ON THE SECOND K-GROUP OF A RATIONAL FUNCTION FIELD

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ABSTRACT. We give an optimal bound on the minimal length of a sum of symbols in the second Milnor K-group of a rational function field in terms of the degree of the ramification.

Keywords: Milnor K-theory, field extension, valuation, ramification

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1. INTRODUCTION

Let E be an arbitrary field and F the function field of the projective line $\mathbb{P}^1$ over E. For $m \in \mathbb{N}$, there is a well-known exact sequence

$$0 \longrightarrow K_2^{(m)}E \longrightarrow K_2^{(m)}F \xrightarrow{\partial} \bigoplus_{x \in \mathbb{P}_0^1} K_1^{(m)}E(x) \longrightarrow K_1^{(m)}E \longrightarrow 0\,, \tag{1.1}$$

due to Milnor and Tate (cf. [6, (2.3)]). Here, $K_1^{(m)}$ and $K_2^{(m)}$ are the functors that associate to a field its first and second K-groups modulo m, respectively, and $\mathbb{P}_0^1$ is the set of closed points of $\mathbb{P}^1$. The map ∂ is called the *ramification map*. By [3, (7.5.4)], for m prime to the characteristic of E, the sequence (1.1) translates into a sequence in Galois cohomology, and the proof of its exactness essentially goes back to Faddeev [2].

In this article we study how for a given element ρ in the image of ∂ one finds a good $\xi \in K_2^{(m)}F$ with $\partial(\xi) = \rho$. Our main result (3.10) states that there is such a ξ that is a sum of r symbols (canonical generators of $K_2^{(m)}F$) where r is bounded by half the degree of the support of ρ. This generalizes results from [4], [7], and [8], where the problem has been studied in terms of Brauer groups in presence of a primitive mth root of unity in E for $m > 0$. Developing further an idea in [8, Prop. 2], we provide examples (4.3) where the bound on r cannot be improved.

2. MILNOR K-THEORY OF A RATIONAL FUNCTION FIELD

We recall the basic terminology of the K-theory for fields as introduced by Milnor [6], with slightly different notation. Let F be a field. For $m, n \in \mathbb{N}$, let

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$K_n^{(m)}F$ denote the abelian group generated by elements called *symbols*, which are of the form $\{a_1,\dots,a_n\}$ with $a_1,\dots,a_n\in F^\times$, subject to the defining relations that $\{*,\dots,*\}:(F^\times)^n\longrightarrow K_n^{(m)}F$ is a multilinear map, that $\{a_1,\dots,a_n\}=0$ whenever $a_i+a_{i+1}=1$ in F for some $i<n$, and that $m\cdot\{a_1,\dots,a_n\}=0$. For $a,b\in F^\times$ we have $\{ab\}=\{a\}+\{b\}$ in $K_1^{(m)}F$. The second relation above is void when $n=1$, hence $K_1^{(m)}F$ is the same as $F^\times/F^{\times m}$, only with different notation for the elements and the group operation. As shown in [6, (1.1) and (1.3)], it follows from the defining relations that, for $a_1,\dots,a_n\in F^\times$, we have $\{a_{\sigma(1)},\dots,a_{\sigma(n)}\}=\varepsilon\{a_1,\dots,a_n\}$ for any permutation σ of the numbers $1,\dots,n$ with signature $\varepsilon=\pm1$, and furthermore $\{a_1,\dots,a_n\}=0$ whenever $a_i+a_{i+1}=0$ for some $i<n$.

With these notations, $K_n^{(0)}F$ is the full Milnor K-group K_nF introduced in [6], and $K_n^{(m)}F$ is its quotient modulo m for $m\geq 1$.

By a $\mathbb{Z}$-*valuation* we mean a valuation with value group $\mathbb{Z}$. Given a $\mathbb{Z}$-valuation v on F we denote by $\mathcal{O}_v$ its valuation ring and by κ_v its residue field. For $a\in\mathcal{O}_v$ let $\overline{a}$ denote the natural image of a in κ_v. By [6, (2.1)], for $n\geq 2$ and a $\mathbb{Z}$-valuation v on F, there is a unique homomorphism $\partial_v: K_n^{(m)}F\longrightarrow K_{n-1}^{(m)}\kappa_v$ such that

$$\partial_v(\{f,g_2,\dots,g_n\})=v(f)\cdot\{\overline{g}_2,\dots,\overline{g_n}\}$$

for $f\in F^\times$ and $g_2,\dots,g_n\in\mathcal{O}_v^\times$. When $n=2$, for $f,g\in F^\times$ we have $f^{-v(g)}g^{v(f)}\in\mathcal{O}_v^\times$ and

$$\partial_v(\{f,g\})=\{(-1)^{v(f)v(g)}\overline{f^{-v(g)}g^{v(f)}}\}\text{ in }K_1^{(m)}\kappa_v\,.$$

We turn to the situation where F is the function field of $\mathbb{P}^1$ over E. By the choice of a generator, we identify F with the rational function field $E(t)$ in the variable t over E. Let $\mathcal{P}$ denote the set of monic irreducible polynomials in $E[t]$. Any $p\in\mathcal{P}$ determines a $\mathbb{Z}$-valuation v_p on $E(t)$ that is trivial on E and such that $v_p(p)=1$. There is further a unique $\mathbb{Z}$-valuation v_∞ on $E(t)$ such that $v_\infty(f)=-\deg(f)$ for any $f\in E[t]\setminus\{0\}$. We set $\mathcal{P}'=\mathcal{P}\cup\{\infty\}$. For $p\in\mathcal{P}'$ we write ∂_p for ∂_{v_p} and we denote by E_p the residue field of v_p. Note that E_p is naturally isomorphic to $E[t]/(p)$ for $p\in\mathcal{P}$, and E_∞ is naturally isomorphic to E.

It follows from [6, Sect. 2] that the sequence

$$0\longrightarrow K_n^{(m)}E\longrightarrow K_n^{(m)}E(t)\overset{\bigoplus\partial_p}{\longrightarrow}\bigoplus_{p\in\mathcal{P}}K_{n-1}^{(m)}E_p\longrightarrow 0 \tag{2.1}$$

is split exact. We are going to reformulate this fact for $n=2$ and to relate (2.1) to (1.1). We set

$$\mathfrak{R}'_m(E)=\bigoplus_{p\in\mathcal{P}'}K_1^{(m)}E_p\,.$$

For $p \in \mathcal{P}'$, the norm map of the finite extension E_p/E yields a group homomorphism $K_1^{(m)}E_p \longrightarrow K_1^{(m)}E$. Summation over these maps for all $p \in \mathcal{P}'$ yields a homomorphism $\mathrm{N} : \mathfrak{R}'_m(E) \longrightarrow K_1^{(m)}E$. Let $\mathfrak{R}_m(E)$ denote the kernel of N. We set $\partial = \bigoplus_{p\in\mathcal{P}'} \partial_p$. By [3, (7.2.4) and (7.2.5)] we obtain an exact sequence

$$0 \longrightarrow K_2^{(m)}E \longrightarrow K_2^{(m)}E(t) \xrightarrow{\partial} \mathfrak{R}'_m(E) \xrightarrow{\mathrm{N}} K_1^{(m)}E \longrightarrow 0\,. \tag{2.2}$$

In particular, $\mathfrak{R}_m(E)$ is equal to the image of $\partial : K_2^{(m)}E(t) \longrightarrow \mathfrak{R}'_m(E)$.

The choice of the generator of F over E fixes a bijection $\phi : \mathbb{P}^1_0 \longrightarrow \mathcal{P}'$ and for $x \in \mathbb{P}^1_0$ a natural isomorphism between $E(x)$ and $E_{\phi(x)}$. This identifies $\bigoplus_{x\in\mathbb{P}^1_0} K_1^{(m)}E(x)$ with $\mathfrak{R}'_m(E)$, and further the sequence (1.1) with (2.2). We will work with (2.2) in the sequel.

For $\rho = (\rho_p)_{p\in\mathcal{P}'} \in \mathfrak{R}'_m(E)$ we denote $\mathrm{Supp}(\rho) = \{p \in \mathcal{P}' \mid \rho_p \neq 0\}$ and $\deg(\rho) = \sum_{p\in\mathrm{Supp}(\rho)}[E_p : E]$, and call this the *support* and the *degree of* ρ. The degree of an element of $\mathfrak{R}'_m(E)$ is invariant under automorphisms of $E(t)/E$.

3. Bound for representation by symbols in terms of the degree

In this section we study the relation between the degree of $\rho \in \mathfrak{R}_m(E)$ to the properties of elements $\xi \in K_2^{(m)}E(t)$ with $\partial(\xi) = \rho$. In (3.10) we will show that there always exists such ξ that is a sum of r symbols where r is the integral part of $\frac{\deg(\rho)}{2}$. In particular, any ramification of degree at most three is realized by a symbol. This settles a question in [4, (2.5)]. In some of the following statements, we consider elements of $\mathfrak{R}'_m(E)$, rather than only of $\mathfrak{R}_m(E)$.

3.1. **Proposition.** *If* $\rho \in \mathfrak{R}_m(E)$ *then* $\deg(\rho) \neq 1$.

Proof. Consider an element $\rho \in \mathfrak{R}'_m(E)$ with $\deg(\rho) = 1$. The support of ρ consists of one rational point $p \in \mathcal{P}'$. Hence $\mathrm{N}(\rho) = \rho_p \neq 0$ in $K_1^{(m)}E$. As $\mathrm{N} \circ \partial = 0$ it follows that $\rho \notin \mathfrak{R}_m(E)$. □

We say that $p \in \mathcal{P}'$ is *rational* if $[E_p : E] = 1$. We call a subset of $\mathcal{P}'$ *rational* if all its elements are rational. We give two examples showing how to realize a given ramification of small degree and with rational support by one symbol.

3.2. **Examples.** (1) Let $a, c \in E^\times$ and $c \notin E^{\times m}$. The symbol $\sigma = \{t - a, c\}$ in $K_2^{(m)}E(t)$ satisfies $\mathrm{Supp}(\sigma) = \{t - a, \infty\}$, $\partial_{t-a}(\sigma) = \{c\}$ and $\partial_\infty(\sigma) = \{c^{-1}\}$.

(2) For $a_1, a_2, c_1, c_2 \in E^\times$ with $a_1 \neq a_2$, we compute the ramification of the symbol $\sigma = \{\frac{t-a_1}{c_2(a_2-a_1)}, \frac{c_1(t-a_2)}{a_1-a_2}\}$ in $K_2^{(m)}E(t)$. It has $\mathrm{Supp}(\sigma) \subseteq \{t-a_1, t-a_2, \infty\}$, $\partial_{t-a_i}(\sigma) = \{c_i\}$ for $i = 1, 2$, and $\partial_\infty(\sigma) = \{(c_1c_2)^{-1}\}$.

A ramification of degree two can under some extra condition be realized by a symbol one of whose entries is a constant.

3.3. **Proposition.** *Let $\rho \in \mathfrak{R}_m(E)$ be such that* $\deg(\rho) = 2$. *If* $\mathrm{Supp}(\rho)$ *is rational or* $\mathrm{char}(E) \neq m = 2$, *there exist $e \in E^\times$ and $f \in E(t)^\times$ such that $\rho = \partial(\{e, f\})$.*

Proof. Suppose first that the support of ρ is rational. We choose $a, e \in E^\times$ such that $t - a \in \mathrm{Supp}(\rho)$ and $\rho_{t-a} = \{e\}$ in $K_1^{(m)}E$. Then $\mathrm{Supp}(\rho) = \{t - a, p\}$ where $p \in \mathcal{P}'$ is rational. As $N(\rho) = 0$ we obtain that $\rho_p = \{e^{-1}\}$ in $K_1^{(m)}E_p$. If $p = \infty$, we set $f = \frac{1}{t-a}$. Otherwise $p = t - b$ for some for $b \in E$, and we set $f = \frac{t-b}{t-a}$. In either case we obtain that $\rho = \partial(\{e, f\})$.

It remains to consider the case where $\mathrm{char}(E) \neq m = 2$ and $\mathrm{Supp}(\rho) = \{p\}$ for a quadratic polynomial $p \in \mathcal{P}$. Then E_p/E is a separable quadratic extension. Let $x \in E_p^\times$ be such that $\rho_p = \{x\}$. As $\mathrm{Supp}(\rho) = \{p\}$ and $\mathrm{N}(\rho) = 0$, we obtain that the norm of x with respect to the extension E_p/E lies in $E^{\times 2}$, and therefore $xE_p^{\times 2} = eE_p^{\times 2}$ for some $e \in E^\times$ (cf. [5, Chap. VII, (3.9)]). Hence, $\rho_p = \{x\} = \{e\}$ in $K_1^{(2)}E_p$, and we obtain that $\rho = \partial(\{e, p\})$. □

In (3.3) the rationality of the support when $m \neq 2$ is not a superfluous condition; the following example was pointed out to us by J.-P. Tignol.

3.4. **Example.** Let k be a field. We consider the rational function field in two variables u and v over k. Let τ denote the k-automorphism of $k(u, v)$ satisfying $\tau(u) = v$ and $\tau(v) = u$. Then τ^2 is the identity map on $k(u, v)$, and $E = \{x \in k(u, v) \mid \tau(x) = x\}$ is a subfield of $k(u, v)$ such that $[k(u, v) : E] = 2$. Consider the element $y = \frac{v}{u} \in k(u, v)$. Since $y \notin E$, the quadratic polynomial $p = (t - y)(t - \tau(y)) = t^2 - \frac{u^2+v^2}{uv}t + 1$ is irreducible over E.

Let m be an odd positive integer. We consider the symbol $\sigma = \{p, t\}$ in $K_2^{(m)}E(t)$. Note that the support of $\partial(\sigma)$ is contained in $\{p\}$ and $\partial_p(\sigma) = \{\overline{t}\}$. Moreover, mapping t to y induces an E-isomorphism $E_p \longrightarrow k(u, v)$. Since y is not an mth power in $k(u, v)$, it follows that $\partial_p(\sigma) \neq 0$. Hence, $\mathrm{Supp}(\partial(\sigma)) = \{p\}$ and $\deg(\partial(\sigma)) = 2$.

We claim that $\partial(\sigma) \neq \partial(\{e, f\})$ for any $e \in E^\times$ and $f \in E(t)^\times$. Suppose on the contrary that there exist $e \in E^\times$ and $f \in E(t)^\times$ such that $\partial_p(\sigma) = \partial_p(\{e, f\})$. Then we obtain that $e^{v_p(f)}y$ is an mth power in $k(u, v)$, and taking norms with respect to the extension $k(u, v)/E$ yields that $e^{2v_p(f)} \in E^{\times m}$. Since m is odd, it follows that $e^{v_p(f)} \in E^{\times m}$, and thus $\partial_p(\{e, f\}) = 0$, a contradiction.

The remainder of this section builds up to our main result (3.10).

3.5. **Lemma.** *Let $\rho \in \mathfrak{R}'_m(E)$ with* $\deg(\rho) \geq 2$. *There exists a symbol σ in $K_2^{(m)}E(t)$ such that* $\deg(\rho - \partial(\sigma)) \leq \deg(\rho) - 1$ *and such that this inequality is strict except possibly when $\partial_\infty(\sigma) \neq 0 = \rho_\infty$ or* $\deg(\rho) = 2$. *More precisely, one may choose $\sigma = \{fh, g\}$ where f is the product of the polynomials in* $\mathrm{Supp}(\rho)$

and $g, h \in E[t] \setminus \{0\}$ *are such that* $\deg(g) < \deg(f)$ *and, either* $\deg(h) < \deg(g)$, *or* $gh \in E^\times$.

Proof. Let f be the product of the polynomials in $\mathrm{Supp}(\rho)$. By the Chinese Remainder Theorem, we may choose $g \in E[t]$ prime to f with $\deg(g) < \deg(f)$ such that $\partial_p(\{f, g\}) = \rho_p$ for all monic irreducible polynomials $p \in \mathrm{Supp}(\rho)$. If g is constant, let $h = 1$. If g is not square-free, let h be the product of the different monic irreducible factors of g. If g is square-free and not constant, then using the Chinese Remainder Theorem we choose $h \in E[t]$ prime to g with $\deg(h) < \deg(g)$ such that $\partial_p(\{f, g\}) - \rho_p = \{\overline{h}\}$ in $K_1^{(m)}E_p$ for every monic irreducible factor p of g. Then g, h and $\sigma = \{fh, g\}$ have the desired properties. $\square$

3.6. **Lemma.** *Let* $d \in \mathbb{N} \setminus \{0\}$ *and* $f \in E[t]$ *non-constant and square-free such that* $\deg(p) \geq d$ *for every irreducible factor* p *of* f. *Let* $F = E[t]/(f)$ *and let* ϑ *denote the class of* t *in* F. *For any* $a \in F^\times$ *there exist nonzero polynomials* $g, h \in E[t]$ *with* $\deg(h) \leq d - 1$ *and* $\deg(g) \leq \deg(f) - d$ *such that* $a = \frac{g(\vartheta)}{h(\vartheta)}$.

Proof. Let $V = \bigoplus_{i=0}^{d-1} E\vartheta^i$ and $W = \bigoplus_{i=0}^{e-d} E\vartheta^i$ where $e = \deg(f)$. By the choice of d and the Chinese Remainder Theorem, we have $V \setminus \{0\} \subseteq F^\times$, where $F^\times$ denotes the group of invertible elements of F. As $a \in F^\times$ we have $\dim_E(Va) = \dim_E(V) = d$ and $\dim_E(Va) + \dim_E(W) = e + 1 > e = [F : E]$, so $Va \cap W \neq 0$. Therefore $h(\vartheta)a = g(\vartheta)$ for certain $h, g \in E[t] \setminus \{0\}$ with $\deg(h) \leq d - 1$ and $\deg(g) \leq e - d$. Thus $h(\vartheta) \in V \setminus \{0\} \subseteq F^\times$ and $a = \frac{g(\vartheta)}{h(\vartheta)}$. $\square$

3.7. **Lemma.** *Let* $\rho \in \mathfrak{R}'_m(E)$ *and* $q \in \mathrm{Supp}(\rho)$ *such that* $\deg(q) = 2n + 1$ *with* $n \geq 1$. *There exists a symbol* σ *in* $K_2^{(m)}E(t)$ *such that* $\deg(\rho - \partial(\sigma)) \leq \deg(\rho) - 2$. *More precisely, one may choose* $\sigma = \{qhf^{-2}g^{-2}, g^{-1}f\}$ *with* $f, g, h \in E[t] \setminus \{0\}$ *such that* $\deg(f), \deg(g) \leq n$ *and* $\deg(h) \leq 2n - 1$.

Proof. Using (3.6) we choose $f, g \in E[t] \setminus \{0\}$ with $\deg(f), \deg(g) \leq n$ such that $\partial_q(\{q, g^{-1}f\}) = \rho_q$. Then q is prime to fg. If fg is constant, let $h = 1$. If fg is not square-free, let h be the product of the different monic irreducible factors of fg. If fg is square-free and not constant, we choose $h \in E[t]$ prime to fg and with $\deg(h) < \deg(fg)$ such that $\partial_p(\{h, g^{-1}f\}) = \partial_p(\{q^{-1}f^2g^2, g^{-1}f\})$ for every monic irreducible factor p of fg. In any case $\deg(h) \leq 2n - 1 = \deg(q) - 2$.

Let $\sigma = \{qhf^{-2}g^{-2}, g^{-1}f\}$. We have $\partial_q(\sigma) = \rho_q$ and $\partial_p(\sigma) = 0$ for every monic irreducible polynomial $p \in E[t]$ prime to h and not contained in $\mathrm{Supp}(\rho)$. It follows that $q \in \mathrm{Supp}(\rho) \setminus \mathrm{Supp}(\rho - \partial(\sigma))$ and that every polynomial in $\mathrm{Supp}(\rho - \partial(\sigma)) \setminus \mathrm{Supp}(\rho)$ divides h. Furthermore, if $\deg(h) = 2n - 1$, then $\deg(f) = \deg(g) = n$, so that $\deg(qh) = 4n = 2\deg(fg)$ and thus $\partial_\infty(\sigma) = 0$. We conclude that $\deg(\rho - \partial(\sigma)) \leq \deg(\rho) - 2$ in any case. $\square$

3.8. **Proposition.** *Let $\rho \in \mathfrak{R}'_m(E)$ with $\deg(\rho) \geq 2$. There exists a symbol σ in $K_2^{(m)}E(t)$ such that $\deg(\rho - \partial(\sigma)) \leq \deg(\rho) - 1$. Moreover, if $\deg(\rho) \geq 3$ and $\mathrm{Supp}(\rho)$ contains an element of odd degree, then there exists a symbol σ in $K_2^{(m)}E(t)$ such that $\deg(\rho - \partial(\sigma)) \leq \deg(\rho) - 2$.*

Proof. In view of (3.5) only the second part of the statement remains to be proven. If $\mathrm{Supp}(\rho)$ contains a non-rational point of odd degree, the statement follows from (3.7). Suppose now that $\mathrm{Supp}(\rho)$ contains a rational point. Note that the statement is invariant under E-automorphisms of $E(t)$. Hence, we may assume that $\infty \in \mathrm{Supp}(\rho)$, in which case the statement follows from (3.5). □

3.9. **Question.** *Given $\rho \in \mathfrak{R}_m(E)$ with $\deg(\rho) \geq 3$, does there always exist a symbol σ in $K_2^{(m)}E(t)$ such that $\deg(\rho - \partial(\sigma)) \leq \deg(\rho) - 2$?*

For $x \in \mathbb{R}$, the unique $z \in \mathbb{Z}$ such that $z \leq x < z + 1$ is denoted $\lfloor x \rfloor$.

3.10. **Theorem.** *For $\rho \in \mathfrak{R}_m(E)$ and $n = \lfloor \frac{\deg(\rho)}{2} \rfloor$, there exist symbols $\sigma_1, \ldots, \sigma_n$ in $K_2^{(m)}E(t)$ such that $\rho = \partial(\sigma_1 + \cdots + \sigma_n)$.*

Proof. We proceed by induction on n. If $n = 0$ then $\rho = 0$ by (3.1) and the statement is trivial. Assume that $n > 0$. We have either $\deg(\rho) = 2n+1$, in which case ρ contains a point of odd degree, or $\deg(\rho) = 2n$. Hence, by (3.8) there exists a symbol σ in $K_2^{(m)}E(t)$ with $\deg(\rho - \partial(\sigma)) \leq 2n-1$. By the induction hypothesis there exist symbols $\sigma_1, \ldots, \sigma_{n-1}$ in $K_2^{(m)}E(t)$ with $\rho - \partial(\sigma) = \partial(\sigma_1 + \cdots + \sigma_{n-1})$. Then $\rho = \partial(\sigma_1 + \cdots + \sigma_{n-1} + \sigma)$. □

If we knew that for $m \geq 1$ every element of $\mathfrak{R}_m(E)$ had a lift to $\mathfrak{R}_0(E)$ of the same degree, it would be sufficient to formulate and prove (3.10) for $m = 0$.

4. Example showing that the bound is sharp

In this section we show that the bound (3.10) is sharp for all m and in all degrees. In order to obtain an example in (4.3) where the bound of (3.10) is an equality, we adapt Sivatski's argument in [8, Prop. 2].

For any $a \in E$, there is a unique homomorphism $s_a : K_n^{(m)}E(t) \longrightarrow K_n^{(m)}E$ such that $s_a(\{f_1, \ldots, f_n\}) = \{f_1(a), \ldots, f_n(a)\}$ for any $f_1, \ldots, f_n \in E[t]$ prime to $t - a$ and such that $s_a(\{t - a, *, \ldots, *\}) = 0$ (cf. [3, (7.1.4)]).

4.1. **Lemma.** *The homomorphism $s = s_0 - s_1 : K_n^{(m)}E(t) \longrightarrow K_n^{(m)}E$ has the following properties:*

(a) $s(K_n^{(m)}E) = 0$,

(b) $s(\{(1-a)t + a, b_2, \ldots, b_n\}) = \{a, b_2, \ldots, b_n\}$ *for any* $a, b_2, \ldots, b_n \in E^\times$,

(c) any symbol in $K_n^{(m)}E(t)$ is mapped under s to a sum of two symbols in $K_n^{(m)}E$.

Proof. Since s_0 and s_1 both restrict to the identity on $K_n^{(m)}E$, part (a) is clear. For $a, b_2, \dots, b_n \in E^\times$ and $\sigma = \{(1-a)t + a, b_2, \dots, b_n\}$, we have $s_1(\sigma) = 0$ and thus $s(\sigma) = s_0(\sigma) = \{a, b_2, \dots, b_n\}$. This shows (b). Part (c) follows from the observation that both s_0 and s_1 map symbols to symbols. $\square$

4.2. **Proposition.** *Let $d \in \mathbb{N}$, $a_1, \dots, a_d \in E^\times$, and $\sigma_1, \dots, \sigma_d$ symbols in $K_{n-1}^{(m)}E$. Assume that $\sum_{i=1}^d \{a_i\} \cdot \sigma_i \in K_n^{(m)}E$ is not equal to a sum of less than d symbols and let*

$$\xi \;=\; \sum_{i=1}^{d} \{(1-a_i)t + a_i\} \cdot \sigma_i \;\in\; K_n^{(m)}E(t)\,.$$

Then $\deg(\partial(\xi)) = d+1$, and if $r \in \mathbb{N}$ is such that $\partial(\xi) = \partial(\tau_1 + \cdots + \tau_r)$ for symbols $\tau_1, \dots, \tau_r$ in $K_n^{(m)}E(t)$, then $r \geq \lfloor \frac{d+1}{2} \rfloor$.

Proof. The hypothesis that ξ cannot be written as a sum of less than d symbols has a few consequences. For $i = 1, \dots, d$, it follows that $\{a_i\} \cdot \sigma_i \neq 0$, so in particular $a_i \neq 1$, and with $p = t - \frac{a_i}{1-a_i}$ we get that $\partial_p(\xi) = \sigma_i \neq 0$ in $K_{n-1}^{(m)}E$. Furthermore, we obtain that $\partial_\infty(\xi) = -\sum_{i=1}^d \sigma_i \neq 0$ in $K_{n-1}^{(m)}E$. Therefore we have $\mathrm{Supp}(\partial(\xi)) = \{t - \frac{a_i}{1-a_i} \mid 1 \leq i \leq d\} \cup \{\infty\}$ and thus $\deg(\partial(\xi)) = d+1$.

Assume now that $r \in \mathbb{N}$ and $\partial(\xi) = \partial(\tau_1 + \cdots + \tau_r)$ for symbols $\tau_1, \dots, \tau_r$ in $K_n^{(m)}E(t)$. Then $\tau_1 + \cdots + \tau_r - \xi$ is defined over E. Let s be the map from (4.1). By (4.1) we obtain that $s(\tau_1 + \cdots + \tau_r - \xi) = 0$ and thus

$$\sum_{i=1}^{d} \{a_i\} \cdot \sigma_i = s(\xi) = s(\tau_1) + \ldots + s(\tau_r) \;\in\; K_n^{(m)}E\,,$$

which is a sum of $2r$ symbols. Hence $2r \geq d$, by the hypothesis on d. $\square$

4.3. **Example.** Let p be a prime dividing m. Let k be a field containing a primitive pth root of unity ω and $a_1, \dots, a_d \in k^\times$ such that the Kummer extension $k(\sqrt[p]{a_1}, \dots, \sqrt[p]{a_d})$ of k has degree p^d. Let $b_1, \dots, b_d$ be indeterminates over k and set $E = k(b_1, \dots, b_d)$. Using [9, (2.10)] and [1, (2.1)], it follows that $\sum_{i=1}^d \{a_i, b_i\}$ is not equal to a sum of less than d symbols in $K_2^{(p)}E$. Since p divides m, it follows immediately that $\sum_{i=1}^d \{a_i, b_i\} \in K_2^{(m)}E$ is not a sum of less than d symbols in $K_2^{(m)}E$. Consider $\xi = \sum_{i=1}^d \{(1-a_i)t + a_i, b_i\}$ in $K_2^{(m)}E(t)$. By (4.2), for $\rho = \partial(\xi)$ we have that $\deg(\rho) = d+1$ and $\rho \neq \partial(\xi')$ for any $\xi' \in K_2^{(m)}E(t)$ that is a sum of less than $r = \lfloor \frac{\deg(\rho)}{2} \rfloor$ symbols.

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