CENTRAL SIMPLE ALGEBRAS OF PRIME EXPONENT AND DIVIDED POWER OPERATIONS

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Abstract. Let p be a prime, F a field of characteristic different from p. We prove triviality of the divided power operations on central simple cyclic algebras of exponent p.

Introduction

Let F be a field, $\mathrm{Br}(F)$ the Brauer group of F, $\alpha \in \mathrm{Br}(F)$. A natural question arises whether α can be represented by a cyclic algebra. For instance, if $\operatorname{ind}\alpha = 2$, then the answer is always positive, since, as is well-known, in this case α is represented by a quaternion algebra. If $\operatorname{char} F \neq 2$, $\exp\alpha = 2$, $\operatorname{ind}\alpha = 4$, and $\sqrt{-1} \in F$, then $\alpha = (a,b) + (c,d)$ is represented by a cyclic algebra of degree 4 if and only if the Pfister form $\langle\langle a,b,c,d\rangle\rangle$ is hyperbolic, or, in other words, $\{a,b,c,d\} = 0 \in K_4(F)/2K_4(F)$ ([RST], Th.3). However, the question of representation of α by a cyclic algebra of *higher* degree was not investigated in [RST]. In the present paper we give necessary conditions for an element $\alpha \in \mathrm{Br}(F)$ of arbitrary prime exponent p to be cyclic, provided all p-primary roots of unity are contained in F.

A few words about the notation. If n is a positive integer, $a, b \in F^*$, $\operatorname{char} F \not| n$, and ξ_n is a fixed primitive root of unity lying in F, then by $(a,b)_n$ we denote the symbol algebra of degree n with the generators i and j, and the relations $i^n = a$, $j^n = b$, and $ij = \xi_n ji$. (If $n = 2$, then we write just (a,b) instead of $(a,b)_2$). Slightly abusing notation, by the same symbol we also denote the corresponding element in the Brauer group. If $n = km$ and $\xi_n \in F$, then $k(a,b)_n = (a,b)_m \in \mathrm{Br}(F)$, where the roots of unity ξ_n and ξ_m are related by the equality $\xi_n^k = \xi_m$.

Given $m \geq 0$, by $K_m(F)$ we denote the mth Milnor K-group of the field F. Let n be a fixed positive integer and $-1 \in F^{*n}$. By $\gamma_k : K_2(F)/n \to K_{2k}(F)/n$ we denote the divided power operation of degree k on $K_2(F)/n$ well defined by the following rule:

$$\gamma_k\Big(\sum_i \alpha_i\Big) = \sum_{i_1 < i_2 < \dots < i_k} \alpha_{i_1}\alpha_{i_2}\dots\alpha_{i_k},$$

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where all α_j are symbols ([K], [Vi]). Obviously, the divided power operations commute with field extension homomorphisms.

Given $n \geq 2$, by ${}_{\mathrm{n}}\mathrm{Br}(F)$ we denote the n-torsion of $\mathrm{Br}(F)$. If $\xi_n \in F$, then the norm residue map $K_2(F)/n \to {}_{\mathrm{n}}\mathrm{Br}(F)$, taking $\{a, b\}$ to $(a, b)_n$ is an isomorphism([MS], Th. 11.5), which allows us to define in the obvious way the divided power operations $\gamma_i : {}_{\mathrm{n}}\mathrm{Br}(F) \to K_{2i}(F)/n$ (provided that $-1 \in F^{*n}$). These operations depend on the choice of ξ_n, and change of ξ_n multiplies γ_i by m^i, where m is a positive integer coprime to n.

We outline briefly the main results from Milnor's K-theory used in the paper. All of them except the norm residue isomorphism theorem (see below) can be found in [GS], Ch.7. One of the main tools is the exact sequence

$$0 \to K_m(F)/n \xrightarrow{\mathrm{res}} K_m(F(t))/n \xrightarrow{\oplus\partial_f} \coprod_f K_{m-1}(F_f)/n \to 0, \qquad (*)$$

where f runs over all monic irreducible polynomials over F, and $F_f = F[t]/(f)$. The residue homomorphism ∂_f here is well defined by the following rule: if $v_f(g_i) = 0$ for all $2 \leq i \leq m$, then $\partial_f\{g_1, g_2, \dots g_m\} = v_f(g_1)\{\overline{g_2}, \dots, \overline{g_m}\}$, where v_f is the valuation of $F(t)$ associated with f, and $\overline{g_i}$ is the image of g_i in F_f. This sequence splits by "the leading coefficient map" $K_m(F(t))/n \xrightarrow{l} K_m(F)/n$, where $l(\{f_1, \dots, f_m\}) = \{l(f_1), \dots, l(f_m)\}$, $f_i \in F[t]$ and $l(f_i)$ is the leading coefficient of f_i. In particular, the sequence

$$0 \to {}_{\mathrm{n}}\mathrm{Br}(F) \xrightarrow{\mathrm{res}} {}_{\mathrm{n}}\mathrm{Br}\, F(t) \xrightarrow{\oplus\partial_f} \coprod_f {F_f}^*/{F_f}^{*n} \to 1,$$

is split exact, provided $\xi_n \in F$ and $\mathrm{char}\, F \not| n$.

To make some computations in the second part of the paper shorter we apply the Weyl reciprocity law for $K_m(F(t))/2$, namely the exact sequence

$$0 \to k_m(F) \xrightarrow{\mathrm{res}} k_m(F(t)) \xrightarrow{\oplus\partial_f} \oplus k_{m-1}(F_f) \xrightarrow{\oplus N_{F_f/F}} k_{m-1}(F) \to 0,$$

where $k_m = K_m/2$, f runs over all monic irreducible polynomials over F together with the infinity point, and $N_{F_f/F}$ are norm maps. Also we need the projection formula $N_{L/F}(\mathrm{res}_{L/F}\, \alpha \cdot \beta) = \alpha \cdot N_{L/F}(\beta)$ for a finite extension L/F, $\alpha \in K_*(F)/p$, $\beta \in K_*(L)/p$. In particular, if L/F is a quadratic extension, and $\alpha \in K_*(F)$, then setting $\beta = \overline{1} \in K_0(L)/2$, we get $N_{L/F}(\mathrm{res}_{L/F}\, \alpha) = 0$.

The only "nonelementary" and indeed, very deep result applied in the paper is the exact sequence

$$K_*(F)/p \xrightarrow{mult.by\{a\}} K_{*+1}(F)/p \xrightarrow{res} K_{*+1}(F(\sqrt[p]{a}))/p,$$

for a cyclic extension $F(\sqrt[p]{a})/F$ provided $\mu_p \in F^*$. Exactness of this sequence can be inferred from the norm residue isomorphism $K_*(F)/p \simeq H^*(F, \mu_p)$, proven by Rost and Voevodsky, ([V], [W]).

1. Cyclic algebras of prime exponent

The first part of the paper deals with cyclic algebras of prime exponent. We show that all divided power operations are trivial on them, provided the ground field contains all p-primary roots of unity. More precisely we have the following

Theorem 1.1. *Let p be a prime, F a field, $\operatorname{char} F \neq p$, m positive integer, $\xi_{p^m} \in F$, $\alpha \in {}_{\mathrm{p}}\operatorname{Br}(F)$. Suppose $a \in F^*$ is such that $F(\sqrt[p^m]{a})$ is a splitting field for α, i.e. $\alpha_{F(\sqrt[p^m]{a})} = 0$. Then $\gamma_i(\alpha) = 0$ for any $i \geq 2$.*

Proof. We will induct on m, the case $m = 1$ being trivial, since α corresponds then to a symbol in $K_2(F)/p$. By the hypothesis $\alpha = (a, b)_{p^m}$ for some $b \in F^*$ ([P], §§ 13.3, 15.1). Since $\exp \alpha = p$, we get $(a, b)_{p^{m-1}} = 0$, which means that $b = N_{F(\sqrt[p^{m-1}]{a})/F}(c)$ for some $c \in F(\sqrt[p^{m-1}]{a})$ ([P], § 15.1). Consider first the generic case, where $x_0, \dots x_{p^{m-1}-1}, a$ are indeterminates over a prime field k, $F = k(x_0, \dots, x_{p^{m-1}-1}, a)$ and $b = N_{F(\sqrt[p^{m-1}]{a})/F}(\sum_{j=0}^{p^{m-1}-1} x_j \sqrt[p^{m-1}]{a^j}) \in k[x_0, \dots, x_{p^{m-1}-1}, a]$ is the norm polynomial of the generic element of $F(\sqrt[p^{m-1}]{a})$. Set $K = k(x_0, \dots, x_{p^{m-1}-1})$, so that $F = K(a)$. The commutative diagram

$$\begin{array}{ccc}
{}_{\mathrm{p}}\operatorname{Br} K(a) & \xrightarrow{\coprod \partial_f} & \coprod_f {K_f}^*/{K_f}^{*p} \\
\Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle p^{m-1}} \\
{}_{\mathrm{p^m}}\operatorname{Br} K(a) & \xrightarrow{\coprod \partial_f} & \coprod_f {K_f}^*/{K_f}^{*p^m}
\end{array}$$

shows that α has nonzero residues at the same polynomials independently of whether it is considered as an element of ${}_{\mathrm{p}}\operatorname{Br} K(a)$ or of ${}_{\mathrm{p^m}}\operatorname{Br} K(a)$. Hence, $\alpha \in {}_{\mathrm{p}}\operatorname{Br} K(a)$ can have nonzeros residues only at the monic polynomials a and $b' = l(b)^{-1}b$, where $l(b)$ is the leading coefficient of b. Moreover, the similar argument shows that the image of α under "the leading coefficient" map $l : {}_{\mathrm{p}}\operatorname{Br} K(a) \to {}_{\mathrm{p}}\operatorname{Br} K$ is zero.

In view of the exact sequence $(*)$ the assertion that $\gamma_i(\alpha) = 0$ is equivalent to that $\partial_f(\gamma_i(\alpha)) = 0$ for any monic polynomial $f \in K[a]$, and $l(\gamma_i(a)) = 0$. This is just what we are going to prove.

Lemma 1.2. *Let $i \geq 2$. Then*

1) *$\partial_f(\gamma_i(\alpha)) = 0$ for any monic irreducible polynomial $f \in K[a]$ distinct from a and b'.*

2) *$l(\gamma_i(\alpha)) = 0$.*

Proof. 1) The element $\alpha \in {}_{\mathrm{p}}\operatorname{Br}(K(a))$ can be written as $\alpha = (f, g)_p + \sum_j (f_j, g_j)_p$ where $g, f_j, g_j \in K[a]$ are polynomials not divisible by f. Since $\partial_f(\alpha) = 0$, we have $g \in {F_f}^{*p}$, which makes the assertion obvious.

2) This follows from the facts that $l(a) = 1$ and the commutative diagram

$$\begin{array}{ccc}
K_2(K(a))/p & \xrightarrow{l} & K_2(K)/p \\
\Big\downarrow{\scriptstyle \gamma_i} & & \Big\downarrow{\scriptstyle \gamma_i} \\
K_{2i}(K(a))/p & \xrightarrow{l} & K_{2i}(K)/p
\end{array}$$

The lemma is proved. □

It remains to show that $\partial_a(\gamma_i(\alpha))=0$ and $\partial_{b'}(\gamma_i(\alpha))=0$. To do this represent α as $\alpha=(a,x)_p+\alpha_1$, where $x=\partial_a(\alpha)\in K$, and $\partial_f(\alpha_1)=0$ for any monic polynomial $f\neq b'$. Obviously, $\alpha_{1F(\sqrt[p^m]{a})}=\alpha_{F(\sqrt[p^m]{a})}=0$, hence $\gamma_{i-1}(\alpha_1)_{F(\sqrt[p^m]{a})}=0$. Since by the projection formula

$$\{a\}\gamma_{i-1}(\alpha_1)=N_{F(\sqrt[p^m]{a})/F}(\sqrt[p^m]{a})\gamma_{i-1}(\alpha_1)_{F(\sqrt[p^m]{a})}=N_{F(\sqrt[p^m]{a})/F}(0)=0,$$

we have

$$\gamma_i(\alpha)=\{a,x\}\gamma_{i-1}(\alpha_1)+\gamma_i(\alpha_1)=\gamma_i(\alpha_1).$$

Since $\partial_a(\alpha_1)=0$, it follows that $\partial_a(\gamma_i(\alpha))=\partial_a(\gamma_i(\alpha_1))=0$. Furthermore, by the induction hypothesis applied to the element $\alpha_{F(\sqrt[p]{a})}$, we have

$$\gamma_i(\alpha)_{F(\sqrt[p]{a})}=\gamma_i(\alpha_{F(\sqrt[p]{a})})=0.$$

By the result of Voevodsky mentioned in the introduction ([V], [W]), we conclude that $\gamma_i(\alpha)=\{a\}\beta$ for some $\beta\in K_{2i-1}(F)/p$.

Therefore,

$$\partial_{b'}(\gamma_i(\alpha))=\partial_{b'}(\{a\}\beta)=\{a\}\partial_{b'}(\beta)=0,$$

since $a\in {K_{b'}}^{*p}$ in view of the equality $(a,b)_p=0$.

Thus, we have proved the theorem in the generic case. In the general case consider any element $\widetilde{\alpha}\in {}_{\mathrm{p}}\operatorname{Br}(F)$. We have $\widetilde{\alpha}=(\widetilde{a},\widetilde{b})_{p^m}$, where $\widetilde{b}=N_{F(\sqrt[p^{m-1}]{\widetilde{a}})/F}(\sum_{j=0}^{p^{m-1}-1}\widetilde{x}_j\sqrt[p^{m-1}]{\widetilde{a}^j})$, and $\widetilde{a}$, $\widetilde{x_j}\in F$. Interpreting the groups ${}_{\mathrm{p}}\operatorname{Br}$ and ${}_{\mathrm{p^m}}\operatorname{Br}$ as K_2/p and K_2/p^m respectively, we obtain the following commutative diagrams

$$\begin{array}{ccc}
{}_{\mathrm{p}}\operatorname{Br}(k(x_0,\dots,x_{p^{m-1}-1},a)) & \xrightarrow{s_p} & {}_{\mathrm{p}}\operatorname{Br}(F)\\
{\scriptstyle id}\big\downarrow & & {\scriptstyle id}\big\downarrow\\
{}_{\mathrm{p^m}}\operatorname{Br}(k(x_0,\dots,x_{p^{m-1}-1},a)) & \xrightarrow{s_{p^m}} & {}_{\mathrm{p^m}}\operatorname{Br}(F)
\end{array}$$

and

$$\begin{array}{ccc}
{}_{\mathrm{p}}\operatorname{Br}(k(x_0,\dots,x_{p^{m-1}-1},a)) & \xrightarrow{s_p} & {}_{\mathrm{p}}\operatorname{Br}(F)\\
{\scriptstyle \gamma_i}\big\downarrow & & {\scriptstyle \gamma_i}\big\downarrow\\
K_{2i}(k(x_0,\dots,x_{p^{m-1}-1},a))/p & \xrightarrow{s_p} & K_{2i}(F)/p
\end{array}$$

where s_p and s_{p^m} are specialization maps taking x_j to $\widetilde{x_j}$ and a to $\widetilde{a}$. Looking at the first diagram we see that $\widetilde{\alpha}=s_p(\alpha)$, where α is the element considered in the generic case. Then the second diagram yields

$$\gamma_i(\widetilde{\alpha})=\gamma_i(s_p(\alpha))=s_p(\gamma_i(\alpha))=s_p(0)=0.$$

This finishes the proof of Theorem 1.1. □

Remark. We do not know how to avoid using the higher norm residue isomorphism to obtain equality $\gamma_i(\alpha)=\{a\}\beta$, which is essential in the proof of Theorem 1.1. Notice also that this proof can not be generalized straightforwardly to the case of an arbitrary p-primary exponent.

We will call an element $\alpha\in\operatorname{Br}(F)$ cyclic if $\alpha_L=0$ for some cyclic field extension L/F.

Corollary 1.3. *Let F be a field, p a prime, $\operatorname{char} F \neq p$. Suppose all p-primary roots of unity are contained in F, and $\alpha \in {}_{\mathrm{p}}\operatorname{Br} F$ is cyclic. Then $\gamma_i(\alpha) = 0$ for any $i \geq 2$.*

Proof. By the hypothesis $\alpha_L = 0$ for some cyclic field extension L/F. Then $\alpha_K = 0$ for some intermediate field K between F and L such that $[K : F] = p^m$ and $[L : K]$ is not divisible by p. Then $K = F(\sqrt[p^m]{a})$ for some $a \in F^*$ and the result follows from Theorem 1.1. □

Corollary 1.4. *Let $\operatorname{char} F \neq 2$, D a biquaternion division algebra over F, m a positive integer, $\xi_{2^m} \in F$, $a \in F^*$. Suppose $D_{F(\sqrt[2^m]{a})} = 0 \in \operatorname{Br}(F)$. Then D is a cyclic algebra.*

Proof. By Theorem 1.1 we have $\gamma_2(D) = 0$, which, on the other hand, is equivalent to the assertion that the algebra D is cyclic ([RST], Th.3). □

The statement converse to Theorem 1.1 is not true, as the following counterexample shows.

Proposition 1.5. *Let p be a prime, k a field, containing all p-primary roots of unity, $\operatorname{char} k \neq p$. Suppose that $a, b, c \in k^*$ are such that $[k(\sqrt[p]{a}, \sqrt[p]{b}, \sqrt[p]{c}) : k] = p^3$. Let further $F = k((x))((y))((z))$ be the iterated Laurent series field over k. Then the following holds:*

1) The element $D = (a, x)_p + (b, y)_p + (c, z)_p$ is not cyclic.

2) If, in addition, $(a, b)_p = (a, c)_p = (b, c)_p = 0$ (this condition holds, in particular, for any field k with $\operatorname{cd}_p k = 1$; for instance, one can put $k = \mathbb{C}(t)$, $a = t$, $b = t + 1$, $c = t + 2$), then $\gamma_i(D) = 0$ for any $i \geq 2$.

Proof. 1) Assume the converse. The p-primary component of any cyclic element is cyclic as well. Furthermore, for any field L of characteristic not p there is a natural isomorphism of groups $L^*/L^{*p} \oplus \mathbb{Z}/p\mathbb{Z} \simeq L((t))^*/L((t))^{*p}$ given by the rule $(\bar{l}, \bar{i}) \to \overline{lt^i}$ $(0 \leq i \leq p - 1)$. Therefore, we may assume that

$$D = (a^{p^{m-1}}, x)_{p^m} + (b^{p^{m-1}}, y)_{p^m} + (c^{p^{m-1}}, z)_{p^m} = (\alpha x^{i_1} y^{j_1} z^{k_1}, \beta x^{i_2} y^{j_2} z^{k_2})_{p^m}$$

for some $\alpha, \beta \in k^*$. Taking the residue at z we get

$$(-1)^{k_1 k_2} (\alpha x^{i_1} y^{j_1})^{k_2} (\beta x^{i_2} y^{j_2})^{-k_1} = c^{p^{m-1}} f^{p^m}$$

for some $f \in k((x))((y))$. It follows that $(-1)^{k_1 k_2} \alpha^{k_2} \beta^{-k_1} = c^{p^{m-1}} u^{p^m}$ for some $u \in k^*$, which gives us, in particular, that $\sqrt[p]{c} \in k(\sqrt[p^m]{\alpha}, \sqrt[p^m]{\beta})$. Moreover, $(a^{p^{m-1}}, x)_{p^m} + (b^{p^{m-1}}, y)_{p^m} = (\alpha x^{i_1} y^{j_1}, \beta x^{i_2} y^{j_2})_{p^m}$. Applying the above argument again, we subsequently get $\sqrt[p]{b} \in k(\sqrt[p^m]{\alpha}, \sqrt[p^m]{\beta})$ and $\sqrt[p]{a} \in k(\sqrt[p^m]{\alpha}, \sqrt[p^m]{\beta})$, i.e. we obtain a field embedding $k(\sqrt[p]{a}, \sqrt[p]{b}, \sqrt[p]{c}) \subset k(\sqrt[p^m]{\alpha}, \sqrt[p^m]{\beta})$. This is a contradiction, since the Galois group of the extension $k(\sqrt[p^m]{\alpha}, \sqrt[p^m]{\beta})/k$ is generated by two elements, but the Galois group of the extension $k(\sqrt[p]{a}, \sqrt[p]{b}, \sqrt[p]{c})/k$ is not.

2) This part of the proposition is obvious in view of the equalities

$$\gamma_2(D) = \{a, b, x, y\} + \{a, c, x, z\} + \{b, c, y, z\} \in K_4(F)/p$$

and $\gamma_3(D) = \{a, b, c, x, y, z\} \in K_6(F)/p$. □

A natural question arises whether it is possible to generalize Theorem 1.1 to arbitrary primary exponent and corresponding divided power operations. We do not know the answer, but the following statement shows that at least this is the case for cyclic algebras of degree 2^n and exponent 2^{n-1}.

Proposition 1.6. *Suppose an element $D \in \mathrm{Br}(F)$ is represented by a cyclic algebra of degree 2^n and exponent 2^{n-1}, and $\xi_{2^n} \in F$. Then $\gamma_i(D) = 0$ for any $i \geq 2$, where $\gamma_i : K_2(F)/2^{n-1} \to K_{2i}(F)/2^{n-1}$ are the corresponding divided power operations.*

Proof. By the hypothesis $D = (a, x^2 - ay^2)_{2^n}$. We have

$$D = (a, x^2 - ay^2)_{2^n} = (a, x^2)_{2^n} + (a, 1 - ax^{-2}y^2)_{2^n} = (a, x^2)_{2^n} + (x^2y^{-2}, 1 - ax^{-2}y^2)_{2^n} + (ax^{-2}y^2, 1 - ax^{-2}y^2)_{2^n} = (a, x)_{2^{n-1}} + (xy^{-1}, 1 - ax^{-2}y^2)_{2^{n-1}}.$$

Therefore,

$$\gamma_2(D) = \{a, x, xy^{-1}, 1 - ax^{-2}y^2\} = \{x^2y^{-2}, x, xy^{-1}, 1 - ax^{-2}y^2\} + \{ax^{-2}y^2, x, xy^{-1}, 1 - ax^{-2}y^2\} = \{x^2y^{-2}, x, xy^{-1}, 1 - ax^{-2}y^2\} = 2\{xy^{-1}, x, xy^{-1}, 1 - ax^{-2}y^2\} = 2\{-1, x, xy^{-1}, 1 - ax^{-2}y^2\} = 0 \in K_4(F)/2^{n-1}$$

since $-1 \in F^{*2^{n-1}}$, and, moreover, $\gamma_i(D) = 0$ if $i \geq 3$, since $D = (a, x)_{2^{n-1}} + (xy^{-1}, 1 - ax^{-2}y^2)_{2^{n-1}}$. The proposition is proved. □

2. Division algebras of index 8 and exponent 2

In the second part of the paper we apply the division power operations γ_2 and γ_3 to studying the structure of a central division algebra D of exponent 2 and index 8 over F. Theorem 1.1 shows that the condition $\gamma_2(D) = \gamma_3(D) = 0$ is necessary for existense of a specific (cyclic) maximal subfield of D. It turns out that the condition $\gamma_3(D) = 0$ is necessary for existence of a specific subfield of D of degree 4. The key point is a certain relationship between $\gamma_2(D)$ and $\gamma_3(D)$.

In the sequel we denote $K_*/2$ by k_*, and all symbols are supposed to be in k_*. The isomorphism $k_2 \simeq_2 \mathrm{Br}$ allows us to multiply elements from ${}_2\mathrm{Br}$ by elements from k_*. We start with the following

Theorem 2.1. *Let F be a field, $\mathrm{char}\, F \neq 2$, $\sqrt{-1} \in F^*$. Suppose $D \in_2 \mathrm{Br}(F)$, $\mathrm{ind}\, D = 8$, and $a \in F^*$ is such that $\mathrm{ind}\, D_{F(\sqrt{a})} = 4$. Then*

1) $\{a\}\gamma_2(D) \in k_5(F)$ is a symbol.

2) There exists $s \in F^*$ such that $\gamma_3(D) = \{a, s\}\gamma_2(D) \in k_6(F)$. In particular, $\gamma_3(D)$ is a symbol.

Proof. 1) It follows from the proof of Theorem 6.2 in [R] that there exist $b, c \in F^*$ such that $D_{F(\sqrt{a}, \sqrt{b}, \sqrt{c})} = 0$. Hence

$$D_{F(\sqrt{a})} = (b, \omega_1) + (c, \omega_2),$$

where $\omega_1, \omega_2 \in F(\sqrt{a})$ ([ELW]), and, consequently, since $N_{F(\sqrt{a})/F} D_{F(\sqrt{a})} = 0$, we get $(b, N_{F(\sqrt{a})/F}\omega_1) = (c, N_{F(\sqrt{a})/F}\omega_2)$. Therefore,

$$N_{F(\sqrt{a})/F}\{b, c, \omega_1\} = \{b, c, N_{F(\sqrt{a})/F}\omega_1\} = \{c, c, N_{F(\sqrt{a})/F}\omega_2\} = 0.$$

By the transfer principle ([EL], 2.2) and the isomorphism $k_3 \simeq I^3/I^4$ it can be easily seen that $\{b, c, \omega_1\} = \{b, c, r\}_{F(\sqrt{a})}$ for some $r \in F^*$. Similarly, $\{b, c, r, \omega_2\} = \{b, c, r, t\}_{F(\sqrt{a})}$ for some $t \in F^*$. Hence

$$\gamma_2(D)_{F(\sqrt{a})} = \{b, c, \omega_1, \omega_2\}_{F(\sqrt{a})} = \{b, c, r, t\}_{F(\sqrt{a})}.$$

It follows that $\gamma_2(D) = \{b, c, r, t\} + \{a\}\theta$ for some $\theta \in k_3(F)$. Therefore, $\{a\}\gamma_2(D) = \{a, b, c, r, t\}$, which proves part 1).

2) The proof of the second part of the theorem was proposed by the referee. Let B be the centralizer of $F(\sqrt{a})$ in D. The trace form $q_B(x) = \operatorname{Trd}(x^2)$ is a 4-fold Pfister form over $F(\sqrt{a})$ corresponding to $\gamma_2(B) = \gamma_2(D_{F(\sqrt{a})})$ under the isomorphism $k_4(F(\sqrt{a})) \simeq I^4(F(\sqrt{a}))/I^5(F(\sqrt{a}))$. As was shown in the first part of Theorem 2.1, we have $\gamma_2(D_{F(\sqrt{a})}) = \{b, c, r, t\}$ for some $b, c, r, t \in F^*$. Hence $T_{F(\sqrt{a})/F}(q_B) = \langle 2\rangle\langle\langle a, b, c, r, t\rangle\rangle$, where

$$T_{F(\sqrt{a})/F} : W(F(\sqrt{a})) \to W(F)$$

is the transfer, corresponding to the trace for the extension $F(\sqrt{a})/F$. It was shown ([BM], proof of Th. 2.10) that the trace form $q_D(x) = \operatorname{Trd}(x^2)$ contains $T_{F(\sqrt{a})/F}(q_B)$ as a subform, and that q_D is a multiple of a 6-fold Pfister form. It follows that $q_D = \langle 2\rangle\langle\langle a, b, c, r, t, s\rangle\rangle$ for some $s \in F^*$. By [BM], Th. 2.10 the form q_D corresponds to $\gamma_3(D)$ under the isomorphism $k_6(F) \simeq I^6(F)/I^7(F)$, hence $\gamma_3(D) = \{a, b, c, r, t, s\}$. Since by the first part of Theorem 2.1 $\{a\}\gamma_2(D) = \{a, b, c, r, t\}$, we conclude that $\gamma_3(D) = \{a, s\}\gamma_2(D)$, proving the second part of Theorem 2.1.

Remarks. 1) The assertion that $\gamma_3(D)$ is a symbol was also proven in [BM] (Th. 2.10) by quite a different method.

2) If the division algebra corresponding to $D_{F(\sqrt{a})}$ is defined over F, then Theorem 2.1 becomes trivial. Indeed, in this case $D \simeq (a, s) \otimes_F (b, r) \otimes_F (c, r)$ for some $b, c, r, s, t \in F^*$, hence $\gamma_3(D) = \{a, b, c, s, r, t\}$ and $\{a\}\gamma_2(D) = \{a, b, c, r, t\}$. However, in general an algebra of exponent 2 and index 8 is not a tensor product of three quaternion algebras ([ART], [Ka], [S1]).

3) It is easy to see that modulo Lemma 2.4 Theorem 2.1 is equivalent to the equation $\{a, c, s\}D = \gamma_2(D)\{c\}$, which looks similar to the equation in Lemma 2.3. However, we do not know any direct proof of it.

In the following corollary of Theorem 2.1 we give a necessary condition for the existence of subfields of D of certain type (here we consider D as a division algebra).

Corollary 2.7. *Let F be a field, $\operatorname{char} F \neq 2$, $\sqrt{-1} \in F^*$, $D \in_2 \operatorname{Br} F$, $\operatorname{ind} D = 8$. Then*

1) *Suppose that there exist $\alpha, \beta, a \in F$ such that $\operatorname{ind} D_{F(\sqrt{(\alpha+\beta\sqrt{a})^2-\sqrt{a}})} = 2$. (In other words, D contains the field $F(\sqrt{(\alpha + \beta\sqrt{a})^2 - \sqrt{a}})$. Then $\gamma_3(D) = 0$.*

2) *Suppose that L/F is a field extension of degree 8, decomposing into a tower of two cyclic extensions, such that $D_L = 0$. Then $\gamma_3(D) = 0$.*

Proof. 1) Obviously, $\operatorname{ind} D_{F(\sqrt{a})} = 4$ and $\gamma_2(D)_{F(\sqrt{(\alpha+\beta\sqrt{a})^2-\sqrt{a}})} = 0$. Hence $\gamma_2(D)_{F(\sqrt{a})} = \{(\alpha+\beta\sqrt{a})^2 - \sqrt{a}, \omega_1, \omega_2, \omega_3\}$ for some $\omega_1, \omega_2, \omega_3 \in F(\sqrt{a})$. This implies

$$a\gamma_2(D) = N_{F(\sqrt{a})/F}\{\sqrt{a}, (\alpha+\beta\sqrt{a})^2 - \sqrt{a}, \omega_1, \omega_2, \omega_3\} = N_{F(\sqrt{a})/F}(0) = 0,$$

since $\{\sqrt{a}, (\alpha+\beta\sqrt{a})^2 - \sqrt{a}\}$=0. By Theorem 2.1 we are done.

2) If $K = F(\sqrt[4]{a})$ and L/K is a quadratic extension, we are done by part 1), setting $\alpha = \beta = 0$. If $a \in F^*$ and $L/F(\sqrt{a})$ is a cyclic extension of degree 4, then $\operatorname{ind}(D)_{F(\sqrt{a})} = 4$. Moreover, by [RST] or Theorem 1.1 we have $\gamma_2(D)_{F(\sqrt{a})} = \gamma_2(D_{F(\sqrt{a})}) = 0$. Hence

$$\{a\}\gamma_2(D) = N_{F(\sqrt{a})}(\{\sqrt{a}\}\gamma_2(D)_{F(\sqrt{a})}) = N_{F(\sqrt{a})}(0) = 0.$$

Again the proof is finished by Theorem 2.1. □

Corollary 2.8. *Under the previous notation suppose that D is a crossed product for a group G different from $(\mathbb{Z}/2\mathbb{Z})^3$. Then $\gamma_3(D) = 0$.*

Proof. We will give two proofs of this statement. The first one is based on the obvious observation that G contains an element, say σ, of order 4. Let L/F be a subfield of D with the Galois group G and $K = G^{<\sigma>}$. Then the extensions K/F and L/K are cyclic, so we are done by Corollary 2.7.

The second proof was proposed by the referee. As was mentioned in the proof of Theorem 2.1, the quadratic form $q_D(x) = \operatorname{Trd}_D(x^2)$ is similar to the 6-fold Pfister form, corresponding to $\gamma_3(D)$ under the isomorphism $k_6(F) \simeq I^6(F)/I^7(F)$ ([BM], Th. 2.10). Hence it suffices to prove that q_D is isotropic. To do this choose as in the first proof an element $\sigma \in G$ of order 4, and $l \in L^*$ such that $\sigma^2(l) = -l$. There exists $x_\sigma \in D^*$ such that $x_\sigma l' = \sigma(l')x_\sigma$ for any $l' \in L$. It follows that $l^{-1}x_\sigma^2 l = -x_\sigma^2$, hence

$$q_D(x_\sigma) = \operatorname{Trd}(x_\sigma^2) = \operatorname{Trd}(l^{-1}x_\sigma^2 l) = \operatorname{Trd}(-x_\sigma^2) = -q_D(x_\sigma),$$

which implies $q_D(x_\sigma) = 0$. □

Finally we pose a few related problems, which seem to be interesting:

Open questions.

1) Can one extend Theorem 1 to cyclic elements of arbitrary exponent ?

2) Suppose $\exp\alpha = p$, $\operatorname{ind}\alpha = p^n$. Is it true that $\gamma_i(\alpha) = 0$ if $i > n$? (Even in the case $p = 2$ the answer is unknown).

3) Given an odd prime number p, are the conditions $\xi_p \in F$ and $\{a, b, c, d\} = 0 \in K_4(F)/p$ sufficient for the algebra $(a, b)_p \otimes_F (c, d)_p$ to be cyclic ?

4) Suppose that D is a division algebra, whose image in the Brauer group is cyclic. Is D a cyclic algebra ? (In view of Corollary 1.4 the answer is positive for biquaternion algebras if all 2-primary roots of unity are contained in F).

5) Suppose that $\exp D = 2$, $\operatorname{ind} D = 8$, $\gamma_3(D) = 0$. Is $\{a\}\gamma_2(D) = 0$ for some $a \in F^*$ such that $\operatorname{ind} D_{F(\sqrt{a})} = 4$?

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