ON THE TORSION OF CHOW GROUPS OF TWISTED SPIN-FLAGS

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Abstract. In the present paper we provide a uniform bound for the annihilators of the torsion of the Chow groups of the variety of Borel subgroups of a strongly inner linear algebraic group of orthogonal type.

1. Introduction

Let X be the variety of Borel subgroups of a semisimple linear algebraic group G over an arbitrary field k. One way to study the geometry of X is to study its Chow groups $\mathrm{CH}^d(X)$ of algebraic cycles modulo the rational equivalence relation. Simple transfer arguments together with the Bruhat-Tits decomposition imply that the Chow groups $\mathrm{CH}^d(X;\mathbb{Q})$ with rational coeffcients of any variety of Borel subgroups X can be identified with $\mathrm{CH}^d(G/B;\mathbb{Q})$ for some split G. Since $\mathrm{CH}^d(G/B)$ is a free Abelian group of rank equal to the number of cells of codimension d in G/B, the problem of determining $\mathrm{CH}^d(X)$ is reduced to determining its torsion part.

The latter seems to be a highly nontrivial question. Only very few partial results are known and most of them concern small codimensions ($d \leq 4$) or the dimension-zero cases: for strongly inner groups and $d = 2, 3$ we refer to [15], [4] and for $d = 4$ to [1]; for inner groups of type A and $d = 2$ see [8]; for quadrics and $d = 2, 3, 4$ see [9], [10]. In particular, in [10] it was shown that the torsion of CH^4 can be infinitely generated.

In the present paper we provide a uniform bound for the annihilator of the torsion of $\mathrm{CH}^d(X)$ for any d and strongly inner orthogonal group G. Namely, we prove the following:

1.1. **Theorem.** *Let G be a strongly inner group of an orthogonal type of rank n over a field of characteristic different from 2. Let X be the respective variety of Borel subgroups. Then for all $2 \leq d \leq n-1$ the integer*

$$M_d = (d-1)! \prod_{i=2}^{d} (i-1)! \cdot [i/2]! \cdot 2^{i+1}$$

annihilates the torsion part of $\mathrm{CH}^d(X)$.

We would like to stress that our bound M_d doesn't depend on the rank n of G but only on the codimension d. Note also that for groups of types A and C there are no non-split strongly inner forms, so the torsion part of $\mathrm{CH}^d(X)$ is trivial. Hence, the only interesting case to consider (of groups that form an infinite series) is the orthogonal case.

The only estimate for the annihilator of the torsion of $\mathrm{CH}^d(X)$ known in the literature is given by the torsion index of G (see [5]). For strongly inner orthogonal groups, i.e. for *Spin* groups, it was computed in [16, Thm.01] and has the property that $t_G \to \infty$ as the rank of G grows.

The paper is organized as follows. In sections 2 and 3 we prove several technical facts (Proposition 2.9 and Corollary 3.4) concerning the ideals of generalized invariants and symmetric functions. These facts are used in section 4 to relate the kernel of the characteristic map with the ideal of invariants (Propositions 4.5 and 4.7). In section 5 we extend the results of the paper [1] by providing a uniform upper bound for all the exponents of the Weyl group action (Proposition 5.6). In the last section we combine the obtained results to prove the main theorem (Theorem 6.1).

2. Divided differences and ideals of invariants

In the present section we provide several basic facts concerning the ring of symmetric polynomials over an arbitrary commutative ring and the associated invariant ideals. We refer to [7] and [11] for details.

2.1. Consider a polynomial ring $R = A[e_1, \dots, e_n]$ over a commutative ring A. The symmetric group S_n acts on R by permutations of variables $\{e_1, \dots, e_n\}$. The subring of invariants R^{S_n} is a polynomial ring in elementary symmetric functions [17, Thm. 1, 2]

$$s_1 = e_1 + \ldots + e_n,\ s_2 = \sum_{i<j} e_i e_j,\ s_3 = \sum_{i<j<k} e_i e_j e_k,\ \ldots, s_n = \prod_{i=1}^{n} e_i.$$

Let $J = (s_1, s_2, \dots, s_n)$ denote the ideal of R generated by symmetric functions. We denote by $\epsilon \colon R \to A$ the augmentation map $e_i \mapsto 0$. Observe that ϵ restricts to $\epsilon \colon R^{S_n} \to A$.

2.2. Following [7, §0] consider divided difference operators Δ_σ, $\sigma \in S_n$. Each of them is an A-linear operator $\Delta_\sigma \colon R^{(m)} \to R^{(m-l(\sigma))}$ decreasing the degree m of a homogeneous polynomial in $e_1, \dots, e_n$ by the length $l(\sigma)$ of permutation σ. It is defined as follows:

We set $\Delta_1 = \mathrm{id}$. If $m < l(\sigma)$, then we set $\Delta_\sigma = 0$. For a (non-trivial) transposition $\tau = (ij)$, we set $\Delta_\tau(f) = (f - f^\tau)/(e_i - e_j)$ for $f \in R^{(m)}$, $m \geq 1$. If σ is a product of transpositions, we define Δ_σ to be the composite of the respective Δ_τ. This doesn't depend on the choice of a reduced decomposition of σ. Observe that if s is a symmetric function, then

$$\Delta_\sigma(s \cdot f) = s \cdot \Delta_\sigma(f).$$

By definition we have

$$R^{S_n} = \{f \in R \mid \Delta_\tau(f) = 0 \text{ for all non-trivial transpositions } \tau\}.$$

2.3. **Definition.** Following [7, p.239] we define the ideal of *generalized invariants* I as

$$I = \{f \in R \mid \Delta_\sigma(f) = 0 \quad \forall \sigma \in S_n \text{ with } l(\sigma) = \deg(f)\}.$$

2.4. **Lemma.** *We have $J \subseteq I$.*

Proof. Follows from the fact that $\Delta_\sigma(s \cdot f) = s \cdot \Delta_\sigma(f) = 0$ for any $s \cdot f \in J^{(m)}$ and $l(\sigma) = m$. □

2.5. **Definition.** Following [11, §1] we define the ideal of *stable invariants* J_∞ inductively as:

Set $R_1 := R$ and $R_{m+1} := R_m \otimes_{R_m^{S_n}} A$ for $m \geq 1$, where A is the $R_m^{S_n}$-module via the augmentation ϵ. Observe that S_n acts on R_{m+1} via the action on R_m and there is a canonical S_n-equivariant surjection $R_m \to R_{m+1}$. Let J_m denote the kernel of the composite $R_1 \to \ldots \to R_m \to R_{m+1}$, i.e. $R_{m+1} = R/J_m$. We then set

$$J_\infty := \bigcup_{m \geq 1} J_m.$$

2.6. **Remark.** The ideal J_m can be also defined inductively as follows:

$$J_m = \begin{cases} (0) & \text{if } m = 0, \\ \{f \in R \mid \Delta_\sigma(f) \in J_{m-1},\ \forall \sigma \in S_n\} & \text{if } m \geq 1. \end{cases}$$

2.7. **Lemma.** *(cf.* [11, Lemma 3.2]*) We have $I \subseteq J_\infty$.*

Proof. We show by induction on m that $I^{(m)} \subseteq J_m$.

If $m = 1$ and $f \in I^{(1)}$, then $\Delta_\tau(f) = 0$ for all $\tau \neq 1$, implies that $f \in R^{S_n} \cap \ker \epsilon$. Therefore, $f \in J_1$.

Suppose that $I^{(m)} \subseteq J_m$. For $m+1$ and $f \in I^{(m+1)}$, if $\Delta_\sigma(f) = 0$ for any reduced decomposition $\sigma = \tau_1\tau_2 \ldots \tau_{m+1}$, then $\Delta_{\tau_1 \ldots \tau_m}\Delta_{\tau_{m+1}}(f) = 0$. So by induction $\Delta_{\tau_{m+1}}(f) \in I^{(m)} \subseteq J_m$ and, therefore, $\Delta_\tau(f) \in J_m$ for all $\tau \neq 1$. By the remark above, $f \in J_{m+1}$. □

The ideal J_∞ is universal in the following sense

2.8. **Lemma.** *(cf.* [11, Lemma 2.1]*) Let $J' \subset R$ be an S_n-stable ideal with $\epsilon(J') = 0$. If $(R/J')^{S_n} = A$, then $J_\infty \subseteq J'$.*

Proof. We prove $J_m \subset J'$ by induction on m. If $m = 1$, since $(R/J')^{S_n} = A$, the compositions $R^{S_n} \hookrightarrow R \to R/J'$ and $R^{S_n} \xrightarrow{\epsilon} A \hookrightarrow R/J'$ coincide, hence, there is a map $R/J_1 = R \otimes_{R^{S_n}} A \to R/J'$, which shows that $J_1 \subset J'$.

Now assume that $J_m \subset J'$. Repeating the above arguments after replacing R (resp. J') by $R_m = R/J_m$ (resp. by the ideal $J'_m = (J')$ in R_m), we see that $J_{m+1} \subset J$. This finishes the proof. □

2.9. **Proposition.** *We have $J = I = J_\infty$ in the polynomial ring $R = A[e_1, \ldots, e_n]$.*

Proof. Since $R^{S_n} = A \otimes \mathbb{Z}[e_1, \ldots, e_n]^{S_n}$, $(R/J)^{S_n} = A$. Therefore, by Lemma 2.8, $J_\infty \subseteq J$. The proposition then follows by combining Lemmas 2.4 and 2.7. □

3. Elementary symmetric functions and power sums

In the present section we prove several technical lemmas which will be used in the subsequent section.

Let $\alpha = (\alpha_1, ..., \alpha_m)$ denote a partition with $1 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_m$ of an integer $|\alpha| = \alpha_1 + \ldots + \alpha_m$. We set $|(0)| = 0$. Let $s_\alpha = \prod_{i=1}^m s_{\alpha_i}$ denote the product of respective elementary symmetric functions (here $s_0 = 1$).

3.1. **Lemma.** *Consider a homogeneous polynomial of degree $d \leq n$ with integer coefficients*

$$P_m = \sum_{\{\alpha \mid d-m \leq |\alpha| \leq d\}} f^\alpha s_\alpha, \quad \text{where } 0 \leq m < d \text{ and } \deg(f^\alpha) = d - |\alpha|.$$

If $M \mid P_m$ for some positive integer M, then for each α there exists a homogeneous polynomial v^α of degree $d-|\alpha|$ such that

$$\sum_{\{\alpha \mid d-m\le|\alpha|\le d\}} v^\alpha s_\alpha = 0 \quad \text{and} \quad M \mid (f^\alpha + v^\alpha).$$

Proof. We proceed by induction on $m \ge 0$.

If $m=0$, then $P_0 = \sum_{|\alpha|=d} f^\alpha s_\alpha$, $f^\alpha \in \mathbb{Z}$. If $M \mid P_0$, then $M \mid f^\alpha$ for each α as $\{s_\alpha\}_{|\alpha|=d}$ are linearly independent in $A[e_1,\ldots,e_n]$ for $A=\mathbb{Z}/M\mathbb{Z}$. So we can take $v^\alpha = 0$ for each α.

Assume it is true for $m-1$, $m\ge 1$. Now we prove it for m. We use β to denote those partitions α with $|\alpha| = d-m$. Observe that $\deg(f^\beta)=m$. Apply to P_m a divided difference operator Δ of length m, we obtain:

$$M \mid \Delta(P_m) = \sum_\beta \Delta(f^\beta)s_\beta, \quad \text{where } \Delta(f^\beta)\in\mathbb{Z}.$$

Similar to $m=0$ case this implies that $M \mid \Delta(f^\beta)$ for every Δ of length m. By Definition 2.3 this means that $f^\beta \in I$, where I is the ideal of generalized invariants for $A=\mathbb{Z}/M\mathbb{Z}$. By Proposition 2.9 we have $J=I$ and, therefore,

$$f^\beta \equiv \sum_{j=1}^m g^\beta_{m-j}s_j \mod M \text{ for some polynomials } g^\beta_{m-j} \text{ of degree } m-j.$$

Plugging it into the original expression for P_m, we obtain

$$P_m \equiv \sum_{\{\alpha|d-m+1\le|\alpha|\le d\}} (f^\alpha + g^{\beta(\alpha)}_{d-|\alpha|})s_\alpha \equiv 0 \mod M,$$

where $\beta(\alpha)$ is the unique β such that $s_\alpha = s_{|\alpha|-|\beta(\alpha)|}s_{\beta(\alpha)}$. By induction, for each α such that $d-m+1\le|\alpha|$ there exists a polynomial v^α such that

$$\sum_{\{\alpha|d-m+1\le|\alpha|\le d\}} v^\alpha s_\alpha = 0 \quad\text{and}\quad M \mid (f^\alpha + g^{\beta(\alpha)}_{d-|\alpha|} + v^\alpha).$$

Now we set $\tilde v^\beta = -\sum_{j=1}^m g^\beta_{m-j}s_j$ and $\tilde v^\alpha = g^{\beta(\alpha)}_{d-|\alpha|} + v^\alpha$ for $|\alpha|\ge d-m+1$. Then $\sum_{d-m\le|\alpha|\le d}\tilde v^\alpha s_\alpha = 0$. So these $\tilde v^\alpha$ satisfy the condition of the lemma. This finishes the proof. ☐

3.2. Let $q_i = e_1^i + e_2^i + \ldots + e_n^i$, $i\ge 1$ denote the power sum symmetric function. Given a partition $\alpha=(\alpha_1,\ldots,\alpha_m)$, let $q_\alpha = \prod_{i=1}^m q_{\alpha_i}$ denote the product of respective power sum functions.

According to [12, Ch I. 2.11] the elementary symmetric function s_i can be written in terms of q_j, $j\le m$ as $s_i = \frac{1}{i!}\sum_{|\alpha|=i} a_\alpha q_\alpha$, $a_\alpha\in\mathbb{Z}$. Since $i! = \max_{\sum i_j = i}\{i_j!\}$, we have

$$s_\alpha = \frac{1}{|\alpha|!}\sum_{\{\beta,\ |\beta|=|\alpha|\}} a_{\alpha,\beta}q_\beta \quad \text{for some } a_{\alpha,\beta}\in\mathbb{Z}.$$

Multiplying by the respective denominators we obtain the following version of Lemma 3.1 for power sum functions:

3.3. **Corollary.** *Assume that $1\le d\le n$ and $d! \mid M$. Consider a homogeneous polynomial $P=\sum_{i=1}^d f_{d-i}q_i$ of degree d with integer coefficients ($\deg(f_{d-i}) = d-i$).*

If $M \mid P$, then there exist $\hat f_{d-i}$, $i = 1 \dots d$ such that

$$\sum_{i=1}^{d} \hat f_{d-i} q_i = P \quad \textit{and} \quad \tfrac{M}{d!} \mid \hat f_{d-i}.$$

Proof. Let $q_i = \sum_{|\alpha|=i} c_\alpha s_\alpha$, $c_\alpha \in \mathbb{Z}$, so we get $M \mid P = \sum_{|\alpha|\geq 1} c_\alpha f_{d-|\alpha|} s_\alpha$. By Lemma 3.1 there exist v^α with $P = \sum (c_\alpha f_{d-|\alpha|} + v^\alpha) s_\alpha$ and $(c_\alpha f_{d-|\alpha|} + v^\alpha)$ all divisible by M. Expressing s_α in terms of q_β using the formula from 3.2 we obtain

$$P = \tfrac{1}{d!} \sum_\alpha (c_\alpha f_{d-|\alpha|} + v^\alpha) \sum_\beta a_{\alpha,\beta} q_\beta = \tfrac{1}{d!} \sum_{i=1}^{d} \tilde f_{d-i} q_i \text{ for some } \tilde f_{d-i}.$$

Now all these $\tilde f_{d-i}$ are integral polynomials in $c_\alpha f_{d-|\alpha|} + v^\alpha$ and, hence, are divisible by M. Therefore, $\hat f_{d-i} \overset{def}{=} \frac{1}{d!} \tilde f_{d-i}$ is divisible by $\frac{M}{d!}$. □

Restricting to power sums of even degree only we obtain

3.4. **Corollary.** *Assume that $1 \leq d \leq 2n + 1$ and let $[d/2]! \mid M$. Consider a homogeneous polynomial $P = \sum_{i=1}^{[d/2]} f_{d-2i} q_{2i}$ of degree d with integer coefficients.*

If $M \mid P$, then there exist $\hat f_{d-2i}$, $i = 1, \dots, [d/2]$ such that

$$\sum_{i=1}^{[d/2]} \hat f_{d-2i} q_{2i} = P \quad \textit{and} \quad \tfrac{M}{[d/2]!} \mid \hat f_{d-2i}.$$

Proof. For each i we express f_{d-2i} as a linear combination

$$f_{d-2i} = \sum_\delta e^\delta f^\delta_{d-2i},$$

where $\delta = (\delta_1, \dots, \delta_n)$ with $\delta_i = 0, 1$, $e^\delta = \prod_{i=1}^n e_i^{\delta_i}$ and f^δ_{d-2i} is a linear combination of even monomials $e_1^{2i_1} e_2^{2i_2} \dots e_n^{2i_n}$ only. Denote $|\delta| = \sum \delta_j$. Collecting the terms with e^δ we obtain

$$P = \sum_\delta e^\delta \sum_{i=1}^{[d/2]} f^\delta_{d-2i} q_{2i} \equiv 0 \mod M.$$

This implies that $M \mid \sum_{i=1}^{[d/2]} f^\delta_{d-2i} q_{2i}$ for each δ. We apply Corollary 3.3 to the polynomial $P_\delta = \sum_{i=1}^{[d/2]} f^\delta_{d-2i} q_{2i}$ viewed as a polynomial in variables e_j^2 of degree $d_\delta = \frac{d-|\delta|}{2}$. We obtain polynomials $\hat f^\delta_{d-2i}$ divisible by $\frac{M}{d_\delta !}$ such that $\sum_{i=1}^{[d/2]} f^\delta_{d-2i} q_{2i} = \sum_{i=1}^{[d/2]} \hat f^\delta_{d-2i} q_{2i}$.

We then set

$$\hat f_{d-2i} = \sum_\delta e^\delta \hat f^\delta_{d-2i}.$$

Since $d_\delta ! \mid [d/2]!$ for all δ, the proof is finished. □

4. Invariants and characteristic map

In the present section we investigate the relationships between the kernel of the characteristic map and the ideal of invariants. We refer to [2] for basic definitions and details.

4.1**.** Consider a crystallographic root system of Dynkin type $\mathfrak{D}$ with the weight lattice Λ. Let G be the associated split simple simply-connected linear algebraic group with a maximal torus T and a Borel subgroup $B \supset T$. Observe that Λ can be identified with the group of characters of T with a basis given by the fundamental weights $\omega_1, \ldots, \omega_n$.

Consider the variety G/B of Borel subgroups of G (conjugate to B). To every character $\lambda \in \Lambda$ we may associate the line bundle $\mathcal{L}(\lambda)$ over G/B. It induces the ring homomorphism from the symmetric algebra $S^*(\Lambda)$ to the Chow ring of G/B called the characteristic map [2, §8]

$$c_a \colon S^*(\Lambda) \to \mathrm{CH}^*(G/B), \quad \lambda \mapsto c_1(\mathcal{L}(\lambda)).$$

4.2**.** The Weyl group W of G acts on the weight lattice Λ by means of simple reflections and, hence, on $S^*(\Lambda)$. Let I_a^W denote the ideal generated by non-constant W-invariants. According to [2, §4 Cor. 2] the kernel of the characteristic map $\ker c_a$ is generated by elements of $S^*(\Lambda)$ such that their multiples are in I_a^W. Moreover, for each homogeneous degree d there exists an integer b_d such that the prime decomposition of b_d consists only of torsion primes of G and

$$b_d \cdot (\ker c_a)^{(d)} \subseteq (I_a^W)^{(d)}.$$

The ideal $(\ker c_a) \otimes \mathbb{Z}[\frac{1}{t_G}] = I_a^W \otimes \mathbb{Z}[\frac{1}{t_G}]$, where t_G is the torsion index of G, is freely generated by the basic polynomial invariants which are homogeneous polynomials in fundamental weights.

The purpose of the present section is to get an upper bound for the integers b_d.

4.3. **Example.** For the type $\mathfrak{D} = \mathrm{A}_n$ there are no torsion primes (the torsion index is 1). This implies that $b_d = 1$ for all d and the characteristic map c_a is surjective with the kernel $\ker c_a = I_a^W$.

4.4. **Example.** According to [6] the basic polynomial invariants for the root system of type $\mathfrak{D} = \mathrm{B}_n$ are given by even power sums

$$q_{2i} = \sum_{j=1}^{n} e_j^{2i}, \quad 1 \leq i \leq n,$$

where $e_1 = \omega_1$, $e_j = \omega_j - \omega_{j-1}$ for $2 \leq j < n$ and $e_n = 2\omega_n - \omega_{n-1}$. Therefore,

$$q_{2i} = \omega_1^{2i} + (\omega_2 - \omega_1)^{2i} + ... + (\omega_{n-1} - \omega_{n-2})^{2i} + (2\omega_n - \omega_{n-1})^{2i}.$$

The basic polynomial invariants for type D_n are given by q_{2i}, $1 \leq i \leq n-1$ and $p_n = e_1 e_2 \ldots e_n$, where $e_1 = \omega_1$, $e_j = \omega_j - \omega_{j-1}$ for $2 \leq j \leq n-1$ and $e_{n-1} = \omega_n - \omega_{n-1}, e_n = \omega_n + \omega_{n-1} - \omega_{n-2}$.

From now on we assume that $\mathfrak{D} = \mathrm{B}_n$. In this case the torsion index of G is a power of 2.

4.5. **Proposition.** *Let* $\mathfrak{D} = \mathrm{B}_n$ *and let* $1 \leq d \leq 2n+1$. *Then* $2^d[d/2]! \cdot (\ker c_a)^{(d)} \subset I_a^W$, *i.e.* $b_d \mid 2^d[d/2]!$.

Proof. Since $(\ker c_a) \otimes \mathbb{Z}[\frac{1}{2}] = I_a^W \otimes \mathbb{Z}[\frac{1}{2}]$ is generated by q_{2i}, $i = 1, ..., n$, given a polynomial $f \in (\ker c_a)^{(d)}$, we can write it as

$$2^r f = \sum_{i=1}^{[d/2]} f_{d-2i} q_{2i} \in I_a^W, \;\; \text{for some } f_{d-2i} \in \mathbb{Z}[\omega_1, ..., \omega_n] \text{ and } r \geq 0.$$

Suppose r is the smallest such integer. To finish the proof it suffices to show that $r \le v_2([d/2]!) + d$, where v_2 denotes the 2-adic valuation.

Assume the contrary, i.e. that $r \ge v_2([d/2]!) + d + 1$. Expressing ω_j's in terms of e_j's, we obtain $f = \frac{1}{2^d}\tilde{f}$ and $f_{d-2i} = \frac{1}{2^{d-2i}}\tilde{f}_{d-2i}$ for some $\tilde{f}, \tilde{f}_{d-2i} \in \mathbb{Z}[e_1, ..., e_n]$. So that

$$M = 2^{d+1}[d/2]! \mid 2^r s \cdot \tilde{f} = \sum_{i=1}^{[d/2]} (2^{2i} s \tilde{f}_{d-2i}) \cdot q_{2i}, \text{ where } s = \tfrac{[d/2]!}{2^{v_2([d/2]!)}}.$$

By Corollary 3.4, there exists $\tilde{h}_{d-2i} \in \mathbb{Z}[e_1, ..., e_n]$ divisible by $\frac{M}{[d/2]!} = 2^{d+1}$ such that $2^r s \cdot \tilde{f} = \sum_{i=1}^{[d/2]} \tilde{h}_{d-2i} q_{2i}$. Expressing e_j's in terms of ω_j's back, we obtain $2^d 2^r s \cdot f = \sum_{i=1}^{[d/2]} \tilde{h}_{d-2i} q_{2i}$, which implies

$$2^{r-1} f = \sum_{i=1}^{[d/2]} \left(\tfrac{1}{2^{d+1}} \tilde{h}_{d-2i} - \tfrac{s-1}{2} f_{d-2i} \right) \cdot q_{2i}.$$

Since $\tilde{h}_{d-2i}$ are divisible by 2^{d+1}, we have $\frac{1}{2^{d+1}}\tilde{h}_{d-2i} \in \mathbb{Z}[\omega_1, ..., \omega_n]$. This contradicts to the minimality assumption on r. □

Assume now that $\mathfrak{D} = \mathrm{D}_n$. In this case we have an additional basic polynomial invarant p_n in degree n, however, this doesn't change the situation much in view of the following slight modifications of Corollaries 3.3 and 3.4:

4.6. **Lemma.** *Assume $d \ge n$ and $n! \mid M$. If $M \mid \sum_{i=1}^n f_{d-i} q_i + g_{d-n} p_n$, then there exists v_{d-i} and u_{d-n} such that $\sum_{i=1}^n v_{d-i} q_i + u_{d-n} p_n = 0$ and $\frac{M}{n!} \mid$ g.c.d.$(f_{d-i} + v_{d-i}, g_{d-n} + u_{d-n})$.*

If $M \mid \sum_{i=1}^n f_{d-2i} q_{2i} + g_{d-n} p_n$, then we can find v_{d-2i}, u_{d-n} such that $\sum_{i=1}^n v_{d-2i} q_{2i} + u_{d-n} p_n = 0$ and $\frac{M}{n!} \mid$ g.c.d.$(f_{d-2i} + v_{d-2i}, g_{d-n} + u_{d-n})$.

Proof. The proof of the first statement follows by the same arguments as the proof of Corollary 3.3. As for the second, multiplying by p_n we obtain

$$M \mid \sum_{i=1}^n \tilde{f}_{d+n-2i} q_{2i} + g_{d-n} p_n^2.$$

Following the proof of 3.4 we can rewrite this as

$$M \mid \sum_{\delta} e^{\delta} (\sum_{i=1}^n \tilde{f}^{\delta}_{d+n-2i} q_{2i} + g^{\delta}_{d-n} p_n^2)$$

and reduce it to the first statement (replacing $e'_j = e_j^2$). □

4.7. **Proposition.** *Let $\mathfrak{D} = D_n$. If $d < n$, then $2^d[d/2]! \cdot (\ker c_a)^{(d)} \subset I_a^W$. If $d \ge n$, then*

$$2^d n! \cdot (\ker c_a)^{(d)} \subset I_a^W.$$

Proof. If $d < n$, then it is similar to the B_n-case. If $d \ge n$, consider the equation

$$2^r f = \sum_{i=1}^n f_{d-2i} q_{2i} + g_{d-n} p_n \in I_a^W.$$

Following the proof of Proposition 4.5 and using Lemma 4.6 we show that the smallest 2^r satisfying the equation will divide $2^d n!$. □

5. Exponents of types B_n and D_n

Following the notation of [1] let $I_m := \ker(\mathbb{Z}[\Lambda] \to \mathbb{Z})$ and $I_a := \ker(S^*(\Lambda) \to \mathbb{Z})$ be the augmentation ideals, where $\mathbb{Z}[\Lambda] \to \mathbb{Z}$ (respectively, $S^*(\Lambda) \to \mathbb{Z}$) is the map sending e^λ to 1 (respectively, any element of positive degree to 0). For any $d \geq 0$, we consider the ring homomorphism

$$\phi^{(d)} : \mathbb{Z}[\Lambda] \to \mathbb{Z}[\Lambda]/I_m^{d+1} \to S^*(\Lambda)/I_a^{d+1} \to S^d(\Lambda),$$

where the first and the last maps are projections and the middle map sends $e^{\sum_{i=1}^n a_i\omega_i}$ to $\prod_{i=1}^n (1-\omega_i)^{-a_i}$. The *dth-exponent* of a root system (denoted by τ_d), as introduced in [1], is the gcd of all nonnegative integers N_d satisfying

$$N_d \cdot (I_a^W)^{(d)} \subseteq \phi^{(d)}(I_m^W),$$

where $I_m^W := \langle \mathbb{Z}[\Lambda]^W \cap I_m\rangle$ (respectively, $I_a^W := \langle S^*(\Lambda)^W \cap I_a\rangle$) denotes the W-invariant augmentation ideal of $\mathbb{Z}[\Lambda]$ (respectively, $S^*(\Lambda)$). For any $d \leq 4$, it was shown that the dth-exponent divides the Dynkin index in [1].

In this section we show that all the remaining exponents of types B_n and D_n divide the Dynkin index 2.

5.1. For any $\lambda \in \Lambda$, we denote by $W(\lambda)$ the W-orbit of λ. For any finite set S of weights, we denote $-S$ the set of opposite weights. By the action of Weyl groups of types B_n and D_n, one has the following decomposition of W-orbits: if $\mathfrak{D} = \mathrm{B}_n$ (respectively, $\mathfrak{D} = \mathrm{D}_n$), then for any $1 \leq k \leq n-1$ (respectively, $1 \leq k \leq n-2$)

$$W(\omega_k) = W_+(\omega_k) \cup -W_+(\omega_k), \tag{5.0.1}$$

where $W_+(\omega_k) = \{e_{i_1} \pm \cdots \pm e_{i_k}\}_{i_1 < \cdots < i_k}$. If n is even, then the W-orbits of the last two fundamental weights of D_n are given by

$$W(\omega_{n-1}) = W_+(\omega_{n-1}) \cup -W_+(\omega_{n-1}) \text{ and } W(\omega_n) = W_+(\omega_n) \cup -W_+(\omega_n), \tag{5.0.2}$$

where $W_+(\omega_{n-1})$ (respectively, $W_+(\omega_n)$) is the subset of $W(\omega_{n-1})$ (respectively, $W(\omega_n)$) containing elements of the positive sign of e_1.

For any $\lambda = \sum_{i=1}^n a_i\omega_i \in \Lambda$ and any integer $m \geq 0$, we set $\lambda(m) = \sum_{i=1}^n a_i\omega_i^m$. We shall need the following lemma:

5.2. **Lemma.** *Let p be a positive integer and $m_1, \cdots, m_p$ nonnegative integers.*

(i) If $\mathfrak{D} = \mathrm{B}_n$ (respectively, D_n), then for odd p and any $1 \leq k \leq n-1$ (respectively, any $1 \leq k \leq n-2$), we have

$$\sum_{\lambda \in W(\omega_k)} \lambda(m_1)\cdots\lambda(m_p) = 0.$$

(ii) Let $\mathfrak{D} = \mathrm{D}_n$. Then we have

$$\sum_{\lambda \in W(\omega_n)} \lambda(m_1)\cdots\lambda(m_p) = \sum_{\lambda \in W(\omega_{n-1})} \lambda(m_1)\cdots\lambda(m_p) \text{ for odd } n \text{ and even } p,$$

$$\sum_{\lambda \in W(\omega_n)} \lambda(m_1)\cdots\lambda(m_p) = \sum_{\lambda \in W(\omega_{n-1})} \lambda(m_1)\cdots\lambda(m_p) = 0 \text{ for odd } p < n.$$

Proof. *(i)* It follows from (5.0.1) that

$$\sum_{\lambda \in W(\omega_k)} \lambda(m_1)\cdots\lambda(m_p) = \sum_{\lambda \in W_+(\omega_k)} \lambda(m_1)\cdots\lambda(m_p) + \sum_{\lambda \in -W_+(\omega_k)} \lambda(m_1)\cdots\lambda(m_p)$$

$$= \sum_{\lambda \in W_+(\omega_k)} \lambda(m_1)\cdots\lambda(m_p) - \sum_{\lambda \in W_+(\omega_k)} \lambda(m_1)\cdots\lambda(m_p) = 0.$$

(ii) If n is odd, then we have $W(\omega_n) = -W(\omega_{n-1})$. Hence, the result immediately follows from the assumption that p is even. If n is even, then the result follows from (5.0.2) by the same argument as in the proof of (i). □

Let p be an even integer and $q \geq 2$ an integer. For any nonnegative integers $m_1, \cdots, m_p$, we define

$$\Lambda(p,q)(m_1,\cdots,m_p) := \sum \lambda_{j_1}(m_1)\cdots\lambda_{j_p}(m_p),$$

where the sum ranges over all different $\lambda_{i_1}, \cdots, \lambda_{i_q} \in W_+(\omega_1)$ and all $\lambda_{i_1}, \cdots, \lambda_{i_p} \in \{\lambda_{i_1}, \cdots, \lambda_{i_q}\}$ such that the numbers of $\lambda_{i_1}, \cdots, \lambda_{i_q}$ appearing in $\lambda_{i_1}, \cdots, \lambda_{i_p}$ are all nonnegative even solutions of $x_1 + \cdots + x_q = p$. If $p < 2q$, then we set $\Lambda(p,q)(m_1,\cdots,m_p) = 0$. We simply write $\Lambda(p,q)$ for $\Lambda(p,q)(m_1,\cdots,m_p)$. For instance, $\Lambda(4,2)$ is the sum of $\lambda_{j_1}(m_1)\lambda_{j_2}(m_2)\lambda_{j_3}(m_3)\lambda_{j_4}(m_4)$ for all $j_1, j_2, j_3, j_4 \in \{i, j\}$ and all $1 \leq i \neq j \leq n$ such that two i's and two j's appear in j_1, j_2, j_3, j_4.

5.3. **Lemma.** *If $\mathfrak{D} = \mathrm{B}_n$ (respectively, D_n), then for any $2 \leq k \leq n-1$ (respectively, $2 \leq k \leq n-2$), any even p, and any nonnegative integers $m_1, \cdots m_p$ we have*

$$\sum_{W(\omega_k)} \lambda(m_1)\cdots\lambda(m_p) = 2^{k-1}\binom{n-1}{k-1}\sum_{W(\omega_1)} \lambda(m_1)\cdots\lambda(m_p) + \sum_{j=2}^{k} 2^k\binom{n-j}{k-j}\Lambda(p,j).$$

Proof. Let L be the LHS of the above equation. For any $\lambda \in W(\omega_1)$, there are $2^k\binom{n-1}{k-1}$ choices of the element containing λ in $W(\omega_k)$, thus we have the term $2^{k-1}\binom{n-1}{k-1}\sum_{W(\omega_1)}\lambda(m_1)\cdots\lambda(m_p)$ in L.

If an element $\lambda \in W(\omega_1)$ appears odd times in a term $\lambda_{i_1}(m_1)\cdots\lambda_{i_p}(m_p)$ of L, where $\lambda_{i_1}, \cdots, \lambda_{i_p} \in W(\omega_1)$, then by the action of Weyl group this term vanishes in L. Hence, the remaining terms in L are linear combinations of $\Lambda(p,j)$ for all $2 \leq j \leq k$ such that $p \geq 2k$. As each term $\Lambda(p,j)$ appears $2^k\binom{n-j}{k-j}$ times in $\sum_{W(\omega_k)}\lambda(m_1)\cdots\lambda(m_p)$, the result follows. □

For any $\lambda \in \Lambda$, we denote by $\rho(\lambda)$ the sum of all elements $e^{\mu} \in \mathbb{Z}[\Lambda]$ over all elements μ of $W(\lambda)$. By the recursive formulas in [1, Section 1], we can let $d! \cdot \phi^{(d)}(e^{\lambda}) = \lambda^d + Q_d$ for any $d \geq 1$, where Q_d is the sum of remaining terms in $d! \cdot \phi^{(d)}(e^{\lambda})$. Hence, for any fundamental weight ω_k we have

$$d! \cdot \phi^{(d)}(\rho(\omega_k)) = \sum_{W(\omega_k)} \lambda^d + \sum_{W(\omega_k)} Q_d. \tag{5.0.3}$$

We view $d! \cdot \phi^{(d)}(e^{\lambda})$ as a polynomial in variables $\lambda, \lambda(m_1), \cdots, \lambda(m_j)$ for some nonnegative integers $m_1, \cdots, m_j$. Let T_d be the sum of monomials in Q_d whose degrees are even.

If $\mathfrak{D} = \mathrm{B}_n$ (respectively, D_n), then by Lemma 5.2(i) the equation (5.0.3) reduces to

$$d! \cdot \phi^{(d)}(\rho(\omega_k)) = \sum_{W(\omega_k)} \lambda^d + \sum_{W(\omega_k)} T_d \tag{5.0.4}$$

for any $1 \leq k \leq n-1$ (respectively $1 \leq k \leq n-2$).

Given p and q, we define $\Omega(p,q) := \sum \Lambda(p,q)(m_1,\cdots,m_p)$, where the sum ranges over all $m_1, \cdots, m_p$ which appear in all monomials of T_d.

5.4. **Example.** If $\mathfrak{D} = \mathrm{B}_n$ $(n \geq 4)$ or D_n $(n \geq 5)$ and $d = 6$, then by (5.0.4) and Lemma 5.3 we have

$$6!\phi^{(6)}(\rho(\omega_1)) = \sum_{W(\omega_1)} \lambda^6 + \sum_{W(\omega_1)} T_6,$$
$$6!\phi^{(6)}(\rho(\omega_2)) = \sum_{W(\omega_2)} \lambda^6 + 2(n-1)\sum_{W(\omega_1)} T_6 + 4\Omega(4,2),$$
$$6!\phi^{(6)}(\rho(\omega_3)) = \sum_{W(\omega_3)} \lambda^6 + 4\binom{n-1}{2}\sum_{W(\omega_1)} T_6 + 8(n-2)\Omega(4,2),$$

which implies that

$$\phi^{(6)}(\rho(\omega_3)) - 2(n-2)\phi^{(6)}(\rho(\omega_2)) + 2(n-1)(n-2)\phi^{(6)}(\rho(\omega_1)) = \sum_{i<j<k} e_i^2 e_j^2 e_k^2.$$

5.5. **Lemma.** *Let $\mathfrak{D} = \mathrm{D}_n$ and let $1 \leq p \leq n-1$ and $m_1, \cdots, m_p$ be nonnegative integers. Then we have*

$$\sum_{W(\omega_n)} \lambda^n - \sum_{W(\omega_{n-1})} \lambda^n = n! e_1 \cdots e_n \text{ and}$$
$$\sum_{W(\omega_n)} \lambda(m_1)\cdots\lambda(m_p) = \sum_{W(\omega_{n-1})} \lambda(m_1)\cdots\lambda(m_p).$$

Proof. Let L be the LHS of the upper equation. First, assume that $n \geq 4$ is even. We show that

$$\sum_{W_+(\omega_n)} \lambda^n - \sum_{W_+(\omega_{n-1})} \lambda^n = (n!/2)e_1 \cdots e_n.$$

As $|W_+(\omega_n)| = |W_+(\omega_{n-1})| = 2^{n-2}$, we have

$$(n!/2^n)2^{n-2}e_1 \cdots e_n - (-(n!/2^n)2^{n-2}e_1 \cdots e_n) = (n!/2)e_1 \cdots e_n$$

in L. If one of the exponents $i_1, \cdots, i_n$ in $e_1^{i_1} \cdots e_n^{i_n}$ (except the case $i_1 = \cdots = i_n = 1$) is odd, then from the orbits $W_+(\omega_n)$ and $W_+(\omega_{n-1})$ this monomial vanishes in each sum of $\sum_{W_+(\omega_n)} \lambda^n - \sum_{W_+(\omega_{n-1})} \lambda^n$. Otherwise, the terms $2^{n-2}\sum_{j=1}^n e_j^n$, $\Lambda(n,2)\cdots, \Lambda(n, n/2)$ with $m_1 = \cdots = m_n = 1$ are in both $\sum_{W_+(\omega_n)} \lambda^n$ and $\sum_{W_+(\omega_{n-1})} \lambda^n$.

Now, we assume that $n \geq 4$ is odd. As $|W(\omega_n)| = |W(\omega_{n-1})| = 2^{n-1}$, we have

$$(n!/2^n)2^{n-1}e_1 \cdots e_n - (-(n!/2^n)2^{n-1}e_1 \cdots e_n) = n!e_1 \cdots e_n$$

in $\sum_{W(\omega_n)} \lambda^n - \sum_{W(\omega_{n-1})} \lambda^n$. By the same argument, if one of the exponents $i_1, \cdots, i_n$ in $e_1^{i_1} \cdots e_n^{i_n}$ (except the case $i_1 = \cdots = i_n = 1$) is odd, then this monomial vanishes in each sum of L. This completes the proof of the first equation.

By Lemma 5.2(ii), it is enough to consider the case where both n and p are even. Let $A = \sum_{W_+(\omega_n)} \lambda(m_1)\cdots\lambda(m_p)$ and $B = \sum_{W_+(\omega_{n-1})} \lambda(m_1)\cdots\lambda(m_p)$. For any p and any $n \geq p+2$, we have $C := 2^{n-2}(\sum_{W_+(\omega_1)} \lambda(m_1)\cdots\lambda(m_p))$ in both A and B. By the action of Weyl group, any term $\lambda_{i_1}(m_1)\cdots\lambda_{i_p}(m_p)$ with $\lambda_{i_j} \in W(\omega_1)$ appearing odd times in either $A-C$ or $B-C$ vanishes. As each term of $\Lambda(p,2), \cdots, \Lambda(p, p/2)$ appears in both A and B, this completes the proof. $\square$

5.6. **Proposition.** *If $\mathfrak{D} = \mathrm{B}_n$ (respectively, D_n), then for any $d \geq 3$ and any $n \geq [d/2]+1$ (respectively, $n \geq [d/2]+2$) the exponent τ_d divides the Dynkin index $\tau_2 = 2$.*

Proof. As $\mathrm{B}_2 = \mathrm{C}_2$ and $\mathrm{D}_3 = \mathrm{A}_3$, we have $1 = \tau_3 \mid 2$ by [1, Theorem 5.4]. If $\mathfrak{D} = \mathrm{D}_n$ for any $n \geq 4$, then by Lemma 5.5 we have

$$p_n = \phi^{(n)}(\rho(\omega_n)) - \phi^{(n)}(\rho(\omega_{n-1})),$$

which implies that the invariant p_n is in the ideal generated by the image of $\phi^{(n)}$. As there are no invariants of odd degree except p_n, we have

$$\tau_{2d+1} \mid \tau_{2d}$$

for all $d \geq 1$. Therefore, it suffices to show that $\tau_{2d} \mid \tau_2$ for any $d \geq 2$.

By Lemma 5.3 together with the same argument as in Example 5.4 we have

$$\phi^{(2d)}(\rho(\omega_d)) + \sum_{j=1}^{d-1} a_j \phi^{(2d)}(\rho(\omega_{d-j})) = \sum_{j_1 < \cdots < j_d} e_{j_1}^2 \cdots e_{j_d}^2, \tag{5.0.5}$$

where the integers $a_1, \cdots, a_{d-1}$ satisfy

$$\Big(\sum_{j=k}^{d-2} 2^{j+1}\binom{n-1-k}{j-k} a_{j+1}\Big) + 2^d \binom{n-1-k}{d-1-k} = 0,$$

for $0 \leq k \leq d-2$. Let r_d be the RHS of (5.0.5). Then this equation implies that r_d is in the image of $\phi^{(2d)}$.

We show that the invariant q_{2d} is in the ideal $\phi^{(2d)}(I_m^W)$ for any $d \geq 2$. We proceed by induction on d. By [1, Lemma 5.3], the case $d = 2$ is obvious. By Newton's identities we have

$$(-1)^{d-1} q_{2d} = d r_d - \sum_{j=1}^{d-1} (-1)^{j-1} r_{d-j} q_{2j}. \tag{5.0.6}$$

By the induction hypothesis, the sum of (5.0.6) is in $\phi^{(2d)}(I_m^W)$. Hence, q_{2d} is in $\phi^{(2d)}(I_m^W)$. □

6. Annihilators of torsion of twisted spin-flags

In the present section we apply Propositions 4.5, 4.7, and 5.6 to prove the main result of the paper:

6.1. **Theorem.** *Let G be a split simple simply-connected linear algebraic group of Dynkin type B_n ($n \geq 3$) or D_n ($n \geq 4$) over a field k of characteristic different from 2, i.e. a Spin group. Let $X = {}_\xi G/B$ be a twisted form of the variety of Borel subgroups of G by means of a cocycle $\xi \in Z^1(k, G)$.*

If G is of type B_n (resp. of type D_n), then for all $2 \leq d \leq 2n-1$ (resp. $2 \leq d \leq n-1$), the integer $M_d = (d-1)! \prod_{i=2}^{d} (i-1)! \cdot [i/2]! \cdot 2^{i+1}$ annihilates the torsion part of $\mathrm{CH}^d(X)$, i.e.

$$M_d \cdot Tors\, \mathrm{CH}^d(X) = 0.$$

6.2. **Remark.** Observe that since 2 is the only torsion prime of G we can replace the integer M_d by its 2-primary part. Note also that the integer M_d depends only on the codimension d but not on the rank n of G.

Proof. We follow the arguments of section 6 of [1]. Following the proof of [1, Thm. 6.5] and using Propositions 4.5, 4.7, and 5.6, we obtain that the integer $m_d = (d-1)! \cdot [d/2]! \cdot 2^{d+1}$ annihilates the torsion of $\gamma^d(G/B)/\gamma^{d+1}(G/B)$.

By [13, Thm.2.2.(2)] the restriction map $K_0(X) \to K_0(G/B)$ on Grothendieck's K_0 is an isomorphism (here we identify $K_0(G/B)$ with the $K_0(X \times_k \bar{k})$ over the algebraic closure $\bar{k}$). Since the characteristic classes commute with restrictions, this induces an isomorphism between the respective γ-filtrations on X and on G/B and, hence, between the respective quotients (see [1, p.12])

$$\gamma^d(X)/\gamma^{d+1}(X) \simeq \gamma^d(G/B)/\gamma^{d+1}(G/B) \quad \text{for every } d \geq 0.$$

This implies that m_d annihilates the torsion of $\gamma^d(X)/\gamma^{d+1}(X)$. Finally, by the proof of [1, Cor.6.8] we obtain that the torsion of $\mathrm{CH}^d(X)$ is annihilated by the product $M_d = (d-1)! \prod_{i=2}^d m_i$. □

6.3. **Remark.** Using the motivic decomposition of [14] one immediately extends this result to any generically split twisted form $X = {}_\xi G/P$, where P is a parabolic subgroup of G. In particular, it holds for any maximal orthogonal Grassmannian of a quadratic form with trivial discriminant and trivial Clifford invariant.

Acknowledgments. The first and the third authors have been partially supported from the Fields Institute and the NSERC DAS grant of the second author. The second author has been supported by the NSERC Discovery grant 385795-2010, NSERC Accelerator Supplement grant 396100-2010 and the Early Researcher Award (Ontario).

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