THE SPECIAL LINEAR VERSION OF THE PROJECTIVE BUNDLE THEOREM

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ABSTRACT. A special linear Grassmann variety $SGr(k,n)$ is the complement to the zero section of the determinant of the tautological vector bundle over $Gr(k,n)$. For a representable ring cohomology theory $A(-)$ with a special linear orientation and invertible stable Hopf map η, including Witt groups and $MSL[\eta^{-1}]$, we have $A(SGr(2,2n+1)) = A(pt)[e]/\langle e^{2n}\rangle$, and $A(SGr(2,2n))$ is a truncated polynomial algebra in two variables over $A(pt)$. A splitting principle for such theories is established. We use the computations for the special linear Grassmann varieties to calculate $A(BSL_n)$ in terms of the homogeneous power series in certain characteristic classes of the tautological bundle.

1. INTRODUCTION.

The basic and most fundamental computation for oriented cohomology theories is the projective bundle theorem (see [Mor1] or [PS, Theorem 3.9]) claiming $A(\mathbb{P}^n)$ to be a truncated polynomial ring over $A(pt)$ with an explicit basis in terms of the powers of a Chern class. Having this result at hand one can define higher characteristic classes and compute the cohomology of Grassmann varieties and flag varieties. In particular, the fact that cohomology of the full flag variety is a truncated polynomial algebra gives rise to a splitting principle, which states that from a viewpoint of the oriented cohomology theory every vector bundle is in a certain sense a sum of linear bundles. For a representable cohomology theory one can deal with an infinite dimensional Grassmannian which is a model for the classifying space BGL_n and obtain even neater answer, the formal power series in the characteristic classes of the tautological vector bundle.

There are analogous computations for symplectically oriented cohomology theories [PW1] with appropriate chosen varieties: quaternionic projective spaces HP^n instead of the ordinary ones and symplectic Grassmannian and flag varieties. The answers are essentially the same, algebras of truncated polynomials in characteristic classes.

These computations have a variety of applications, for example theorems of Conner and Floyd's type [CF] describing the K-theory and hermitian K-theory as quotients of certain universal cohomology theories [PPR1, PW4].

In the present paper we establish analogous results for the cohomology theories with special linear orientations. The notion of such orientation was introduced in [PW3, Definition 5.1]. At the same preprint there was constructed a universal example of a cohomology theory with a special linear orientation, namely the algebraic special linear cobordisms MSL, [PW3, Definition 4.2]. A more down to earth example is derived Witt groups defined by Balmer [Bal1] and oriented via Koszul complexes [Ne2]. A comprehensive survey on the Witt groups could be found in [Bal2]. Of course, every oriented cohomology theory admits a special linear orientation, but it will turn out that we are not interested in such examples. We will deal with representable cohomology theories and work in the unstable $H_\bullet(k)$ and stable $\mathcal{SH}(k)$ motivic homotopy categories introduced by Morel and Voevodsky [MV, V]. We recall all the necessary constructions and notions in sections 2-4 as well as provide preliminary calculations with special linear orientations.

Then we need to choose an appropriate version of "projective space" analogous to $\mathbb{P}^n$ and HP^n. Natural candidates are SL_{n+1}/SL_n and $\mathbb{A}^{n+1}-\{0\}$. There is no difference which one to choose since the first one is an affine bundle over the latter one, so they have the same cohomology. We take $\mathbb{A}^{n+1}-\{0\}$ since it looks prettier from the geometric point of view. There is a calculation for the Witt groups of this space [BG, Theorem 8.13] claiming that $W^*(\mathbb{A}^{n+1}-\{0\})$ is a free module of rank two over $W^*(pt)$ with an explicit basis. The fact that it is a free module of rank two is not surprising since $\mathbb{A}^{n+1}-\{0\}$ is a sphere in the stable homotopy category $\mathcal{SH}_\bullet(k)$ and $W^*(-)$ is representable [Hor]. The interesting part is the basis. Let $\mathcal{T}=\mathcal{O}^{n+1}/\mathcal{O}(-1)$ be the tautological rank n bundle over $\mathbb{A}^{n+1}-\{0\}$. Then for $n=2k$ the basis consists of the element 1 and the class of a Koszul complex. The latter one is the Euler class $e(\mathcal{T})$ in the Witt groups. Unfortunately, for the odd n the second term of the basis looks more complicated. Moreover, for the oriented cohomology theory even in the case of $n=2k$ the corresponding Chern class vanishes, so one can not expect that 1 and $e(\mathcal{T})$ form a basis for every cohomology theory with a special linear orientation.

Here we introduce another principle. The maximal compact subgroup of $SL_n(\mathbb{R})$ is $SO_n(\mathbb{R})$, so over $\mathbb{R}$ the notion of a special linear orientation of a vector bundle derives to the usual topological orientation of the bundle. The Euler classes of oriented vector bundles in topology behave themselves well only after inverting 2 in the coefficients, so we want to invert in the algebraic setting something analogous to 2. There are two interesting elements in the stable cohomotopy groups $\pi^{*,*}(pt)$ that go to 2 after taking $\mathbb{R}$-points, a usual $2\in\pi^{0,0}(pt)$ and the stable Hopf map $\eta\in\pi^{-1,-1}(pt)$ arising from the morphism $\mathbb{A}^2-\{0\}\to\mathbb{P}^1$. In general 2 is not invertible in the Witt groups, so we will invert η. Moreover, recall a theorem due to Morel [Mor2] claiming that for a perfect field there is an isomorphism $\oplus_n\pi^{n,n}(\operatorname{Spec} k)[\eta^{-1}]\cong W^0(k)[\eta,\eta^{-1}]$, so in a certain sense η is invertible in the Witt groups. In sections 5-6 we do some computations justifying the choice of η.

In this paper we deal mainly with the cohomology theories obtained as follows. Take a commutative monoid $(A,m,e:\mathbb{S}\to A)$ in the stable homotopy category $\mathcal{SH}(k)$ and fix a special linear orientation on the cohomology theory $A^{*,*}(-)$. The unit $e:\mathbb{S}\to A$ of the monoid (A,m,e) induces a morphism of cohomology theories $\pi^{*,*}(-)\to A^{*,*}(-)$ making $A^{*,*}(X)$ an algebra over the stable cohomotopy groups. Set $\eta=1$, that is

$$A^*(X)=A^{*,*}(X)/\langle 1-\eta\rangle.$$

It is an easy observation that $A^*(-)$ is still a cohomology theory, see Section 5. For these cohomology theories we have a result analogous to the case of the Witt groups.

Theorem. *There is an isomorphism*
$$A^*(\mathbb{A}^{2n+1}-\{0\}) \cong A^*(pt) \oplus A^{*-2n}(pt)e(\mathcal{T}).$$
The relative version of this statement is Theorem 3 in section 7. Note that there is no similar result for $\mathbb{A}^{2n}-\{0\}$.

In the next section we consider another family of varieties, called special linear Grassmannians $SGr(2,n) = SL_n/P_2'$, where P_2' stands for the derived group of the parabolic subgroup P_2, i.e. P_2' is the stabilizer of $e_1 \wedge e_2$ in the exterior square of the regular representation of SL_n. There are tautological bundles $\mathcal{T}_1$ and $\mathcal{T}_2$ over $SGr(2,n)$ of ranks 2 and $n-2$ respectively. We have the following theorem which seems to be the correct version of the projective bundle theorem for the special linear orientation.

Theorem. *For the special linear Grassmann varieties we have the next isomorphisms.*
$$A^*(SGr(2,2n)) \cong \bigoplus_{i=0}^{2n-2} A^{*-2i}(pt)e(\mathcal{T}_1)^i \oplus A^{*-2n+2}(pt)e(\mathcal{T}_2),$$
$$A^*(SGr(2,2n+1)) \cong \bigoplus_{i=0}^{2n-1} A^{*-2i}(pt)e(\mathcal{T}_1)^i.$$
Recall that there is a recent computation of the twisted Witt groups of Grassmannians [BC]. The twisted groups are involved since the authors use pushforwards that exist only in the twisted case. We deal with the varieties with a trivialized canonical bundle and closed embeddings with a special linear normal bundle in order to avoid these difficulties. In fact we are interested in the relative computations that could be extended to the Grassmannian bundles, so we look for a basis consisting of characteristic classes rather then pushforwards of certain elements. It turns out that such bases exist only for the special linear flag varieties with all but at most one dimension step being even, i.e. we can handle $SGr(1,7)$, $\mathcal{SF}(2,4,6)$ and $\mathcal{SF}(2,5,7)$ but not $SGr(3,6)$. Nevertheless it seems that one can construct the basis for the latter case in terms of pushforwards.

Section 9 deals with symmetric polynomials and algebras of coinvariants which appear in section 10 as the cohomology rings of maximal SL_2 flag varieties,
$$\mathcal{SF}(2n) = SL_{2n}/P'_{2,4,\dots,2n-2}, \quad \mathcal{SF}(2n+1) = SL_{2n+1}/P'_{2,4,\dots,2n}.$$
We obtain an analogue of the splitting principle in Theorem 7 and its relative version.

Theorem. *For $n \geq 1$ consider*
$$s_i = \sigma_i(e_1^2, e_2^2, \dots, e_n^2), \quad t = \sigma_n(e_1, e_2, \dots, e_n)$$
with σ_i being the elementary symmetric polynomials in n variables. Then we have the following isomorphisms

(1) $A^*(\mathcal{SF}(2n)) \cong A^*(pt)[e_1, e_2, \dots, e_n]\big/\langle s_1, s_2, \dots, s_{n-1}, t\rangle,$

(2) $A^*(\mathcal{SF}(2n+1)) \cong A^*(pt)[e_1, e_2, \dots, e_n]\big/\langle s_1, s_2, \dots, s_n\rangle.$

Note that one can substitute the $SL_n/(SL_2)^{[n/2]}$ instead of $\mathcal{SF}(n)$. These answers and the choice of commuting SL_2 in SL_n perfectly agree with our principle that $SL_n(\mathbb{R})$ stands for $SO_n(\mathbb{R})$, since $SL_2(\mathbb{R})$ stands for the compact torus S^1, and the choice of maximal number of commuting SL_2 is parallel to the choice of the maximal compact torus. We get the coinvariants for the Weyl groups $W(B_n)$ and $W(D_n)$ and it is what one gets computing the cohomology of $SO_n(\mathbb{R})/T$.

At the end, in section 11, we assemble the calculations for the special linear Grassmannians and compute in Theorem 8 the cohomology of the classifying spaces in terms of the homogeneous formal power series.

Theorem. *We have the following isomorphisms.*

$$A^*(BSL_{2n}) \cong A^*(pt)\,[[b_1,\ldots,b_{n-1},e]]_h\,,$$
$$A^*(BSL_{2n+1}) \cong A^*(pt)\,[[b_1,\ldots,b_n]]_h\,.$$

Finally, we leave for the forthcoming paper [An] the careful proof of the fact that Witt groups arise from the hermitian K-theory in the described above fashion, that is $W^*(X) \cong \mathbf{BO}^{*,*}(X)/\langle 1-\eta\rangle$. We give only a sketch of the proof in Proposition 3. In the same paper we are going to prove the following special linear version of the motivic Conner and Floyd theorem.

Theorem. *Let k be a field of characteristic different from 2. Then for all small pointed motivic spaces Y over k there is an isomorphism*

$$MSL^{*,*}(Y)\otimes_{MSL^{4*,2*}(pt)} W^{2*}(pt) \cong W^*(Y).$$

Another application of the developed technique lies in the field of the equivariant Witt groups and we are going to address it in another paper.

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2. Preliminaries on $\mathcal{SH}(k)$ and ring cohomology theories.

Let k be a field of characteristic different from 2 and let Sm/k be the category of smooth varieties over k.

A motivic space over k is a simplicial presheaf on Sm/k. Each $X \in Sm/k$ defines an unpointed motivic space $Hom_{Sm/k}(-,X)$ constant in the simplicial direction. We will often write pt for the $\operatorname{Spec} k$ regarded as a motivic space.

We use the injective model structure on the category of the pointed motivic spaces $M_\bullet(k)$. Inverting the weak motivic equivalences in $M_\bullet(k)$ gives the pointed motivic unstable homotopy category $H_\bullet(k)$.

Let $T = \mathbb{A}^1/(A^1-\{0\})$ be the Morel-Voevodsky object. A T-spectrum M [Jar] is a sequence of pointed motivic spaces $(M_0, M_1, M_2, \ldots)$ equipped with the structural maps $\sigma_n\colon T\wedge M_n \to M_{n+1}$. A map of T-spectra is a sequence of maps of pointed motivic spaces which is compatible with the structure maps. We write $MS(k)$ for the category of T-spectra. Inverting the stable motivic weak equivalences as in [Jar] gives the motivic stable homotopy category $\mathcal{SH}(k)$.

A pointed motivic space X gives rise to a suspension T-spectrum $\Sigma_T^\infty X$. Set $\mathbb{S} = \Sigma_T^\infty(pt_+)$ for the spherical spectrum. Both $H_\bullet(k)$ and $\mathcal{SH}(k)$ are equipped with symmetric monoidal structures $(\wedge, pt_+)$ and $(\wedge, \mathbb{S})$ respectively and

$$\Sigma_T^\infty\colon H_\bullet(k) \to \mathcal{SH}(k)$$

is a strict symmetric monoidal functor. We will usually omit the subscript T and write Σ^∞ for this functor.

Recall that there are two spheres in $M_\bullet(k)$, the simplicial one $S^{1,0} = S^1_s = \Delta^1/\partial(\Delta^1)$ and $S^{1,1} = (\mathbb{G}_m, 1)$. We write $S^{p,q}$ for $(S^1_s)^{\wedge p-q} \wedge (\mathbb{G}_m, 1)^{\wedge q}$ and $\Sigma^{p,q}$ for the suspension functor $- \wedge S^{p,q}$. In the motivic homotopy category there is a canonical isomorphism $T \cong S^{2,1}$.

Any T-spectrum A defines a bigraded cohomology theory on the category of pointed motivic spaces. Namely, for a pointed space (X, x) one sets

$$A^{p,q}(X, x) = Hom_{\mathcal{SH}(k)}(\Sigma^\infty(X, x), \Sigma^{p,q} A)$$

and $A^{*,*}(X, x) = \bigoplus_{p,q} A^{p,q}(X, x)$. In case of $j, i - j \geq 0$ one has a canonical suspension isomorphism $A^{p,q}(X, x) \cong A^{p+i,q+j}(\Sigma^{i,j}(X, x))$. For an unpointed space X we set $A^{p,q}(X) = A^{p,q}(X_+, +)$ with $A^{*,*}(X)$ defined accordingly. Set $\pi^{i,j}(X) = \mathbb{S}^{i,j}(X)$ to be the stable cohomotopy groups of X.

We can regard smooth varieties as unpointed motivic spaces and obtain the groups $A^{p,q}(X)$. Given a closed embedding $i\colon Z \to X$ of varieties we write $Th(i)$ for $X/(X - Z)$. For a vector bundle $E \to X$ set $Th(E) = E/(E - X)$ to be the Thom space of E.

A commutative ring T-spectrum is a commutative monoid (A, m, e) in $(\mathcal{SH}(k), \wedge, \mathbb{S})$. The cohomology theory defined by a commutative T-spectrum is a ring cohomology theory satisfying a certain bigraded commutativity condition described by Morel.

We recall the essential properties of the cohomology theories represented by a commutative ring T-spectrum A.

(1) *Localization:* for a closed embedding of varieties $i\colon Z \to X$ with a smooth X and an open complement $j\colon U \to X$ we have a long exact sequence

$$\dots \xrightarrow{\partial} A^{*,*}(Th(i)) \xrightarrow{z^A} A^{*,*}(X) \xrightarrow{j^A} A^{*,*}(U) \xrightarrow{\partial} A^{*+1,*}(Th(i)) \xrightarrow{z^A} \dots$$

It is a special case of the cofiber long exact sequence.

(2) *Nisnevich excision:* consider a Cartesian square of smooth varieties

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d, "f'"] & X' \arrow[d, "f"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

where i is a closed embedding, f is etale and f' is an isomorphism. Then for the induced morphism $g\colon Th(i') \to Th(i)$ the corresponding morphism $g^A\colon A^{*,*}(Th(i)) \to A^{*,*}(Th(i'))$ is an isomorphism. It follows from the fact that g is an isomorphism in the homotopy category.

(3) *Homotopy invariance:* for an $\mathbb{A}^n$-bundle $p\colon E \to X$ over a variety X the induced homomorphism $p^A\colon A^{*,*}(X) \to A^{*,*}(E)$ is an isomorphism.

(4) *Mayer-Vietoris:* if $X = U_1 \cup U_2$ is a union of two open subsets U_1 and U_2 then there is a natural long exact sequence

$$\ldots \to A^{*,*}(X) \to A^{*,*}(U_1) \oplus A^{*,*}(U_2) \to A^{*,*}(U_1 \cap U_2) \to A^{*+1,*}(X) \to \ldots$$

(5) *Cup-product:* for a motivic space Y we have a functorial graded ring structure

$$\cup\colon A^{*,*}(Y) \times A^{*,*}(Y) \to A^{*,*}(Y).$$

Also, for closed subsets $i_1\colon Z_1 \to X$ and $i_2\colon Z_2 \to X$ set $i_{12}\colon Z_1 \cap Z_2 \to X$, then we have functorial, bilinear and associative cup-product

$$\cup\colon A^{*,*}(Th(i_1)) \times A^{*,*}(Th(i_2)) \to A^{*,*}(Th(i_{12})).$$

In particular, setting $Z_1 = X$ we obtain an $A^{*,*}(X)$-module structure on $A^{*,*}(Th(i_2))$. All the morphisms in the localization sequence are homomorphisms of $A^{*,*}(X)$-modules.

(6) *Module structure over stable cohomotopy groups:* for every motivic space Y we have a homomorphism of graded rings $\pi^{*,*}(Y) \to A^{*,*}(Y)$, which defines a $\pi^{*,*}(pt)$-module structure on $A^{*,*}(Y)$. For a smooth variety X the ring $A^{*,*}(X)$ is a graded $\pi^{*,*}(pt)$-algebra via $\pi^{*,*}(pt) \to \pi^{*,*}(X) \to A^{*,*}(X)$.

(7) *Graded ϵ-commutativity* [Mor1]: let $\epsilon \in \pi^{0,0}(pt)$ be the element corresponding under the suspension isomorphism to the morphism $T \to T, x \mapsto -x$. Then for every motivic space X and $a \in A^{i,j}(X), b \in A^{p,q}(X)$ we have

$$a \cup b = (-1)^{ip}\epsilon^{jq} \cup b \cup a.$$

Recall that $\epsilon^2 = 1$.

3. Special linear orientation.

In this section we recall the notion of a special linear orientation introduced in [PW3] and establish some of its basic properties.

Definition 1. A *special linear bundle* over a variety X is a pair (E, λ) with $E \to X$ a vector bundle and $\lambda\colon \det E \xrightarrow{\simeq} \mathcal{O}_X$ an isomorphism of line bundles. An isomorphism $\phi\colon (E, \lambda) \xrightarrow{\simeq} (E', \lambda')$ of special linear vector bundles is an isomorphism $\phi\colon E \xrightarrow{\simeq} E'$ of vector bundles such that $\lambda' \circ (\det \phi) = \lambda$.

Notation 1. Consider a trivialized rank n bundle $\mathcal{O}_X^n$ over a smooth variety X. There is a canonical trivialization $\det \mathcal{O}_X^n \xrightarrow{\simeq} \mathcal{O}_X$. We denote the corresponding special linear bundle by $(\mathcal{O}_X^n, 1)$.

Lemma 1. *Let (E, λ) be a special linear bundle over a smooth variety X such that $E \cong \mathcal{O}_X^n$. Then there exists an isomorphism of special linear bundles*

$$\phi\colon (E, \lambda) \xrightarrow{\simeq} (\mathcal{O}_X^n, 1).$$

Proof. An exact sequence of algebraic groups

$$1 \to SL_n \to GL_n \xrightarrow{\det} \mathbb{G}_m \to 1$$

induces an exact sequence of pointed sets

$$H^0(X, GL_n) \xrightarrow{p} H^0(X, \mathbb{G}_m) \to H^1(X, \mathrm{SL}_n) \xrightarrow{i} H^1(X, GL_n)$$

There is a splitting $\mathbb{G}_m \to GL_n$ for det, so p is surjective. Hence we have $\ker i = \{*\}$ and this means that, up to an isomorphism of special linear bundles, there exists only one trivialization $\lambda\colon \det \mathcal{O}_X^n \to \mathcal{O}_X$. □

Lemma 2. *Let E_1 be a subbundle of a vector bundle E over a smooth variety X. Then there are canonical isomorphisms*

1. $\det E_1 \otimes \det(E/E_1) \cong \det E$,
2. $\det E^\vee \cong (\det E)^\vee$.

Proof. These isomorphisms are induced by the corresponding vector space isomorphisms. In the first case we have $\Lambda^m V_1 \otimes \Lambda^n(V/V_1) \xrightarrow{\simeq} \Lambda^{m+n}V$ with

$$v_1 \wedge \ldots \wedge v_m \otimes \overline{w}_1 \wedge \ldots \wedge \overline{w}_n \mapsto v_1 \wedge \ldots \wedge v_m \wedge w_1 \wedge \ldots \wedge w_n.$$

For the second isomorphism consider the perfect pairing

$$\phi\colon \Lambda^n V \times \Lambda^n V^\vee \to k$$

defined by

$$\phi(v_1 \wedge \ldots \wedge v_n, f_1 \wedge \ldots \wedge f_n) = \sum_{\sigma \in S_n} sign(\sigma) f_{\sigma(1)}(v_1) \cdot \ldots \cdot f_{\sigma(n)}(v_n).$$ □

Definition 2. Let $\mathcal{T} = (E, \lambda_E)$ be a special linear bundle over a smooth variety X. By Lemma 2 there is a canonical trivialization $\lambda_{E^\vee}\colon \det E^\vee \xrightarrow{\simeq} \mathcal{O}_X$. The special linear bundle $\mathcal{T}^\vee = (E^\vee, \lambda_{E^\vee})$ is called the *dual special linear bundle.*

Definition 3. Let $A^{*,*}(-)$ be a cohomology theory represented by a T-spectrum A. A *(normalized) special linear orientation* on $A^{*,*}(-)$ is a rule which assigns to every special linear bundle (E, λ) of rank n over a smooth variety X a class $th(E, \lambda) \in A^{2n,n}(Th(E))$ satisfying the following conditions [PW3, Definition 5.1]:

1. For an isomorphism $f\colon (E, \lambda) \xrightarrow{\simeq} (E', \lambda')$ we have $th(E, \lambda) = f^A th(E', \lambda')$.
2. For a morphism $r\colon Y \to X$ we have $r^A th(E, \lambda) = th(r^*(E, \lambda))$ in $A^{2n,n}(Th(r^*E))$.
3. The maps $- \cup th(E, \lambda)\colon A^{*,*}(X) \to A^{*+2n,*+n}(Th(E))$ are isomorphisms.
4. We have
$$th(E_1 \oplus E_2, \lambda_1 \otimes \lambda_2) = q_1^A th(E_1, \lambda_1) \cup q_2^A th(E_2, \lambda_2),$$
where q_1, q_2 are projections from $E_1 \oplus E_2$ onto its summands. Moreover, for the zero bundle $\mathbf{0} \to pt$ we have $th(\mathbf{0}) = 1 \in A^{0,0}(pt)$.
5. (normalization) For the trivial line bundle over a point we have $th(\mathcal{O}_{pt}, 1) = \Sigma^{2,1} 1 \in A^{2,1}(T)$.

The isomorphism $- \cup th(E, \lambda)$ is a *Thom isomorphism.* The class $th(E, \lambda)$ is a *Thom class* of the special linear bundle, and

$$e(E, \lambda) = z^A th(E, \lambda) \in A^{2n,n}(X)$$

with natural $z\colon X \to Th(E)$ is its *Euler class.*

Remark 1. For a rank $2n$ special linear bundle (E, λ) over a variety X we have $th(E, \lambda) \in A^{4n,2n}(Th(E))$ and $e(E, \lambda) \in A^{4n,2n}(X)$, so this classes are universally central.

Recall that a symplectic bundle is a special linear bundle in a natural way, so having a special linear orientation we have the Thom classes also for symplectic bundles, thus a cohomology theory with a special linear orientation is also symplectically oriented. We recall the definition of the Pontryagin classes theory [PW1, Definition 14.1] which is equivalent to the symplectic orientation via Thom classes.

Definition 4. Let $A^{*,*}(-)$ be a cohomology theory represented by a T-spectrum A. A *Pontryagin classes theory* on $A^{*,*}(-)$ is a rule which assigns to every symplectic bundle (E, ϕ) over every smooth variety X a system of *Pontryagin classes* $p_i(E, \phi) \in A^{4i,2i}(X)$ for all $i \geq 1$ satisfying

(1) For $(E_1, \phi_1) \cong (E_2, \phi_2)$ we have $p_i(E_1, \phi_1) = p_i(E_2, \phi_2)$ for all i.
(2) For a morphism $r\colon Y \to X$ and a symplectic bundle (E, ϕ) over X we have $r^A(p_i(E, \phi)) = p_i(r^*(E, \phi))$ for all i.
(3) For the tautological rank 2 symplectic bundle (E, ϕ) over
$$HP^1 = Sp_4/(Sp_2 \times Sp_2)$$
the elements $1, p_1(E, \phi)$ form a $A^{*,*}(pt)$-basis of $A^{*,*}(HP^1)$.
(4) For a rank 2 symplectic bundle (V, ϕ) over pt we have $p_1(V, \phi) = 0$.
(5) For an orthogonal direct sum of symplectic bundles $(E, \phi) \cong (E_1, \phi_1) \perp (E_2, \phi_2)$ we have
$$p_i(E, \phi) = p_i(E_1, \phi_1) + \sum_{j=1}^{i-1} p_{i-j}(E_1, \phi_1) p_j(E_2, \phi_2) + p_i(E_2, \phi_2)$$
for all i.
(6) For (E, ϕ) of rank $2r$ we have $p_i(E, \phi) = 0$ for $i > r$.

We set $p_*(E, \phi) = 1 + \sum_{j=1}^{\infty} p_i(E, \phi) t^i$ to be the *total Pontryagin class.*

Every oriented cohomology theory possesses a special linear orientation via $th(E, \lambda) = th(E)$, so one can consider K-theory or algebraic cobordism represented by MGL as examples. We have two main instances of the theories with a special linear orientation but without a general one. The first one is hermitian K-theory [Sch] represented by the spectrum **BO** [PW2]. The special linear orientation on $\mathbf{BO}^{*,*}$ via Koszul complexes could be found in [PW2]. The second one is universal in the sense of [PW3, Theorem 5.9] and represented by the algebraic special linear cobordism spectrum MSL [PW3, Definition 4.2].

Notation 2. From now on $A^{*,*}(-)$ is a ring cohomology theory represented by a commutative monoid in $\mathcal{SH}(k)$ with a fixed special linear orientation.

Lemma 3. *Let X be a smooth variety. Then $th(\mathcal{O}_X^n, 1) = \Sigma^{2n,n} 1$ and $th(\mathcal{O}_X, -1) = \Sigma^{2,1} \epsilon$.*

Proof. It follows immediately from the conditions (4) and (5) and functoriality. □

Lemma 4. *Let (E, λ_E) be a special linear bundle over a smooth variety X. Then*
$$e(E, \lambda_E) = \epsilon \cup e(E, -\lambda_E).$$

Proof. Consider the bundle $E \oplus \mathcal{O}_X$ and denote the projections onto the summands by q_1, q_2. We have
$$(E \oplus \mathcal{O}_X, \lambda_E \otimes 1) = (E \oplus \mathcal{O}_X, (-\lambda_E) \otimes -1),$$
hence
$$q_1^* th(E, \lambda_E) \cup q_2^* \Sigma 1 = q_1^* th(E, -\lambda_E) \cup q_2^* \Sigma \epsilon.$$
By the suspension isomorphism we obtain
$$th(E, \lambda_E) = th(E, -\lambda_E) \cup \epsilon,$$
hence $e(E, \lambda_E) = \epsilon \cup e(E, -\lambda_E)$. □

Lemma 5. *Let $\mathcal{T}$ be a rank 2 special linear bundle over a smooth variety X. Then $\mathcal{T} \cong \mathcal{T}^\vee$ and $e(\mathcal{T}) = e(\mathcal{T}^\vee)$.*

Proof. Set $\mathcal{T} = (E, \lambda_E)$. The trivialization $\lambda_E \colon \Lambda^2 E \xrightarrow{\simeq} \mathcal{O}_X$ defines a symplectic form on E and an isomorphism $\phi \colon E \xrightarrow{\simeq} E^\vee$, thus it is sufficient to check that
$$\lambda_{E^\vee} \circ \det \phi = \lambda_E.$$
It could be checked locally, so we can suppose that $E \cong \mathcal{O}_X^2$ and, in view of Lemma 1, $(E, \lambda_E) \cong (\mathcal{O}_X^2, 1)$. Fixing a basis $\{e_1, e_2\}$ such that $e_1 \wedge e_2 = 1$ and taking the dual basis $\{e_1^\vee, e_2^\vee\}$ for $(\mathcal{O}_X^2)^\vee$ we have
$$\phi(e_1) = (e_1 \wedge -) = e_2^\vee, \qquad \phi(e_2) = (e_2 \wedge -) = -e_1^\vee.$$
Thus we obtain
$$\det \phi(e_1 \wedge e_2) = e_2^\vee \wedge (-e_1^\vee) = e_1^\vee \wedge e_2^\vee$$
and
$$\lambda_{E^\vee} \det \phi(e_1 \wedge e_2) = \lambda_{E^\vee}(e_1^\vee \wedge e_2^\vee) = 1.$$ □

Notation 3. For a vector bundle E we denote by E^0 the complement to the zero section. For a special linear bundle $\mathcal{T} = (E, \lambda)$ we set $\mathcal{T}^0 = E^0$.

Definition 5. Let $\mathcal{T}$ be a rank n special linear bundle over a smooth variety X. The *Gysin sequence* is a long exact sequence
$$\ldots \xrightarrow{\partial} A^{*-2n,*-n}(X) \xrightarrow{\cup e(\mathcal{T})} A^{*,*}(X) \to A^{*,*}(\mathcal{T}^0) \xrightarrow{\partial} A^{*-2n+1,*-n}(X) \to \ldots$$
obtained from the localization sequence for the zero section $X \to \mathcal{T}$ via homotopy invariance and Thom isomorphism.

Lemma 6. *Let (E, λ_E) be a special linear bundle over a smooth variety X.*

(1) *Let λ'_E be any other trivialization of* $\det E$. *Then one has*
$$A^{0,0}(X) \cup e(E, \lambda_E) = A^{0,0}(X) \cup e(E, \lambda'_E).$$

(2) *For the dual special linear bundle $(E^\vee, \lambda_{E^\vee})$ one has*
$$A^{0,0}(X) \cup e(E, \lambda_E) = A^{0,0}(X) \cup e(E^\vee, \lambda_{E^\vee}).$$

Proof. Set $n = \operatorname{rank} E$ and denote the projections $E^0 \to X$ and $E^{\vee 0} \to X$ by p and p' respectively.

(1) Consider the Gysin sequences corresponding to the trivializations λ_E and λ'_E.

$$\begin{tikzcd} \cdots \arrow[r] & A^{0,0}(X) \arrow[r, "\cup e(E,\lambda_E)"] & A^{2n,n}(X) \arrow[r, "p^A"] \arrow[d, "="] & A^{2n,n}(E^0) \arrow[r] \arrow[d, "="] & \cdots \\ \cdots \arrow[r] & A^{0,0}(X) \arrow[r, "\cup e(E,\lambda'_E)"] & A^{2n,n}(X) \arrow[r, "p^A"] & A^{2n,n}(E^0) \arrow[r] & \cdots \end{tikzcd}$$

We have

$$A^{0,0}(X) \cup e(E, \lambda_E) = \ker p^A = A^{0,0}(X) \cup e(E, \lambda'_E).$$

(2) Consider

$$Y = \{ (v, f) \in E \times_X E^\vee \mid f(v) = 1 \}.$$

Projections $p_1 \colon Y \to E^0$ and $p_2 \colon Y \to E^{\vee 0}$ have fibres isomorphic to $\mathbb{A}^{n-1}$, thus

$$A^{*,*}(E^0) \cong A^{*,*}(Y) \cong A^{*,*}(E^{\vee 0})$$

and there is a canonical isomorphism $A^{*,*}(E^0) \cong A^{*,*}(E^{\vee 0})$ over $A^{*,*}(X)$. Now proceed as in the first part and consider the Gysin sequences.

$$\begin{tikzcd} \cdots \arrow[r] & A^{0,0}(X) \arrow[r, "\cup e(E,\lambda_E)"] & A^{2n,n}(X) \arrow[r, "p^A"] \arrow[d, "="] & A^{2n,n}(E^0) \arrow[r] \arrow[d, "\cong"] & \cdots \\ \cdots \arrow[r] & A^{0,0}(X) \arrow[r, "\cup e(E^\vee,\lambda_{E^\vee})"] & A^{2n,n}(X) \arrow[r, "p'^A"] & A^{2n,n}(E^{\vee 0}) \arrow[r] & \cdots \end{tikzcd}$$

We have

$$A^{0,0}(X) \cup e(E, \lambda_E) = \ker p^A = \ker p'^A = A^{0,0}(X) \cup e(E^\vee, \lambda_{E^\vee}). \qquad \square$$

Lemma 7. *Let $\mathcal{T}$ be a special linear bundle over a smooth variety X such that there exists a nowhere vanishing section $s \colon X \to \mathcal{T}$. Then $e(\mathcal{T}) = 0$.*

Proof. Set $\operatorname{rank} \mathcal{T} = n$ and consider the Gysin sequence

$$\ldots \to A^{0,0}(X) \xrightarrow{\cup e(\mathcal{T})} A^{2n,n}(X) \xrightarrow{j^A} A^{2n,n}(\mathcal{T}^0) \to \ldots$$

The section s induces a splitting s^A for j^A, thus j^A is injective and

$$e(\mathcal{T}) = 1 \cup e(\mathcal{T}) = 0. \qquad \square$$

4. Pushforwards along closed embeddings.

In this section we give the construction of the pushforwards along the closed embeddings with special linear normal bundles for a cohomology theory with a special linear orientation. It is quite similar to the construction of such pushforwards for oriented [PS, Ne1] or symplectically oriented [PW1] cohomology theories and twisted Witt groups [Ne2].

Definition 6. Let $i \colon Z \to X$ be a closed embedding of smooth varieties. The *deformation space* $D(Z, X)$ is obtained as follows.

(1) Consider $X \times \mathbb{A}^1$.
(2) Blow-up it along $Z \times 0$.

(3) Remove the blow-up of $X \times 0$ along $Z \times 0$.

This construction produces a smooth variety $D(Z,X)$ over $\mathbb{A}^1$. The fiber over 0 is canonically isomorphic to N_i while the fiber over 1 is isomorphic to X and we have the corresponding closed embeddings $i_0 \colon N_i \to D(Z,X)$ and $i_1 \colon X \to D(Z,X)$. There is a closed embedding $z \colon Z \times \mathbb{A}^1 \to D(Z,X)$ such that over 0 it coincides with the zero section $s \colon Z \to N_i$ of the normal bundle and over 1 it coincides with the closed embedding $i \colon Z \to X$. At last, we have a projection $p \colon D(Z,X) \to X$.

Thus we have homomorphisms of $A^{*,*}(X)$-modules (via p^A)

$$A^{*,*}(Th(N_i)) \xleftarrow{i_0^A} A^{*,*}(Th(z)) \xrightarrow{i_1^A} A^{*,*}(Th(i)).$$

These homomorphisms are isomorphisms, since in the homotopy category $H_\bullet(k)$ we have isomorphisms $i_0 \colon Th(N_i) \cong Th(z)$ and $i_1 \colon Th(i) \cong Th(z)$ [MV, Theorem 2.23]. We set

$$d_i^A = i_1^A \circ (i_0^A)^{-1} \colon A^{*,*}(Th(N_i)) \to A^{*,*}(Th(i))$$

to be the *deformation to the normal bundle isomorphism*. The functoriality of the deformation space $D(Z,X)$ makes the deformation to the normal bundle isomorphism functorial.

Definition 7. For a closed embedding $i \colon Z \to X$ of smooth varieties a *special linear normal bundle* is a pair (N_i, λ) with N_i the normal bundle and $\lambda \colon \det N_i \xrightarrow{\simeq} \mathcal{O}_Z$ an isomorphism of line bundles.

Definition 8. Let $i \colon Z \to X$ be a closed embedding of smooth varieties with a rank n special linear normal bundle (N_i, λ). Denote by $\tilde{\imath}_A$ the composition of the Thom and deformation to the normal bundle isomorphisms,

$$\tilde{\imath}_A = d_i^A \circ (- \cup th(N_i, \lambda)) \colon A^{*-2n,*-n}(Z) \xrightarrow{\simeq} A^{*,*}(Th(i)).$$

For the inclusion $z \colon X \to Th(i)$ the composition

$$i_A = z^A \circ \tilde{\imath}_A \colon A^{*-2n,*-n}(Z) \to A^{*,*}(X)$$

is the *pushforward map*. Note that in general i_A depends on the trivialization of $\det N_i$.

Remark 2. We have an analogous definition of the pushforward map for a closed embedding $i \colon Z \to X$ in every cohomology theory possessing a Thom class for the normal bundle N_i. In particular, we have pushforwards in the stable cohomotopy groups for closed embeddings with a trivialized normal bundle (N_i, θ), where $\theta \colon N_i \xrightarrow{\simeq} \mathcal{O}_Z^n$ is an isomorphism of vector bundles, since there is a Thom class $th(\mathcal{O}_Z^n) = \Sigma^{2n,n}1$ and suspension isomorphism

$$\pi^{*-2n,*-n}(Z) \xrightarrow{\cup \Sigma^{2n,n}1} \pi^{*,*}(Th(\mathcal{O}_Z^n)).$$

Notation 4. Let $i \colon Z \to X$ be a closed embedding of smooth varieties with a rank n special linear normal bundle. Then using the notation of pushforward maps the localization sequence boils down to

$$\ldots \xrightarrow{\partial} A^{*-2n,*-n}(Z) \xrightarrow{i_A} A^{*,*}(X) \xrightarrow{j^A} A^{*,*}(X-Z) \xrightarrow{\partial} A^{*-2n+1,*-n}(Z) \xrightarrow{i_A} \ldots$$

In the rest of this section we sketch some properties of the pushforward maps. The next lemma is similar to [PW1, Proposition 7.4].

Lemma 8. *Consider the following pullback diagram with all the involved varieties being smooth.*

$$\begin{tikzcd} X' = X \times_Y Y' \arrow[r, "i'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$

Let i, i' be the closed embeddings with special linear normal bundles (N_i, λ) and $(N_{i'}, \lambda') \cong (g'^* N_i, g'^* \lambda)$. Then we have $g^A \tilde{\imath}_A = \tilde{\imath}' g'^A$.

Proof. It follows from the functoriality of the deformation to the normal bundle and the functoriality of Thom classes. □

The next proposition is an analogue of [PW1, Proposition 7.6].

Proposition 1. *Let $\mathcal{T}$ be a special linear bundle over a smooth variety X with a section $s\colon X \to \mathcal{T}$ meeting the zero section r transversally in Y. Then for the inclusion $i\colon Y \to X$ and all $b \in A^{*,*}(X)$ we have*

$$i_A i^A(b) = b \cup e(\mathcal{T}).$$

Proof. Let $z^A\colon A^{*,*}(Th(i)) \to A^{*,*}(X)$ and $\overline{z}^A\colon A^{*,*}(Th(\mathcal{T})) \to A^{*,*}(\mathcal{T})$ be the extension of supports maps and let $p\colon \mathcal{T} \to X$ be the structure map for the bundle. Consider the following diagram.

$$\begin{tikzcd} A^{*,*}(X) \arrow[r, "\tilde{r}_A", "\cup th(\mathcal{T})"'] \arrow[d, "i^A"] & A^{*,*}(Th(\mathcal{T})) \arrow[r, "z^A"] \arrow[d, "s^A"] & A^{*,*}(\mathcal{T}) \arrow[d, shift right, "r^A"'] \arrow[d, shift left, "s^A"] \\ A^{*,*}(Y) \arrow[r, "\tilde{\imath}_A"] & A^{*,*}(Th(i)) \arrow[r, "\overline{z}^A"] & A^{*,*}(X) \arrow[u, bend right, "p^A"'] \end{tikzcd}$$

The pullbacks along the two section of p are inverses of the same isomorphism p^A, so $s^A = r^A$. The right-hand square consists of pullbacks thus it is commutative. The left-hand square commutes by Lemma 8. Hence we have

$$i_A i^A(b) = \overline{z}^A \tilde{\imath}_A i^A(b) = r^A z^A(b \cup th(\mathcal{T})) = b \cup e(\mathcal{T}). \qquad \square$$

The pushforward maps are compatible with the compositions of the closed embeddings. The following proposition is similar to [Ne2, Proposition 5.1] and the same reasoning works out, so we omit the proof.

Proposition 2. *Let $Z \xrightarrow{i} Y \xrightarrow{j} X$ be closed embeddings of smooth varieties with special linear normal bundles $(N_{ji}, \lambda_{ji}), (N_i, \lambda_i), (i^* N_{ji}/N_i, \lambda_j)$ such that $\lambda_i \otimes \lambda_j = \lambda_{ji}$. Then*

$$j_A i_A = (ji)_A.$$

5. Inverting the stable Hopf map.

The Hopf map is the canonical morphism of varieties

$$H\colon \mathbb{A}^2 - \{0\} \to \mathbb{P}^1$$

defined via $H(x,y) = [x,y]$. Pointing $\mathbb{A}^2 - \{0\}$ by $(1,1)$ and $\mathbb{P}^1$ by $[1,1]$ and taking the suspension spectra we obtain the corresponding morphism

$$\Sigma^\infty H \in Hom_{\mathcal{SH}(k)}(\Sigma^\infty(\mathbb{A}^2 - \{0\}), \Sigma^\infty \mathbb{P}^1).$$

Recall that one has canonical isomorphisms $\mathbb{A}^2 - \{0\} \xrightarrow{\phi} \mathbb{G}_m \wedge T$ and $T \cong \mathbb{P}^1/\mathbb{A}^1 \cong \mathbb{P}^1$ in $\mathcal{H}_\bullet(k)$ (see Lemma 10 for the first one, the latter one is given by $x \mapsto [x:1]$), thus, using the suspension isomorphism, we obtain an element

$$\eta \in \pi^{-1,-1}(pt) \cong Hom_{\mathcal{SH}(k)}(\Sigma^\infty(\mathbb{A}^2 - \{0\}), \Sigma^\infty \mathbb{P}^1)$$

such that $\Sigma^{3,2}\eta = \Sigma^\infty H$.

Let $A^{*,*}(-)$ be a bigraded ring cohomology theory represented by a commutative monoid $A \in \mathcal{SH}(k)$. Inverting $\eta \in A^{-1,-1}(\mathrm{pt})$ we obtain a new cohomology theory with $(2i,i)$ groups isomorphic to $(2i+n, i+n)$ ones by means of the cup product with η^{-n}. We identify these groups setting $\eta = 1$ and obtaining a graded cohomology theory:

$$\overline{A}^*(X) = A^{*,*}(X) \otimes_{A^{*,*}(pt)} \left(A^{*,*}(pt)/\langle 1-\eta\rangle\right),$$

$$\overline{A}^i(X) \cong \left(A^{*,*}(X) \otimes_{A^{*,*}(pt)} A^{*,*}(pt)[\eta^{-1}]\right)^{2i,i}.$$

For the K-theory represented by BGL [PPR2] this construction gives $\overline{BGL}^*(pt) = 0$ since we have $\eta \in BGL^{-1,-1}(pt) = K_{-1}(pt) = 0$. As we will see in Corollary 1 it is always the case that an oriented cohomology theory produces a trivial cohomology theory. Thus we are interested in cohomology theories with a special linear orientation but without a general one. Our running example is hermitian K-theory represented by the spectrum $\mathbf{BO}$ that derives to the Witt groups.

Proposition 3. *For every smooth variety X we have a natural isomorphism*

$$\overline{\mathbf{BO}}^i(X) \cong W^i(X).$$

Proof. We give only a sketch of the proof, for the detailed version see [An]. We have a canonical isomorphism $\mathbf{BO}^{p,q}(X) \cong KO^{[q]}_{2q-p}(X)$ and in case of $2q-p<0$ we have $KO^{[q]}_{2q-p}(X) = W^{p-q}(X)$. Thus for $2q-p<0$ there is a natural isomorphism $\phi\colon \mathbf{BO}^{p,q}(X) \xrightarrow{\simeq} W^{p-q}(X)$. One can show that it is multiplicative, i.e. for every (p,q) and (p',q') such that $2q-p, 2q'-p' < 0$ the diagram

$$\begin{tikzcd}
\mathbf{BO}^{p,q}(X) \times \mathbf{BO}^{p',q'}(X) \arrow[r, "\cup"] \arrow[d, "\phi\times\phi"] & \mathbf{BO}^{p+p',q+q'}(X) \arrow[d, "\phi"] \\
W^{p-q}(X) \times W^{p'-q'}(X) \arrow[r, "\cup"] & W^{p+p'-q-q'}(X)
\end{tikzcd}$$

is commutative. Moreover, for $\eta \in \mathbf{BO}^{-1,-1}(pt)$ we have $\phi(\eta) = 1$. Define $\psi\colon \mathbf{BO}^{*,*}(X) \to W^*(X)[\eta,\eta^{-1}]$ in the following way. For $\alpha \in \mathbf{BO}^{p,q}(X)$ set

$$\psi(\alpha) = \begin{cases} \phi(\alpha)\eta^{-p+2q}, & 2q-p<0 \\ \phi(\alpha \cup \eta^{p-2q+1})\eta^{-p+2q}, & 2q-p \geq 0. \end{cases}$$

Regarding $W^p(X)$ as $W^{2p,p}(X)$ and setting on the right-hand side $\deg \eta = (-1,-1)$ we turn ψ into a homomorphism of bigraded algebras. For $2q-p<$

0 we know that $\psi\colon \mathbf{BO}^{p,q}(X) \to W^{p-q}(X)\eta^{-p+2q}$ is an isomorphism. Thus inverting η we obtain an isomorphism

$$\widetilde{\psi}\colon \mathbf{BO}^{*,*}(X)[\eta^{-1}] \xrightarrow{\simeq} W^*(X)[\eta,\eta^{-1}]. \qquad \square$$

For the stable cohomotopy groups we have the following result by Morel [Mor2].

Theorem 1. *There exists a canonical isomorphism* $\overline{\pi}^0(pt) \xrightarrow{\simeq} W^0(pt)$.

Notation 5. From now on $A^*(-)$ denotes a graded ring cohomology theory obtained from a bigraded ring cohomology theory represented by a commutative monoid $A \in \mathcal{SH}(k)$ with a fixed special linear orientation. Hence we have Thom and Euler classes and all the machinery of theories with a special linear orientation, including the Gysin sequences and pushforwards.

Remark 3. Note that from ϵ-commutativity we have $\eta\cup\eta = -\epsilon\cup(\eta\cup\eta)$, thus inverting η we obtain $\overline{\epsilon} = -1$.

Definition 9. Let E be a vector bundle over a smooth variety X. The total Pontryagin class

$$b_*(E) = p_*\left(E\oplus E^\vee, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = \sum p_i\left(E\oplus E^\vee, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) t^i$$

is called the *total Borel class* of the vector bundle E. We denote the even Pontryagin classes by b_i,

$$b_i(E) = p_{2i}\left(E\oplus E^\vee, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right),$$

and refer to them as *Borel classes.*

We defined Borel classes for arbitrary vector bundles without any additional structure. For special linear bundles there is an interconnection between the Borel classes and the Euler class. The following lemma shows it in the case of rank 2 bundles and the general case would be dealt with in Corollary 4.

Lemma 9. *Let $\mathcal{T}$ be a rank 2 special linear bundle. Then*

$$b_*(\mathcal{T}) = 1 - e(\mathcal{T})^2t^2$$

Proof. Set $\mathcal{T} = (E,\lambda)$. Let ϕ be the symplectic form on E corresponding to λ. There exists an isomorphism [Bal2, Examples 1.1.21, 1.1.22]

$$\left(E\oplus E^\vee, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \cong \left(E\oplus E, \begin{pmatrix} \phi & 0 \\ 0 & -\phi \end{pmatrix}\right),$$

so we have

$$b_*(E) = p_*(E,\phi)p_*(E,-\phi) = (1+p_1(E,\phi)t)(1+p_1(E,-\phi)t) =$$
$$= (1+e(E,\lambda)t)(1+e(E,-\lambda)t).$$

By Lemma 4 and Remark 3 we have $e(E,-\lambda) = -e(E,\lambda)$, thus

$$b_*(E) = (1+e(\mathcal{T})t)(1-e(\mathcal{T})t) = 1 - e(\mathcal{T})^2t^2. \qquad \square$$

6. Preliminary computation in the stable cohomotopy groups.

We are going to do preliminary computations involving $\pi^{*,*}$. Recall that for this cohomology theory we have canonical Thom classes for the trivialized vector bundles $th(\mathcal{O}_X^n) = \Sigma^{2n,n}1$ and pushforwards i_π for the closed embeddings with a trivialized normal bundle (N_i, θ).

We fix the following notation. For $n \geq 1$ let $i\colon \mathbb{G}_m \to \mathbb{A}^{n+1} - \{0\}$ be a closed embedding to the zeroth coordinate with $t \mapsto (t, 0, \ldots, 0)$. Identify the normal bundle

$$N_i \cong U = \mathbb{G}_m \times \mathbb{A}^n \subset \mathbb{A}^{n+1} - \{0\}$$

with the Zariski neighbourhood U of $\mathbb{G}_m$ and consider the trivialization $\theta\colon U \xrightarrow{\simeq} \mathcal{O}^n_{\mathbb{G}_m}$ via

$$\theta(t, x_1, \ldots, x_n) = (t, x_1/t, x_2, \ldots, x_n).$$

There is a pushforward map

$$i_\pi\colon \pi^{0,0}(\mathbb{G}_m) \to \pi^{2n,n}(\mathbb{A}^{n+1} - \{0\})$$

induced by the trivialization θ. Let

$$\partial\colon \pi^{2n,n}(\mathbb{A}^{n+1} - \{0\}) \to \pi^{2n+1,n}(T^{\wedge n+1})$$

be the connecting homomorphism in the localization sequence for the embedding $\{0\} \to \mathbb{A}^{n+1}$.

Set $X = \mathbb{A}^{n+1} - \{0\}$ and let $x = (1, 1, 0, \ldots, 0)$ be a point on X. We need the following well-known result.

Lemma 10. *There is a canonical isomorphism in the homotopy category*

$$\phi\colon (X, x) \xrightarrow{\simeq} (\mathbb{G}_m, 1) \wedge T^{\wedge n}$$

that is a composition $\phi = \phi_2^{-1}\phi_1$ for isomorphisms

$$(X, x) \xrightarrow{\phi_1} X/(X - (\mathbb{G}_m - \{1\})) \xleftarrow{\phi_2} (\mathbb{G}_m, 1) \wedge T^{\wedge n},$$

where ϕ_1 is induced by the identity map on X and ϕ_2 is induced by the inclusion $\mathbb{G}_m \times \mathbb{A}^n \subset X$.

Proof. The first map ϕ_1 is an isomorphism since $X - (\mathbb{G}_m - \{1\})$ is $\mathbb{A}^1$-contractible. The second isomorphism is induced by the excision isomorphism $(\mathbb{G}_{m+}, +) \wedge T^{\wedge n} \cong X/(X - \mathbb{G}_m)$. □

Proposition 4. *In the above notation we have $\partial i_\pi(1) = \Sigma^{2n+2,n+1}\eta$.*

Proof. From the construction of the pushforward map we have

$$i_\pi(1) = z^\pi d_i^\pi(th(U, \theta))$$

with z^π being a support extension and d_i^π a deformation to the normal bundle isomorphism. Represent i as a composition

$$i\colon \mathbb{G}_m \xrightarrow{i_1} U \xrightarrow{i_2} X$$

and let $s\colon Th(i_1) \xrightarrow{\simeq} Th(i)$ be the induced isomorphism in the homotopy category. Recall that for the total space of a vector bundle U there is a natural isomorphism [Ne2, proof of Proposition 3.1] $D(\mathbb{G}_m, U) \cong U \times \mathbb{A}^1$ and $d^\pi_{i_1} = id$.

By the functoriality of the deformation construction we have $d_i^\pi = (s^\pi)^{-1}$, so we need to compute

$$\partial z^\pi (s^\pi)^{-1}(th(U,\theta)).$$

The choice of the point x on X induces a map $r\colon (X_+,+) \to (X,x)$ such that r^π splits $\phi^\pi\Sigma^{-1,0}\partial$, i.e. $\phi^\pi\Sigma^{-1,0}\partial r^\pi = id$.

$$\begin{tikzcd}
\pi^{2n,n}(X) \arrow[d, "\partial"'] & & \\
\pi^{2n+1,n}(T^{\wedge n+1}) \arrow[r, "\Sigma^{-1,0}"'] & \pi^{2n,n}((\mathbb{G}_m,1)\wedge T^{\wedge n}) \arrow[r, "\phi^\pi"'] & \pi^{2n,n}(X,x) \arrow[llu, "r^\pi"']
\end{tikzcd}$$

Decomposing z in

$$z\colon (X_+,+) \xrightarrow{r} (X,x) \xrightarrow{z_1} Th(i)$$

we obtain

$$\begin{aligned}\partial z^\pi (s^\pi)^{-1}(th(U,\theta)) = \Sigma^{1,0}(\phi^\pi)^{-1}\phi^\pi\Sigma^{-1,0}\partial r^\pi z_1^\pi (s^\pi)^{-1}(th(U,\theta)) = \\ = \Sigma^{1,0}(\phi^\pi)^{-1} z_1^\pi (s^\pi)^{-1}(th(U,\theta)).\end{aligned}$$

We can represent the Thom class $th(U,\theta) \in \pi^{2n,n}(Th(i_1))$ by the map

$$(t, x_1, x_2, \ldots, x_n) \mapsto (x_1/t, x_2, \ldots, x_n).$$

Identifying one copy of T with $\mathbb{P}^1/\mathbb{A}^1$ we rewrite the above map in the following way

$$\widetilde{H}_1\colon (t, x_1, x_2, \ldots, x_n) \mapsto ([t, x_1], x_2, \ldots, x_n).$$

Consider the following diagram.

$$\begin{tikzcd}
 & (\mathbb{P}^1/\mathbb{A}^1)\wedge T^{\wedge n-1} & \\
Th(i_1) \arrow[ru, "\widetilde{H}_1"] \arrow[r, "s_1", "\simeq"'] & ((\mathbb{A}^2-\{0\})/(\mathbb{A}^2-\mathbb{A}^1))\wedge T^{\wedge n-1} \arrow[u, "\widetilde{H}_2"] \arrow[r, "s_2", "\simeq"'] & Th(i) \\
(\mathbb{A}^2-\{0\},(1,1))\wedge T^{\wedge n-1} \arrow[ru, "j"] \arrow[r, "\psi_1", "\simeq"'] & X/(X-(\mathbb{G}_m-\{1\})) \arrow[ru, "j'"] & (X,x) \arrow[l, "\phi_1"', "\simeq"] \arrow[u, "z_1"']
\end{tikzcd}$$

Here $\widetilde{H}_2$ is defined by the same formula as $\widetilde{H}_1$ and all the other maps are given by the tautological inclusions, i.e. s_1 is induced by the inclusion $U \subset (\mathbb{A}^2-\{0\})\times\mathbb{A}^{n-1}$, s_2 and ψ_1 are induced by $(\mathbb{A}^2-\{0\})\times\mathbb{A}^{n-1} \subset X$, j' is given by the identity map on X and j is given by identity map on $\mathbb{A}^2-\{0\}$. One can easily check that this diagram is commutative.

We can represent $z_1^\pi(s^\pi)^{-1}(th(U,\theta))$ by the morphism

$$\widetilde{H}_2 j \psi_1^{-1}\phi_1 = (\Sigma^{2n-2,n-1}H)\psi_1^{-1}\phi_1.$$

At last, there is the following commutative diagram consisting of isomorphisms.

$$\begin{tikzcd}
(\mathbb{A}^2-\{0\},(1,1))\wedge T^{\wedge n-1} \arrow[r, "\psi_1"] \arrow[d, "\psi_2"'] & X/(X-(\mathbb{G}_m-\{1\})) \\
(\mathbb{A}^2-\{0\})/(\mathbb{A}^2-(\mathbb{G}_m-\{1\}))\wedge T^{\wedge n-1} \arrow[ru] & (\mathbb{G}_m,1)\wedge T\wedge T^{\wedge n-1} \arrow[l, "\psi_3"'] \arrow[u, "\phi_2"']
\end{tikzcd}$$

As above, all the maps in the diagram are induced by the tautological inclusions, ψ_3 is induced by $\mathbb{G}_m \times \mathbb{A}^1 \subset \mathbb{A}^2 - \{0\}$ and ψ_2 is given by the identity map on $\mathbb{A}^2 - \{0\}$.

Thus we can represent $\Sigma^{1,0}(\phi^\pi)^{-1} z_1^\pi (s^\pi)^{-1}(th(U,\theta))$ by the morphism

$$\Sigma^{1,0}((\Sigma^{2n-2,n-1}H)\psi_1^{-1}\phi_1\phi_1^{-1}\phi_2) = \Sigma^{1,0}((\Sigma^{2n-2,n-1}H)\psi_1^{-1}\phi_2) = \\ = \Sigma^{1,0}((\Sigma^{2n-2,n-1}H)\psi_2^{-1}\psi_3).$$

It remains to notice that $\psi_2^{-1}\psi_3$ is a suspension of the canonical isomorphism $(\mathbb{G}_m, 1) \wedge T \cong (\mathbb{A}^2 - \{0\}, (1,1))$ we used to define η, so we obtain $\Sigma^{2n+2,n+1}\eta$. □

7. Complement to the zero section.

In this section we compute the cohomology of the complement to the zero section of a special linear vector bundle. It turns out that there is a good answer in terms of the characteristic classes only in the case of odd rank.

Recall that for a special linear bundle $\mathcal{T}$ we denote by $\mathcal{T}^0$ the complement to the zero section. We start from the following lemma concerning the case of a special linear bundle possessing a section.

Notation 6. We denote an operator of the $\cup$-product with an element by the symbol of the element.

Lemma 11. *Let $\mathcal{T}$ be a rank k special linear bundle over a smooth variety X with a nowhere vanishing section $s \to \mathcal{T}$. Then for some $\alpha \in A^{k-1}(X)$ we have an isomorphism*

$$(1,\alpha)\colon A^*(X) \oplus A^{*+1-k}(X) \to A^*(\mathcal{T}^0).$$

Proof. Consider the Gysin sequence

$$\ldots \to A^{*-k}(X) \xrightarrow{0} A^*(X) \xrightarrow{j^A} A^*(\mathcal{T}^0) \xrightarrow{\partial_A} A^{*-k+1}(X) \xrightarrow{0} \ldots$$

The section s induces a splitting s^A for j^A hence gives a splitting r for ∂_A. We have the claim for $\alpha = r(1)$. □

We want to obtain an isomorphism which does not depend on the choice of the section, so we act as in the projective bundle theorem for oriented cohomology theories: take a certain special linear bundle over $\mathcal{T}^0$ and compute its Euler class.

Definition 10. Let $p\colon E \to X$ be a vector bundle over a smooth variety X. The tautological line subbundle L_E of $(p^*E)|_{E^0}$ could be trivialized by means of the diagonal section $\Delta\colon E^0 \to E^0 \times_X E$. Hence, by lemma 2, for a special linear bundle (E,λ) there exists a canonical trivialization

$$\lambda_{\mathcal{T}_E}\colon \det(p^*E|_{E^0}/L_E) \xrightarrow{\simeq} \mathcal{O}_{E^0}.$$

We obtain a special linear bundle $\mathcal{T}_E = ((p^*E|_{E^0}/L_E), \lambda_{\mathcal{T}_E})$ over E^0.

For the Witt groups there is a result by Balmer and Gille [BG, Theorem 8.13].

Theorem 2. *Let $(E,\lambda)=(\mathcal{O}_{\mathrm{pt}}^{2n+1},1)$ be a trivialized special linear bundle of odd rank over a point with $n\geq 1$. Then for $e=e(\mathcal{T}_E)\in W^{2n}(E^0)$ we have an isomorphism*

$$(1,e)\colon W^*(pt)\oplus W^{*-2n}(pt)\xrightarrow{\simeq} W^*(E^0).$$

We can derive an analogous result for $A^*(-)$ from our computation in stable cohomotopy groups.

Lemma 12. *Let $(E,\lambda)=(\mathcal{O}_{\mathrm{pt}}^{2n+1},1)$, $n\geq 1$, be a trivialized special linear bundle over a point. Then for $e=e(\mathcal{T}_E)\in A^{2n}(E^0)$ we have an isomorphism*

$$(1,e)\colon A^*(pt)\oplus A^{*-2n}(pt)\xrightarrow{\simeq} A^*(E^0).$$

Proof. Consider the Gysin sequence

$$\ldots\to A^{*-2n-1}(pt)\xrightarrow{0} A^*(pt)\to A^*(E^0)\xrightarrow{\partial_A} A^{*-2n}(pt)\xrightarrow{0}\ldots$$

The bundle E is trivial hence $e(E,\lambda)=0$ and the Gysin sequence consists of short exact sequences.

Consider the dual special linear bundle $\mathcal{T}_E^\vee$. Taking the dual trivialization of $E^\vee$ we obtain

$$\mathcal{T}_E^\vee=\{(x_0,\ldots,x_{2n},y_0,\ldots,y_{2n})\in E^0\times E^\vee\mid x_0y_0+\cdots+x_{2n}y_{2n}=0\}.$$

There is a section $s\colon E^0\to\mathcal{T}_E^\vee$ with

$$s(x_0,x_1,x_2,\ldots,x_{2n-1},x_{2n})=(x_0,x_1,\ldots,x_{2n},0,x_2,-x_1,\ldots,x_{2n},-x_{2n-1}).$$

This section meets the zero section in $\mathbb{G}_m\cong\{(t,0,\ldots,0)\mid t\neq 0\}$. Proposition 1 states that $e(\mathcal{T}_E^\vee)=i_A(1)$ for the inclusion $i\colon\mathbb{G}_m\to\mathbb{A}^{2n+1}-\{0\}$ with the trivialization of $\det N_i$ arising from the trivialization of $\det\mathcal{T}_E^\vee$. Identify $N_i\cong\mathcal{T}_E^\vee|_{\mathbb{G}_m}$ with $U=\mathbb{G}_m\times\mathbb{A}^{2n}\subset E^0$ via

$$(t,0,\ldots,0,0,y_1\ldots,y_{2n})\mapsto(t,y_1,\ldots,y_{2n}).$$

The isomorphism $\lambda_{\mathcal{T}_E^\vee}\colon\det\mathcal{T}_E^\vee|_{\mathbb{G}_m}\xrightarrow{\simeq}\mathcal{O}_{\mathbb{G}_m}$ arises from the canonical trivialization of $E^\vee|_{\mathbb{G}_m}$ and morphism $\phi\colon E^\vee\to L_E^\vee\cong\mathcal{O}_{\mathbb{G}_m}$ with

$$\phi(t,y_0,y_1,\ldots,y_{2n})=(t,ty_0).$$

Thus over t for $\mathbf{y}^i=(y_1^i,y_2^i,\ldots,y_{2n}^i)$ we have

$$\lambda_{\mathcal{T}_E^\vee}(\mathbf{y}^1\wedge\mathbf{y}^2\wedge\cdots\wedge\mathbf{y}^{2n})=\det\begin{pmatrix}1/t&0&0&\ldots&0\\0&y_1^1&y_1^2&\ldots&y_1^{2n}\\0&y_2^1&y_2^2&\ldots&y_2^{2n}\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&y_{2n}^1&y_{2n}^2&\ldots&y_{2n}^{2n}\end{pmatrix}$$

and $\theta(U,\lambda_{\mathcal{T}_E^\vee})\xrightarrow{\simeq}(\mathcal{O}_{\mathbb{G}_m}^{2n},1)$ with $\theta(t,y_1,y_2,\ldots,y_{2n})=(t,y_1/t,y_2,\ldots,y_{2n})$ is an isomorphism of special linear bundles.

Consider the following diagram with i_π being a pushforward in stable cohomotopy groups for the closed embedding i with the trivialization θ of the normal bundle.

$$\begin{tikzcd}
A^0(\mathbb{G}_m) \arrow[r, "i_A"] & A^{2n}(E^0) \arrow[r, "\partial_A"] & A^0(pt) \\
\overline{\pi}^0(\mathbb{G}_m) \arrow[r, "i_\pi"] \arrow[u] & \overline{\pi}^{2n}(E^0) \arrow[r, "\partial_\pi"] \arrow[u] & \overline{\pi}^0(pt) \arrow[u]
\end{tikzcd}$$

The left-hand side commutes since θ is an isomorphism of special linear bundles. The right-hand side of the diagram consist of the structure morphisms for A and the boundary maps for the Gysin sequences of the inclusion $\{0\} \to E$ hence commutes as well. Proposition 4 states that $\partial_\pi i_\pi(1) = 1$, hence

$$\partial_A(e(\mathcal{T}_E^\vee)) = \partial_A i_A(1) = 1$$

and $\{1, e(\mathcal{T}_E^\vee)\}$ forms a basis of $A^*(E^0)$ over $A^*(pt)$. There is a nowhere vanishing section of $\mathcal{T}_E^\vee \oplus \mathcal{T}_E^\vee$ constructed analogous to s defined above, so

$$e(\mathcal{T}_E^\vee)^2 = e(\mathcal{T}_E^\vee \oplus \mathcal{T}_E^\vee) = 0.$$

By Lemma 6 for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in A^*(pt)$ we have

$$e = (\alpha_1 + \beta_1 \cup e(\mathcal{T}_E^\vee)) \cup e(\mathcal{T}_E^\vee) = \alpha_1 \cup e(\mathcal{T}_E^\vee),$$

$$e(\mathcal{T}_E^\vee) = (\alpha_2 + \beta_2 \cup e(\mathcal{T}_E^\vee)) \cup e = \alpha_2 \cup \alpha_1 \cup e(\mathcal{T}_E^\vee).$$

We already know that $\{1, e(\mathcal{T}_E^\vee)\}$ is a basis, then $\alpha_2 \cup \alpha_1 = 1$ and α_1 is invertible. Hence $\{1, \alpha_1 \cup e(\mathcal{T}_E^\vee)\} = \{1, e\}$ is a basis as well.

$\square$

Corollary 1. *Let $A^{*,*}(-)$ be a oriented cohomology theory represented by a commutative monoid $A \in \mathcal{SH}(k)$. Then $\overline{A}^*(pt) = 0$.*

Proof. There is a natural special linear orientation on $A^{*,*}(-)$ obtained by setting $th(E, \lambda) = th(E)$ with the latter Thom class arising from the orientation on $A^{*,*}(-)$. Hence for a rank n special linear bundle we have $e(E, \lambda) = c_n(E)$. By the above lemma, for $E = \mathcal{O}_{pt}^3$ there is an isomorphism

$$(1, c_2(\mathcal{T}_E)) \colon \overline{A}^*(pt) \oplus \overline{A}^{*-2}(pt) \xrightarrow{\simeq} \overline{A}^*(E^0).$$

Applying the Cartan formula we obtain $c_*(\mathcal{O}_{E^0}) c_*(\mathcal{T}_E) = c_*(\mathcal{O}_{E^0}^3)$, thus $c_2(\mathcal{T}_E) = 0$. The above isomorphism yields $\overline{A}^*(pt) = 0$. $\square$

Having a canonical basis for a trivial bundle we can glue it into a basis in the cohomology of the complement to the zero section of an arbitrary special linear bundle of odd rank.

Theorem 3. *Let (E, λ) be a special linear bundle of rank $2n+1, n \geq 1$, over a smooth variety X. Then for $e = e(\mathcal{T}_E)$ we have an isomorphism*

$$(1, e) \colon A^*(X) \oplus A^{*-2n}(X) \to A^*(E^0).$$

Proof. The general case is reduced to the case of the trivial vector bundle E via the usual Mayer-Vietoris arguments. In the latter case we have a commutative diagram of the Gysin sequences

$$\begin{tikzcd}
0 \arrow[r] & A^*(X) \arrow[r] & A^*(E^0) \arrow[r, "\partial_A"] & A^{*-2n}(X) \arrow[r] & 0 \\
0 \arrow[r] & A^*(pt) \arrow[r] \arrow[u] & A^*(E'^0) \arrow[r, "\partial_A"] \arrow[u] & A^{*-2n}(pt) \arrow[r] \arrow[u, "p^A"] & 0
\end{tikzcd}$$

with $E' = \mathcal{O}_{pt}^{2n+1}$. By Lemma 12 the element $\partial_A e\left(\mathcal{T}_{E'}\right)$ generates $A^{*-2n}(pt)$ as a module over $A^*(pt)$, thus for a certain $\alpha \in A^*(pt)$ we have $\alpha \cup \partial_A e\left(\mathcal{T}_{E'}\right) = 1$. Using $E = p^* E'$ we obtain

$$\alpha \cup \partial_A e\left(\mathcal{T}_E\right) = \alpha \cup p^A \partial_A e\left(\mathcal{T}_{E'}\right) = 1,$$

so $\partial_A\left(e\left(\mathcal{T}_E\right)\right)$ generates $A^{*-2n}(X)$ over $A^*(X)$. Hence $(1, e)$ is an isomorphism. $\square$

Remark 4. In case of $\operatorname{rank} E = 1$ one still has an isomorphism: a special linear bundle of rank one is a trivialized line bundle, hence there is an isomorphism

$$A^*(X) \oplus A^*(X) \cong A^*(E^0) = A^*(X \times \mathbb{G}_m)$$

induced by the isomorphism $A^*(pt) \oplus A^*(pt) \cong A^*(\mathbb{G}_m)$.

Corollary 2. *Let $\mathcal{T}$ be a special linear bundle of odd rank over a smooth variety X. Then $e(\mathcal{T}) = 0$.*

Proof. Set $\operatorname{rank} \mathcal{T} = 2n+1$ and $e = e(\mathcal{T})$. Consider the Gysin sequence

$$\ldots \to A^0(X) \xrightarrow{e} A^{2n+1}(X) \xrightarrow{j^A} A^{2n+1}(\mathcal{T}^0) \to A^1(X) \to \ldots$$

The above calculations show that j^A is injective hence $e = 0$. $\square$

8. Special linear projective bundle theorem.

Definition 11. For $k < n$ consider the group

$$P'_k = \begin{pmatrix} SL_k & * \\ 0 & SL_{n-k} \end{pmatrix}.$$

The quotient variety $SGr(k, n) = SL_n / P'_k$ is called a *special linear Grassmann variety.*

Notation 7. Denote by $\mathcal{T}_1$ and $\mathcal{T}_2$ the tautological special linear bundles over $SGr(k, n)$ with $\operatorname{rank} \mathcal{T}_1 = k$ and $\operatorname{rank} \mathcal{T}_2 = n - k$.

Remark 5. We have a projection $SL_n/P'_k \to SL_n/P_k$ identifying the special linear Grassmann variety with the complement to the zero section of the determinant of the tautological vector bundle over the ordinary Grassmann variety $Gr(k, n)$. This yields the following geometrical description of $SGr(k, n)$: fix a vector space V of dimension n. Then

$$SGr(k, n) = \{(U \leq V, \lambda \in (\Lambda^k U)^0) \mid \dim U = k\}.$$

In particular, we have $SGr(1, n) \cong \mathbb{A}^n - \{0\}$.

Theorem 4. *For the special linear Grassmann varieties we have the following isomorphisms.*

$$(1, e_1, ..., e_1^{2n-2}, e_2)\colon \bigoplus_{i=0}^{2n-2} A^{*-2i}(pt) \oplus A^{*-2n+2}(pt) \to A^*(SGr(2, 2n)),$$

$$(1, e_1, e_1^2, ..., e_1^{2n-1})\colon \bigoplus_{i=0}^{2n-1} A^{*-2i}(pt) \to A^*(SGr(2, 2n+1)),$$

with $e_1 = e(\mathcal{T}_1)$*,* $e_2 = e(\mathcal{T}_2)$*.*

Proof. We are going to deal with several special linear Grassmann varieties at once, so we will use $\mathcal{T}_i(r,k)$ for $\mathcal{T}_i$ over $SGr(r,k)$ and abbreviate $e(\mathcal{T}_i(r,k))$ to $e_i(r,k)$ and $e(\mathcal{T}_i(r,k)^\vee)$ to $e_i^\vee(r,k)$. The proof is done by induction on the Grassmannian's dimension.

The base case. We have $SGr(2,3) \cong SGr(1,3) \cong \mathbb{A}^3 - \{0\}$ and under these isomorphisms the bundle $\mathcal{T}_1(2,3)^\vee$ goes to $\mathcal{T}_2(1,3)$ which goes to $\mathcal{T}_{\mathcal{O}^3_{pt}}$ in the notation of definition 10. Note that $\operatorname{rank} \mathcal{T}_1(2,3) = 2$, thus $\mathcal{T}_1(2,3) \cong \mathcal{T}_1(2,3)^\vee$ and $e(\mathcal{T}_1(2,3)) = e\left(\mathcal{T}_1(2,3)^\vee\right)$. Hence Lemma 12 gives the claim for $SGr(2,3)$.

Basic geometry. Fix a vector space V of dimension $k+1$, a subspace $W \le V$ of codimension one and forms $\mu_1 \in (\Lambda^{k+1}V)^0, \mu_2 \in (\Lambda^k W)^0$. Then we have the following diagram constructed in the same vein as in the case of ordinary Grassmannians:

$$\begin{tikzcd} SGr(2,k) \arrow[r, "i"] & SGr(2,k+1) & Y \arrow[l, "j"'] \arrow[d, "p"] \\ & & SGr(1,k) \end{tikzcd}$$

the inclusion i corresponds to the pairs $(U, \mu \in (\Lambda^2 U)^0)$ with $U \le W$, $\dim U = 2$; the open complement Y consists of the pairs $(U, \mu \in (\Lambda^2 U)^0)$ with $\dim U = 2, \dim U \cap W = 1$; the projection p is given by $p(U,\mu) = (U \cap W, \mu')$ where μ' is given by the isomorphism $(U \cap W) \otimes V/W \cong \Lambda^2 U$. Here i is a closed embedding, j is an open embedding and p is an $\mathbb{A}^k$-bundle. Take an arbitrary $f \in V^\vee$ such that $\ker f = W$. It gives rise to a constant section of the trivial bundle $\left(\mathcal{O}^{k+1}_{SGr(2,k+1)}\right)^\vee$ hence a section of $\mathcal{T}_1(2,k+1)^\vee$. The latter section vanishes exactly over $i(SGr(2,k))$. Note that we have $\operatorname{rank} \mathcal{T}_1(2,k+1) = 2$ hence $e_1^\vee(2,k+1) = e_1(2,k+1)$.

k=2n-1. Consider the localization sequence.

$$\ldots \to A^{*-2}(SGr(2,2n-1)) \xrightarrow{i_A} A^*(SGr(2,2n)) \xrightarrow{j^A} A^*(SGr(1,2n-1)) \to \ldots$$

Lemma 12 states that $\{1, e_2(1,2n-1)\}$ is a basis of $A^*(SGr(1,2n-1))$ over $A^*(pt)$. We have $j^*\mathcal{T}_2(2,2n) \cong p^*\mathcal{T}_2(1,2n-1)$ and

$$j^A(e_2(2,2n)) = e_2(1,2n-1)$$

hence j^A is a split surjection (over $A^*(pt)$) with the splitting defined by

$$1 \mapsto 1,\ e_2(1,2n-1) \mapsto e_2(2,2n).$$

Then i_A is injective. Hence to obtain a basis of $A^*(SGr(2,2n))$ it is sufficient to calculate the pushforward for a basis of $A^{*-2}(SGr(2,2n-1))$ and combine it with $\{1, e_2(2,2n)\}$. Using the induction we know that

$$\{1, e_1(2,2n-1), \ldots, e_1(2,2n-1)^{2n-3}\}$$

is a basis of $A^*(SGr(2,2n-1))$. We have $i^*(\mathcal{T}_1(2,2n)) \cong \mathcal{T}_1(2,2n-1)$ hence

$$e_1(2,2n-1) = i^A(e_1(2,2n)).$$

By Proposition 1 we have

$$i_A(e_1(2,2n-1)^l) = e_1(2,2n)^{l+1}$$

obtaining the desired basis

$$\{e_1(2,2n), e_1(2,2n)^2, \ldots, e_1(2,2n)^{2n-2}, 1, e_2(2,2n)\}$$

of $A^*(SGr(2,2n))$ over $A^*(pt)$.

k=2n. Consider the localization sequence.

$$\ldots \xrightarrow{\partial_A} A^{*-2}(SGr(2,2n)) \xrightarrow{i_A} A^*(SGr(2,2n+1)) \xrightarrow{j^A} A^*(SGr(1,2n)) \xrightarrow{\partial_A} \ldots$$

Using the induction we know a basis of $A^*(SGr(2,2n))$, namely

$$\{1, e_1(2,2n), e_1(2,2n)^2, \ldots, e_1(2,2n)^{2n-2}, e_2(2,2n)\}$$

and Lemma 11 gives us a non-canonical basis $\{1, \alpha\}$ for $A^*(SGr(1,2n))$. Examine $i_A(e_2(2,2n))$. It can't be computed using Proposition 1 since it seems that $e_2(2,2n)$ can not be pullbacked from $A^*(SGr(2,2n+1))$, so we use the following argument. Consider a nontrivial vector $w \in W$. It induces constant sections of $\mathcal{O}^{2n}_{SGr(2,2n)}$ and $\mathcal{O}^{2n+1}_{SGr(2,2n+1)}$ and sections of $\mathcal{T}_2(2,2n)$ and $\mathcal{T}_2(2,2n+1)$. The latter sections vanish over $SGr(1,2n-1)$ and $SGr(1,2n)$ respectively. Here $SGr(1,2n-1)$ corresponds to the vectors in $W/\langle w\rangle$ and $SGr(1,2n)$ corresponds to the vectors in $V/\langle w\rangle$. Hence we have the following commutative diagram consisting of closed embeddings.

$$\begin{tikzcd}
SGr(1,2n-1) \arrow[r, "r'"] \arrow[d, "i'"] & SGr(2,2n) \arrow[d, "i"] \\
SGr(1,2n) \arrow[r, "r"] & SGr(2,2n+1)
\end{tikzcd}$$

By Proposition 1 we have $e_2(2,2n) = r'_A(1)$, so, using Proposition 2, we obtain $i_A(e_2(2,2n)) = r_A i'_A(1)$. Notice that $N_{i'}$ is a trivial bundle of rank one. In fact, there is a section of trivial bundle $\mathcal{T}_1(1,2n)^\vee$ over $SGr(1,2n)$ constructed using the same element f such that $\ker f = W$ and this section meets the zero section exactly at $SGr(1,2n-1)$. So we have

$$i_A(e_2(2,2n)) = r_A i'_A(1) = r_A(e_1^\vee(1,2n)) = r_A(0) = 0.$$

We claim that $\ker i_A = A^*(pt) \cup e_2(2,2n)$ and $\operatorname{Im} j^A = A^*(pt) \cup 1$. We have $j^A(1) = 1$, hence $\partial_A(1) = 0$ and

$$\ker i_A = \operatorname{Im} \partial_A = A^*(pt) \cup \partial_A(\alpha).$$

The localization sequence is exact, so we have

$$e_2(2,2n) = \partial_A(y \cup \alpha) = y \cup \partial_A(\alpha)$$

for some $y \in A^*(pt)$ and since $e_2(2,2n)$ is an element of the basis, y is not a zero divisor. Consider the presentation of $\partial_A(\alpha)$ with respect to the chosen basis:

$$\partial_A(\alpha) = x_0 \cup 1 + x_1 \cup e_1(2,2n) + \cdots + x_{2n-2} \cup e_1(2,2n)^{2n-2} + z \cup e_2(2,2n).$$

We have $y \cup \partial_A(\alpha) = e_2(2,2n)$, hence $y \cup z = 1$ and every $y \cup x_i = 0$, hence $x_i = 0$. Then $\partial_A(\alpha) = z \cup e_2(2,2n)$ and

$$\ker i_A = \operatorname{Im} \partial_A = A^*(pt) \cup \partial_A(\alpha) = A^*(pt) \cup e_2(2,2n).$$

We have

$$\partial_A(x_0 \cup 1 + x_1 \cup \alpha) = x_1 \cup \partial_A(\alpha) = x_1 \cup z \cup e_2(2,2n),$$

hence $\operatorname{Im} j^A = \ker \partial_A = A^*(pt) \cup 1$.

There is an obvious splitting for $A^*(SGr(2,2n+1)) \xrightarrow{j^A} \operatorname{Im} j^A$, $1 \mapsto 1$. Then calculating by the same vein as in the odd-dimensional case the pushforwards for the basis of $\operatorname{Coker} \partial_A$, $\{e_1(2,2n)^l\}$, and adding to them $\{1\}$, we obtain the desired basis of $SGr(2,2n+1)$

$$\{e_1(2,2n+1), \ldots, e_1(2,2n+1)^{2n-1}, 1\}. \qquad \square$$

Corollary 3. *There is an isomorphism:*

$$A^*(pt)[e] \big/ \langle e^{2n-2} \rangle \xrightarrow{\simeq} A^*(SGr(2,2n-1)),$$

induced by sending e to $e(\mathcal{T}_1)$.

Proof. Keep the notations from the proof of the theorem. It is sufficient to show that $e_1(2,2n-1)^{2n-2} = 0$.

Consider a vector space V of dimension $2n-1$ and a collection of subspaces $W_i \le V$ of codimension one such that $\dim \bigcap_{i=1}^{2n-2} W_i = 1$. Every subspace W_i defines a section of $\left(\mathcal{O}^{2n-1}_{SGr(2,2n-1)}\right)^\vee$ and a section s_i of $\mathcal{T}_1(2,2n-1)^\vee$ in the same vein as in the proof of the theorem. The section of $(\mathcal{T}_1(2,2n-1)^\vee)^{\oplus 2n-2}$ defined by $(s_1, \ldots, s_{2n-2})$ vanishes nowhere, hence by Lemma 7 we have

$$e_1(2,2n-1)^{2n-2} = e((\mathcal{T}_1(2,2n-1)^\vee)^{\oplus 2n-2}) = 0.$$

$\square$

Definition 12. Let $\mathcal{T}$ be a special linear bundle over a smooth variety X. Then we define the *relative special linear Grassmann variety* $SGr(k,\mathcal{T})$ in an obvious way. This variety is a $SGr(k, \operatorname{rank} \mathcal{T})$-bundle over X. Similarly to the above, we denote by $\mathcal{T}_1$ and $\mathcal{T}_2$ the tautological special linear bundles over $SGr(k,\mathcal{T})$.

Theorem 5. *Let $\mathcal{T}$ be a special linear bundle over a smooth variety X.*

(1) *If* $\operatorname{rank} \mathcal{T} = 2n$ *then there is an isomorphism*

$$(1, e_1, ..., e_1^{2n-2}, e_2) \colon \bigoplus_{i=0}^{2n-2} A^{*-2i}(X) \oplus A^{*-2n+2}(X) \xrightarrow{\simeq} A^*(SGr(2,\mathcal{T})),$$

with $e_1 = e(\mathcal{T}_1)$, $e_2 = e(\mathcal{T}_2)$.

(2) *If* $\operatorname{rank} \mathcal{T} = 2n+1$ *then there is an isomorphism*

$$(1, e, e^2, ..., e^{2n-1})\colon \bigoplus_{i=0}^{2n-1} A^{*-2i}(X) \xrightarrow{\simeq} A^*(SGr(2, \mathcal{T})),$$

with $e = e(\mathcal{T}_1)$.

Proof. The general case is reduced to the case of the trivial bundle $\mathcal{T}$ via the usual Mayer-Vietoris arguments. The latter case follows from Theorem 4. □

9. Symmetric polynomials.

In this section we deal with the polynomials invariant under the action of the Weyl group $W(B_n)$ or $W(D_n)$ and obtain certain spanning sets for the polynomial rings. Our method is an adaptation of the one used in [Fu, § 10, Proposition 3]. The proof is quite straightforward but a bit messy.

Consider $\mathbb{Z}^n$ and fix a usual basis $\{e_1, \ldots, e_n\}$. Let

$$W(B_n) = \{\phi \in Aut(\mathbb{Z}^n) \mid \phi(e_i) = \pm e_j\}$$

be the Weyl group of the root system B_n and let

$$W(D_n) = \{\phi \in Aut(\mathbb{Z}^n) \mid \phi(e_i) = (-1)^{k_i} e_j,\ (-1)^{\sum k_i} = 1\}$$

be the Weyl group of the root system D_n. Identifying $R = \mathbb{Z}[e_1, \ldots, e_n]$ with the symmetric algebra $Sym^*((\mathbb{Z}^n)^\vee)$ in a usual way, we obtain the actions of these Weyl groups on R. Let $R_B = R^{W(B_n)}$ and $R_D = R^{W(D_n)}$ be the algebras of invariants.

For the elementary polynomials $\sigma_i \in \mathbb{Z}[x_1, \ldots, x_n]$ consider

$$s_i = \sigma_i(e_1^2, \ldots, e_n^2), \quad t = \sigma_n(e_1, \ldots, e_n).$$

One can easily check that $R_B = \mathbb{Z}[s_1, \ldots, s_n]$ and $R_D = \mathbb{Z}[s_1, \ldots, s_{n-1}, t]$.

In order to compute spanning sets for R over R_B and R_D we need "decreasing degree" equalities provided by the following lemma.

Lemma 13. *There exist homogeneous polynomials* $g_i, h_i \in R$ *such that*

$$e_1^{2n} = \sum_{i=1}^{n} g_i s_i, \quad e_1^{2n-1} = \sum_{i=1}^{n-1} h_i s_i + h_n t.$$

Proof. Let $I_B = \langle s_1, \ldots, s_n \rangle$ and $I_D = \langle s_1, \ldots, s_{n-1}, t \rangle$ be the ideals generated by the homogeneous invariant polynomials of positive degree. We need to show that $e_1^{2n} \in I_B$ and $e_1^{2n-1} \in I_D$. Set $S_B = R/I_B$, $S_D = R/I_D$.

Consider $S_B\,[[x]]$. Since all the s_i belong to I_B we have

$$(1 - \bar{e}_1^2 x)(1 - \bar{e}_2^2 x)\ldots(1 - \bar{e}_n^2 x) = 1,$$

hence

$$(1 - \bar{e}_2^2 x)(1 - \bar{e}_3^2 x)\ldots(1 - \bar{e}_n^2 x) = 1 + \bar{e}_1^2 x + \bar{e}_1^4 x^2 + \ldots$$

Comparing the coefficients at x^n we obtain $\bar{e}_1^{2n} = 0$, thus $e_1^{2n} \in I_B$.

Consider $S_D\,[[x]]$. As above, we have

$$(1 - \bar{e}_1^2 x^2)(1 - \bar{e}_2^2 x^2)\ldots(1 - \bar{e}_n^2 x^2) = 1,$$

hence

$$(1 + \bar{e}_1 x)(1 - \bar{e}_2^2 x^2)(1 - \bar{e}_3^2 x^2)\ldots(1 - \bar{e}_n^2 x^2) = 1 + \bar{e}_1 x + \bar{e}_1^2 x^2 + \ldots$$

Comparing the coefficients at x^{2n-1} we obtain

$$\bar{e}_1^{2n-1} = (-1)^{n-1}\bar{e}_1\bar{e}_2^2\ldots\bar{e}_n^2 = (-1)^{n-1}\bar{t}\bar{e}_2\bar{e}_3\ldots\bar{e}_n = 0,$$

thus $e_1^{2n-1} \in I_D$. $\square$

Proposition 5. *In the above notation we have the following spanning sets:*

(1) $\mathcal{B}_1 = \{e_1^{m_1}e_2^{m_2}\ldots e_n^{m_n} \mid 0 \le m_i \le 2n-2i+1\}$ *spans R over R_B.*

(2) $\mathcal{B}_2 = \left\{ u_1u_2\ldots u_{n-1} \;\middle|\; u_i = \left[\begin{array}{l} e_i^{m_i},\ 0 \le m_i \le 2n-2i \\ e_{i+1}e_{i+2}\ldots e_n \end{array}\right. \right\}$ *spans R over R_D.*

Proof. In both cases proceed by induction on n. The base case of $n=1$ is clear. Denote by $\mathcal{B}_1'$ and $\mathcal{B}_2'$ the spanning sets in $R' = \mathbb{Z}[e_2,\ldots,e_n]$ and let $s_i', t' \in R'$ be the corresponding invariant polynomials. Note that $s_i = e_1s_{i-1}' + s_i'$ and $t = e_1t'$.

It is sufficient to show that every monomial is a R_B (or R_D) linear combination of the monomials of lesser total degree and monomials from the corresponding spanning set.

(1) Consider a monomial $f = e_1^{k_1}e_2^{k_2}\ldots e_n^{k_n} \in R$. In case of $k_1 \ge 2n$ we can use the preceding lemma and substitute $\sum g_is_i$ for e_1^{2n} obtaining

$$f = \sum s_ig_ie_1^{k_1-2n}e_2^{k_2}\ldots e_n^{k_n}$$

with $\deg g_ie_1^{k_1-2n}e_2^{k_2}\ldots e_n^{k_n} < \deg f$, so we get the claim. Now suppose that $k_1 < 2n$. By the induction we have

$$e_2^{k_2}e_3^{k_3}\ldots e_n^{k_n} = \sum \alpha_j(s_1',\ldots,s_{n-1}')b_j'$$

for some $b_j' \in \mathcal{B}_1'$ and $\alpha_j \in \mathbb{Z}[x_1,\ldots,x_{n-1}]$. We can assume that all the summands at the right-hand side are homogeneous of total degree $k_2+\ldots+k_n$. Since $s_i = e_1s_{i-1}' + s_i'$ one has

$$\alpha_j(s_1,\ldots,s_{n-1}) = \alpha_j\left(s_1',\ldots,s_{n-1}'\right) + \sum_{l>0} e_1^l\beta_{jl}$$

for some $\beta_{jl} \in R_B$. Thus we obtain

$$f = \sum_j e_1^{k_1}\alpha_j(s_1,\ldots,s_{n-1})b_j' - \sum_{j,l} e_1^{k_1+l}\beta_{jl}b_j'.$$

Note that $e_1^{k_1}b_j' \in \mathcal{B}_1$, so the first sum is a R_B-linear combination of the monomials from the spanning set. If $\deg\beta_{jl} > 0$ then $\deg e_1^{k_1+l}b_j' < \deg f$ and it is the case of the linear combination with lesser total degree. At last, in case of $\deg\beta_{jl} = 0$ there are two variants: if $k_1+l < 2n$ then we have $e_1^{k_1+l}b_j' \in \mathcal{B}_1$, otherwise $k_1+l \ge 2n$ and we can lower the total degree by using the preceding lemma.

(2) Consider a monomial $f = e_1^{k_1}e_2^{k_2}\ldots e_n^{k_n} \in R$. As above, in case of $k_1 \ge 2n-1$ we can use the preceding lemma and lower the total degree, so suppose that $k_1 < 2n-1$. By induction we have

$$e_2^{k_2}e_3^{k_3}\ldots e_n^{k_n} = \sum \alpha_j(s_1',\ldots,s_{n-2}',t')b_j'$$

for some $b'_j \in \mathcal{B}'_2$ and $\alpha_j \in \mathbb{Z}[x_1, \ldots, x_{n-1}]$. One has $t'^2 = s'_{n-1}$ hence

$$\alpha_j(s'_1, \ldots, s'_{n-2}, t) = \widetilde{\alpha}_j(s'_1, \ldots, s'_{n-2}, s'_{n-1}) + t'\widehat{\alpha}_j(s'_1, \ldots, s'_{n-2}, s'_{n-1}).$$

As above, we can substitute s_i into $\widetilde{\alpha}_j$ and $\widehat{\alpha}_j$ and obtain some $\widetilde{\beta}_{jl}, \widehat{\beta}_{jl} \in R_D$. Thus we have

$$f = \sum_j e_1^{k_1}\widetilde{\alpha}_j(s_1, \ldots, s_{n-1})b'_j + \sum_j e_1^{k_1}\widehat{\alpha}_j(s_1, \ldots, s_{n-1})t'b'_j -$$

$$- \sum_{j,l} e_1^{k_1+l}\widetilde{\beta}_{jl}b'_j - \sum_{j,l} e_1^{k_1+l}\widehat{\beta}_{jl}t'b'_j.$$

In the first sum we have $e_1^{k_1}b'_j \in \mathcal{B}_2$. One has $t'b'_j \in \mathcal{B}_2$, so in case of $k_1 = 0$ the second sum is a linear combination of the elements from the spanning set, otherwise, if $k_1 \geq 1$, one can lower the total degree by carrying out $t = e_1t'$. The third sum is dealt with like the second one in (1), in case of $\deg \widetilde{\beta}_{jl} = 0$ we use that $e_1^{k_1+l}b'_j \in \mathcal{B}_2$ or lower degree using the preceding lemma, otherwise we lower degree by carrying out $\widetilde{\beta}_{jl}$. At last, in the forth sum we lower degree by carrying out $t = e_1t'$.

□

10. A splitting principle.

In this section we assert a splitting principle for the cohomology theories with a special linear orientation and inverted stable Hopf map. The principle states that from the viewpoint of such cohomology theories every special linear bundle is a direct sum of rank 2 special linear bundles and at most one trivial linear bundle.

Definition 13. For $k_1 < k_2 < \cdots < k_m$ consider the group

$$P'_{k_1,\ldots,k_{m-1}} = \begin{pmatrix} SL_{k_1} & * & \ldots & * \\ 0 & SL_{k_2-k_1} & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & SL_{k_m-k_{m-1}} \end{pmatrix}$$

and define a *special linear flag variety* as the quotient

$$\mathcal{SF}(k_1, \ldots, k_m) = SL_{k_m}/P'_{k_1,\ldots,k_{m-1}}.$$

In particular, we are interested in the following varieties:

$$\mathcal{SF}(2n) = \mathcal{SF}(2, 4, ..., 2n),\ \mathcal{SF}(2n+1) = \mathcal{SF}(2, 4, ..., 2n, 2n+1).$$

These varieties are called *maximal SL_2 flag varieties.*

For every special linear flag variety $\mathcal{SF}(k_1, k_2, \ldots, k_m)$ there is an affine variety

$$\widetilde{\mathcal{SF}}(k_1, k_2, \ldots, k_m) = SL_{k_m}/(SL_{k_1} \times SL_{k_2-k_1} \times \cdots \times SL_{k_m-k_{m-1}}).$$

Note that $\widetilde{\mathcal{SF}}(k_1, k_2, \ldots, k_m)$ is an $\mathbb{A}^r$-bundle over $\mathcal{SF}(k_1, k_2, \ldots, k_m)$.

Remark 6. The projection

$$\mathcal{SF}(k_1, k_2, \dots, k_m) = SL_n/P'_{k_1,\dots,k_m} \to SL_n/P_{k_1,\dots,k_m} = \mathcal{F}(k_1, k_2, \dots, k_m)$$

yields the following geometrical description of the special linear flag varieties. Consider a vector space V of dimension k_m. Then we have

$$\mathcal{SF}(k_1, k_2, \dots, k_m) = \\ = \left\{ (V_1 \le \dots \le V_{m-1} \le V,\ \lambda_1, \dots, \lambda_{m-1}) \;\middle|\; \dim V_j = k_j,\ \lambda_j \in \left(\Lambda^{k_j} V_j\right)^0 \right\}$$

Notation 8. Denote by $\mathcal{T}_i$ the tautological special linear bundles over $\mathcal{SF}(k_1, k_2, \dots, k_m)$ with $\operatorname{rank} \mathcal{T}_i = k_i - k_{i-1}$.

Definition 14. Let $\mathcal{T}$ be a special linear bundle over a smooth variety X. Then we define the *relative special linear flag variety* $\mathcal{SF}_\mathcal{T}(k_1, k_2, \dots, k_m)$ with $k_m = \operatorname{rank} \mathcal{T}$ in an obvious way. This variety is a $\mathcal{SF}(k_1, k_2, \dots, k_m)$-bundle over X. We also define relative version of the maximal SL_2 flag variety, $\mathcal{SF}(\mathcal{T})$, and relative versions for the affine coverings, $\widetilde{\mathcal{SF}}_\mathcal{T}(k_1, k_2, \dots, k_m)$ and $\widetilde{\mathcal{SF}}(\mathcal{T})$.

Theorem 6. *Let $\mathcal{T}$ be a rank k special linear bundle over a smooth variety X. Then $A^*(\mathcal{SF}_\mathcal{T}(2, 4, \dots, 2n, k))$ is a free module over $A^*(X)$ with the following basis:*

- *k is odd:*
$$\left\{ e_1^{m_1} e_2^{m_2} \dots e_n^{m_n} \mid 0 \le m_i \le k - 2i \right\},$$
- *k is even:*
$$\left\{ u_1 u_2 \dots u_n \;\middle|\; u_i = \left[\begin{array}{l} e_i^{m_i},\ 0 \le m_i \le k - 2i \\ e_{i+1} e_{i+2} \dots e_{n+1} \end{array} \right. \right\},$$

where $e_i = e(\mathcal{T}_i, \lambda_{\mathcal{T}_i})$.

Proof. Proceed by induction on n. For $n = 1$ the claim follows from Theorem 5.

Consider the projection

$$p\colon Y = \mathcal{SF}_\mathcal{T}(2, 4, \dots, 2n, k) \to \mathcal{SF}_\mathcal{T}(2, 4, \dots, 2n-2, k) = Y_1$$

that forgets about the last subspace. Denote the tautological bundles over Y by $\mathcal{T}_i$ and the tautological bundles over Y_1 by $\mathcal{T}'_i$.

k is odd. Using an isomorphism $Y \cong SGr(2, \mathcal{T}'_n)$ and Theorem 5 we obtain that $A^*(Y)$ is a free module over $A^*(Y_1)$ with the basis

$$\mathcal{B} = \left\{ 1, e_n, \dots, e_n^{k-2n} \right\}.$$

Using the induction we have the following basis for $A^*(Y_1)$:

$$\mathcal{B}_1 = \left\{ e_1'^{m_1} e_2'^{m_2} \dots e_{n-1}'^{m_{n-1}} \;\middle|\; 0 \le m_i \le k - 2i \right\},$$

with $e'_i = e(\mathcal{T}'_i)$. One has $p^*(\mathcal{T}'_i) \cong \mathcal{T}_i$ and $p^A(e'_i) = e_i$ for $i \le n-1$. Computing the pullback for $\mathcal{B}_1$ and multiplying it with $\mathcal{B}$ we obtain the desired basis.

k is even. This case is completely analogous to the previous one. We have an isomorphism $Y \cong SGr(2, \mathcal{T}'_n)$ then by Theorem 5 obtain that $A^*(Y)$ is a free module over $A^*(Y_1)$ with the basis

$$\mathcal{B} = \left\{ u_n \,\middle|\, u_n = \left[\begin{array}{l} e_n^{m_n},\ 0 \le m_n \le k-2n \\ e_{n+1} \end{array} \right. \right\}.$$

Using the induction we have the following basis for $A^*(Y_1)$:

$$\mathcal{B}_1 = \left\{ u_1 u_2 \ldots u_{n-1} \,\middle|\, u_i = \left[\begin{array}{l} e_i'^{m_i},\ 0 \le m_i \le k-2i \\ e'_{i+1} e'_{i+2} \ldots e'_n \end{array} \right. \right\}$$

with $e'_i = e(\mathcal{T}'_i)$. Note that $p^*(\mathcal{T}'_i) \cong \mathcal{T}_i$ and $p^A(e'_i) = e_i$ for $i \le n-1$. In order to compute $p^A(e'_n)$ pass to $\widetilde{\mathcal{SF}}_E(2, 4, \ldots, 2n, k)$, there we have $p^*\mathcal{T}'_n \cong \mathcal{T}_n \oplus \mathcal{T}_{n+1}$ and $p^A(e'_n) = e_n e_{n+1}$. Computing the pullback for $\mathcal{B}_1$ and multiplying it with $\mathcal{B}$ we obtain the desired basis of Y. ☐

Corollary 4. *Let $\mathcal{T}$ be a rank $2n$ special linear bundle over a smooth variety X. Then we have*

(1) $e(\mathcal{T}) = e(\mathcal{T}^\vee)$,
(2) $b_n(\mathcal{T}) = (-1)^n e(\mathcal{T})^2$.

Proof. Consider $p\colon \widetilde{\mathcal{SF}}(\mathcal{T}) \to X$. From the theorem we have that p^A is an injection. Also we have that $p^*\mathcal{T} \cong \bigoplus_i \mathcal{T}_i$ and $p^*\mathcal{T}^\vee \cong \bigoplus_i \mathcal{T}_i^\vee$. Note that $\operatorname{rank} \mathcal{T}_i = 2$, hence $(\mathcal{T}_i, \lambda_{\mathcal{T}_i}) \cong (\mathcal{T}_i^\vee, \lambda_{\mathcal{T}_i^\vee})$ and we obtain $p^*\mathcal{T} \cong p^*\mathcal{T}^\vee$, so $p^A e(\mathcal{T}) = p^A e(\mathcal{T}^\vee)$. By Lemma 9 we have

$$b_*(p^*\mathcal{T}) = \prod (1 - e(\mathcal{T}_i)^2 t^2),$$

thus $p^A b_n(\mathcal{T}) = (-1)^n \prod e(\mathcal{T}_i)^2 = (-1)^n (p^A e(\mathcal{T}))^2$. ☐

Having the above corollary at hand we can write down the relations for $SGr(2, 2n)$. Recall that the odd-dimensional case was computed in Corollary 3.

Corollary 5. *There is an isomorphism*

$$A^*(pt)[e_1, e_2] \Big/ \langle e_1 e_2,\ e_1^{2n-2} + (-1)^n e_2^2 \rangle \xrightarrow{\simeq} A^*(SGr(2, 2n)),$$

induced by sending e_1 and e_2 to $e(\mathcal{T}_1)$ and $e(\mathcal{T}_2)$ respectively.

Proof. In view of Theorem 4 it is sufficient to show that the relations from the left-hand side hold. Passing to $\widetilde{SGr}(2, 2n)$ we obtain

$$0 = e\left(\mathcal{O}^{2n}_{\widetilde{SGr}(2,2n)}\right) = e\left(\mathcal{T}_1\right) e\left(\mathcal{T}_2\right).$$

For the second relation compute the total Borel class:

$$1 = b_*\left(\mathcal{O}^{2n}_{\widetilde{SGr}(2,2n)}\right) = b_*(\mathcal{T}_1) b_*(\mathcal{T}_2) = \left(1 - e(\mathcal{T}_1)^2 t^2\right) b_*(\mathcal{T}_2).$$

Expanding $b_*(\mathcal{T}_2)$ we have $b_k(\mathcal{T}_2) = e(\mathcal{T}_1)^{2k}$ for $k \le n-1$. Thus, by the above corollary, $e(\mathcal{T}_1)^{2n-2} = (-1)^{n-1} e(\mathcal{T}_2)^2$. ☐

We finish this section with an explicit answer for the cohomology of maximal SL_2 flag variety. Note that the answer looks like the ring of coinvariants for the groups $W(B_n)$ and $W(D_n)$ rather then $W(A_{n-1})$, although we deal with the special linear group SL_n.

Theorem 7. *For $n \geq 1$ consider*

$$s_i = \sigma_i(e_1^2, e_2^2, ..., e_n^2), \quad t = \sigma_n(e_1, e_2, \dots, e_n)$$

with σ_i being the elementary symmetric polynomials in n variables. Then we have the following isomorphisms

(1) $\phi_1\colon A^*(pt)[e_1, e_2, ..., e_n] \big/ \langle s_1, s_2, \dots, s_n\rangle \xrightarrow{\simeq} A^*(\mathcal{SF}(2n+1))$,

(2) $\phi_2\colon A^*(pt)[e_1, e_2, ..., e_n] \big/ \langle s_1, s_2, \dots, s_{n-1}, t\rangle \xrightarrow{\simeq} A^*(\mathcal{SF}(2n))$,

induced by sending e_i to $e(\mathcal{T}_i)$.

Proof. First of all we show that the claimed relations on the Euler classes hold. Passing to $\widetilde{\mathcal{SF}}(2n+1)$ and using Lemma 9 we obtain

$$b_*\left(\mathcal{O}_{\widetilde{\mathcal{SF}}}^{2n+1}\right) = \prod_{i=1}^{n+1} b_*(\mathcal{T}_i) = \prod_{i=1}^{n} \left(1 - e(\mathcal{T}_i)^2 t^2\right),$$

hence

$$\phi_1(s_i) = (-1)^i b_i\left(\mathcal{O}_{\widetilde{\mathcal{SF}}}^{2n+1}\right) = 0.$$

In the even case we can do the same calculations in order to obtain

$$\phi_2(s_i) = (-1)^i b_i\left(\mathcal{O}_{\widetilde{\mathcal{SF}}}^{2n}\right) = 0,$$

moreover, we have

$$\phi_2(t) = \prod_{i=1}^{n} e(\mathcal{T}_i) = e(\mathcal{O}_{\widetilde{\mathcal{SF}}}^{2n}) = 0.$$

Hence the homomorphisms ϕ_1 and ϕ_2 are well-defined.

To finish the proof note that by Proposition 5 and Theorem 6 the spanning set from the left-hand side goes to the basis of the right-hand side, so ϕ_1 and ϕ_2 are isomorphisms. $\square$

11. The cohomology of BSL_n.

This section is devoted to the computation of the cohomology ring of the classifying spaces

$$BSL_n = \varinjlim_{m\in\mathbb{N}} SGr(n, m).$$

The case of BSL_2 easily follows from Corollary 3. Then we deal with BSL_{2n} using the calculations for the relative maximal SL_2 flag varieties, and in the end using certain Gysin sequences compute the cohomology of BSL_{2n+1}.

Recall that A^* is constructed from a representable cohomology theory. In this setting we have the following proposition relating the cohomology groups of a limit space to the limit of the cohomology groups [PPR2, Lemma A.5.10].

Proposition 6. *For any sequence of motivic spaces $X_1 \xrightarrow{i_1} X_2 \xrightarrow{i_2} X_3 \xrightarrow{i_3} \dots$ and any p we have an exact sequence of abelian groups*

$$0 \to \varprojlim{}^1 A^{p-1}(X_k) \to A^p(\varinjlim X_k) \to \varprojlim A^p(X_k) \to 0.$$

Moreover, if all the i_k^A are surjective, then $\lim^1$ vanishes and we have

$$A^p(\varinjlim X_k) \cong \varprojlim A^p(X_k).$$

Lemma 14. *For every $m, n \in \mathbb{N}$ the $A^*(pt)$-algebra $A^*(SGr(2n, 2m+1))$ is generated by the classes $b_1(\mathcal{T}_1), \dots, b_{n-1}(\mathcal{T}_1), e(\mathcal{T}_1)$.*

Proof. Consider the covering

$$p\colon Y = \widetilde{\mathcal{SF}}(2, 4, \dots, 2n, 2m+1) \to SGr(2n, 2m+1)$$

splitting $\mathcal{T}_1$ into a sum of the rank 2 special linear bundles. Denote the tautological bundles over Y by $\mathcal{T}_1', \mathcal{T}_2', \dots, \mathcal{T}_{n+1}'$ and the corresponding Euler classes by $e_1, e_2, \dots, e_{n+1}$.

By Theorem 6 applied to $\mathcal{T}_1$ we have that p^A is injective. Also one has $p^*\mathcal{T}_1 \cong \bigoplus_{i=1}^n \mathcal{T}_i'$ hence

$$p^A e(\mathcal{T}_1) = \sigma_n(e_1, e_2, \dots, e_n) = t$$

and, by Lemma 9,

$$p^A b_i(\mathcal{T}_1) = \sigma_i(e_1^2, e_2^2, \dots, e_n^2) = s_i$$

with σ_i being the elementary symmetric polynomials.

By Theorem 6 we also have that the set

$$\mathcal{B} = \left\{ u_1 u_2 \dots u_{n-1} \;\middle|\; u_i = \left[\begin{array}{l} e_i^{m_i},\, 0 \le m_i \le k - 2i \\ e_{i+1} e_{i+2} \dots e_n \end{array} \right. \right\},$$

forms a basis of $A^*(Y)$ over $p^A A^*(SGr(2n, 2m+1))$. Note that by the same theorem $A^*(Y)$ is generated as an $A^*(pt)$-algebra by $e_1, e_2 \dots, e_n$, thus by Proposition 5 we know that $\mathcal{B}$ spans $A^*(Y)$ over $A^*(pt)[s_1, s_2, \dots, s_{n-1}, t]$. Since we have

$$A^*(pt)[s_1, s_2, \dots, s_{n-1}, t] \subset p^A A^*(SGr(2n, 2m+1))$$

it follows that $A^*(pt)[s_1, s_2, \dots, s_{n-1}, t] = p^A A^*(SGr(2n, 2m+1))$. $\square$

Consider the sequence of embeddings

$$\dots \to SGr(2n, 2m+1) \xrightarrow{i_{2m+1}} SGr(2n, 2m+3) \to \dots$$

By the above lemma we know that i_{2m+1}^A are surjective hence

$$A^p(BSL_{2n}) \cong \varprojlim A^p(SGr(2n, 2m+1)) \cong \varprojlim A^p(SGr(2n, m)).$$

The sequence of tautological special linear bundles $\mathcal{T}_1(2n, m)$ over $SGr(2n, m)$ gives rise to a bundle $\mathcal{T}$ over BSL_{2n}. We have a sequence of embeddings of the Thom spaces

$$\dots \to Th(\mathcal{T}_1(2n, 2m+1)) \xrightarrow{j_{2m+1}} Th(\mathcal{T}_1(2n, 2m+3)) \to \dots$$

Since all the considered morphisms $\mathcal{T}_1(2n, k) \to \mathcal{T}_1(2n, l)$ are inclusions there is a canonical isomorphism $Th(\mathcal{T}) \cong \varinjlim \mathcal{T}_1(2n, m)$. For every k we have an isomorphism

$$A^{*-2n}(SGr(2n, k)) \xrightarrow{\cup th(\mathcal{T}_1(2n,k))} A^*(Th(\mathcal{T}_1(2n, k))),$$

so j_{2m+1}^A are surjective and

$$A^p(Th(\mathcal{T})) \cong \varprojlim A^p(\mathcal{T}_1(2n, m)).$$

Notation 9. Let $\mathcal{T}$ be the tautological special linear bundle over BSL_{2n}. Denote by $b_i(\mathcal{T}), e(\mathcal{T}) \in A^*(BSL_{2n})$ and $th(\mathcal{T}) \in A^*(Th(\mathcal{T}))$ the elements corresponding to the sequences of classes of the tautological bundles,

$$b_i(\mathcal{T}) = (\dots, b_i(\mathcal{T}_1(2n,m)), b_i(\mathcal{T}_1(2n,m+1)), \dots),$$

$$e(\mathcal{T}) = (\dots, e(\mathcal{T}_1(2n,m)), e(\mathcal{T}_1(2n,m+1)), \dots),$$

$$th(\mathcal{T}) = (\dots, th(\mathcal{T}_1(2n,m)), th(\mathcal{T}_1(2n,m+1)), \dots),$$

with $\mathcal{T}_1(2n,m)$ being the tautological special linear bundles over $SGr(2n,m)$.

The above considerations show that we have a Gysin sequence for the tautological bundle over the classifying space BSL_{2n}.

Lemma 15. *Let $\mathcal{T}$ be the tautological special linear bundle over BSL_{2n}. Then there exists a long exact sequence*

$$\dots \to A^{*-2n}(BSL_{2n}) \xrightarrow{\cup e} A^*(BSL_{2n}) \xrightarrow{j^A} A^*(BSL_{2n-1}) \xrightarrow{\partial} \dots$$

Proof. For the zero section inclusion of motivic spaces $BSL_{2n} \to \mathcal{T}$ we have the following long exact sequence.

$$\dots \to A^*(Th(\mathcal{T})) \to A^*(\mathcal{T}) \to A^*(\mathcal{T}^0) \xrightarrow{\partial} \dots$$

The isomorphisms

$$A^{*-2n}(SGr(2n,k)) \xrightarrow{\cup th(\mathcal{T}_1(2n,k))} Th(\mathcal{T}_1(2n,k)),$$

induce an isomorphism $A^{*-2n}(BSL_{2n}) \xrightarrow{\cup th(\mathcal{T})} A^*(Th(\mathcal{T}))$, so we can substitute $A^{*-2n}(BSL_{2n})$ for the first term in the above sequence. Using homotopy invariance we exchange $\mathcal{T}$ for BSL_{2n}. By the definition of $e(\mathcal{T})$ the first arrow represents the cup product $\cup e(\mathcal{T})$.

We have isomorphisms

$$\mathcal{T}^0 \cong \varinjlim SGr(1, 2n-1, m) \cong \varinjlim SGr(2n-1, 1, m).$$

The sequence of projections

$$\begin{array}{ccccccc} \cdots \longrightarrow & SGr(2n-1,1,m) & \longrightarrow & SGr(2n-1,1,m+1) & \longrightarrow \cdots \\ & \downarrow & & \downarrow & \\ \cdots \longrightarrow & SGr(2n-1,m+1) & \longrightarrow & SGr(2n-1,m+2) & \longrightarrow \cdots \end{array}$$

induces a morphism $\mathcal{T}^0 \xrightarrow{r} BSL_{2n-1}$. Note that

$$SGr(2n-1,1,m) \cong \mathcal{T}_2(2n-1,m+1)^0,$$

and $\mathcal{T}^0$ is an $\mathbb{A}^\infty - \{0\}$-bundle over BSL_{2n-1}, so by [MV, Section 4, Proposition 2.3] r is an isomorphism in homotopy category and we can substitute $A^*(BSL_{2n-1})$ for the third term in the long exact sequence. $\square$

Definition 15. For a graded ring R^* let $R^*\,[[t]]_h$ be the homogeneous power series ring, i.e. a graded ring with

$$R^*\,[[t]]_h^k = \left\{ \sum a_i t^i \mid \deg a_i + i \deg t = k \right\}.$$

Note that $R^* [[t]]_h = \varprojlim R^*[t]/t^n$, where the limit is taken in the category of graded algebras. For example, considering $R^* = \mathbb{Z}[x]$ and degrees $\deg x = 1, \deg t = 1$ we have

$$\mathbb{Z}[x] [[t]]_h = \bigcup_{k\in\mathbb{Z}} \left\{ \sum p_i(x)t^i \in \mathbb{Z}[x] [[t]] \mid \deg p_i + \deg t \le k \right\}.$$

We set $R^* [[t_1, \dots, t_n]]_h = R^* [[t_1, \dots, t_{n-1}]]_h [[t_n]]_h$.

Theorem 8. *For* $\deg e = 2n, \deg b_i = 2i$ *we have isomorphisms*

$$A^*(pt) [[b_1, \dots, b_{n-1}, e]]_h \xrightarrow{\simeq} A^*(BSL_{2n}),$$

$$A^*(pt) [[b_1, \dots, b_n]]_h \xrightarrow{\simeq} A^*(BSL_{2n+1}).$$

Proof. The case of BSL_2 follows from Corollary 3 and Proposition 6, since for the sequence

$$\dots \to SGr(2, 2m + 1) \to SGr(2, 2m + 3) \to \dots$$

the pullbacks are surjective and $\lim^1$ vanishes yielding

$$A^*(BSL_2) \cong \varprojlim A^*(SGr(2, 2m + 1)) = \varprojlim A^*(pt)[e]/\langle e^{2m}\rangle = A^*(pt) [[e]]_h .$$

For the even case consider the sequences

$$\begin{tikzcd} \cdots \arrow[r] & \mathcal{SF}(\mathcal{T}_1(2n, 2m + 1)) \arrow[r, "j_m"] \arrow[d, "p_m"] & \mathcal{SF}(\mathcal{T}_1(2n, 2m + 3)) \arrow[r] \arrow[d, "p_{m+1}"] & \cdots \\ \cdots \arrow[r] & SGr(2n, 2m + 1) \arrow[r, "i_m"] & SGr(2n, 2m + 3) \arrow[r] & \cdots \end{tikzcd}$$

with $\mathcal{T}_1(2n, m)$ being the tautological rank $2n$ special linear bundle over $SGr(2n, m)$. We have $\mathcal{SF}(\mathcal{T}_1(2n, 2m + 1)) \cong \mathcal{SF}(2, 4, \dots , 2n, 2m + 1)$. By Theorem 6 the pullbacks j_m^A are surjective, so $\lim^1$ vanishes. By the same theorem $A^*(\mathcal{SF}(\mathcal{T}_1(2n, 2m+1)))$ is generated by the Euler classes of the tautological bundles and for every polynomial $f(e_1, e_2, \dots , e_n)$ and $2m+1 > \deg f$ this polynomial is nonzero in $A^*(\mathcal{SF}(\mathcal{T}_1(2n, 2m + 1)))$. Thus we obtain

$$A^*(\varinjlim \mathcal{SF}(\mathcal{T}_1(2n, 2m + 1))) \cong A^*(pt) [[e_1, e_2, \dots , e_n]]_h ,$$

$\deg e_i = 2$.

On the other hand we know that $A^*(BSL_{2n}) \cong \varprojlim A^*(SGr(2n, 2m + 1))$. By Lemma 14 and Theorem 6 $p_m^A A^*(SGr(2n, 2m + 1))$ is the subalgebra of $A^*(\mathcal{SF}(\mathcal{T}_1(2n, 2m + 1)))$ generated by

$$p_m^A b_i(\mathcal{T}_1) = s_i = \sigma_i(e_1^2, e_2^2, \dots , e_n^2), \quad p_m^A e(\mathcal{T}) = t = \sigma_n(e_1, e_2, \dots , e_n).$$

Passing to the limit we obtain

$$p^*A^*(BSL_{2n}) = A^*(pt) [[s_1, \dots, s_{n-1}, t]]_h \subset A^*(pt) [[e_1, e_2, \dots , e_n]]_h ,$$

where $s_i = p^*b_i(\mathcal{T})$, $t = p^*e(\mathcal{T})$.

For the odd case consider the Gysin sequence from Lemma 15 for BSL_{2n+2}. By the above calculations $e(\mathcal{T})$ is not a zero divisor, so the the map $\cup e(\mathcal{T})$ is injective and we have a short exact sequence

$$0 \to A^{*-2n-2}(BSL_{2n+2}) \xrightarrow{\cup e(\mathcal{T})} A^*(BSL_{2n+2}) \to A^*(BSL_{2n+1}) \to 0.$$

Identifying $A^*(BSL_{2n+2})$ with the homogeneous power series and removing e we obtain the desired result. $\square$

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