} : W(A \otimes_F L, \sigma \otimes \mathrm{id}) \longrightarrow W(A, \sigma),\ (M, h) \longmapsto (M, \mathrm{Tr}_{A \otimes_F L} \circ h),$$

cf. [1, p. 362].

The following theorem is an extension of a result due to Knebusch [8, Proposition 5.2], [20, Chapter 3, Theorem 4.5] to F-algebras with involution. The proof follows the general lines of Knebusch's original proof.

Theorem 5.1. *Let $P \in X_F$. Let L/F be a finite extension of ordered fields and let h be a hermitian form over $(A \otimes_F L, \sigma \otimes \mathrm{id})$. Fix a real closure $\iota : F \to F_P$ and a Morita equivalence $\mathscr{M}$ as in* (2)*. Then, with notation as in Section 5.2,*

$$\mathrm{sign}_\iota^{\mathscr{M}} (\mathrm{Tr}^*_{A \otimes_F L} h) = \sum_{i=1}^{r} \mathrm{sign}^{(\omega_i \circ \iota_0)(\mathscr{M})}_{\omega_i \circ \iota_1} h.$$

Proof. By definition of signature we have

$$\mathrm{sign}_\iota^{\mathscr{M}} (\mathrm{Tr}^*_{A \otimes_F L} h) = \mathrm{sign}\, \mathscr{M}[(\mathrm{Tr}^*_{A \otimes_F L} h) \otimes_F F_P]. \tag{6}$$

Consider the commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\mathrm{Tr}_{L/F}} & F \\ \downarrow{\scriptstyle \otimes_F F_P} & & \downarrow{\scriptstyle \otimes_F F_P} \\ L \otimes_F F_P & \xrightarrow{\mathrm{Tr}_{L\otimes F_P/F_P}} & F_P \end{array}$$

It induces a commutative diagram

$$\begin{array}{ccc} A \otimes_F L & \xrightarrow{\mathrm{Tr}_{A\otimes L}} & A \\ \downarrow{\scriptstyle \otimes_F F_P} & & \downarrow{\scriptstyle \otimes_F F_P} \\ A \otimes_F L \otimes_F F_P & \xrightarrow{\mathrm{id}_A \otimes \mathrm{Tr}_{L\otimes F_P/F_P}} & A \otimes_F F_P \end{array}$$

which in turn induces a commutative diagram of Witt groups

$$\begin{array}{ccc} W(A \otimes_F L, \sigma \otimes \mathrm{id}) & \xrightarrow{\mathrm{Tr}^*_{A\otimes L}} & W(A, \sigma) \\ \downarrow & & \downarrow \\ W(A \otimes_F L \otimes_F F_P, \sigma \otimes \mathrm{id} \otimes \mathrm{id}) & \xrightarrow{(\mathrm{id}_A \otimes \mathrm{Tr}_{L\otimes F_P/F_P})^*} & W(A \otimes_F F_P, \sigma \otimes \mathrm{id}) \end{array}$$

where the vertical arrows are the canonical restriction maps. Thus

$$\operatorname{sign} \mathscr{M}[(\mathrm{Tr}^*_{A\otimes_F L} h) \otimes_F F_P] = \operatorname{sign} \mathscr{M}[(\mathrm{id}_A \otimes \mathrm{Tr}_{L\otimes F_P/F_P})^*(h \otimes_F F_P)]. \tag{7}$$

With reference to Section 5.2 consider the diagram

$$\begin{array}{ccccc} A \otimes_F L \otimes_F F_P & \xrightarrow{\mathrm{id}_A \otimes \omega} & A \otimes_F (F_1 \times \cdots \times F_t) & \xrightarrow{\sim} & (A \otimes_F F_1) \times \cdots \times (A \otimes_F F_t) \\ \downarrow{\scriptstyle \mathrm{id}_A \otimes_F \mathrm{Tr}_{L\otimes F_P/F_P}} & & \downarrow{\scriptstyle \mathrm{id}_A \otimes \mathrm{Tr}_{F_1 \times \cdots \times F_t/F_P}} & & \downarrow{\scriptstyle \sum_{i=1}^t \mathrm{id}_A \otimes \mathrm{Tr}_{F_i/F_P}} \\ A \otimes_F F_P & \xrightarrow{\mathrm{id}} & A \otimes_F F_P & \xrightarrow{\mathrm{id}} & A \otimes_F F_P \end{array}$$

where commutativity of the first square follows from the isomorphism (5) of F_P-algebras, whereas commutativity of the second square follows from [4, p. 137]. We push $h \otimes F_P$ through the induced commutative diagram of Witt groups:

$$\begin{array}{ccccc} h \otimes F_P & \xmapsto{(\mathrm{id}_A \otimes \omega)^*} & h' & \longmapsto & h'_1 \perp \ldots \perp h'_t \\ \downarrow & & & & \downarrow \\ (\mathrm{id}_A \otimes \mathrm{Tr}_{L\otimes F_P/F_P})^*(h \otimes F_P) & & \xmapsto{\mathrm{id}} & & \sum_{i=1}^t (\mathrm{id}_A \otimes \mathrm{Tr}_{F_i/F_P})^*(h'_i) \end{array}$$

where the image of h' equals the orthogonal sum $h'_1 \perp \ldots \perp h'_t$ by Section 5.1. Thus

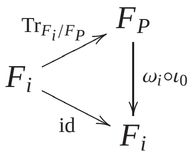
$$\operatorname{sign} \mathscr{M}[(\mathrm{id}_A \otimes \mathrm{Tr}_{L\otimes F_P/F_P})^*(h \otimes_F F_P)] = \sum_{i=1}^{t} \operatorname{sign} \mathscr{M}[(\mathrm{id}_A \otimes \mathrm{Tr}_{F_i/F_P})^*(h'_i)]. \tag{8}$$

We have to consider the following two cases:

Case 1: Assume that $1 \leq i \leq r$. Observe that $h_i' = (\mathrm{id}_A \otimes (\omega_i \circ \iota_1))^*(h)$. Then

$$\operatorname{sign} \mathscr{M}[(\mathrm{id}_A \otimes \mathrm{Tr}_{F_i/F_P})^*(h_i')] = \operatorname{sign} \mathscr{M}[(\mathrm{id}_A \otimes (\mathrm{Tr}_{F_i/F_P}))^* \circ (\mathrm{id}_A \otimes (\omega_i \circ \iota_1))^*(h)].$$

The form $(\mathrm{id}_A \otimes (\omega_i \circ \iota_1))^*(h)$ is defined over $A \otimes_F F_i$, and the commutative diagram



together with Proposition 3.3 gives

$$\begin{aligned}&\operatorname{sign} \mathscr{M}[(\mathrm{id}_A \otimes (\mathrm{Tr}_{F_i/F_P}))^* \circ (\mathrm{id}_A \otimes (\omega_i \circ \iota_1))^*(h)] \\ &\qquad = \operatorname{sign}(\omega_i \circ \iota_0)(\mathscr{M})[(\mathrm{id}_A \otimes (\omega_i \circ \iota_1))^*(h)] \qquad (9)\end{aligned}$$

Case 2: Assume that $r + 1 \leq i \leq t$. Since F_i is algebraically closed, it follows from Morita theory that the Witt group $W(A \otimes_F F_i, \sigma \otimes \mathrm{id})$ is torsion and so h_i' is a torsion form. Therefore $(\mathrm{id}_A \otimes \mathrm{Tr}_{F_i/F_P})^*(h_i')$ is also a torsion form and thus has signature zero.

We conclude that equations (6)-(9) yield the Knebusch Trace Formula. ∎

6. Existence of Forms with Nonzero Signature

Theorem 6.1. *Let (A, σ) be an F-algebra with involution and let $P \in X_F \setminus \mathrm{Nil}[A, \sigma]$. Fix a real closure $\iota : F \to F_P$ and a Morita equivalence $\mathscr{M}$ as in* (2). *There exists a hermitian form h over (A, σ) such that $\operatorname{sign}_\iota^{\mathscr{M}} h \neq 0$.*

Proof. Let $P \in X_F \setminus \mathrm{Nil}[A, \sigma]$. Then $A \otimes_F F_P \simeq M_m(D_P)$ for some $m \in \mathbb{N}$, where $D_P = F_P$, H_P or $F_P(\sqrt{-1})$ if σ is orthogonal, symplectic or unitary, respectively. In each case there exists a positive involution τ on $M_m(D_P)$ (namely, transposition, conjugate transposition and quaternion conjugate transposition, respectively, cf. Example 4.1). By Lemma 4.6 the hermitian form $\langle 1 \rangle_\tau$ over $(M_m(D_P), \tau)$ has nonzero signature since τ is the adjoint involution of the form $\langle 1 \rangle_\tau$. After scaling we obtain a rank one hermitian form h_0 over $(A \otimes_F F_P, \sigma \otimes \mathrm{id})$ such that $\operatorname{sign} \mathscr{M}(h_0) \neq 0$ by Proposition 3.4. The form h_0 is already defined over a finite extension L of $\iota(F)$, contained in F_P. Thus we can consider h_0 as a form over $(A \otimes_F L, \sigma \otimes \mathrm{id})$ and we have $\operatorname{sign} \mathscr{M}(h_0 \otimes F_P) \neq 0$. In other words, if Q_1 is the ordering $L \cap F_P^{\times 2}$ on L, then for any real closure $\kappa_1 : L \to L_1$ of L at Q_1 and any Morita equivalence $\mathscr{M}_1$ as in (2), but starting from $\mathscr{H}(A \otimes_F L_1, \sigma \otimes \mathrm{id})$, we have

$$\operatorname{sign}_{\kappa_1}^{\mathscr{M}_1} h_0 \neq 0$$

by Corollary 3.5.

Let $X = \{Q \in X_L \mid P \subset Q\}$. By [20, Chapter 3, Lemma 2.7], X is finite, say $X = \{Q_1, Q_2, \ldots, Q_r\}$. Thus there exist $a_2, \ldots, a_r \in L^\times$ such that

$$\{Q_1\} = H(a_2, \ldots, a_r) \cap X.$$

Consider the Pfister form $q := \langle\langle a_2, \ldots, a_r \rangle\rangle = \langle 1, a_2 \rangle \otimes \cdots \otimes \langle 1, a_r \rangle$. Then $\operatorname{sign}_{Q_1} q = 2^{r-1}$ and $\operatorname{sign}_{Q_\ell} q = 0$ for $\ell \neq 1$. It follows that $\operatorname{sign}_{\kappa_1}^{\mathscr{M}_1}(q \otimes h_0) = \operatorname{sign}_{Q_1} q \cdot \operatorname{sign}_{\kappa_1}^{\mathscr{M}_1} h_0 \neq 0$ and

$\mathrm{sign}_{\kappa_\ell}^{\mathscr{M}_\ell}(q \otimes h_0) = 0$ for $\ell \neq 1$, where $\kappa_\ell : L \to L_\ell$ is any real closure of L at Q_ℓ and $\mathscr{M}_\ell$ is any Morita equivalence as in (2), but starting from $\mathscr{H}(A \otimes_F L_\ell, \sigma \otimes \mathrm{id})$.

Now $\mathrm{Tr}^*_{A\otimes_F L}(q \otimes h_0)$ is a hermitian form over (A, σ). By the trace formula, Theorem 5.1, we have

$$\mathrm{sign}_\iota^{\mathscr{M}} \mathrm{Tr}^*_{A\otimes_F L}(q \otimes h_0) = \sum_{i=1}^{r} \mathrm{sign}_{\omega_i \circ \iota_1}^{(\omega_i \circ \iota_0)(\mathscr{M})}(q \otimes h_0) = \mathrm{sign}_{\omega_1 \circ \iota_1}^{(\omega_1 \circ \iota_0)(\mathscr{M})}(q \otimes h_0) \neq 0.$$

Taking $h := \mathrm{Tr}^*_{A\otimes_F L}(q \otimes h_0)$ proves the theorem. ∎

Corollary 6.2. *Let (A_1, σ_1) and (A_2, σ_2) be F-algebras with involution of the same type. Assume that A_1 and A_2 are Brauer equivalent. Let $P \in X_F$, let $\iota : F \to F_P$ be a real closure of F at P and let $\mathscr{M}_\ell$ be any Morita equivalence as in* (2)*, but starting from $\mathscr{H}(A_\ell \otimes_F F_P, \sigma_\ell \otimes \mathrm{id})$ for $\ell = 1, 2$. Then the following statements are equivalent:*

(*i*) $\mathrm{sign}_\iota^{\mathscr{M}_1} h = 0$ *for all hermitian forms h over (A_1, σ_1);*
(*ii*) $\mathrm{sign}_\iota^{\mathscr{M}_2} h = 0$ *for all hermitian forms h over (A_2, σ_2);*
(*iii*) $\mathrm{sign}_P \vartheta = 0$ *for all involutions ϑ on A_1 of the same type as σ_1;*
(*iv*) $\mathrm{sign}_P \vartheta = 0$ *for all involutions ϑ on A_2 of the same type as σ_2.*

Proof. By Theorem 6.1, the first two statements are equivalent to $P \in \mathrm{Nil}[A_1, \sigma_1] = \mathrm{Nil}[A_2, \sigma_2]$. Thus (*i*) ⇔ (*ii*).

(*i*) ⇒ (*iii*) Let ϑ be as in (*iii*). Then $\vartheta = \mathrm{ad}_{\langle 1\rangle_\vartheta}$ and $\vartheta = \mathrm{Int}(a) \circ \sigma_1$ for some invertible $a \in \mathrm{Sym}(A_1, \sigma_1)$. Thus, with notation as in Remark 3.13 and using Lemma 4.6 and Proposition 3.4 we have for any Morita equivalence $\mathscr{M}$,

$$\begin{aligned}\mathrm{sign}_P \vartheta &= \lambda_P |\mathrm{sign}_\iota^{\mathscr{M}} \langle 1\rangle_\vartheta| \\ &= \lambda_P |\mathrm{sign}_\iota^{\mathscr{M}\circ(\zeta_a \otimes \mathrm{id})} \zeta_a^{-1}(\langle 1\rangle_\vartheta)| \\ &= \lambda_P |\mathrm{sign}_\iota^{\mathscr{M}_1} \zeta_a^{-1}(\langle 1\rangle_\vartheta)| \\ &= \lambda_P |\mathrm{sign}_\iota^{\mathscr{M}_1} \langle a^{-1}\rangle_{\sigma_1}|\end{aligned}$$

which is zero by the assumption.

(*ii*) ⇒ (*iv*): This is the same proof as (*i*) ⇒ (*iii*) after replacing (A_1, σ_1) by (A_2, σ_2).

For the remainder of the proof we may assume without loss of generality that A_2 is a division algebra and that $A_1 \simeq M_m(A_2)$ for some $m \in \mathbb{N}$.

(*iii*) ⇒ (*iv*): Let ϑ be any involution on A_2. By the assumption, $\mathrm{sign}_P(t \otimes \vartheta) = 0$, where t denotes the transpose involution. Since t is a positive involution, it follows that $\mathrm{sign}_P \vartheta = 0$, cf. Remark 4.2.

(*iv*) ⇒ (*ii*): The assumption implies that $\mathrm{sign}_\iota^{\mathscr{M}_2} h = 0$ for every hermitian form h of dimension 1 over (A_2, σ_2), which implies (*ii*) since all hermitian forms over (A_2, σ_2) can be diagonalized. ∎

Remark 6.3. Note that statement (*iii*) in Corollary 6.2 is equivalent to: $\mathrm{sign}_\iota^{\mathscr{M}_1} h = 0$ for all diagonal hermitian forms h of rank one over (A_1, σ_1).

Theorem 6.4. *Let (A, σ) be an F-algebra with involution. There exists a finite set $H = \{h_1, \dots, h_s\}$ of diagonal hermitian forms of rank one over (A, σ) such that for* *every* $P \in X_F \setminus \mathrm{Nil}[A,\sigma]$*, real closure* $\iota : F \to F_P$ *and Morita equivalence* $\mathscr{M}$ *as in* (2) *there exists* $h \in H$ *such that* $\mathrm{sign}_\iota^{\mathscr{M}} h \neq 0$.

Proof. For every $P \in X_F$, choose a real closure $\iota_P : F \to F_P$ and a Morita equivalence $\mathscr{M}_P$ as in (2). By Corollary 3.5 we may assume without loss of generality and for the sake of simplicity that the map ι_P is an inclusion. The algebra $A \otimes_F F_P$ is isomorphic to a matrix algebra over D_P, where $D_P \in \{F_P, H_P, F_P(\sqrt{-1}), F_P \times F_P\}$. There is a finite extension L_P of F, $L_P \subset F_P$ such that $A \otimes_F L_P$ is isomorphic to a matrix algebra over E_P, where $E_P \in \{L_P, (-1,-1)_{L_P}, L_P(\sqrt{-1}), L_P \times L_P\}$, and P extends to L_P. Let

$$U_P := \{Q \in X_F \mid Q \text{ extends to } L_P\}.$$

Since $P \in U_P$ we can write

$$X_F = \bigcup_{P \in X_F} U_P.$$

We know from [20, Chapter 3, Lemma 2.7, Theorem 2.8] that $\mathrm{sign}_Q(\mathrm{Tr}^*_{L_P/F}\langle 1\rangle)$ is the number of extensions of Q to L_P. Thus,

$$U_P = \left(\mathrm{sign}(\mathrm{Tr}^*_{L_P/F}\langle 1\rangle)\right)^{-1}(\{1,\dots,k\}),$$

where k is the dimension of the quadratic form $\mathrm{Tr}^*_{L_P/F}\langle 1\rangle$, and so U_P is clopen in X_P (and in particular compact). Therefore, and since X_F is compact, there exists a finite number of orderings $P_1,\dots,P_\ell$ in X_F such that

$$X_F = \bigcup_{i=1}^{\ell} U_{P_i}.$$

Now let $P \in \{P_1,\dots,P_\ell\}$ and let L_P be as before. By Theorem 6.1 we have that for every $Q \in U_P \setminus \mathrm{Nil}[A,\sigma]$ there exists a hermitian form h_Q over (A,σ) such that $\mathrm{sign}_{\iota_Q}^{\mathscr{M}_Q} h_Q \neq 0$. By Corollary 6.2 and Remark 6.3 we may assume that h_Q is diagonal of rank one. Consider the total signature map

$$\mu_Q : X_F \longrightarrow \mathbb{Z},\ P \longmapsto \mathrm{sign}_{\iota_P}^{\mathscr{M}_P} h_Q.$$

Then

$$U_P \setminus \mathrm{Nil}[A,\sigma] = \bigcup_{Q \in U_P \setminus \mathrm{Nil}[A,\sigma]} \mu_Q^{-1}(\mathbb{Z} \setminus \{0\}). \tag{10}$$

Consider the continuous map

$$\lambda_P : X_{L_P} \longrightarrow X_F, R \longmapsto R \cap F.$$

We have $Q \in U_P \setminus \mathrm{Nil}[A,\sigma]$ if and only if some extension Q' of Q to L_P is in $X_{L_P} \setminus \mathrm{Nil}[A \otimes_F L_P, \sigma \otimes \mathrm{id}]$ (this follows from the fact that the ordered fields (F,Q) and (L_P,Q') have a common real closure) if and only if $Q \in \lambda(X_{L_P} \setminus \mathrm{Nil}[A \otimes_F L_P, \sigma \otimes \mathrm{id}])$.

We observe that $X_{L_P} \setminus \mathrm{Nil}[A \otimes_F L_P, \sigma \otimes \mathrm{id}]$ is clopen and compact since $\mathrm{Nil}[A \otimes_F L_P, \sigma \otimes \mathrm{id}]$ is either $\varnothing$ or the whole of X_{L_P}, which follows from the fact that $A \otimes_F L_P$ is a matrix algebra over one of $L_P, (-1,-1)_{L_P}, L_P(\sqrt{-1}), L_P \times L_P$.

Hence,

$$U_P \setminus \mathrm{Nil}[A,\sigma] = \lambda(X_{L_P} \setminus \mathrm{Nil}[A \otimes_F L_P, \sigma \otimes \mathrm{id}])$$

is compact and thus closed. Thus $U_P \cap \mathrm{Nil}[A,\sigma]$ is open in U_P. Using (10) we can write

$$U_P = (U_P \cap \mathrm{Nil}[A,\sigma]) \cup \bigcup_{Q \in U_P \setminus \mathrm{Nil}[A,\sigma]} \mu_Q^{-1}(\mathbb{Z} \setminus \{0\}).$$

Now $\mu_Q^{-1}(\mathbb{Z} \setminus \{0\}) = (\mathrm{sign}\,\mathrm{ad}_{h_Q})^{-1}(\mathbb{Z} \setminus \{0\})$ by Lemma 4.6, which is open since $\mathrm{sign}\,\mathrm{ad}_{h_Q}$ is continuous by Remark 4.4. Thus, since U_P is compact, there exist $Q_1, \ldots, Q_t \in U_P \setminus \mathrm{Nil}[A,\sigma]$ such that

$$U_P = (U_P \cap \mathrm{Nil}[A,\sigma]) \cup \bigcup_{i=1}^{t} \mu_{Q_i}^{-1}(\mathbb{Z} \setminus \{0\}).$$

In other words, for every $Q \in U_P \setminus \mathrm{Nil}[A,\sigma]$ one of $\mathrm{sign}_{\iota_Q}^{\mathscr{M}_Q} h_{Q_i}$ ($i = 1, \ldots, t$) is nonzero. Now let $H_P = \{h_{Q_1}, \ldots, h_{Q_t}\}$. Letting $H = \bigcup_{i=1}^{\ell} H_{P_i}$ finishes the proof. ∎

Corollary 6.5. *Let (A,σ) be an F-algebra with involution. The set* $\mathrm{Nil}[A,\sigma]$ *is clopen in X_F.*

Proof. By Theorem 6.4 we have $\mathrm{Nil}[A,\sigma] = \bigcap_{i=1}^{s} \{P \in X_F \mid \mathrm{sign}_{\iota_P}^{\mathscr{M}_P} h_i = 0\}$. The result follows from Lemma 4.6 and Remark 4.4. ∎

At this stage we have established all results that are needed for the definition of the H-signature in Definion 3.9. In the final two sections we show that the total H-signature of a hermitian form is continuous and we reformulate the Knebusch Trace Formula in terms of the H-signature.

7. Continuity of the Total H-Signature Map of a Hermitian Form

Let h be a hermitian form over (A,σ). With reference to Definition 3.9 we denote by $\mathrm{sign}^H h$ the total H-signature map of h:

$$X_F \longrightarrow \mathbb{Z},\ P \longmapsto \mathrm{sign}_P^H h.$$

Lemma 7.1. *Let $H = (h_1, \ldots, h_s)$ be as in Definition 3.9. There is a finite partition of X_F into clopens*

$$X_F = \mathrm{Nil}[A,\sigma]\ \dot\cup \dot{\bigcup_{i=1}^{\ell}} Z_i,$$

such that for every $i \in \{1, \ldots, \ell\}$ one of the total H-signature maps $\mathrm{sign}^H h_1, \ldots, \mathrm{sign}^H h_s$ *is constant non-zero on Z_i.*

Proof. For $r = 1, \ldots, s$, let

$$Y_r := \{P \in X_F \mid \mathrm{sign}_P^H h_i = 0,\ i = 1, \ldots, r\}.$$

By Lemma 4.6 we have

$$Y_r = \bigcap_{i=1}^{r} \{P \in X_F \mid \operatorname{sign}_P \operatorname{ad}_{h_i} = 0\}.$$

Thus each Y_r is clopen.

We have $Y_0 := X_F \supseteq Y_1 \supseteq \cdots \supseteq Y_{s-1} \supseteq Y_s = \operatorname{Nil}[A,\sigma]$ and therefore,

$$X_F = (Y_0 \setminus Y_1) \dot\cup (Y_1 \setminus Y_2) \dot\cup \cdots \dot\cup (Y_{s-1} \setminus Y_s) \dot\cup \operatorname{Nil}[A,\sigma].$$

Let $r \in \{0,\dots,s-1\}$ and consider $Y_r \setminus Y_{r+1}$. By the definition of $Y_1,\dots,Y_s$ the map $\operatorname{sign}^H h_{r+1}$ is never 0 on $Y_r \setminus Y_{r+1}$. Furthermore, since the rank of h_{r+1} is finite, $\operatorname{sign}^H h_{r+1}$ only takes a finite number of values $k_1,\dots,k_m$.

Now observe that there exists a $\lambda \in \{1,2\}$ such that

$$\operatorname{sign}^H h_{r+1} = \frac{1}{\lambda} \operatorname{sign} \operatorname{ad}_{h_{r+1}}$$

on $Y_r \setminus Y_{r+1}$. This follows from Lemma 4.6 and Definition 3.9 for $P \in Y_r \setminus Y_{r+1}$.

Therefore,

$$(\operatorname{sign}^H h_{r+1})^{-1}(k_i) \cap (Y_r \setminus Y_{r+1}) = (\operatorname{sign} \operatorname{ad}_{h_{r+1}})^{-1}(\lambda k_i) \cap (Y_r \setminus Y_{r+1}),$$

which is clopen by Remark 4.4. It follows that $Y_r \setminus Y_{r+1}$ is covered by finitely many disjoint clopen sets on which the map $\operatorname{sign}^H h_{r+1}$ has constant non-zero value. The result follows since the sets $Y_r \setminus Y_{r+1}$ for $r = 0,\dots,s-1$ form a partition of $X_F \setminus \operatorname{Nil}[A,\sigma]$. ■

Theorem 7.2. *Let h be a hermitian form over (A,σ). The total H-signature of h,*

$$\operatorname{sign}^H h : X_F \longrightarrow \mathbb{Z},\ P \longmapsto \operatorname{sign}^H_P h$$

is continuous.

Proof. We use the notation and the conclusion of Lemma 7.1. Since $\operatorname{Nil}[A,\sigma]$ and the sets Z_i are clopen, it suffices to show that $(\operatorname{sign}^H h)|_{Z_i}$ is continuous for every $i = 1,\dots,\ell$.

Let $i \in \{1,\dots,\ell\}$, $k_i \in \mathbb{Z} \setminus \{0\}$ and $j \in \{1,\dots,s\}$ be such that $\operatorname{sign}^H h_j = k_i$ on Z_i. Let $k \in \mathbb{Z}$. Then

$$\begin{aligned}((\operatorname{sign}^H h)|_{Z_i})^{-1}(k) &= \{P \in Z_i \mid \operatorname{sign}^H_P h = k\}\\ &= \{P \in Z_i \mid k_i \operatorname{sign}^H_P h = k_i k\}\\ &= \{P \in Z_i \mid k_i \operatorname{sign}^H_P h = k \operatorname{sign}^H_P h_j\}\\ &= \{P \in Z_i \mid \operatorname{sign}^H_P (k_i \times h \perp k \times h_j) = 0\}.\end{aligned}$$

It follows from Lemma 4.6 that

$$((\operatorname{sign}^H h)|_{Z_i})^{-1}(k) = \{P \in Z_i \mid \operatorname{sign}_P \operatorname{ad}_{k_i \times h \perp k \times h_j} = 0\},$$

which is clopen by Remark 4.4. ■

8. The Knebusch Trace Formula for H-Signatures

Theorem 8.1. *Let $H = (h_1, \dots, h_s)$ be as in Definition 3.9. Let $P \in X_F$. Let L/F be a finite extension of ordered fields and let h be a hermitian form over $(A \otimes_F L, \sigma \otimes \mathrm{id})$. Then, with $H \otimes L := (h_1 \otimes L, \dots, h_s \otimes L)$, we have*

$$\mathrm{sign}^H_P(\mathrm{Tr}^*_{A\otimes_F L} h) = \sum_{P\subseteq Q\in X_L} \mathrm{sign}^{H\otimes L}_Q h.$$

Proof. By Theorem 5.1 (and using its notation), we know that

$$\mathrm{sign}^{\mathscr{M}}_\iota(\mathrm{Tr}^*_{A\otimes_F L} h) = \sum_{i=1}^{r} \mathrm{sign}^{(\omega_i\circ\iota_0)(\mathscr{M})}_{\omega_i\circ\iota_1} h,$$

for any ι and $\mathscr{M}$. Fix a real closure $\iota : F \to F_P$ and choose a Morita equivalence $\mathscr{M}$ such that $\mathrm{sign}^{\mathscr{M}}_\iota = \mathrm{sign}^H_P$ (cf. Remark 3.14). We only have to check that $\mathrm{sign}^{(\omega_i\circ\iota_0)(\mathscr{M})}_{\omega_i\circ\iota_1} = \mathrm{sign}^{H\otimes L}_{Q_i}$ for $i = 1, \dots, r$.

By definition of $\mathscr{M}$, there is a $k \in \{1, \dots, s\}$ such that $\mathrm{sign}^{\mathscr{M}}_\iota h_j = 0$ for $1 \le j \le k-1$ and $\mathrm{sign}^{\mathscr{M}}_\iota h_k > 0$. To check that $\mathrm{sign}^{(\omega_i\circ\iota_0)(\mathscr{M})}_{\omega_i\circ\iota_1} = \mathrm{sign}^{H\otimes L}_{Q_i}$ for $i = 1, \dots, r$, it suffices to check that $\mathrm{sign}^{(\omega_i\circ\iota_0)(\mathscr{M})}_{\omega_i\circ\iota_1}(h_j \otimes L) = 0$ for $j = 1, \dots, k-1$ and $\mathrm{sign}^{(\omega_i\circ\iota_0)(\mathscr{M})}_{\omega_i\circ\iota_1}(h_k \otimes L) > 0$. This follows from the fact that

$$\mathrm{sign}^{(\omega_i\circ\iota_0)(\mathscr{M})}_{\omega_i\circ\iota_1}(h_\ell \otimes L) = \mathrm{sign}^{\mathscr{M}}_\iota h_\ell \text{ for every } 1 \le \ell \le s,$$

which we verify in the remainder of the proof.

By definition,

$$\mathrm{sign}^{(\omega_i\circ\iota_0)(\mathscr{M})}_{\omega_i\circ\iota_1}(h_\ell \otimes L) = \mathrm{sign}(\omega_i \circ \iota_0)(\mathscr{M})[(\mathrm{id}_A \otimes (\omega_i \circ \iota_1))^*(h_\ell \otimes L)].$$

Consider the commutative diagram

$$\begin{array}{ccccccc}
 & & F_P & \xrightarrow{\iota_0} & L\otimes_F F_P & \xrightarrow{\omega_i} & F_i \\
 & \overset{\iota}{\nearrow} & & & & & \downarrow{\scriptstyle \mathrm{id}} \\
F & & & & & & \\
 & \searrow & & & & & \\
 & & L & \xrightarrow[\iota_1]{} & L\otimes_F F_P & \xrightarrow[\omega_i]{} & F_i
\end{array}$$

Thus, by Proposition 3.3,

$$\mathrm{sign}(\omega_i \circ \iota_0)(\mathscr{M})[(\mathrm{id}_A \otimes (\omega_i \circ \iota_1))^*(h_\ell \otimes L)] = \mathrm{sign}(\omega_i \circ \iota_0)(\mathscr{M})[(\mathrm{id}_A \otimes (\omega_i \circ \iota_0 \circ \iota))^*(h_\ell)].$$

Finally the commutative diagram

$$\begin{array}{ccc}
 & & F_i \\
 & \overset{\omega_i\circ\iota_0\circ\iota}{\nearrow} & \\
F & & \downarrow{\scriptstyle (\omega_i\circ\iota_0)^{-1}} \\
 & \underset{\iota}{\searrow} & \\
 & & F_P
\end{array}$$

together with Proposition 3.3 yields

$$\mathrm{sign}(\omega_i \circ \iota_0)(\mathscr{M})[(\mathrm{id}_A \otimes (\omega_i \circ \iota_0 \circ \iota))^*(h_\ell)] = \mathrm{sign}\,\mathscr{M}[(\mathrm{id}_A \otimes \iota)^*(h_\ell)] = \mathrm{sign}^{\mathscr{M}}_\iota h_\ell,$$

which concludes the proof. ∎

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References

[1] E. Bayer-Fluckiger and H. W. Lenstra, Jr. Forms in odd degree extensions and self-dual normal bases. *Amer. J. Math.*, **112** (1990), no. 3, 359–373.

[2] E. Bayer-Fluckiger, R. Parimala, Galois cohomology of the classical groups over fields of cohomological dimension ≤ 2, *Invent. Math.* **122** (1995), no. 2, 195–229.

[3] E. Bayer-Fluckiger, R. Parimala, Classical groups and the Hasse principle, *Ann. of Math. (2)* **147** (1998), no. 3, 651–693.

[4] N. Bourbaki, *Éléments de mathématique. 23. Première partie: Les structures fondamentales de l'analyse. Livre II: Algèbre. Chapitre 8: Modules et anneaux semi-simples*. Actualités Sci. Ind. no. 1261. Hermann, Paris (1958).

[5] A. Fröhlich and A.M. McEvett, Forms over rings with involution. *J. Algebra* **12** (1969), 79–104.

[6] N. Grenier-Boley, Groupe de Witt d'une algèbre simple centrale à involution, PhD thesis, Université de Franche-Comté (2004).

[7] N. Jacobson, A note on hermitian forms, *Bull. Amer. Math. Soc.* **46** (1940), 264–268.

[8] M. Knebusch, On the uniqueness of real closures and the existence of real places. *Comment. Math. Helv.* **47** (1972), 260–269.

[9] M. Knebusch, A. Rosenberg, and R. Ware, Structure of Witt rings and quotients of Abelian group rings. *Amer. J. Math.* **94** (1972), 119–155.

[10] M.-A. Knus, *Quadratic and Hermitian forms over rings*. Grundlehren der Mathematischen Wissenschaften **294**, Springer-Verlag, Berlin (1991).

[11] M.-A. Knus, A.S. Merkurjev, M. Rost, J.-P. Tignol, *The Book of Involutions*, Coll. Pub. **44**, Amer. Math. Soc., Providence, RI (1998).

[12] T.Y. Lam, *Introduction to quadratic forms over fields*, Graduate Studies in Mathematics **67**, American Mathematical Society, Providence, RI (2005).

[13] D.W. Lewis, Forms over real algebras and the multisignature of a manifold, *Advances in Math.* **23** (1977), no. 3, 272–284.

[14] D.W. Lewis, The isometry classification of Hermitian forms over division algebras, *Linear Algebra Appl.* **43** (1982), 245–272.

[15] D.W. Lewis, J.-P. Tignol, On the signature of an involution, *Arch. Math.* **60** (1993), 128–135.

[16] D.W. Lewis, T. Unger, A local-global principle for algebras with involution and hermitian forms, *Math. Z.* **244** (2003), 469–477.

[17] D.W. Lewis, T. Unger, Hermitian Morita theory: a matrix approach, *Irish Math. Soc. Bull.* **62** (2008), 37–41.

[18] A. Quéguiner. Signature des involutions de deuxième espèce. *Arch. Math.* **65** (1995), no. 5, 408–412.

[19] W. Scharlau, Induction theorems and the structure of the Witt group, *Invent. Math.* **11** (1970), 37–44.

[20] W. Scharlau, *Quadratic and Hermitian Forms*, Grundlehren der Mathematischen Wissenschaften **270**, Springer-Verlag, Berlin (1985).

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