DERIVED CATEGORY OF SOME TWISTED PROJECTIVE HOMOGENEOUS VARIETIES

MARK BLUNK

ABSTRACT. In this paper we construct a tilting sheaf for Severi-Brauer Varieties and Involution Varieties. This sheaf relates the derived category of each variety to the derived category of modules over a ring whose semisimple component consists of the Tits algebras of the corresponding linear algebraic group.

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1. INTRODUCTION

The origin of this direction of research is the classic paper [5] of Beilinson, where it is shown that the derived category of the projective space $\mathbb{P}_{\mathbb{C}}(V)$ that the sheaves $\mathcal{O}, \mathcal{O}(-1), \ldots \mathcal{O}(-\dim(V)+1)$ form a simple set of generators, (what is called a *strong exceptional collection*). In [16], Kapranov generalized these calculations to form strong

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exceptional collections for Grassmannians and Quadrics over $\mathbb{C}$. All of these varieties are *Projective Homogeneous Varieties*. That is, they are of the form G/P, where G is a semisimple linear algebraic group and P is a parabolic subgroup. It is a conjecture of Catanese that every G/P should possess such a collection, and there have been several results in this direction (see [9],[18], [19], and [12], among others).

In this paper, we will work in another direction. The varieties considered here are *twisted* projective homogeneous varieties. These are varieties X defined over a field which, after extension of scalars to a separable closure, become isomorphic to some G/P. Examples of such varieties are Severi-Brauer Varieties and Involution Varieties. The group G is of type A for Severi-Brauer varieties and of type D for Involution Varieties.

Examples of such varieties are Severi-Brauer Varieties and Involution Varieties. Instead of producing exceptional collections, in this paper we follow [3] and produce *tilting sheaves* (see Section 2 for a definition). Our main result is to produce a locally free sheaf $\mathcal{T}$ on X which induces a derived equivalence

$$D^b(\mathrm{Coh}(X)) \to D^b(\quad \mathrm{mod} \ - \mathrm{End}(\mathcal{T})).$$

This paper is heavily influenced by [22], where the Quillen K-theory of twisted Projective Homogeneous Varieties is computed. In [8], the author produced a locally sheaf $\mathcal{F}$ on a del Pezzo surface S of degree 6 which was used to describe the K-theory of S. Later, in [7], it was shown that the same sheaf $\mathcal{F}$ is a tilting sheaf for the surface S. This paper completes the chain by showing that the sheaves produced in [22] can be used to develop derived equivalences for some twisted Projective Homogeneous Varieties. The arguments in this paper follow a similar line of reasoning to that found in [7].

In Section 2, we recall the definition of a tilting sheaf, our main tool for constructing the desired derived equivalences. In Section 3, we recall some needed facts about modules and generation of categories and in 4, we briefly recall some basic properties of semisimple linear algebraic groups. In particular we state the Borel-Weil-Bott Theorem (Theorem 4.1), which we will use repeatedly. In the remainder of the paper, we construct tilting sheaves for Severi-Brauer Varieties, Generalized Severi-Brauer Varieties, and Involution Varieties. We conclude in Section 8 with an application on computing the Quillen K-theory of these varieties.

1.1. **Notation.** Let us fix some notation. We will let F denote a field, and $\overline{F}$ a fixed separable closure of F. The group $\Gamma := \mathrm{Gal}(\overline{F}/F)$ is the

Galois group. By a variety X over F we mean a reduced scheme of finite type over $\mathrm{Spec}(F)$. The abelian category of quasicoherent sheaves on X will be denoted $\mathrm{Qcoh}(X)$, and $\mathrm{Coh}(X)$ is the abelian subcategory of sheaves on X. For a ring R, $\mod -R$ (resp. $R- \mod$) is the abelian category of finitely generated right (resp. left) R-modules.

If $\mathcal{A}$ is an abelian category, then $D(\mathcal{A})$ will denote the corresponding derived category $\mathcal{A}$ (confer [13, Chapter III]). This is a triangulated category objects are complexes with terms in $\mathcal{A}$, and maps homomorphisms of chain complexes, modulo homotopy equivalences, and localizing the set of quasi-isomorphisms. The subcategory of bounded complexes will be denoted $D^b(\mathcal{A})$. If $M \in \mathcal{A}$, by abuse of notation we will use the same symbol to denote the complex in $D^b(\mathcal{A})$ with M concentrated in degree 0 and every other term equal zero.

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2. Tilting Sheaf

We say that a sheaf $\mathcal{T}$ on a smooth variety X is a *tilting sheaf* if the following conditions hold:

- The sheaf $\mathcal{T}$ has no self extensions, i.e. $\mathrm{RHom}_{D^b(X)}(\mathcal{T}[i], \mathcal{T}) = 0$, for $i > 0$.
- The algebra $\mathrm{End}_{\mathcal{O}_X}(\mathcal{T})$ of global endomorphisms of the sheaf $\mathcal{T}$ has finite global dimension (see Section 3 for a definition of global dimension of a ring).
- There is no proper thick subcategory of $D^b(\mathrm{Coh}(X))$ containing the element $\mathcal{T}$ (see Section 3.1 for a definition of thick).

Our main tool in this paper is the following theorem.

Theorem 2.1 (Theorem 3.12 of [3])**.** *Let X be a smooth variety, and let $M \in \mathrm{Coh}(X)$ be a tilting sheaf with $S := End_{\mathcal{O}_X}(M)$. Then the functors*

$$F := \mathrm{Hom}(M, -) : \mathrm{Coh}(X) \to \mod -S$$
$$G := - \otimes M : \mod -S \to \mathrm{Coh}(X)$$

induce equivalences of categories,

$$RF : D^b(\mathrm{Coh}(X)) \to D^b(\mod -S)$$
$$LG : D^b(\mod -S) \to D^b(\mathrm{Coh}(X)),$$

inverse to each other.

3. Global Dimension

Let R be a ring. The projective dimension of an arbitrary left R-module T is denoted by $\mathrm{pdim}_R T$. The *global (homological) dimension* of R is supremum of $\mathrm{pdim}_R T$ over all such modules T.

Proposition 3.1 ([2], Prop III.2.7)**.** *Let R and S be artinian F-algebras , and M an R-S-bimodule, finitely generated over F. If S is a semisimple ring, then*

$$\mathrm{gldim}\begin{pmatrix} R & M \\ 0 & S \end{pmatrix} = \max\{\mathrm{pdim}_R M + 1, \mathrm{gldim}\, R\}.$$

3.1. Generation and thick subcategories.

We recall some properties of generation of triangulated categories (confer [21], [7, Section 4]).

Let $\mathcal{D}$ be a triangulated category, and let $\mathcal{E}$ denote a subset of objects of $\mathcal{D}$. A triangulated category is equipped with a shift operator. If $M \in \mathcal{D}$, the shift of M will be denoted by $M[1]$.

- A subcategory of $\mathcal{C} \subset \mathcal{D}$ is said to be *thick* (épaisse) if it is closed under isomorphisms, shift, taking cones of morphisms, and taking direct summands of objects in $\mathcal{C}$.
- An object $C \in \mathcal{D}$ is said to be *compact* if $\mathrm{Hom}_{\mathcal{D}}(C, -)$ commutes with direct sums. Let $\mathcal{D}^c \subset \mathcal{D}$ denote the subcategory of compact objects in $\mathcal{D}$.
- We define $\langle \mathcal{E} \rangle$ to be the smallest thick full subcategory of $\mathcal{D}$ containing the elements of $\mathcal{E}$.
- We define $\mathcal{E}^{\perp}$ to be the subcategory of $\mathcal{D}$ consisting of all objects M such that $\mathrm{Hom}_{\mathcal{D}}(E[i], M) = 0$, for all $i \in \mathbb{Z}$ and all $E \in \mathcal{E}$.

We say that $\mathcal{E}$ *generates* $\mathcal{D}$ if $\mathcal{E}^{\perp} = 0$. If $\mathcal{D}^c$ generates $\mathcal{D}$, then we say $\mathcal{D}$ is *compactly generated.*

If $\mathcal{D}$ is compactly generated and $\mathcal{E} \subset \mathcal{D}^c$. It's clear that if $\langle \mathcal{E} \rangle = \mathcal{D}^c$, then $\mathcal{E}$ generates $\mathcal{D}$. The following theorem tells us that the converse is also true.

Theorem 3.2 (Ravenel and Neeman [21]. Also see Thm. 2.1.2 in [10])**.** *Let $\mathcal{D}$ be a compactly generated triangulated category. Then a set of objects $E \subset \mathcal{D}^c$ generates $\mathcal{D}$ if and only if $\langle E \rangle = \mathcal{D}^c$.*

4. Groups, Quotients, and Associated Sheaves

We briefly summarize some definitions and properties of linear algebraic groups, which we will need in the paper. Some references for this section are [11], [15, I.5], [23], and [17, Chapter 24].

Let G denote a split, semisimple linear algebraic group over F, and $T \subset G$ a fixed maximal torus. The group $\mathrm{Lie}(T)^* := \mathrm{Hom}(T, \mathbb{G}_m)$ will denote the *character lattice* of G. It is a free $\mathbb{Z}$-module of finite rank. The set $R \subset \mathrm{Lie}(T)^*$ will denote a root system corresponding to T, R^+ the set of positive roots, and $R^- := -R^+$ the set of negative roots, and the set $S = \{\alpha_1, \ldots, \alpha_r\} \subset R^+$ is a basis of simple roots.

For $\alpha \in \mathrm{Lie}(T)^*$, let $\mathrm{Lie}(G)^\alpha$ denote the eigenspace of $\mathrm{Lie}(G)$ corresponding to α. The Lie subalgebra

$$\mathrm{Lie}(T) \bigoplus_{\alpha \in R^+} \mathrm{Lie}(G)^\alpha$$

is $\mathrm{Lie}(B)$ for a fixed Borel subgroup B of G determined by T and S. If $I \subset S$, then the Lie subalgebra

$$\mathrm{Lie}(B) \oplus \bigoplus_{\alpha \in R^-(I)} \mathrm{Lie}(G)^\alpha,$$

where

$$R^-(I) := \{\alpha \in R^- | \quad \alpha = \Sigma_{i=1}^r a_i \alpha_i \text{ with } a_i \leq 0, \quad a_i = 0 \ \forall \ \alpha_i \in I\},$$

is $\mathrm{Lie}(P_I)$, for a unique parabolic subgroup $P_I \supset B$ of G. Every such parabolic subgroup (intermediate between B and G) arises in this fashion. For example, $P_\emptyset = G$ and $P_S = B$.

Let $\{\lambda_1, \ldots, \lambda_r\}$ be the set of *fundamental weights* determined by the simple roots α_i, and $\Lambda =: \mathbb{Z}[\lambda_i]$ be the *weight lattice*. If P_I is a parabolic subgroup of G, then we say a weight λ is *dominant for* P_I if

$$\lambda = \Sigma_{i \in I} n_i \lambda_i + \Sigma_{j \notin I} n_j \lambda_j, \quad \text{where } n_j \geq 0.$$

Also, let $\rho = \Sigma_i \lambda_i$.

The Weyl group W is the group generated by the simple reflections s_α corresponding to $\alpha \in S$. For $w \in W$, the *length* of w is the least number of factors in a decomposition of w as a product of simple reflections. There is an an action of W on the weight lattice Λ. We will need another action, called the *dot* or *affine* action on Λ:

$$w.\lambda := w(\lambda + \rho) - \rho.$$

We say that a weight λ is *singular* if there is some non-trivial $w \in W$ such that $w.\lambda = \lambda$.

If $P \subset G$ is a parabolic subgroup, there exists a decomposition, called the Levi decomposition, of P into a semisimple or Levi factor L_P and a unipotent subgroup $R_u(P)$. If $\phi : P \to \mathbf{GL}(V)$ is an irreducible finite-dimensional representation, $R_u(P)$ acts trivially, and so ϕ descends to a representation of the Levi factor L_P. Each such representation posses a unique highest weight $\lambda \in \Lambda$ which determines the representation.

Moreover this weight is dominant for P. For a weight λ, we will denote the corresponding representation vector space by $V(\lambda)$.

Finally, if $\phi : P \to \mathbf{GL}(V)$ is as in previous paragraph, with corresponding weight λ, then we define a locally free sheaf of rank $\dim(V)$ on the projective homogeneous variety G/P as follows:

$$G \times_r V := G \times V / \{(g, v) \sim (gp^{-1}, \phi(p)(v)) \mid p \in P,\ g \in G,\ v \in V\}.$$

The projection $G \times V \to G$ induces a map $G \times_r V \to G/P$, defining a vector bundle over G/P. The corresponding locally free sheaf on G/P will be denoted $\mathcal{O}_{G/P}(\lambda)$. A section of this sheaf can be thought of as a function $F : G \to V$ satisfying $F(gp) = \phi(p^{-1})F(g)$.

Let us recall the celebrated Borel-Weil-Bott theorem, which relates representations on P to the cohomology of the induced sheaf on the variety G/P.

Theorem 4.1 (Theorem 5.0.1 of [4])**.** *Let G be a simply connected split semisimple algebraic group, $P \subset G$ be a parabolic subgroup, and $\lambda \in \Lambda$ is dominant with respect to P. Consider the corresponding sheaf $\mathcal{O}_{G/P}(\lambda)$ on G/P. Then:*

a *If λ is singular,*

$$H^r(G/P, \mathcal{O}_{G/P}(\lambda)) = 0, \quad \forall r.$$

b *If λ is not singular, then there exists a unique $w \in W$ such that $w.\lambda$ is dominant. Moreover,*

$$H^i(G/P, \mathcal{O}_{G/P}(\lambda)) = 0,\ i \neq l(w)$$
$$H^{l(w)}(G/P, \mathcal{O}_{G/P}(\lambda)) = V(w.\lambda).$$

5. Severi-Brauer Varieties

We recall some well known facts about central simple algebras (confer [1] and [17, Section I.1.B]). Let A be a *central simple algebra* over F of degree n. The algebra A has dimension over F equal to n^2, has no non-trivial two sided ideals, and center equal to F. Equivalently, $A \otimes_F \overline{F}$ is isomorphic to $\mathrm{End}_{\overline{F}}(V')$, for some $\overline{F}$-vector space V' of dimension n. The algebraic group $\mathrm{SL}_1(A)$ is of type A_n. Every finitely generated right A-module has dimension, as an F-vector space, divisible by n, and we define the *reduced dimension* of M by

$$\mathrm{rdim}_A(M) := \frac{\dim_F(M)}{n}.$$

Finally, we say that the algebra A is *split* if $A = \mathrm{End}_F(V)$, where V is a finite dimensional F-vector space.

Let $X := \mathbf{SB}(A)$ be the *Severi-Brauer* variety of the algebra A. This is an irreducible, smooth, projective variety of dimension $n-1$, whose points consist of right ideals of A which have reduced dimension 1. If E/F is a field extension, $\mathbf{SB}(A)_E = \mathbf{SB}(A \otimes_F E)$. When $A = \mathrm{End}_F(V)$ for some F-vector space V of dimension n, every right ideal of A of reduced dimension 1 is determined by a unique 1-dimensional subspace of V. Thus $\mathbf{SB}(\mathrm{End}_F(V)) = \mathbb{P}(V)$. In particular, Since $P(V) = G/P$ for $G = \mathbf{SL}(V)$ and $P = P_{\alpha_1}$ the stabilizer of a line in V, we see that $\mathbf{SB}(A)$ is a twisted form of a projective homogeneous variety.

We define the 'tautological' sheaf $\mathcal{I}$ on X (confer [1], [22, section 10.2]), a subsheaf of the consist sheaf A. The fiber over a closed point $x \in X(\overline{F})$ consists of the elements $a \in A_{\overline{F}}$ such that $a \in x \subset A_{\overline{F}}$ (here x is a right ideal of $A_{\overline{F}}$ of reduced dimension 1). This defines a locally free sheaf of rank n on X. This sheaf $\mathcal{I}$ is locally free, and has an induced right A action, and the algebra $\mathrm{End}(\mathcal{I})$ is isomorphic to the algebra A. Finally, If $A = \mathrm{End}_F(V)$ is split, then $\mathcal{I} = \mathcal{O}_{\mathbb{P}(V)}(-\lambda_1) \otimes_F V^*$.

Finally, let

$$\mathcal{T} = \mathcal{O}_X \oplus \mathcal{I}^1 \oplus \mathcal{I}^2 \oplus \cdots \oplus \mathcal{I}^{\otimes(n-1)}.$$

We will show that $\mathcal{T}$ is a tilting sheaf for X.

Theorem 5.1. *The sheaf $\mathcal{T}$ has no self extensions, i.e.,*

$$\mathrm{RHom}_{D^b(\mathrm{Coh}(X))}(\mathcal{T}[i], \mathcal{T}) = 0,$$

for $i > 0$.

Proof. It suffices to extend scalars to $\overline{F}$, so we may assume that F is separably closed. In this case, $X = \mathbb{P}(V)$, and the sheaf $\mathcal{T}$ decomposes into a sum of invertible sheaves of the form $\mathcal{O}_{\mathbb{P}(V)}(-j\lambda_1)$, where $j = 0, 1, \ldots, n-1$.

So it suffices to show that

$$\mathrm{RHom}_{D^b(\mathrm{Coh}(\mathbb{P}(V)))}(\mathcal{O}_{\mathbb{P}(V)}(-j_1\lambda_1)[i], \mathcal{O}_{\mathbb{P}(V)}(-j_2\lambda_1)) = 0,$$

for $i > 0$ and $j_1, j_2 = 0, \ldots n-1$. But

$$\begin{aligned} \mathrm{RHom}_{D^b(\mathrm{Coh}(\mathbb{P}(V)))}(\mathcal{O}_{\mathbb{P}(V)}(-j_1\lambda_1)[i], \mathcal{O}_{\mathbb{P}(V)}(-j_2\lambda_1)) = \\ \mathrm{Ext}^i_{\mathbb{P}(V)}(\mathcal{O}_{\mathbb{P}(V)}(-j_1\lambda_1), \mathcal{O}_{\mathbb{P}(V)}(-j_2\lambda_1)) = \\ H^i(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}((j_1 - j_2)\lambda_1)) = 0, \end{aligned}$$

for $-n < j_1 - j_2 \leq n$ and $i > 0$. This follows from Theorem 4.1, since the corresponding weight in each case is either dominant or singular. $\square$

Remark 5.2. Also, note that

$$\mathrm{Hom}_{\mathbb{P}(V)}(\mathcal{O}_{\mathbb{P}(V)}, \mathcal{O}_{\mathbb{P}(V)}(-i\lambda_1)) = H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(-i\lambda_1)) = 0,$$

since the weight $-i\lambda_1$ is singular when $1 \leq i \leq n-1$. We will need this in the proof of Theorem 5.3.

Theorem 5.3. *The algebra* $\mathrm{End}(\mathcal{T})$ *has finite global dimension.*

Proof. We prove by induction on n. The base case $n = 1$ is trivial, as the field $F = \mathrm{End}(\mathcal{O}_{pt})$ has global dimension 0.

Because $\mathcal{T} = \oplus_{i=0}^{n-1} \mathcal{I}^{\otimes i}$, we have the following matrix presentation for $\mathrm{End}(\mathcal{T})$ (by Remark 5.2, this matrix is lower triangular).

$$\mathrm{End}(\mathcal{T}) = \begin{pmatrix} F & \mathrm{Hom}(\mathcal{I}, \mathcal{O}_X) & \mathrm{Hom}(\mathcal{I}^{\otimes 2}, \mathcal{O}_X) & \cdots & * \\ 0 & A & * & \cdots & * \\ 0 & 0 & A^{\otimes 2} & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A^{\otimes n-1} \end{pmatrix}.$$

So we can write

$$\mathrm{End}_X(\mathcal{T}) = \begin{pmatrix} R & B \\ 0 & (A^{\otimes n-1}) \end{pmatrix}, \quad \text{where}$$

$$R = \begin{pmatrix} F & * & * & \cdots & * \\ 0 & A & * & \cdots & * \\ 0 & 0 & A^{\otimes 2} & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A^{\otimes n-2} \end{pmatrix},$$

$$B = \begin{pmatrix} \mathrm{Hom}(\mathcal{O}_X, \mathcal{I}^{\otimes n-1}) \\ \vdots \\ \mathrm{Hom}(\mathcal{I}^{\otimes n-2}, \mathcal{I}^{\otimes n-1}) \end{pmatrix}.$$

By induction, the ring R has finite global dimension, and in particular, $\mathrm{pdim}_R(B)$ is finite. Since $A^{\otimes}(n-1)$ is semisimple, we have by Proposition 3.1 that $\mathrm{gldim}(\mathrm{End}_X(\mathcal{T})) = \max\{\mathrm{pdim}_R(B)+1, \mathrm{gldim}(R)\}$, and we conclude that $\mathrm{End}_X(\mathcal{T})$ has finite global dimension. $\square$

Lemma 5.4. *Assume that the algebra A is split. The sheaf $\mathcal{T}$ generates* $D(\mathrm{Qcoh}(\mathbf{SB}(A)))$*, and* $\langle \mathcal{T} \rangle = D^b(\mathrm{Coh}(\mathbf{SB}(A)))$.

Proof. Since $A = \mathrm{End}_F(V)$ is split, $\mathbf{SB}(A) = \mathbb{P}(V)$, and $\mathcal{I} = \mathcal{O}_{\mathbb{P}(V)}(-\lambda_1) \otimes V^*$. The sheaf $\mathcal{O}_{\mathbb{P}(V)}(-i\lambda_1)$ is a summand of $\mathcal{T}$, for $i = 0, \cdots = n-1$. By [5] (or [16, Section 3]), we know that these invertible sheaves form a strong exceptional collection on $\mathbb{P}(V)$. In particular,

$$\langle \{ \mathcal{O}_{\mathbb{P}(V)}(-i\lambda_1) \ \ 0 \le i \le n-1 \} \rangle = D^b(\mathrm{Coh}(\mathbb{P}(V))).$$

This forces $\langle \mathcal{T} \rangle = D^b(\mathrm{Coh}(\mathbb{P}(V)))$, and hence the sheaf $\mathcal{T}$ generates $D(\mathrm{Qcoh}(\mathbb{P}(V)))$. □

Proposition 5.5. *The sheaf* $\mathcal{T}$ *generates* $D(\mathrm{Qcoh}(\mathbf{SB}(A)))$.

Proof. Let $\mathcal{M} \in D(\mathrm{Qcoh}(\mathbf{SB}(A)))$ and assume that

$$\mathrm{RHom}_{D(\mathrm{Qcoh}(SB(A)))}(\mathcal{T}, \mathcal{M}) = 0.$$

Since $\mathcal{T}$ is locally free, $\mathcal{H}om(\mathcal{T}, -)$ and $T^* \otimes -$ are exact functors on $\mathrm{Qcoh}(X)$. (Similarly for $\mathcal{H}om(\overline{\mathcal{T}}, -)$ and $\overline{T}^* \otimes -$ on $\mathrm{Qcoh}(\overline{X})$.) Thus, for example, $\mathcal{R}Hom_X(\mathcal{T}, \mathcal{M})$ can be computed on $D(\mathrm{Qcoh}(X))$ by applying $\mathcal{H}om(\mathcal{T}, -)$ to each individual term in $\mathcal{M}$.

Consider the following cartesian square:

$$\begin{tikzcd} \mathbf{SB}(A)_{\overline{F}} \arrow[r, "v"'] \arrow[d, "q"'] & \mathbf{SB}(A) \arrow[d, "q"'] \\ \mathrm{Spec}(\overline{F}) \arrow[r, "u"'] & \mathrm{Spec}(F) \end{tikzcd} .$$

Since u (and thus v) is flat, it follows that the natural map

$$u^* Rp_* \to Rq_* v^*$$

is an isomorphism of functors (see [14, (3.18)]).

$$\begin{aligned} 0 &= u^*(\mathrm{RHom}_X(\mathcal{T}, \mathcal{M})) \\ &= u^* Rp_* \mathcal{R}Hom_X(\mathcal{T}, \mathcal{M}) \quad \text{By [14], page 85} \\ &= Rq_* v * \mathcal{R}Hom_X(\mathcal{T}, \mathcal{M}) \\ &= Rq_* v * (\mathcal{T}^\vee \otimes^L_X \mathcal{M}) \\ &= Rq_*(\overline{\mathcal{T}}^\vee \otimes^L_{\overline{X}} v^* \mathcal{M}) \\ &= Rq_* \mathcal{R}Hom_{\overline{X}}(v^* \mathcal{T}, v^* \mathcal{M}) \\ &= \mathrm{RHom}_{\overline{X}}(v^* \mathcal{T}, v^* \mathcal{M}). \end{aligned}$$

The algebra $A_{\overline{F}}$ splits, and thus $v^* \mathcal{T} = \overline{\mathcal{T}}$ generates $D(\mathrm{Qcoh}(\overline{X}))$, by Lemma 5.4. This implies that $v^*(\mathcal{M}) = 0$. Since v is flat, this forces $\mathcal{M} = 0$. Hence, $\mathcal{T}$ generates $D(\mathrm{Qcoh}(X))$. □

We conclude by collecting our results to prove the main theorem of this section.

Theorem 5.6. *The map*

$$R\operatorname{Hom}(\mathcal{T},-) : D^b(\operatorname{Coh}(X)) \to D^b(\operatorname{End}(\mathcal{T}) - \quad \text{mod })$$

is an equivalence of categories.

Proof. The sheaf $\mathcal{T}$ generates $D(\operatorname{Qcoh}(X))$ by Proposition 5.5, and since $D(\operatorname{Qcoh}(X))$ is compactly generated, it follows that $\langle \mathcal{T} \rangle = D^b(\operatorname{Coh}(X))$ by Theorem 3.2. The sheaf $\mathcal{T}$ has no-self extensions by Theorem 5.1. The algebra $A = \operatorname{End}(\mathcal{T})$ has finite global dimension, by Theorem 5.3. So $\mathcal{T}$ is a tilting sheaf, and the theorem follows from [3, Theorem 3.1.2]. □

Remark 5.7. In [6], the author produces a semi-orthogonal decomposition for the derived category of a Severi-Brauer scheme over the derived category of the base, by producing a collection of twisted sheaves which satisfy the necessary properties. In the case where the base is $\operatorname{Spec}(F)$, this is equivalent to result here, since the twisting data comes from a Brauer class, i.e. the central simple algebra A.

6. Generalized Severi-Brauer Varieties

As in the previous section, let A be a central simple algebra of degree n. Let $X = SB(r, A)$ be the *Generalized Severi-Brauer Variety*, for some $0 < r < n$. The points of X are the right ideals $I \subset A$ of reduced dimension r. Obviously, $SB(1, A)$ is just the usual Severi-Brauer variety.

Let $\mathcal{I}$ be the tautological sheaf of $\mathbf{SB}(r, A)$, defined in an analogous fashion to the sheaf in Section 5. This is a locally free sheaf of rank rn. When $A = \operatorname{End}(V)$ is split, $\mathbf{SB}(r, A) = \mathbf{SL}(V)/P_{\alpha_r}$, and $\mathcal{I} = \mathcal{O}_{G/P}(-\lambda_r) \otimes V^*$. Let

$$\mathcal{T} = \bigoplus_a \Sigma^a(\mathcal{I}).$$

Here $a = (a_1, \ldots, a_r) \in \mathbb{N}^r$, subject to the condition $n \geq a_1 \geq a_2 \geq \cdots \geq a_r \geq 0$ (that is, Young diagrams with at most $n - r$ rows and at most r columns) and Σ is the Schur Functor corresponding to a. Finally, let $d(a) = a_1 + \cdots + a_n$.

Theorem 6.1. *The sheaf $\mathcal{T}$ has no self extensions.*

Proof. The proof is the same as in the proof of Theorem 5.1. The only difference is that the weight that appears in the argument is λ_r, instead of λ_1. □

Theorem 6.2. *The ring* $\operatorname{End}(\mathcal{T})$ *has finite global dimension.*

Proof. The ring $\mathrm{End}(\mathcal{T})$ is upper triangular, with diagonal entries of the form $A^{d(a)}$, which are all semisimple. The proof follows as in the proof of 5.3. $\square$

Proposition 6.3. *The sheaf $\mathcal{T}$ generates $D(\mathrm{Qcoh}(X))$, and hence $\langle \mathcal{T} \rangle = D^b(\mathrm{Coh}(X))$.*

Proof. It suffices to show that $\mathcal{T}$ generates $D(\mathrm{Qcoh}(X))$ in the case where A is split. In that case, $\mathcal{T}$ contains terms of the form $\Sigma^a(\mathcal{O}_{G/P_{\alpha_r}}(-\lambda_r))$. In [16, Theorem 3.4], it is shown that these sheaves form a strong exceptional collection for G/P_{α_r}. In particular,

$$\langle \left\{ \Sigma^a(\mathcal{O}_{G/P_{\alpha_r}}(-\lambda_r)), a \right\} \rangle = D^b(\mathrm{Coh}(G/P_{\alpha_r})).$$

It follows that $\mathcal{T}$ generates $D(\mathrm{Qcoh}(X))$, and by Theorem 3.2, $\langle \mathcal{T} \rangle = D^b(\mathrm{Coh}(X))$. $\square$

Theorem 6.4. *The sheaf $\mathcal{T}$ is a tilting bundle, and thus*

$$\mathcal{R}Hom(\mathcal{T}, -) : D^b(\mathrm{Coh}(X)) \to D^b(\quad \mathrm{mod} \quad - \mathrm{End}(\mathcal{T}))$$

is an equivalence of derived categories.

Proof. The sheaf $\mathcal{T}$ has no self-extensions by Theorem 6.1, $\langle \mathcal{T} \rangle = D^(\mathrm{Coh}(X))$ by Proposition 6.3, and the algebra $\mathrm{End}(\mathcal{T})$ has finite global dimension by PROPOSITION 6.3. Thus $\mathcal{T}$ is a tilting sheaf, and the result follows by 2.1. $\square$

7. Involution Varieties

In this section we assume that $\mathrm{char}(F) \neq 2$.

Let (A, σ) be a central simple algebra of degree $2n$, equipped with an orthogonal involution σ. Recall ([17, Proposition 2.6] that σ is *orthogonal* if

$$\dim_F \mathrm{Sym}(A, \sigma) = \frac{2n(2n+1)}{2},$$
$$\dim_F \mathrm{Skew}(A, \sigma) = \frac{2n(2n-1)}{2}.$$

Let $X := I(A, \sigma)$ be the *involution variety* of A (confer [20], [24]). This is a codimension 1 subvariety of $\mathbf{SB}(A)$, whose points consists of the right ideals $I \in \mathbf{SB}(A)$ which are *orthogonal*, i.e. $\sigma(I) \cdot I = 0$. We will let $\mathcal{I} \in \mathrm{Coh}(I(A, \sigma))$ to denote the pullback of the tautological bundle of $\mathbf{SB}(A)$ to $I(A, \sigma)$.

Recall from that introduction that $\Gamma = \mathrm{Gal}(\overline{F}/F)$. If $\mathcal{F}$ is a sheaf for the étale topology, we have the Hochschild-Serre Spectral Sequence:

[1, Section 2]

$$H^p(\Gamma, H^q(\overline{Y}, \mathcal{F})) \to H^{p+q}(Y, \mathcal{F}).$$

If $Y = \mathbf{SB}(A)$, then $\overline{Y} = \mathbb{P}(V')$ for some vector space V' over $\overline{F}$. By looking at the edge terms of the spectral sequence, we have the following exact sequence:

$$0 \to \mathrm{Pic}(\mathbf{SB}(A)) \to \mathrm{Pic}(\mathbb{P}(V'))^\Gamma \xrightarrow{f} \mathrm{Br}(F).$$

Here $\mathrm{Pic}(\mathbb{P}(V_{\overline{F}}))^\Gamma = \mathbb{Z}^\Gamma = \mathbb{Z}$, generated by the invertible sheaf $\mathcal{O}_{\mathbb{P}(V_{\overline{F}})}(-\lambda_1)$. The map f sends $\mathcal{O}_{\mathbb{P}(V_{\overline{F}})}(\lambda_1)$ to the class of A in the Brauer group $\mathrm{Br}(F)$. Since A has an orthogonal involution σ, the exponent of A divides 2. It follows that $\mathcal{O}_{\mathbb{P}(V_{\overline{F}})}(-2\lambda_1) \in \mathrm{Ker}(f)$ descends to an invertible sheaf on $\mathbf{SB}(A)$. Restricting to X, we get an invertible sheaf which we will label $\mathcal{O}_X(-2\lambda_1)$.

Let $C(A, \sigma)$ denote the *Clifford Algebra* associated to the pair (A, σ) (confer [17, II.8.7]). Since $C(A, \sigma)$ is defined as a quotient of the tensor algebra of A, there exists a canonical F-linear map $c : A \to C(A, \sigma)$ (confer [17, (II.8.13)]).

We define a subsheaf $\mathcal{J}$ of the constant sheaf $C(A, \sigma)$ on $I(A, \sigma)$. If $I \in I(A, \sigma)$ is an isotropic right ideal of reduced dimension 1, then the fiber over I is the right ideal of $C(A, \sigma)$ generated by $c(I)$. The endomorphism ring $\mathrm{End}_X(\mathcal{J})$ is isomorphic to $C(A, \sigma)$

Remark 7.1. If $A = \mathrm{End}(V)$ for some finite dimensional vector space V over F, then $\sigma = \sigma_q$ is the adjoint involution with respect to some non-singular quadratic form $q \in S^2(V^*)$. In this case, The isomorphism $\mathbf{SB}(\mathrm{End}(V)) = \mathbb{P}(V)$ identifies $I(\mathrm{End}(V), \sigma_q)$ with the quadric $Z(q)$. Also, if q is maximally isotropic (i.e. there exists an isotropic subspace of dimension n in V), then $I(\mathrm{End}(V), \sigma_q) = G/P$, where $G = \mathbf{Spin}(q)$ is a split group of type D_n, and $P = P_{\alpha_1}$. So $I(A, \sigma)$ is a twisted projective homogeneous variety. We say that (A, σ) is split if $A = \mathrm{End}(V)$ and $\sigma = \sigma_q$, where q is maximally isotropic. Finally, when $X = G/P$, $\mathcal{J} = \mathcal{O}_{G/P}(-\lambda_{n-1}) \otimes_F W_+^* \oplus \mathcal{O}_{G/P}(-\lambda_n) \otimes_F W_-^*$, where $W_+ = V(-\lambda_{n-1})$ and $W_- = V(-\lambda_n)$, the two half-spin representation spaces associated to the weights $-\lambda_{n-1}$ and $-\lambda_n$ (confer [22, page 574]).

Let

$$\mathcal{T} := \bigoplus_{i=0}^{n-2} \left(\mathcal{O}_X(-2\lambda_1)^{\otimes i} \oplus \mathcal{I} \otimes \mathcal{O}_X(-2\lambda_1)^{\otimes i} \right) \bigoplus \mathcal{J} \otimes \mathcal{O}_X(-2\lambda_1)^{\otimes(n-1)}.$$

We will show that $\mathcal{T}$ is a tilting bundle for $I(A, \sigma)$.

Proposition 7.2. *The sheaf $\mathcal{T}$ generates $D(\mathrm{Qcoh}(X))$*

Proof. This is similar to the proof of Proposition 5.5. We first check that the proposition is true in the split case, where the argument is similar to Lemma 5.4. So we may assume that $A = \mathrm{End}_F(V)$ and $\sigma = \sigma_q$ for a maximally isotropic quadratic form $q \in S^2(V^*)$, $I(A,\sigma) = Z(q) = \mathbf{Spin}(q)/P_{\alpha_1}$. The sheaf $\mathcal{T}$ decomposes into a sum with terms $\mathcal{O}_{G/P}(-i\lambda)$, $\mathcal{O}_{G/P}(-2(n-1)\lambda_1 - \lambda_{n-1})$ and $\mathcal{O}_{G/P}(-2(n-1)\lambda_1 - \lambda_n)$, where $0 \le i \le 2n-3$. It is shown in [16, section 4] that these sheaves form a strong exceptional collection. In particular,

$$\Big\langle \mathcal{O}_{G/P}(-i\lambda_1), \mathcal{O}_{G/P}(-(2n-2)\lambda_1 - \lambda_{n-1}), \mathcal{O}_{G/P}(-(2n-2)\lambda_1 - \lambda_n), \Big\rangle \\ = D^b(\mathrm{Coh}(Z(q))).$$

Thus we conclude that $\mathcal{T}$ generates $D(\mathrm{Qcoh}(X))$ in the split case. The arbitrary case follows by extending scalars to $\overline{F}$, and reasoning as in Proposition 5.5. □

Theorem 7.3. *The sheaf $\mathcal{T}$ has no self-extensions.*

Proof. We extend scalars to $\overline{F}$, arguing as in Theorem 5.1. The sheaf $\overline{\mathcal{T}}$ decomposes into a sum of sheaves with terms $\mathcal{O}_{G/P}(-(2n-2)\lambda_1-\lambda_{n-1})$, $\mathcal{O}_{G/P}(-(2n-2)\lambda_1 - \lambda_n)$, and $\mathcal{O}_X(j\lambda_1)$, where $-(2n-2) < j \le 0$. So it suffices to check that

$$\begin{aligned} H^i(\overline{X}, \mathcal{O}_X(\pm j\lambda_1)) &= 0, \\ H^i(\overline{X}, \mathcal{J}_1^* \otimes \mathcal{O}_X(j\lambda_1)) &= 0, \\ H^i(\overline{X}, \mathcal{J}_1 \otimes \mathcal{O}_X(-j\lambda_1)) &= 0, \\ H^i(\overline{X}, \mathcal{J}_2^* \otimes \mathcal{O}_X(j\lambda_1)) &= 0, \\ H^i(\overline{X}, \mathcal{J}_2 \otimes \mathcal{O}_X(-j\lambda_1)) &= 0, \\ H^i(\overline{X}, \mathcal{J}_1 \otimes \mathcal{J}_2^*) &= 0, \\ H^i(\overline{X}, \mathcal{J}_2 \otimes \mathcal{J}_1^*) &= 0, \end{aligned}$$

for $i > 0$ and $-(2n-2) < j \le 0$.

The highest weights of the corresponding P-representations for these sheaves are, respectively,

$$\begin{aligned} &\pm j\lambda_1, \\ &\pm(-\lambda_n + j\lambda_1), \\ &\pm(-\lambda_{n-1} + j\lambda_1), \\ &\pm(\lambda_{n-1} - \lambda_n). \end{aligned}$$

All of these weights are either singular or dominant. Thus by Theorem 4.1, all of the non-zero cohomology above vanishes. □

Remark 7.4. As in the Severi-Brauer case, all of the weights $j\lambda_i$ are singular for $j < 0$, so $H^0(\overline{X}, \mathcal{O}_{\overline{X}}(j\lambda_1)) = 0$. It follows that we have the following upper triangular presentation of the global endomorphism rings $\mathrm{End}(\mathcal{T})$:

$$\mathrm{End}(\mathcal{T}) = \begin{pmatrix} F & * & * & * & \cdots & * \\ 0 & A & * & * & \cdots & * \\ 0 & 0 & F & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A & 0 \\ 0 & 0 & 0 & \cdots & 0 & C(A,\sigma) \end{pmatrix}.$$

As in the Severi-Brauer case, we see that $\mathrm{End}(\mathcal{T})$ is upper triangular, with the Tits Algebras appearing along the diagonal terms.

Theorem 7.5. *The ring* $\mathrm{End}(\mathcal{T})$ *has finite global dimension.*

Proof. The proof follows the same line of reasoning as in Theorem 5.3, since all diagonal terms are semisimple algebras. □

Theorem 7.6. *The sheaf* $\mathcal{T}$ *induces a natural equivalence*

$$\mathrm{RHom}(\mathcal{T}, -) : D^b(\mathrm{Coh}(X)) \to D^b(\mod - \mathrm{End}(\mathcal{T})).$$

Proof. We need to verify that $\mathcal{T}$ is a tilting sheaf. It has no self extensions by Theorem 7.3. By Proposition 7.2 and Theorem 3.2, we see that $\langle \mathcal{T} \rangle = D^b(\mathrm{Coh}(X))$. The ring $\mathrm{End}(\mathcal{T})$ has finite global dimension by Theorem 7.5, and so the statement follows from Theorem 2.1.

□

8. K-theory

In each case discussed above, the ring $\mathrm{End}(\mathcal{T})$ has an upper triangular presentation, with the Tits Algebras of the corresponding simply connected linear algebraic group appearing along the diagonal. Thus in each case, $\mathrm{End}(\mathcal{T})$ a nilpotent ideal I, consisting of the strictly upper triangular terms. By applying the K-theory functor (confer [25, Theorem 1.98]) to the natural equivalences found in Theorems 5.6, 6.4, and 7.6, we can express the Quillen K-theory of each variety as sum of the K-theory of the algebras appearing along the diagonal. The K-theory is not affected if we replace $\mathrm{End}(\mathcal{T})$ by $\mathrm{End}(\mathcal{T})/I$, and we recovers results found in [22, 10.2, 10.3].

Theorem 8.1. *The tilting bundles induce the following isomorphisms:*

$$K_*(\mathbf{SB}(A)) \xrightarrow{\sim} \bigoplus_{i=0}^{n-1} K_*(A^{\otimes i})$$
$$K_*(\mathbf{SB}(r,A)) \xrightarrow{\sim} \bigoplus_{a} K_*(A^{\otimes d(a)})$$
$$K_*(I(A,\sigma)) \xrightarrow{\sim} \Big(\bigoplus_{i=0}^{n-2} K_*(F) \oplus K_*(A)\Big) \bigoplus K_*(C_0(A,\sigma)).$$

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Department of Mathematics, Univ. of British Columbia, Vancouver, BC V6T 1Z2, Canada

E-mail address: `mblunk@math.ubc.ca`