WEAKLY HYPERBOLIC INVOLUTIONS

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Abstract. An exposition is given of the theory of central simple algebras with involution over (formally) real fields. This includes a new proof of Pfister's Local-Global Principle in this setting and the study of related problems.

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1. Introduction

Pfister's Local-Global Principle says that a regular quadratic form over a (formally) real field represents a torsion element in the Witt ring if and only if its signature at each ordering of the field is zero. This result has been extended in [13] to central simple algebras with involution.

The theory of central simple algebras with involution is a natural extension of quadratic form theory. On the one hand many concepts and related results associated to quadratic forms have been extended to algebras with involution. Examples include isotropy, hyperbolicity, cohomological invariants and signatures. On the other hand quadratic forms are used as tools in the study of algebras with involution. Examples include involution trace forms and spaces of similitudes.

In this article we are interested in weakly hyperbolic algebras with involution, a natural generalization of torsion quadratic forms considered first in [20, Chap. 5]. In [13] such algebras with involution were characterized as those having trivial signature at all orderings of the base field, thus generalizing Pfister's Local-Global Principle.

We aim to give a new exposition of this result including several new aspects and extensions. We attempt to minimize the use of hermitian forms and treat algebras with involution as direct analogues of quadratic forms.

The structure of this article is as follows. In Section 2 we give a self-contained presentation of Pfister's Local-Global Principle for quadratic forms in a generalized version, relative to a preordering. Along the way we will set up the necessary background material from the theory of quadratic forms and ordered fields. This corresponds to the material covered in [10, Chap. 1]. Our approach makes crucial use of Lewis' annihilating polynomials, enabling us to touch on the quantitative aspect of the relation between nilpotence and torsion.

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In Sections 3, 4 and 5 we recall the basic terminology for algebras with involution, consider their relations to quaternion algebras and quadratic forms and study involution trace forms.

In Section 6 we treat the notion of hyperbolicity for algebras with involution and cite the relevant results about hyperbolicity behaviour over field extensions.

In Section 7 we turn to the study of algebras with involution over ordered fields. In (7.2) we obtain a classification over real closed fields. We then provide a uniform definition of signatures for involutions of both kinds with respect to an ordering. Signatures of involutions were introduced in [12] for involutions of the first kind and in [16] for involutions of the second kind, and both cases are treated in [8, (11.10), (11.25)].

In Section 8 we give a new proof of the main result of [13], an analogue of Pfister's Local-Global Principle for algebras with involution. As we present this result in (8.5) it further covers an observation due to Scharlau in [18] on the torsion part of Witt groups. In (8.7) we extend this result to a local-global principle for T-hyperbolicity with respect to a preordering T. Some of the essential ideas contained in Sections 7 and 8 germinated in the MSc thesis of Beatrix Bernauer [2], prepared under the guidance of the first named author.

In its original version for quadratic forms as well as in the generalized version for algebras with involution Pfister's Local-Global Principle relates the hyperbolicity of tensor powers to the hyperbolicity of multiples. For quadratic forms this corresponds to the relation between nilpotence and torsion for an element of the Witt ring. In Section 9 we touch on the quantitative aspect of this relation in the setting of algebras with involution.

2. Pfister's Local-Global Principle

We refer to [9] and [17] for the foundations of quadratic form theory over fields. Let K be a field of characteristic different from 2. We denote by $K^\times$ the multiplicative group of K, by $K^{\times 2}$ the subgroup of nonzero squares, and by $\sum K^2$ the subgroup of nonzero sums of squares in K. If $\sum K^2 = K^{\times 2}$ then K is said to be *pythagorean*.

By a *quadratic form over* K we mean a pair (V, B) consisting of a finite-dimensional K-vector space V and a regular symmetric K-bilinear form $B : V \times V \longrightarrow K$. We mostly use a (single) lower case Greek letter to denote such a pair and often say 'form' instead of 'quadratic form'. If $\varphi = (V, B)$ is a form over K, we say that $a \in K^\times$ is *represented by* φ if $a = B(x, x)$ for some $x \in V$, and we write $\mathsf{D}_K(\varphi)$ for the elements of $K^\times$ represented by φ. Up to isometry a form of dimension n is given by a diagonalization $\langle a_1, \dots, a_n\rangle$, where $a_1, \dots, a_n \in K^\times$ are the values represented on some orthogonal basis. Given $n \in \mathbb{N}$ and $a_1, \dots, a_n \in K^\times$ we write $\langle\langle a_1, \dots, a_n\rangle\rangle$ to denote the form $\langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$ and call this an *n-fold Pfister form*. By [17, Chap. 4, (1.5)] a Pfister form is either anisotropic or hyperbolic. We consider quadratic forms up to isometry and use the equality sign to indicate that two forms are isometric.

Let WK denote the Witt ring of K and IK its fundamental ideal, which consists of the classes of even-dimensional quadratic forms over K. For $n \in \mathbb{N}$ we write I^nK for $(IK)^n$, the nth power of IK. Recall that I^nK is generated as a group by the Witt equivalence classes of the n-fold Pfister forms. We sometimes write $[\varphi]$ to denote the class in WK given by a form φ.

For $n \in \mathbb{N}$ let

$$L_n(X) \;=\; \prod_{i=0}^{n}(X - n + 2i)\,.$$

Note that $L_n(-X) = (-1)^{n+1} \cdot L_n(X)$. In [11], Lewis showed that these polynomials have a crucial property relating to quadratic forms, and that this fact can be applied to study the structure of Witt rings.

2.1. **Theorem** (Lewis)**.** *Let $n \in \mathbb{N}$ and let φ be a quadratic form of dimension n over K. Then $L_n([\varphi]) = 0$ in WK.*

For completeness we include a proof due to K.H. Leung, also given in [11].

Proof. Note that $(a\varphi)^{\otimes 2} = \varphi^{\otimes 2}$ for all $a \in K^\times$. Thus we may scale φ and assume that $\varphi = \varphi' \perp \langle 1\rangle$ where φ' is a form of dimension $n-1$. Using the induction hypothesis for φ' we obtain that $L_{n-1}([\varphi] - 1) = L_{n-1}([\varphi']) = 0$. Since $L_n(X) = (X+n) \cdot L_{n-1}(X-1)$ we conclude that $L_n([\varphi]) = 0$. □

2.2. **Corollary.** *Let $n \in \mathbb{N}$ and let φ be a quadratic form of dimension $2n$ over K. Then $2^{2n-1}n!(n-1)! \cdot [\varphi]$ is a multiple of $[\varphi]^2$ in WK.*

Proof. We may scale φ and assume that $\varphi = \langle 1\rangle \perp \varphi'$ where φ' is a form of dimension $2n-1$. Then $L_{2n-1}([\varphi']) = 0$ by (2.1). It follows that $[\varphi]$ is a zero of the polynomial

$$L_{2n-1}(X-1) = (X-2n)X\prod_{i=1}^{n-1}(X^2 - 4i^2).$$

This implies the statement. □

2.3. **Corollary.** *Let J be an ideal of WK contained in IK and such that WK/J is torsion free. Then J is a radical ideal.*

Proof. For $\alpha \in WK$ with $\alpha^2 \in J$ we obtain by (2.2) that $m\alpha \in J$ for some $m \geq 1$, and as WK/J is torsion free we conclude that $\alpha \in J$. This implies the statement. □

An *ordering of* K is a set $P \subseteq K$ that is additively and multiplicatively closed and that satisfies $P \cup -P = K$ and $P \cap -P = 0$. Any such set P is the *positive cone* $\{x \in K \mid x \geq 0\}$ for a unique total order relation $\leq$ on K that is compatible with the field operations. Let X_K denote the set of orderings of K; it can be equipped with the *Harrison topology* (cf. [9, Chap. VIII, Sect. 6]), but this is not relevant in the sequel.

Let $T \subseteq K$ be additively and multiplicatively closed with $K^{\times 2} \cup \{0\} \subseteq T$. Then $T + xT = \{s + xt \mid s, t \in T\}$ is additively and multiplicatively closed for any $x \in K$. Moreover $T^\times = T \setminus \{0\}$ is a subgroup of $K^\times$ containing $\sum K^2$. If further $-1 \notin T$, then T is called a *preordering of* K. Any ordering is a preordering. Furthermore, if T is a preordering of K, then so is $T + xT$ for any $x \in K \setminus -T$.

2.4. **Proposition.** *Any preordering is contained in an ordering.*

Proof. Using Zorn's Lemma, we obtain that any preordering is contained in a maximal preordering. For a preordering T of K that is not an ordering, there exists an element $x \in K \setminus (T \cup -T)$ and then $T + xT$ is a preordering of K that strictly contains T. Hence, any maximal preordering is an ordering. $\square$

If the field K has an ordering we say that it is *real*, otherwise *nonreal.*

2.5. **Theorem** (Artin-Schreier). *The field K is real if and only if $-1 \notin \sum K^2$.*

Proof. The set $\sum K^2 \cup \{0\}$ is a preordering of K if and only if $-1 \notin \sum K^2$. Since any ordering of K contains $\sum K^2 \cup \{0\}$, the statement follows from (2.4). $\square$

For a preordering T of K we set $X_T = \{P \in X_K \mid T \subseteq P\}$.

2.6. **Theorem** (Artin). *Assume that T is a preordering of K. Then $T = \bigcap_{P \in X_T} P$.*

Proof. For $x \in K \setminus T$ the set $T - xT$ is a preordering of K, hence by (2.4) contained in some ordering P, which then contains T but not x. $\square$

2.7. **Corollary.** *If K is real, then $\sum K^2 \cup \{0\}$ is a preordering and equal to $\bigcap_{P \in X_K} P$.*

Proof. This is clear from (2.5) and (2.6). $\square$

Any $P \in X_K$ determines a unique ring homomorphism $\mathrm{sign}_P : WK \longrightarrow \mathbb{Z}$ that maps the class of $\langle a \rangle$ to 1 for all $a \in P^\times$, called the *signature at* P. Furthermore, any form φ over K induces a map $\widehat{\varphi} : X_K \longrightarrow \mathbb{Z}, P \longmapsto \mathrm{sign}_P(\varphi)$ (cf. [17, Chap. 2, §4]. We obtain a ring homomorphism

$$\mathrm{sign} : WK \longrightarrow \mathbb{Z}^{X_K}, \varphi \longmapsto \widehat{\varphi}$$

called the *total signature.* If K is nonreal, then $X_K = \emptyset$ and $\mathbb{Z}^{X_K}$ is the ring with one element.

Let T be a fixed preordering of K. We write

$$\mathrm{sign}_T : WK \longrightarrow \mathbb{Z}^{X_T}, \varphi \longmapsto \widehat{\varphi}|_{X_T}$$

and we denote the kernel of this homomorphism by $I_T K$.

Let φ be a quadratic form over K. We say that φ is *T-positive* if φ is nontrivial and $\mathsf{D}_K(\varphi) \subseteq T^\times$. If $a_1, \ldots, a_n \in K^\times$ are such that $\varphi = \langle a_1, \ldots, a_n \rangle$, then φ is T-positive if and only if $a_1, \ldots, a_n \in T^\times$. Hence, orthogonal sums and tensor products of T-positive forms are again T-positive. We say that φ is *T-isotropic* or *T-hyperbolic* if there exists a T-positive form ϑ over K such that $\vartheta \otimes \varphi$ is isotropic or hyperbolic, respectively. We write $\mathsf{D}_T(\varphi)$ for the union of the sets $\mathsf{D}_K(\vartheta\otimes\varphi)$ where ϑ runs over all T-positive forms over K.

2.8. **Proposition.** *Let $n\in\mathbb{N}$ and $a_1,\dots,a_n\in K^\times$. The form $\langle a_1,\dots,a_n\rangle$ is T-isotropic if and only if $\langle t_1a_1,\dots,t_na_n\rangle$ is isotropic for certain $t_1,\dots,t_n\in T^\times$. For $a\in K^\times$, we have that $a\in\mathsf{D}_T\langle a_1,\dots,a_n\rangle$ if and only if $a\in\mathsf{D}_K\langle t_1a_1,\dots,t_na_n\rangle$ for certain $t_1,\dots,t_n\in T^\times$.*

Proof. Let $\varphi=\langle a_1,\dots,a_n\rangle$. For $t_1,\dots,t_n\in T^\times$ the form $\vartheta=\langle t_1,\dots,t_n\rangle$ is T-positive, and $\langle t_1a_1,\dots,t_na_n\rangle$ is a subform of $\vartheta\otimes\varphi$. This shows the right-to-left implications. To show the left-to-right implications, consider a T-positive form ϑ and an element $a\in K$ that is non-trivially represented by $\vartheta\otimes\varphi$. Since $\vartheta\otimes\varphi=a_1\vartheta\perp\cdots\perp a_n\vartheta$ it follows that there exist $s_1,\dots,s_n\in\mathsf{D}_K(\vartheta)\cup\{0\}$, not all equal to zero, such that $a=a_1s_1+\cdots+a_ns_n$. Letting $t_i=1$ if $s_i=0$ and $t_i=s_i$ otherwise for $1\le i\le n$, we have that $t_1,\dots,t_n\in T^\times$ and that a is represented nontrivially by $\langle t_1a_1,\dots,t_na_n\rangle$. □

2.9. **Proposition.** *Let $\mathfrak{p}$ be a prime ideal of WK different from IK. The set $P=\{t\in K^\times\mid[\langle 1,-t\rangle]\in\mathfrak{p}\}\cup\{0\}$ is an ordering of K, and $I_PK\subseteq\mathfrak{p}$.*

Proof. For $s,t\in P\setminus\{0\}$ we have that $[\langle 1,-st\rangle]=[\langle t\rangle]\cdot([\langle 1,-s\rangle]-[\langle 1,-t\rangle])\in\mathfrak{p}$ and thus $st\in P$. Therefore P is a multiplicatively closed subset of K. For $t\in K^\times$ we have $[\langle 1,-t\rangle]\otimes[\langle 1,t\rangle]=0\in\mathfrak{p}$ and thus $[\langle 1,-t\rangle]\in\mathfrak{p}$ or $[\langle 1,t\rangle]\in\mathfrak{p}$, showing that $K=P\cup-P$. Since $\mathfrak{p}$ is different from IK, which is a maximal ideal of WK and generated by the elements $[\langle 1,-a\rangle]$ with $a\in K^\times$, we obtain that $P\subsetneq K$. Since $K=P\cup-P$ it follows that $-1\notin P$ and $P\cap-P=0$. To show that P is additively closed, we consider $s,t\in P\setminus\{0\}$. As $s^{-1}t\in P$ we have $s+t\neq 0$. Using [9, Chap. I, (5.1)] we see that $[\langle 1,s+t\rangle]\cdot[\langle 1,st\rangle]=[\langle 1,s\rangle]\cdot[\langle 1,t\rangle]$. As $-s,-t\notin P$, the elements $[\langle 1,s\rangle]$ and $[\langle 1,t\rangle]$ do not lie in $\mathfrak{p}$, thus neither does their product, for $\mathfrak{p}$ is prime. We conclude that $[\langle 1,s+t\rangle]\notin\mathfrak{p}$ and thus $s+t\in K\setminus-P=P\setminus\{0\}$. Hence P is additively closed. This shows that P is an ordering of K.

The ideal I_PK is generated by the classes of forms $\langle 1,-t\rangle$ with $t\in P^\times$, and these belong to $\mathfrak{p}$. So $I_PK\subseteq\mathfrak{p}$. □

The following statement is a generalization of Pfister's Local-Global Principle, relative to a preordering (cf. [10, (1.26)]).

2.10. **Theorem** (Pfister)**.** *Let T be a preordering of K. The ideal I_TK is generated by the classes of binary forms $\langle 1,-t\rangle$ with $t\in T^\times$. Moreover, for a quadratic form φ over K the following statements are equivalent:*

(*i*) *We have* $\mathrm{sign}_T(\varphi)=0$.
(*ii*) *The form φ is T-hyperbolic.*
(*iii*) *There exists a T-positive Pfister form τ over K such that $\tau\otimes\varphi$ is hyperbolic.*
(*iv*) *There exist $r\ge 0$, $a_1,\dots,a_r\in K^\times$ and $t_1,\dots,t_r\in T^\times$ such that φ is Witt equivalent to $\langle a_1,-a_1t_1\rangle\perp\cdots\perp\langle a_r,-a_rt_r\rangle$.*

Proof. Let J denote the ideal of WK generated by the classes of the binary forms $\langle 1, -t\rangle$ with $t \in T^\times$. Obviously, $J \subseteq I_TK$. Note that I_TK is equal to the intersection of prime ideals $\bigcap_{P\in X_T} I_PK$ and contained in IK. Given any prime ideal $\mathfrak{p}$ of WK such that $J \subseteq \mathfrak{p} \neq IK$, the set $P = \{t \in K^\times \mid \langle 1, -t\rangle \in \mathfrak{p}\} \cup \{0\}$ is an ordering of K containing T, so that $J \subseteq I_TK \subseteq I_PK \subseteq \mathfrak{p}$ by (2.9). This shows that $J \subseteq I_TK \subseteq \sqrt{J}$.

Note that a quadratic form φ over K is T-isotropic if and only if $\varphi \equiv \psi \bmod J$ for a quadratic form ψ over K with $\dim(\psi) < \dim(\varphi)$. In particular, φ is T-hyperbolic if and only if $\varphi \in J$. From this, it follows immediately that WK/J is torsion free. Hence J is a radical ideal by (2.3), and we conclude that $I_TK = J$.

This shows that $(i) \iff (iv)$. The implications $(iii) \implies (ii) \implies (i)$ are obvious. Finally we have $(iv) \implies (iii)$, since, with elements given as in (iv), we may choose $\tau = \langle 1, t_1\rangle \otimes \cdots \otimes \langle 1, t_r\rangle$. □

Let φ be a quadratic form over K and $m \in \mathbb{N}$. We write $m \times \varphi$ for the m-fold orthogonal sum $\varphi \perp \cdots \perp \varphi$. We abbreviate $\mathsf{D}_K(m) = \mathsf{D}_K(m \times \langle 1\rangle)$, which is the set of nonzero sums of m squares in K.

A quadratic form φ over K is said to be *torsion* or *weakly hyperbolic* if $m \times \varphi$ is hyperbolic for some positive integer m. The following is [15, Satz 22].

2.11. **Corollary** (Pfister)**.** *Assume that K is real. For a quadratic form φ over K the following statements are equivalent:*

- (i) *We have* $\mathrm{sign}(\varphi) = 0$.
- (ii) *The quadratic form φ is weakly hyperbolic.*
- (iii) *There exists $m \in \mathbb{N}$ such that $2^m \times \varphi$ is hyperbolic.*
- (iv) *There exists $n \in \mathbb{N}$ such that $\varphi^{\otimes n}$ is hyperbolic.*
- (v) *There exist $r \geq 0$, $a_1, \ldots, a_r \in K^\times$ and $s_1, \ldots, s_r \in \sum K^2$ such that φ is Witt equivalent to $\langle a_1, -a_1s_1\rangle \perp \cdots \perp \langle a_r, -a_rs_r\rangle$.*

Proof. We consider the preordering $S = \sum K^2 \cup \{0\}$. By (2.10) we have that (i) and (v) are equivalent. Using (2.9), it follows that I_SK is the intersection of all prime ideals of WK and thus the nilradical of WK. This yields the equivalence of (i) and (iv). Clearly (iii) implies (ii), which in turn implies (i). We conclude by showing that (v) implies (iii). Given elements $s_1, \ldots, s_r \in \sum K^2$ such that φ is Witt equivalent to $\langle a_1, -a_1s_1\rangle \perp \cdots \perp \langle a_r, -a_rs_r\rangle$, we choose $m \in \mathbb{N}$ such that $s_1, \ldots, s_r \in \mathsf{D}_K(2^m)$ and then have that $2^m \times \varphi$ is hyperbolic. □

2.12. **Corollary** (Scharlau)**.** *The order of any torsion element in WK is a 2-power.*

Proof. If K is real, this is a rephrasing of the equivalence $(ii) \iff (iii)$ of (2.11). If K is nonreal, then $-1 \in \mathsf{D}_K(2^n)$ for some $n \in \mathbb{N}$, and then $2^{n+1}WK = 0$, which yields the statement. □

2.13. **Corollary** (Scharlau)**.** *Any zero-divisor of WK lies in IK.*

Proof. Let $\alpha \in WK \setminus IK$. By (2.1) there exists $n \in \mathbb{N}$ such that α is a zero of $L_{2n+1}(X) = \prod_{i=0}^{n}(X^2 - (2i+1)^2)$. Hence, for $\beta \in WK$ with $\alpha\beta = 0$ we have $m\beta = 0$ for the odd integer $m = \prod_{i=0}^{n}(2i+1)^2$, thus $\beta = 0$ by (2.12). $\square$

For $n \in \mathbb{N}$ we denote by $d(n)$ the number of occurrences of the digit 1 in the binary representation of n. Note that $d(2n) = d(n)$ and $d(2n+1) = d(n)+1$. In [4, §4.4] the following observation is attributed to Legendre.

2.14. **Proposition.** *For $n \in \mathbb{N}$ the largest 2-power dividing $n!$ is $2^{n-d(n)}$.*

Proof. Let $n \in \mathbb{N}$. The largest 2-power dividing n is 2^m where m is the number of consecutive digits 1 at the end of the binary representation of $n-1$, whereby $m = d(n-1) - d(n) + 1$. Hence the largest 2-power dividing $n!$ is 2^k, where $k = \sum_{i=1}^{n}\big(d(i-1) - d(i) + 1\big) = n - d(n)$. $\square$

For $n \geq 1$ we set $\Delta(n) = 2n - 1 - d(n) - d(n-1)$.

2.15. **Theorem.** *Let φ and π be quadratic forms over K such that $\varphi \otimes \varphi \otimes \pi$ is hyperbolic. Then $2^{\Delta(n)} \times \varphi \otimes \pi$ is hyperbolic for $n = \dim(\varphi)$.*

Proof. Assume that π is not hyperbolic, as otherwise the statement is trivial. Then we have $n = \dim(\varphi) = 2k$ for some $k \in \mathbb{N}$ by (2.13). It follows from (2.2) that $2^{2k-1}k!(k-1)! \times \varphi \otimes \pi$ is hyperbolic. Since we have

$$\Delta(n) = 4k - 1 - d(2k) - d(2k-1) = (2k-1) + k - d(k) + k - 1 - d(k-1)$$

we conclude by (2.12) that $2^{\Delta(n)} \times \varphi \otimes \pi$ is hyperbolic. $\square$

2.16. **Remark.** In view of (2.15) we may define a function $g : \mathbb{N} \longrightarrow \mathbb{N}$ in the following way: for $k \in \mathbb{N}$, let $g(k)$ be the smallest number $m \in \mathbb{N}$ such that, for any quadratic form φ of dimension $2k$ over an arbitrary field of characteristic different from 2 for which $\varphi \otimes \varphi$ is hyperbolic, also $2^m \times \varphi$ is hyperbolic. Applying (2.15) with $\pi = \langle 1 \rangle$ yields that $g(k) \leq \Delta(2k) = 4k - 2 - d(k) - d(k-1)$. This bound, however, does not seem to be optimal for $k > 1$. In fact, it is not difficult to show that $g(2) = g(3) = 2$.

3. Algebras with involution

Our general references for the theory of algebras with involution are [8] and [17, Chap. 8]. We recall the terminology that is used in the sequel.

Let K be a field of characteristic different from two and let A be a K-algebra. We denote by $Z(A)$ the centre of A. A *K-involution on A* is a K-linear map $\sigma : A \longrightarrow A$ such that $\sigma(xy) = \sigma(y)\sigma(x)$ for all $x, y \in A$ and $\sigma \circ \sigma = \mathrm{id}_A$.

A *K-algebra with involution* is a pair (A, σ) where A is a finite-dimensional K-algebra and σ is a K-involution on A such that $K = \{x \in Z(A) \mid \sigma(x) = x\}$ and such that either A is simple or A is a product of two simple K-algebras that are mapped to each other by σ. We will often denote a K-algebra with involution by a single capital Greek letter.

Let (A, σ) be a K-algebra with involution. We have either $Z(A) = K$ or $Z(A)$ is a quadratic étale extension of K. We say that (A, σ) is of the *first* or *second kind*, depending on whether $[Z(A) : K]$ is 1 or 2, respectively. Note that A is simple if and only if $Z(A)$ is a field. If $Z(A)$ is not a field then (A, σ) is *degenerate*; this can only occur if (A, σ) is of the second kind.

We have $\dim_K(A) = [Z(A) : K] \cdot n^2$ for a positive integer $n \in \mathbb{N}$, called the *degree of* A and denoted $\deg(A)$. If A is nondegenerate, then $\deg(A)$ is the degree of A as a central simple $Z(A)$-algebra.

We say that $x \in A$ is *symmetric* or *skew-symmetric* (with respect to σ) if $\sigma(x) = x$ or $\sigma(x) = -x$, respectively. We let $\mathrm{Sym}(A, \sigma) = \{x \in A \mid \sigma(x) = x\}$ and $\mathrm{Skew}(A, \sigma) = \{x \in A \mid \sigma(x) = -x\}$. These are K-linear subspaces of A satisfying

$$A = \mathrm{Sym}(A, \sigma) \oplus \mathrm{Skew}(A, \sigma).$$

There exists $\varepsilon \in \{-1, 0, +1\}$ such that

$$\dim_K(\mathrm{Sym}(A, \sigma)) = \tfrac{1}{2}n(n + \varepsilon) \quad \text{and} \quad \dim_K(\mathrm{Sym}(A, \sigma)) = \tfrac{1}{2}n(n - \varepsilon)\,.$$

If (A, σ) is of the first kind, then $\varepsilon = \pm 1$, and we say that (A, σ) is *orthogonal* if $\varepsilon = 1$ and *symplectic* if $\varepsilon = -1$. If (A, σ) is of the second kind, then $\varepsilon = 0$, and we say that (A, σ) is *unitary.* The integer ε is called the *type of* (A, σ) and denoted $\mathrm{type}(A, \sigma)$.

Following [8, §12], given a K-algebra with involution (A, σ), we denote

$$\begin{aligned} \mathrm{Sim}(A, \sigma) &= \{x \in A^\times \mid \sigma(x)x \in K^\times\} \quad \text{and} \\ \mathsf{G}(A, \sigma) &= \{\sigma(x)x \mid x \in \mathrm{Sim}(A, \sigma)\}\,; \end{aligned}$$

note that these are subgroups of $A^\times$ and $K^\times$, respectively.

We denote by $\mathrm{Br}(K)$ the Brauer group of K. The group operation in $\mathrm{Br}(K)$ is written additively. A central simple K-algebra A is *split* if $[A] = 0$ in $\mathrm{Br}(K)$.

Let Ψ denote the K-algebra with involution (A, σ). If Ψ is non-degenerate, let $[\Psi] = \pm[A]$ in $\mathrm{Br}(L)$ for $L = Z(A)$. Here, $\pm[A]$ denotes the element $[A]$ if this is of order at most 2 and otherwise the unordered pair of $[A]$ and $-[A]$. Recall that if Ψ is of the first kind, then $[A] + [A] = 0$ in $\mathrm{Br}(K)$. If Ψ is degenerate unitary, then $A \simeq A_1 \times A_2$ for two central simple K-algebras A_1 and A_2 with $\sigma(A_1) = A_2$, so that $[A_1] + [A_2] = 0$ in $\mathrm{Br}(K)$, and we set $[\Psi] = \pm[A_i]$ for $i = 1, 2$. If A is simple, let $\mathrm{ind}(\Psi)$ denote the Schur index of A as a central simple $Z(A)$-algebra, otherwise let $\mathrm{ind}(\Psi)$ be the Schur index of the two simple components of A (which is the same). We also write $Z(\Psi)$ to refer to the centre of A.

For any field extension L/K the L-algebra with involution $(A \otimes_K L, \sigma \otimes \mathrm{id}_L)$ is denoted by Ψ_L. Note that $\mathrm{type}(\Psi_L) = \mathrm{type}(\Psi)$. If Ψ is non-degenerate unitary, then Ψ_L is non-degenerate if and only if L is linearly disjoint to $Z(\Psi)$ over K.

We now consider two K-algebras with involution $\Psi = (A, \sigma)$ and $\Theta = (B, \vartheta)$. A *homomorphism of algebras with involution* $\Psi \longrightarrow \Theta$ is a K-homomorphism $f : A \longrightarrow B$ satisfying $\vartheta \circ f = f \circ \sigma$. A homomorphism is called an *embedding* if it is injective and an *isomorphism* if it is bijective. We write $\Psi \simeq \Theta$ if there exists and isomorphism $\Psi \longrightarrow \Theta$. (This occurs if and only if either Ψ and Θ are adjoint to two similar hermitian or skew-hermitian forms over some K-division algebra with involution or Ψ and Θ are both degenerate unitary of the same degree.)

We further write $\Psi \sim \Theta$ to indicate that $\mathrm{type}(\Psi) = \mathrm{type}(\Theta)$ and $[\Psi] = [\Theta]$. (This occurs if and only if Ψ and Θ are either both degenerate unitary or both adjoint algebras with involution of hermitian or skew-hermitian forms over a common K-division algebra with involution.)

Except when Ψ and Θ are both unitary with different centres, we can define their tensor product $\Psi \otimes \Theta$. If Ψ and Θ are not both unitary, let $\Psi \otimes \Theta$ denote the K-algebra with involution $(A \otimes_K B, \sigma \otimes \vartheta)$. If Ψ and Θ are both unitary and with same centre L, then $\Psi \otimes \Theta$ is the unitary K-algebra with involution $(A \otimes_L B, \sigma \otimes \vartheta)$, whose centre is also L. Note that in each of the cases where we defined $\Psi \otimes \Theta$, we have

$$\mathrm{type}(\Psi \otimes \Theta) = \mathrm{type}(\Psi) \cdot \mathrm{type}(\Theta) \quad \text{and} \quad \deg(\Psi \otimes \Theta) = \deg(\Psi) \cdot \deg(\Theta) .$$

For a positive integer n the tensor power $\Psi^{\otimes n}$ of a K-algebra with involution Ψ is now well-defined.

4. Algebras with involution of small index

Involutions on central simple algebras are often considered as adjoint to hermitian or skew-hermitian forms (cf. [8, §4]). We will only need this approach for algebras with involution of small index. We fix some notation for the split case.

Let $\varphi = (V, B)$ be a quadratic form over K. Consider the split central simple K-algebra $\mathrm{End}_K(V)$. Let $\sigma : \mathrm{End}_K(V) \longrightarrow \mathrm{End}_K(V)$ denote the involution determined by the formula

$$B(f(u), v) = B(u, \sigma(f)(v)) \quad \text{for all } u, v \in V \text{ and } f \in \mathrm{End}_K(V).$$

We denote this involution σ by ad_B and call it the *adjoint involution of* φ. Furthermore, we call $(\mathrm{End}_K(V), \mathrm{ad}_B)$ the *adjoint algebra with involution of* φ and denote it by $\mathsf{Ad}(\varphi)$. Note that it is split orthogonal and that φ is determined up to similarity by $\mathsf{Ad}(\varphi)$.

4.1. **Example.** Let n be a positive integer and $\varphi = n \times \langle 1 \rangle$, the n-dimensional form $\langle 1, \ldots, 1 \rangle$ over K. Then $\mathsf{Ad}(\varphi) \simeq (\mathsf{M}_n(K), \tau)$ where $\mathsf{M}_n(K)$ is the K-algebra of $n \times n$-matrices over K and τ is the transpose involution.

4.2. **Proposition.** *For quadratic forms φ and ψ over K we have*

$$\mathsf{Ad}(\varphi \otimes \psi) \simeq \mathsf{Ad}(\varphi) \otimes \mathsf{Ad}(\psi) .$$

Proof. Denoting V and W the underlying vector spaces of φ and ψ, respectively, the natural K-algebra isomorphism $\mathrm{End}_K(V) \otimes_K \mathrm{End}_K(W) \longrightarrow \mathrm{End}_K(V \otimes_K W)$ yields the required identification for the adjoint involutions. $\square$

For a finite-dimensional K-algebra A we denote by $\mathrm{Trd}_A : A \longrightarrow Z(A)$ its reduced trace map (cf. [8, p. 5 and p. 22]).

A *K-quaternion algebra* is a central simple K-algebra of index 2. Given a K-quaternion algebra Q, the map $\sigma : Q \longrightarrow K, x \longmapsto x - \mathrm{Trd}_Q(x)$ is a K-involution, called the *canonical involution on* Q and denoted by can_Q; this is the unique symplectic K-involution on Q. If L is a quadratic étale extension of K we denote by can_L the unique non-trivial K-automorphism of L. We further set $\mathrm{can}_K = \mathrm{id}_K$.

4.3. **Proposition.** *Let (A, σ) be a K-algebra with involution. We have that $\mathrm{Sym}(A, \sigma) = K$ if and only if A is either K, a quadratic étale extension of K, or a K-quaternion algebra, and if further $\sigma = \mathrm{can}_A$.*

Proof. If $\mathrm{Sym}(A, \sigma) = K$ then $\dim_K(A) = 2^{1-\varepsilon}$ for $\varepsilon = \mathrm{type}(A, \sigma)$, so that A is either K, a quadratic étale extension of K, or a K-quaternion algebra. In any of these three cases, can_A is the unique involution of type ε on A, and $\mathrm{Sym}(A, \mathrm{can}_A) = K$. □

In the cases characterized by (4.3) we call (A, σ) a *K-algebra with canonical involution*.

4.4. **Proposition.** *Let Ψ be a K-algebra with involution. Then $\Psi \simeq \Phi \otimes \mathsf{Ad}(\varphi)$ for a K-algebra with canonical involution Φ and a quadratic form φ over K if and only if Ψ is either split or symplectic of index 2.*

Proof. Clearly, any K-algebra with canonical involution Φ is either split or symplectic of index 2, and thus so is $\Phi \otimes \mathsf{Ad}(\varphi)$ for any quadratic form φ over K. Assume now that Ψ is either split or symplectic of index 2. Then $\Psi \sim \Phi$ for a K-algebra with canonical involution Φ. It follows that Ψ is adjoint to a hermitian form over Φ. Any hermitian form over Φ has a diagonalisation with entries in $\mathrm{Sym}(\Psi) = K$. Therefore $\Psi \simeq \Phi \otimes \mathsf{Ad}(\varphi)$ for a form φ over K. □

For computational purposes we augment the classical notation for quaternion algebras in terms of pairs of field elements to take into account an involution. Let $a, b \in K^\times$ and let Q be the K-algebra with basis $(1, i, j, k)$, where $i^2 = a$, $j^2 = b$ and $ij = -ji = k$. This quaternion algebra is denoted by $(a, b)_K$. For $\delta, \varepsilon \in \{+1, -1\}$ there is a unique K-involution σ on Q such that $\sigma(i) = \delta i$ and $\sigma(j) = \varepsilon j$. We denote the pair (Q, σ) by

$$\begin{array}{lll} (a \mid b)_K & \text{if} & \delta = +1,\ \varepsilon = +1, \\ (a \cdot\!\mid b)_K & \text{if} & \delta = -1,\ \varepsilon = +1, \\ (a \mid\!\cdot b)_K & \text{if} & \delta = +1,\ \varepsilon = -1, \\ (a \cdot\!\mid\!\cdot b)_K & \text{if} & \delta = -1,\ \varepsilon = -1. \end{array}$$

In particular, $(a \cdot\!\mid\!\cdot b)_K$ denotes the quaternion algebra $(a, b)_K$ together with its canonical involution. Any K-quaternion algebra with orthogonal involution is isomorphic to $(a \cdot\!| \, b)_K$ for some $a, b \in K^\times$. Note that $(a \mid b)_K \simeq (-ab \cdot\!| \, b)_K$ and $(a \,|\!\cdot b)_K \simeq (b \cdot\!| \, a)_K$ for any $a, b \in K^\times$.

4.5. **Proposition.** *Let Q be a K-quaternion algebra and let $a, b \in K^\times$ be such that $Q \simeq (a, b)_K$. Then $(Q, \mathrm{can}_Q) \simeq (a \cdot\!|\!\cdot b)_K$. Moreover, for $i \in Q^\times \setminus K^\times$ with $i^2 = a$ and $\tau = \mathrm{Int}(i) \circ \mathrm{can}_Q$ we have $(Q, \tau) \simeq (a \cdot\!| \, b)_K$.*

Proof. Since can_Q is the only symplectic involution on Q, any K-isomorphism $Q \longrightarrow (a, b)_K$ is also an isomorphism of K-algebras with involution. Hence, $(Q, \mathrm{can}_Q) \simeq (a \cdot\!|\!\cdot b)_K$. Choose an element $i \in Q^\times \setminus K^\times$ with $i^2 = a$. Then $V = \{j \in Q \mid ij + ji = 0\}$ is the orthogonal complement of $K[i]$ in Q with respect to the symmetric K-bilinear form $B : Q \times Q \longrightarrow K, (x, y) \longmapsto \mathrm{can}_Q(x) \cdot y$. By [17, Chap. 2, (11.4)] we have $(Q, B) \simeq \langle\langle a, b\rangle\rangle$. Since $(K[i], B|_{K[i]}) \simeq \langle\langle a\rangle\rangle$ it follows that $(V, B|_V) \simeq -b\langle\langle a\rangle\rangle$. As $B(j, j) = -j^2$ for any $j \in V$ there exists $j \in V$ with $j^2 = b$. For $\tau = \mathrm{Int}(i) \circ \mathrm{can}_Q$ we obtain that $\tau(j) = j$, whereby $(Q, \tau) \simeq (a \cdot\!| \, b)_K$. □

For $a \in K^\times$, let $(a)_K$ denote the unitary K-algebra with canonical involution (L, can_L) where $L = K[X]/(X^2 - a)$; it is degenerate if and only if $a \in K^{\times 2}$.

4.6. **Corollary.** *Let Φ be a K-quaternion algebra with involution. If Φ is orthogonal there exist $a, b \in K^\times$ such that $\Phi \simeq (a \cdot\!| \, b)_K$. If Φ is symplectic there exist $a, b \in K^\times$ such that $\Phi \simeq (a \cdot\!|\!\cdot b)_K$. If Φ is unitary, there exist $a, b, c \in K^\times$ such that $\Phi \simeq (a \cdot\!|\!\cdot b)_K \otimes (c)_K$.*

Proof. Assume that Φ is of the first kind and let $\Phi = (Q, \sigma)$. If Φ is symplectic, then $\sigma = \mathrm{can}_Q$ and we choose $a, b \in Q^\times$ such that $Q \simeq (a, b)_K$ to obtain by (4.5) that $\Phi \simeq (a \cdot\!|\!\cdot b)_K$. If Φ is orthogonal, we choose $i \in \mathrm{Skew}(Q, \sigma) \cap Q^\times$ and $a, b \in K^\times$ with $a = i^2$ and $Q \simeq (a, b)_K$, and obtain that $\sigma = \mathrm{Int}(i) \circ \mathrm{can}_Q$, so that $\Phi \simeq (a \cdot\!| \, b)_K$ by (4.5).

Assume now that Φ is of the second kind. From [8, (2.22)] we obtain that $\Phi \simeq (Q, \mathrm{can}_Q) \otimes (c)_K$ for a K-quaternion algebra Q and an element $c \in K^\times$, and by the above there exist $a, b \in K^\times$ such that $(Q, \mathrm{can}_Q) \simeq (a \cdot\!|\!\cdot b)_K$. □

4.7. **Proposition.** *Let $a, b, c, d \in K^\times$. We have $(a \cdot\!| \, b)_K \simeq (c \cdot\!| \, d)_K$ if and only if $aK^{\times 2} = cK^{\times 2}$ and $bd \in \mathsf{D}_K\langle\langle a\rangle\rangle$.*

Proof. We set $(Q, \tau) = (a \cdot\!| \, b)_K$. There exists $i \in \mathrm{Skew}(Q, \tau)$ and $j \in \mathrm{Sym}(Q, \tau)$ with $i^2 = a$, $j^2 = b$, and $ij + ji = 0$.

Assuming that $bd \in \mathsf{D}_K\langle\langle a\rangle\rangle$, we may write $d = b(u^2 - av^2)$ with $u, v \in K$ and obtain for $g = uj + vij$ that $g \in \mathrm{Sym}(Q, \tau)$, $gi + ig = 0$, and $g^2 = d$. If further $cK^{\times 2} = aK^{\times 2}$, then $c = f^2$ for some $f \in iK^\times$, and we have $f \in \mathrm{Skew}(Q, \tau)$ and $gf + gh = 0$, and conclude that $(Q, \tau) \simeq (c \cdot\!| \, d)_K$.

For the converse, suppose that $(Q, \tau) \simeq (c \cdot\!| \, d)_K$. There exist $f \in \mathrm{Skew}(Q, \tau)$ and $g \in \mathrm{Sym}(Q, \tau)$ with $f^2 = c$, $g^2 = d$, and $fg + gf = 0$. It follows that $iK = \mathrm{Skew}(Q, \tau) = fK$, so that $aK^{\times 2} = cK^{\times 2}$. Moreover, $ig + gi = 0$ and

$jgi = ijg$. As $K[i]$ is a maximal commutative K-subalgebra of Q, we obtain that $jg \in K[i]$. Writing $jg = x + iy$ with $x, y \in K$, we obtain that

$$bd = j^2g^2 = j(x+iy)g = (x-iy)jg = (x-iy)(x+iy) = x^2 - ay^2\,,$$

whence $bd \in D_K\langle 1, -a\rangle$. □

4.8. **Proposition.** *For $a, b, c, d \in K^\times$ we have*

$$(a \cdot\!| \; b)_K \otimes (c \cdot\!| \; d)_K \simeq (a \cdot\!|\!\cdot bc)_K \otimes (c \cdot\!|\!\cdot ad)_K \quad \textit{and}$$
$$(a \cdot\!| \; b)_K \otimes (c \cdot\!|\!\cdot d)_K \simeq (a \cdot\!|\!\cdot bc)_K \otimes (c \cdot\!| \; ad)_K\,.$$

Proof. Let $(A,\sigma) = (a \cdot\!| \; b)_K \otimes (c \cdot\!| \; d)_K$. Then there exist elements $i, j, f, g \in A^\times$ such that $\sigma(i) = -i$, $\sigma(j) = j$, $\sigma(f) = -f$, $\sigma(g) = g$, $i^2 = a$, $j^2 = b$, $f^2 = c$, $g^2 = d$, $ij + ji = fg + gf = 0$, and each of i and j commutes with each of f and g. Set $j' = fj$ and $g' = ig$. Then $\sigma(i) = -i$, $\sigma(j') = -j'$, $\sigma(f) = -f$, $\sigma(g') = -g'$, $i^2 = a$, $j'^2 = bc$, $f^2 = c$, $g'^2 = ad$, $ij' + j'i = fg' + g'f = 0$, and each of i and j' commutes with each of f and g'. The K-subalgebra Q of A generated by i and j' commutes elementwise with the K-subalgebra Q' of A generated by f and g', and Q and Q' are σ-stable. Hence

$$(A,\sigma) \simeq (Q, \sigma|_Q) \otimes (Q', \sigma|_{Q'}) \simeq (a \cdot\!|\!\cdot bc)_K \otimes (c \cdot\!|\!\cdot ad)_K\,.$$

This shows the first isomorphism. The proof of the second isomorphism is almost identical, with the only difference that $\sigma(g) = -g$ and $\sigma(g') = g'$. □

4.9. **Proposition.** *For $a, b \in K^\times$, we have*

$$\mathsf{G}\,(a \cdot\!|\!\cdot b)_K = \mathsf{D}_K\langle\langle a, b\rangle\rangle \qquad \textit{and} \qquad \mathsf{G}\,(a \cdot\!| \; b)_K = \mathsf{D}_K\langle\langle a\rangle\rangle \cup b\mathsf{D}_K\langle\langle a\rangle\rangle.$$

Proof. Let $Q = (a,b)_K$ and $u, v \in Q^\times$ with $u^2 = a$, $v^2 = b$ and $uv + vu = 0$. Then $\mathrm{Sim}(Q, \mathrm{can}_Q) = Q^\times$ and thus $\mathsf{G}(Q, \mathrm{can}_Q) = \mathsf{D}_K\langle\langle a, b\rangle\rangle$. For $\tau = \mathrm{Int}(u) \circ \mathrm{can}_Q$ we obtain $\mathrm{Sim}(Q,\tau) = K(u)^\times \cup vK(u)^\times$ and thus $\mathsf{G}(Q,\tau) = \mathsf{D}_K\langle\langle a\rangle\rangle \cup b\mathsf{D}_K\langle\langle a\rangle\rangle$. □

5. Involution trace forms

A K-algebra with involution (A,σ) with centre L gives rise to a regular hermitian form $T_{(A,\sigma)} : A \times A \longrightarrow L$ over (L, can_L) defined by $T_{(A,\sigma)}(x,y) = \mathrm{Trd}_A(\sigma(x)y)$; this follows from [8, (2.2) and (2.16)]. We further obtain a regular symmetric K-bilinear form $T_\sigma : A \times A \longrightarrow K$ defined by $T_\sigma(x,y) = \frac{1}{2}\mathrm{Trd}_A(\sigma(x)y + \sigma(y)x)$. Note that if $L = K$ then $T_\sigma = T_{(A,\sigma)}$, otherwise $2T_\sigma = T \circ T_{(A,\sigma)}$ where T is the trace of $Z(A)/K$. (Here, 2φ denotes the form obtained by scaling the form φ by 2, which ought not to be confused with the form $2 \times \varphi = \varphi \perp \varphi$.)

Given a K-algebra with involution $\Psi = (A,\sigma)$, we denote by $\mathsf{Tr}(\Psi)$ the quadratic form (A, T_σ) over K. Note that $\dim(\mathsf{Tr}(A,\sigma)) = \dim_K(A)$.

5.1. **Example.** For $a \in K^\times$ we have $\mathsf{Tr}\,(a)_K = \langle\langle a\rangle\rangle$. For $a, b \in K^\times$ we have $\mathsf{Tr}\,(a \cdot\!| \; b)_K = 2\langle\langle a, -b\rangle\rangle$ and $\mathsf{Tr}\,(a \cdot\!|\!\cdot b)_K = 2\langle\langle a, b\rangle\rangle$.

5.2. **Proposition.** *Let Ψ and Θ be K-algebras with involution. If Ψ is of the first kind, then* $\mathsf{Tr}(\Psi \otimes \Theta) \simeq \mathsf{Tr}(\Psi) \otimes \mathsf{Tr}(\Theta)$. *If Ψ and Θ are both unitary with same centre, then* $2 \times \mathsf{Tr}(\Psi \otimes \Theta) \simeq \mathsf{Tr}(\Psi) \otimes \mathsf{Tr}(\Theta)$.

Proof. Let K' denote the centre of $\Psi = (A, \sigma)$ and L the centre of $\Theta = (B, \tau)$. In view of the claims we may assume that $K' \subseteq L$. For $a \in A$ and $b \in B$ we have $\mathrm{Trd}_{A \otimes_{K'} B}(a \otimes b) = \mathrm{Trd}_A(a) \cdot \mathrm{Trd}_B(b)$, as one verifies by reduction to the split case. Hence, $T_{\Psi \otimes \Theta}$ and $T_\Psi \otimes T_\Theta$ coincide as hermitian forms on $A \otimes_{K'} B$ with respect to (L, can_L). If $L = K$ then we are done. Assume now that $(L, \mathrm{can}_L) \simeq (c)_K$ where $c \in K^\times$. Then $\mathsf{Tr}(\Theta) \simeq \langle\langle c \rangle\rangle \otimes \vartheta$ for a form ϑ over K. If now $K' = K$ then $\mathsf{Tr}(\Psi) \otimes \mathsf{Tr}(\Theta) \simeq \langle\langle c \rangle\rangle \otimes (\mathsf{Tr}(\Psi) \otimes \vartheta) \simeq \mathsf{Tr}(\Psi \otimes \Theta)$. In the remaining case $K' = L$ and $\mathsf{Tr}(\Psi) \simeq \langle\langle c \rangle\rangle \otimes \psi$ for a quadratic form ψ over K, and we obtain that $\mathsf{Tr}(\Psi) \otimes \mathsf{Tr}(\Theta) \simeq \langle\langle c, c \rangle\rangle \otimes \psi \otimes \vartheta \simeq 2 \times \langle\langle c \rangle\rangle \otimes (\psi \otimes \vartheta) \simeq 2 \times \mathsf{Tr}(\Psi \otimes \Theta)$. □

5.3. **Proposition.** *For any form φ over K we have* $\mathsf{Tr}(\mathsf{Ad}(\varphi)) \simeq \varphi \otimes \varphi$.

Proof. See [8, (11.4)]. □

Let A be a finite-dimensional K-algebra. For $a \in A$ let $\lambda_a \in \mathrm{End}_K(A)$ be given by $\lambda_a(x) = ax$ for $x \in A$. The K-algebra homomorphism $\lambda : A \longrightarrow \mathrm{End}_K(A)$, $a \longmapsto \lambda_a$ thus obtained is called the *left regular representation of A*.

5.4. **Proposition.** *Let $\Psi = (A, \sigma)$ be a K-algebra with involution. The left regular representation of A yields an embedding of Ψ into* $\mathsf{Ad}(\mathsf{Tr}(\Psi))$.

Proof. For $a, x, y \in A$ we have that $T_\sigma(x, \lambda_{\sigma(a)}(y)) = T_\sigma(\lambda_a(x), y)$. Thus λ identifies σ with the restriction to $\lambda(A)$ of the involution adjoint to T_σ. □

5.5. **Proposition.** *Let $\Psi = (A, \sigma)$ be a K-algebra with involution of the first kind. Then* $\Psi \otimes \Psi \simeq \mathsf{Ad}(\mathsf{Tr}(\Psi))$.

Proof. We expand the proof of [8, (11.1)]. Consider the K-algebra homomorphism $\sigma_* : A \otimes_K A \longrightarrow \mathrm{End}_K(A)$ determined by $\sigma_*(a \otimes b)(x) = ax\sigma(b)$ for all $a, b, x \in A$. As $A \otimes_K A$ is simple and of the same dimension as $\mathrm{End}_K(A)$, σ_* is an isomorphism. For $a, b, x, y \in A$ we have $T_\sigma(x, \sigma_*(\sigma(a) \otimes \sigma(b))(y)) = T_\sigma(\sigma_*(a \otimes b)(x), y)$. Thus σ_* identifies the involution $\sigma \otimes \sigma$ with the adjoint involution of T_σ. □

6. Hyperbolicity

Following [1, (2.1)], we say that the K-algebra with involution (A, σ) is *hyperbolic* if there exists an element $e \in A$ with $e^2 = e$ and $\sigma(e) = 1 - e$. If (A, σ) is adjoint to a hermitian form over a K-division algebra with involution, then it is hyperbolic if and only if the hermitian form is hyperbolic.

6.1. **Proposition.** *The K-algebra with involution (A, σ) is hyperbolic if and only if there exists $f \in \mathrm{Skew}(A, \sigma)$ with $f^2 = 1$, if and only if $(1)_K$ embeds into (A, σ).*

Proof. The second equivalence is obvious. To prove the first equivalence, given $e \in A$ with $e^2 = e$ and $\sigma(e) = 1 - e$, we see that $f = 2e - 1$ satisfies $\sigma(f) = -f$ and $f^2 = 1$, and conversely, for $f \in A$ with these properties, $e = \frac{1}{2}(f - 1)$ satisfies $e^2 = 1$ and $\sigma(e) = 1 - e$. □

6.2. **Corollary.** *Let Ψ be a split symplectic or degenerate unitary K-algebra with involution. Then Ψ is hyperbolic.*

Proof. Using (4.4) we have that $\Psi \simeq \mathsf{Ad}(\varphi) \otimes \Phi$ for a K-algebra with canonical involution Φ, and conclude that $\Phi \simeq (1 \cdot | \cdot 1)_K$ or $\Phi \simeq (1)_K$. In either case Ψ contains $(1)_K$ and thus is hyperbolic by (6.1). □

Let Ψ and Θ denote K-algebras with involution.

6.3. **Proposition.** *If Ψ and Θ are hyperbolic with $\Psi \sim \Theta$ and* $\deg(\Psi) = \deg(\Theta)$, *then $\Psi \simeq \Theta$.*

Proof. If Ψ and Θ are degenerate unitary, the statement follows from [8, (2.14)]. Otherwise Ψ and Θ are adjoint to hyperbolic hermitian or skew-hermitian forms of the same dimension over a common K-division algebra with involution, and these are necessarily isometric. □

6.4. **Proposition.** *If Ψ is hyperbolic, then $\Psi \otimes \Theta$ is hyperbolic.*

Proof. This is obvious. □

6.5. **Proposition.** *If Ψ is hyperbolic, then $\mathsf{Tr}(\Psi)$ is hyperbolic.*

Proof. By (5.4) Ψ embeds into $\mathsf{Ad}(\mathsf{Tr}(\Psi))$, which implies the statement. □

6.6. **Proposition.** *Let $a \in K^\times$. We have that $a \in \mathsf{G}(\Psi)$ if and only if $\mathsf{Ad}\langle\langle a \rangle\rangle \otimes \Psi$ is hyperbolic.*

Proof. See [8, (12.20)]. □

6.7. **Theorem** (Bayer-Fluckiger, Lenstra)**.** *Let L/K be a finite field extension of odd degree. Then Ψ_L is hyperbolic if and only if Ψ is hyperbolic.*

Proof. See [8, (6.16)]. □

The following is a reformulation of the main result in [6].

6.8. **Theorem** (Jacobson)**.** *Let Φ be a K-algebra with canonical involution and φ a quadratic form over K. Then $\mathsf{Ad}(\varphi) \otimes \Phi$ is hyperbolic if and only if $\varphi \otimes \mathsf{Tr}(\Phi)$ is hyperbolic.*

Proof. Let $\varphi = (V, B)$ and $\Phi = (A, \sigma)$ with $\sigma = \mathrm{can}_A$. Then $T_\sigma(x, x) = x + \sigma(x) \in K$ for $x \in A$. The K-algebra with involution $\mathsf{Ad}(\varphi) \otimes \Phi$ is adjoint to the hermitian form (V_A, h) over Φ obtained from φ, with $V_A = V \otimes_K A$ and $h : V_A \times V_A \to A$ determined by $h(a \otimes v, b \otimes w) = \sigma(a)B(v, w)b$ for $a, b \in A$ and $v, w \in V$. Then $(V_A, T_\sigma \circ h)$ is a quadratic form over K isometric to $\varphi \otimes \mathsf{Tr}(\Phi)$. The isotropic vectors for h and for $T_\sigma \circ h$ coincide. It follows that a maximal totally isotropic K-subspace for $T_\sigma \circ h$ is the same as a maximal totally isotropic A-subspace for h. This implies the statement. $\square$

6.9. **Theorem** (Bayer-Fluckiger, Shapiro, Tignol)**.** *Let $a \in K^\times \setminus K^{\times 2}$. Then $\Psi_{K(\sqrt{a})}$ is hyperbolic if and only if $(a)_K$ embeds into Ψ or $\Psi \simeq \mathsf{Ad}(\varphi)$ for a quadratic form φ over K whose anisotropic part is a multiple of $\langle\langle a\rangle\rangle$.*

Proof. Assume first that Ψ is split orthogonal, so that $\Psi \simeq \mathsf{Ad}(\varphi)$ for a form φ over K. Then $\Psi_{K(\sqrt{a})}$ is hyperbolic if and only if $\varphi_{K(\sqrt{a})}$ is hyperbolic, which by [9, Chap. VII, (3.2)] is if and only if the anisotropic part of φ is a multiple of $\langle\langle a\rangle\rangle$.

In the remaining cases, the statement is proven in [1, (3.3)] for involutions of the first kind, and an adaptation of the argument for involutions of the second kind is provided in [13, (3.6)]. $\square$

6.10. **Remark.** There is an overlap in the two cases of the characterization given in (6.9). Assume that $a \in K^\times \setminus K^{\times 2}$ and φ is a form over K. Then $(a)_K$ embeds into $\mathsf{Ad}(\varphi)$ if and only if φ is a multiple of $\langle\langle a\rangle\rangle$, if and only if the anisotropic part φ_{an} of φ is a multiple of $\langle\langle a\rangle\rangle$ and $\varphi \simeq \varphi_{\mathrm{an}} \perp 2m \times \mathbb{H}$ for some $m \in \mathbb{N}$.

6.11. **Corollary.** *For any $a \in K^\times \setminus K^{\times 2}$ such that $\Psi_{K(\sqrt{a})}$ is hyperbolic, we have that $\mathsf{D}_K\langle\langle a\rangle\rangle \subseteq \mathsf{G}(\Psi)$.*

Proof. As $\mathsf{D}_K\langle\langle a\rangle\rangle = \mathsf{G}\,(a)_K$, the statement follows immediately from (6.9). $\square$

6.12. **Proposition.** *Let Q_1 and Q_2 be K-quaternion algebras. The K-algebra with involution $(Q_1, \mathrm{can}_{Q_1}) \otimes (Q_2, \mathrm{can}_{Q_2})$ is hyperbolic if and only if one of Q_1 and Q_2 is split.*

Proof. Let $(A, \sigma) = (Q_1, \mathrm{can}_{Q_1}) \otimes (Q_2, \mathrm{can}_{Q_2})$. If one of the factors is split, it is hyperbolic, and thus (A, σ) is hyperbolic. Assume now that (A, σ) is hyperbolic. Then by (6.1) there exists $f \in \mathrm{Skew}(\sigma)$ with $f^2 = 1$. We identify Q_1 and Q_2 with K-subalgebras of A that commute with each other elementwise and such that $\sigma|_{Q_i} = \mathrm{can}_{Q_i}$ for $i = 1, 2$. Then $\mathrm{Skew}(\sigma) = Q_1' \oplus Q_2'$ where Q_i' is the K-subspace of pure quaternions of Q_i for $i = 1, 2$. Writing $f = f_1 + f_2$ with $f_i \in Q_i'$ for $i = 1, 2$, we obtain that $1 = f^2 = f_1^2 + f_2^2 + 2f_1f_2$. As $f_1^2, f_2^2 \in K$, we conclude that $f_1f_2 \in K$. This is only possible if $f_1f_2 = 0$, that is, if either $f_1 = 0$ or $f_2 = 0$. If, say, $f_2 = 0$, then $f = f_1$, which then is a hyperbolic element with respect to σ contained in Q_1, whereby Q_1 is split. Hence, one of Q_1 and Q_2 is split. $\square$

6.13. **Theorem** (Karpenko, Tignol)**.** *Let Ψ be a non-hyperbolic K-algebra with involution such that $\Psi \otimes \Psi$ is split. There exists a field extension L/K such that Ψ_L is not hyperbolic and, either Ψ_L is split or Ψ is symplectic and* $\mathrm{ind}(\Psi_L) = 2$.

Proof. See [7, (1.1)] for the orthogonal case and [19, (A.1) and (A.2)] for the other cases. $\square$

Note that the condition in (6.13) that $\Psi \otimes \Psi$ be split is trivially satisfied if Ψ is a K-algebra with involution of the first kind.

We mention separately the following special case of (6.13), which was obtained earlier by more classical methods. It will be used in (9.4).

6.14. **Theorem** (Dejaiffe, Parimala, Sridharan, Suresh)**.** *Let $a, b \in K^\times$ and let L be the function field of the conic $aX^2 + bY^2 = 1$ over K. Let Ψ be a K-algebra with orthogonal involution such that $\Psi \sim (a \cdot\!| \ b)_K$. Then Ψ is hyperbolic if and only if Ψ_L is hyperbolic.*

Proof. If the conic $aX^2+bY^2 = 1$ is split over K, then L is a rational function field over K and the statement is obvious. Otherwise $\Phi = (a \cdot\!|\!\cdot b)_K$ is a K-quaternion division algebra with involution and Ψ is adjoint to a skew-hermitian form over Φ, in which case the statement follows alternatively from [3] or [14, (3.3)]. □

7. Algebras with involution over real closed fields

Let Ψ be a K-algebra with involution. For $n \geq 1$ we set $n \times \Psi = \mathsf{Ad}(n \times \langle 1 \rangle) \otimes \Psi$.

7.1. **Proposition.** *Assume that K is pythagorean and $\Psi \sim (-1 \cdot\!| -1)_K$. Then $\Psi \simeq \mathsf{Ad}(\varphi) \otimes (-1 \cdot\!| -1)_K$ for a form φ over K. Moreover, $2 \times \Psi$ is hyperbolic.*

Proof. Let $Q = (-1, -1)_K$. We may identify Ψ with $(\mathrm{End}_Q(V), \sigma)$ where V is a finite-dimensional right Q-vector space and σ is the involution adjoint to a regular skew-hermitian form $h : V \times V \longrightarrow Q$ with respect to can_Q. Since K is pythagorean, any maximal subfield of Q is K-isomorphic to $K(\sqrt{-1})$. We fix a pure quaternion $i \in Q$ with $i^2 = -1$ and obtain that any invertible pure quaternion in Q is conjugate to an element of $iK^\times$. This yields that h has a diagonalization with entries in $iK^\times$. It follows that $ih : V \times V \longrightarrow Q$ is a hermitian form with respect to the involution $\tau = \mathrm{Int}(i) \circ \mathrm{can}_Q$ and has a diagonalization with entries in $K^\times$. This yields that $\Psi \simeq \mathsf{Ad}(\varphi) \otimes (Q, \tau)$ for a form φ over K. Moreover, $(Q, \tau) \simeq (-1 \cdot\!| -1)_K$. This shows the first claim.

As $\mathsf{Ad}\langle 1, 1 \rangle \simeq (-1 \cdot\!| \ 1)_K$ we obtain using (4.8) that

$$2 \times (Q, \tau) \simeq (-1 \cdot\!| \ 1)_K \otimes (-1 \cdot\!| -1)_K \simeq (-1 \cdot\!|\!\cdot -1)_K \otimes (-1 \cdot\!|\!\cdot 1)_K \,.$$

By (6.1) this K-algebra with involution is hyperbolic, and thus so is $2 \times \Psi$. □

7.2. **Theorem.** *Assume that K is real closed.*

(a) If Ψ is split orthogonal, then $\Psi \simeq \mathsf{Ad}(r \times \langle 1 \rangle \perp \eta)$ for a hyperbolic form η over K and $r \in \mathbb{N}$ such that $\mathrm{sign}(\mathsf{Tr}(\Psi)) = r^2$.

(b) If Ψ is non-split orthogonal, then $\Psi \simeq r \times (-1 \cdot\!| -1)_K$ for a positive integer r, the form $\mathsf{Tr}(\Psi)$ is hyperbolic, and Ψ is hyperbolic if and only if r is even.

(c) If Ψ is split symplectic, then $\Psi \simeq r \times (1 \cdot\!|\!\cdot 1)_K$ for a positive integer r, and both Ψ and $\mathsf{Tr}(\Psi)$ are hyperbolic.

(d) If Ψ is non-split symplectic, then $\Psi \simeq \mathsf{Ad}(r \times \langle 1 \rangle \perp \eta) \otimes (-1 \cdot\!|\!\cdot -1)_K$ for a hyperbolic form η over K and $r \in \mathbb{N}$ such that $\mathrm{sign}(\mathsf{Tr}(\Psi)) = 4r^2$.

(*e*) *If* Ψ *is non-degenerate unitary, then* $\Psi \simeq \mathsf{Ad}(r \times \langle 1 \rangle \perp \eta) \otimes (-1)_K$ *for a hyperbolic form* η *over* K *and* $r \in \mathbb{N}$ *such that* $\mathrm{sign}(\mathsf{Tr}(\Psi)) = 2r^2$.

(*f*) *If* Ψ *is degenerate unitary, then* $\Psi \simeq r \times (1)_K$ *for a positive integer, and both* Ψ *and* $\mathsf{Tr}(\Psi)$ *are hyperbolic.*

These cases are mutually exclusive and cover all possibilities, and the integer r *is unique in each case.*

Proof. It is clear that exactly one of the conditions in (*a*)–(*f*) is satisfied. As K is real closed, the only finite-dimensional K-division algebras are K, $K(\sqrt{-1})$, and $(-1,-1)_K$. Therefore we have $\Psi \sim \Phi$ for the K-algebra with involution

$$\Phi = \begin{cases} (K, \mathrm{id}_K) & \text{if } \Psi \text{ is split orthogonal,} \\ (-1 \cdot\!| -1)_K & \text{if } \Psi \text{ is non-split orthogonal,} \\ (1 \cdot\!|\!\cdot 1)_K & \text{if } \Psi \text{ is split symplectic,} \\ (-1 \cdot\!|\!\cdot -1)_K & \text{if } \Psi \text{ is non-split symplectic,} \\ (-1)_K & \text{if } \Psi \text{ is split non-degenerate unitary,} \\ (1)_K & \text{if } \Psi \text{ is degenerate unitary.} \end{cases}$$

If Ψ is split-symplectic or degenerate unitary, then Ψ is hyperbolic by (6.2), whence $\mathsf{Tr}(\Psi)$ is hyperbolic by (6.5), and using (6.3) it follows that $\Psi \simeq r \times \Phi$ for some $r \in \mathbb{N}$. This shows (*c*) and (*f*).

Next, suppose that Ψ is non-split orthogonal. Then by (7.1) we have $\Psi \simeq \mathsf{Ad}(\varphi) \otimes (-1 \cdot\!| -1)_K$ for a form φ over K, and as $\mathsf{G}\,(-1 \cdot\!| -1)_K = K^{\times 2} \cup -K^{\times 2} = K^\times$ we may choose φ to be $r \times \langle 1 \rangle$ for some $r \in \mathbb{N}$. We thus have $\Psi \simeq r \times \Phi$ with $r \in \mathbb{N}$ such that $\deg(\Psi) = 2r$. Furthermore, $\mathsf{Tr}(\Phi)$ is hyperbolic by (5.1), and thus so is $\mathsf{Tr}(\Psi) \simeq r^2 \times \mathsf{Tr}(\Phi)$. This shows (*b*).

In each of the remaining cases (*a*), (*d*), and (*e*), by (4.4) we have that $\Psi \simeq \mathsf{Ad}(\varphi) \otimes \Phi$ for a form φ over K. Since K is real closed and φ is determined up to a scalar factor, we choose φ to be $r \times \langle 1 \rangle \perp \eta$ for some $r \in \mathbb{N}$ and a hyperbolic form η over K. It further follows that $\mathsf{Tr}(\Psi) \simeq \varphi \otimes \varphi \otimes \mathsf{Tr}(\Phi)$ by (5.2) and (5.3) and thus $\mathrm{sign}(\mathsf{Tr}(\Psi)) = r^2 \cdot \mathrm{sign}(\mathsf{Tr}(\Phi))$. As in either case $\mathsf{Tr}(\Phi)$ is positive definite by (5.1), we have that $\mathrm{sign}(\mathsf{Tr}(\Phi)) = \dim_K(\Phi)$. This establishes the cases (*a*), (*d*), and (*e*).

Finally, note that in each case the non-negative integer r is determined by $\deg(\Psi)$ or $\dim(\mathsf{Tr}(\Psi))$, respectively. $\square$

7.3. **Corollary.** *Assume* K *is real closed. Then* $\mathsf{Tr}(\Psi)$ *is hyperbolic if and only if* $2 \times \Psi$ *is hyperbolic, if and only if either* Ψ *is hyperbolic or* $\Psi \simeq r \times (-1 \cdot\!| -1)_K$ *where* $r \in \mathbb{N}$ *is odd.*

Proof. We shall refer to the cases in (7.2). In each of the cases (*b*), (*c*), or (*f*), both $\mathsf{Tr}(\Psi)$ and $2 \times \Psi$ are hyperbolic. Assume that we are in one of the cases (*a*), (*d*), or (*e*), and let r be the integer occurring in the statement for that case. Then $\mathsf{Tr}(\Psi)$ is hyperbolic if and only if $r = 0$, if and only if Ψ is hyperbolic. $\square$

7.4. **Corollary.** *Let P be an ordering of K and Ψ a K-algebra with involution. Then* $\mathrm{sign}_P(\mathsf{Tr}(\Psi)) = [Z(\Psi) : K] \cdot s^2$ *for some* $s \in \mathbb{N}$.

Proof. By (7.2) the statement holds in the case where K is real closed and P is the unique ordering of K. The general case follows immediately by extending scalars to the real closure of K at P. □

Let P be an ordering of K. The integer s occurring in (7.4) is called the *signature of* Ψ *at* P and denoted $\mathrm{sign}_P(\Psi)$. With $k = [Z(\Psi) : K]$ we thus have

$$\mathrm{sign}_P(\Psi) = \sqrt{\tfrac{1}{k}\mathrm{sign}_P(\mathsf{Tr}(\Psi))}\,.$$

7.5. **Proposition.** *Let Ψ and Θ be two K-algebras with involution. If Ψ and Θ are both unitary, assume that they have the same centre. For every ordering P of K we have that* $\mathrm{sign}_P(\Psi \otimes \Theta) = \mathrm{sign}_P(\Psi) \cdot \mathrm{sign}_P(\Theta)$.

Proof. This follows immediately from (5.2). □

7.6. **Proposition.** *Let φ be a quadratic form over K. For every ordering P of K we have that* $\mathrm{sign}_P(\mathsf{Ad}(\varphi)) = |\mathrm{sign}_P(\varphi)|$.

Proof. This is clear as $\mathsf{Tr}(\mathsf{Ad}(\varphi)) \simeq \varphi \otimes \varphi$ by (5.3). □

8. Local-global principle for weak hyperbolicity

We say that the algebra with involution Ψ is *weakly hyperbolic* if there exists a positive integer n such that $n \times \Psi$ is hyperbolic. We say that Ψ *has trivial signature* and write $\mathrm{sign}(\Psi) = 0$ to indicate that $\mathrm{sign}_P(\Psi) = 0$ for every $P \in X_K$.

8.1. **Lemma.** *Assume that there exists $a \in K^\times \setminus \pm K^{\times 2}$ such that $\Psi_{K(\sqrt{a})}$ and $\Psi_{K(\sqrt{-a})}$ are hyperbolic. Then $2 \times \Psi$ is hyperbolic.*

Proof. Let $a \in K^\times \setminus \pm K^{\times 2}$ be such that $\Psi_{K(\sqrt{a})}$ and $\Psi_{K(\sqrt{-a})}$ are hyperbolic. By (6.11) a and $-a$ both belong to $\mathsf{G}(\Psi)$. As $\mathsf{G}(\Psi)$ is a group, we conclude that $-1 \in \mathsf{G}(\Psi)$, so $2 \times \Psi \simeq \mathsf{Ad}\langle\langle -1\rangle\rangle \otimes \Psi$ is hyperbolic by (6.6). □

8.2. **Proposition.** *Assume that K is nonreal and let $n \in \mathbb{N}$ be such that -1 is a sum of 2^n squares in K. Then $2^{n+1} \times \Psi$ is hyperbolic.*

Proof. By the assumption, the Pfister form $\pi = 2^{n+1} \times \langle 1\rangle$ over K is isotropic, whereby it is hyperbolic. Hence, $2^{n+1} \times \Psi \simeq \mathsf{Ad}(\pi) \otimes \Psi$ is hyperbolic. □

8.3. **Lemma.** *Assume that $2^n \times \Psi$ is not hyperbolic for any $n \in \mathbb{N}$, and that for every proper finite extension L/K there exists $n \in \mathbb{N}$ such that $2^n \times \Psi_L$ is hyperbolic. Then K is real closed and* $\mathrm{sign}(\Psi) \neq 0$.

Proof. By (8.2) the field K is real, by (8.1) its only quadratic field extension is $K(\sqrt{-1})$, and by (6.7) K has no proper finite field extension of odd degree. Thus K is real closed, by [17, Chap. 3, (2.3)]. It follows from (7.2) that $\Psi \simeq \mathsf{Ad}(\varphi) \otimes \Phi$ for a form φ over K and a non-degenerate K-algebra with canonical involution

Φ. As K is real closed, it follows with (5.1) that $\mathsf{Tr}(\Phi)$ is positive definite, and thus $\mathrm{sign}(\Psi)$ is equal to $|\mathrm{sign}(\varphi)|$ or to $2 \cdot |\mathrm{sign}(\varphi)|$. As Ψ is not hyperbolic, φ is not hyperbolic, and we conclude that $\mathrm{sign}(\Psi) \neq 0$. $\square$

8.4. **Lemma.** *Assume that Ψ is split and let $r \in \mathbb{N}$. Then $\Psi^{\otimes 2r}$ is hyperbolic if and only if $\mathsf{Tr}(\Psi)^{\otimes r}$ is hyperbolic.*

Proof. Replacing Ψ by $\Psi^{\otimes r}$ we may in view of (5.2) assume that $r = 1$. If Ψ is symplectic then Ψ and $\mathsf{Tr}(\Psi)$ are hyperbolic by (6.2) and (6.5). If Ψ is orthogonal, then $\Psi^{\otimes 2} \simeq \mathsf{Ad}(\mathsf{Tr}(\Psi))$ by (5.5), implying the statement. Assume now that Ψ is unitary. Then $\Psi \simeq \mathsf{Ad}(\varphi) \otimes (a)_K$ for a form φ over K and some $a \in K^\times$. We obtain that $\Psi^{\otimes 2} \simeq \mathsf{Ad}(\varphi \otimes \varphi) \otimes (a)_K$ and $\mathsf{Tr}(\Psi) \simeq \varphi \otimes \varphi \otimes \langle\langle a \rangle\rangle$. Using (6.8) we conclude that $\Psi^{\otimes 2}$ is hyperbolic if and only if $\mathsf{Tr}(\Psi)$ is hyperbolic. $\square$

8.5. **Theorem.** *The following are equivalent:*
- *(i)* $\mathrm{sign}(\Psi) = 0$*;*
- *(ii)* Ψ *is weakly hyperbolic;*
- *(iii)* $2^n \times \Psi$ *is hyperbolic for some* $n \in \mathbb{N}$*;*
- *(iv) either $\Psi^{\otimes m}$ is hyperbolic for some $m \geq 1$, or K is nonreal and Ψ is split orthogonal of odd degree.*

These conditions are trivially satisfied if K is nonreal.

Proof. Trivially *(iii)* implies *(ii)*, and by (7.5) any of the conditions implies *(i)*.

Suppose that $2^n \times \Psi$ is not hyperbolic for any $n \in \mathbb{N}$. By Zorn's Lemma there exists a maximal algebraic extension L/K such that $2^n \times \Psi_L$ is not hyperbolic for any $n \in \mathbb{N}$. By (8.3) L is real closed and Ψ_L has nonzero signature at the unique ordering of L. For the ordering $P = L^2 \cap K$ of K we obtain that $\mathrm{sign}_P(\Psi) \neq 0$. This shows that *(i)* implies *(iii)*.

To finish the proof we show that *(i)* implies *(iv)*. We may assume that Ψ is simple as otherwise Ψ is hyperbolic. Replacing Ψ by $\Psi^{\otimes e}$ for some positive integer e and using (7.5), we may further assume that Ψ is split. Assuming *(i)* we have $\mathrm{sign}(\mathsf{Tr}(\Psi)) = 0$. Note further that $\dim_K(\Psi) = \dim_K(\mathsf{Tr}(\Psi))$. If $\dim(\mathsf{Tr}(\Psi))$ is odd, we conclude that K is nonreal and Ψ is split orthogonal of odd degree. If $\dim(\mathsf{Tr}(\Psi))$ is even, we obtain by (2.11) that $\mathsf{Tr}(\Psi)^{\otimes r}$ is hyperbolic for some positive integer r, and then $\Psi^{\otimes 2r}$ is hyperbolic by (8.4). $\square$

8.6. **Corollary.** *Assume that $Z(\Psi) \simeq K(\sqrt{a})$ with $a \in \sum K^2$ in case Ψ is unitary, and otherwise that $\Psi_R \sim (-1, -\mathrm{type}(\Psi))_R$ for every real closure R of K. Then Ψ is weakly hyperbolic.*

Proof. In view of the hypothesis, we obtain from (7.2) that $\mathsf{Tr}(\Psi)$ becomes hyperbolic over every real closure of K. Therefore $\mathrm{sign}(\Psi) = 0$, and it follows by (8.5) that Ψ is weakly hyperbolic. $\square$

Let T be a preordering of K. We say that a K-algebra with involution Ψ is *T-hyperbolic* if there exists a T-positive quadratic form τ over K such that

$\mathsf{Ad}(\tau)\otimes\Psi$ is hyperbolic. It is clear from (4.2) that a quadratic form φ over K is T-hyperbolic if and only if $\mathsf{Ad}(\varphi)$ is T-hyperbolic.

8.7. **Theorem.** *Let T be a preordering of K. We have $\mathrm{sign}_P(\Psi)=0$ for every $P\in X_T(K)$ if and only if Ψ is T-hyperbolic. Moreover, in this case there exists a T-positive Pfister form ϑ over K such that $\mathsf{Ad}(\vartheta)\otimes\Psi$ is hyperbolic.*

Proof. Assume first that Ψ is T-hyperbolic. Let ϑ be a T-positive form over K such that $\mathsf{Ad}(\vartheta)\otimes\Psi$ is hyperbolic. By (7.5) then $\mathrm{sign}_P(\vartheta)\cdot\mathrm{sign}_P(\Psi)=0$ for any ordering P of K. For any $P\in X_T$ we have $\mathrm{sign}_P(\vartheta)>0$ as ϑ is T-positive, and we conclude that $\mathrm{sign}_P(\Psi)=0$.

Assume now that $\mathrm{sign}_P(\Psi)=0$ for every $P\in X_T$. Then $\vartheta\otimes\mathsf{Tr}(\Psi)$ is hyperbolic for some T-positive Pfister form ϑ over K, by (2.10). By (5.2) and (5.3) we have $\mathsf{Tr}(\mathsf{Ad}(\vartheta)\otimes\Psi)\simeq\vartheta\otimes\vartheta\otimes\mathsf{Tr}(\Psi)$. We conclude that $\mathsf{Ad}(\vartheta)\otimes\Psi$ has trivial total signature. By (8.5) there exists $n\in\mathbb{N}$ such that $2^n\times\mathsf{Ad}(\vartheta)\otimes\Psi$ is hyperbolic. Hence, the isomorphic K-algebra with involution $\mathsf{Ad}(2^n\times\vartheta)\otimes\Psi$ is hyperbolic. As $2^n\times\vartheta$ is a T-positive Pfister form, this shows the statement. □

9. Bounds on the torsion order

By (8.5), for a K-algebra with involution Ψ such that $\Psi^{\otimes n}$ is hyperbolic for some $n\in\mathbb{N}$, we have that $2^m\times\Psi$ is hyperbolic for some $m\in\mathbb{N}$. In this situation, one may want to bound m in terms of n and the degree of Ψ. We restrict to the case $n=2$, that is, where $\Psi^{\otimes 2}$ is hyperbolic, and use the function $\Delta:\mathbb{N}\longrightarrow\mathbb{N}$ introduced in Section 2 to bound m.

9.1. **Theorem.** *Let Ψ be a K-algebra with involution such that $\Psi^{\otimes 2}$ is split hyperbolic. Let $m=\deg(\Psi)$ if σ is orthogonal or unitary, and $m=\frac{1}{2}\deg(\Psi)$ if σ is symplectic. Then $2^{\Delta(m)}\times\Psi$ is hyperbolic.*

Proof. In view of (6.13) it suffices to consider the situation where Ψ is either split orthogonal, or split unitary, or symplectic of index 2. Then by (4.4) we have $\Psi\simeq\mathsf{Ad}(\varphi)\otimes\Phi$ for a form φ over K with $\dim(\varphi)=m$ and a K-algebra with canonical involution Φ. As $\Psi^{\otimes 2}$ is hyperbolic, it follows from (8.4) in the split case and from (5.5) in the non-split case that $\mathsf{Tr}(\Psi)$ is hyperbolic. By (5.2) and (5.3) we have $\mathsf{Tr}(\Psi)\simeq\varphi\otimes\varphi\otimes\mathsf{Tr}(\Phi)$. Hence, (2.15) yields that $(2^{\Delta(m)}\times\varphi)\otimes\mathsf{Tr}(\Phi)$ is hyperbolic. We conclude using (6.8) that $2^{\Delta(m)}\times\Psi\simeq\mathsf{Ad}(2^{\Delta(m)}\times\varphi)\otimes\Phi$ is hyperbolic. □

9.2. **Theorem.** *Let Ψ be a K-quaternion algebra with involution and $m\in\mathbb{N}$. If $2^m\times\Psi^{\otimes 2}$ is hyperbolic, then $2^{m+1}\times\Psi$ is hyperbolic. Moreover, the converse holds in case Ψ is split.*

Proof. Suppose first that Ψ is split. If Ψ is symplectic then it is hyperbolic. Assume that Ψ is orthogonal or unitary. Then either $\Psi\simeq\mathsf{Ad}\langle\langle a\rangle\rangle$ for some $a\in K^\times$ or $\Psi\simeq\mathsf{Ad}\langle\langle a\rangle\rangle\otimes(b)_K$ for some $a,b\in K^\times$. Either way, as $\langle\langle a,a\rangle\rangle\simeq 2\times\langle\langle a\rangle\rangle$ it follows that $\Psi^{\otimes 2}\simeq 2\times\Psi$. This yields the claimed equivalence in the split case.

We derive the implication claimed in general for Ψ orthogonal or unitary by reduction to the split case by means of (6.13). Assume finally that Ψ is symplectic. Then $\Psi^{\otimes 2} \simeq \mathsf{Ad}(\mathsf{Tr}(\Psi))$ by (5.5) and thus $2^m \times \Psi^{\otimes 2} \simeq \mathsf{Ad}(2^m \times \mathsf{Tr}(\Psi))$ by (4.2). Hence, if $2^m \times \Psi^{\otimes 2}$ is hyperbolic, then $2^m \times \mathsf{Tr}(\Psi)$ is hyperbolic, and it follows by (6.8) that $2^m \times \Psi$ is hyperbolic. $\square$

The following example shows that the converse in (9.2) does not hold in general.

9.3. **Example.** Let $m \in \mathbb{N}$. Assume that K is either $k(t)$ or $k((t))$ for a field k. Let $a \in \mathsf{D}_k(2^{m+1}) \setminus \mathsf{D}_k(2^m)$. The form $2^m \times \langle\langle a, -t\rangle\rangle$ over K is anisotropic. Hence, $2^m \times (a \cdot| \ t)_K^{\otimes 2} \simeq \mathsf{Ad}(2^m \times \langle\langle a, -t\rangle\rangle)$ is anisotropic, whereas $2^{m+1} \times (a \cdot| \ t)_K$ is hyperbolic.

9.4. **Theorem.** *Let $a, b \in K^\times$. Then $2 \times (a \cdot| \ b)_K$ is hyperbolic if and only if $a \in \mathsf{D}_K\langle 1,1\rangle \cup \mathsf{D}_K\langle 1,b\rangle$. For $n \in \mathbb{N}$ with $n \geq 2$ we have that $2^n \times (a \cdot| \ b)_K$ is hyperbolic if and only if $a = x(y+b)$ with $x \in \mathsf{D}_K(2^n-1)$ and $y \in \mathsf{D}_K(2^n-1) \cup \{0\}$.*

Proof. Note that $2 \times (a \cdot| \ b)_K \simeq (-1 \cdot| \ 1)_K \otimes (a \cdot| \ b)_K \simeq (-1 \cdot|\cdot \ a)_K \otimes (a \cdot|\cdot -b)_K$ by (4.8). Hence, by (6.12), $2 \times (a \cdot| \ b)_K$ is hyperbolic if and only if one of $(-1, a)_K$ and $(a, -b)_K$ is split, which happens if and only if $a \in \mathsf{D}_K\langle 1,1\rangle \cup \mathsf{D}_K\langle 1,b\rangle$.

Let $n \geq 2$. Let L denote the function field of the conic $aX^2 + bY^2 = 1$ over K. Note that $(a,b)_L$ is split and thus $2^n \times (a \cdot| \ b)_L \simeq \mathsf{Ad}(2^n \times \langle\langle a\rangle\rangle_L)$. Using (6.14) $2^n \times (a \cdot| \ b)_K$ is hyperbolic if and only if $2^n \times (a \cdot| \ b)_L$ is hyperbolic, which is the case if and only if $2^n \times \langle\langle a\rangle\rangle_L$ is hyperbolic. Using [9, Chap. X, (4.28)] we conclude that this happens if and only if $\langle 1, -a, -b\rangle$ is a subform of $2^n \times \langle\langle a\rangle\rangle$ over K, which is, if and only if $(2^n - 1) \times \langle\langle a\rangle\rangle \perp \langle b\rangle$ is isotropic. Finally, this occurs if and only if $a = x(y+b)$ for some $x \in \mathsf{D}_K(2^n - 1)$ and $y \in \mathsf{D}_K(2^n-1) \cup \{0\}$. $\square$

9.5. **Question.** *If K is pythagorean and* $\operatorname{sign}(\Psi) = 0$*, is then $2 \times \Psi$ necessarily hyperbolic?*

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References

[1] E. Bayer-Fluckiger, D.B. Shapiro, J.-P. Tignol. Hyperbolic involutions. *Math. Z.* **214** (1993): 461–476.

[2] B. Bernauer. *Ein Lokal-Global-prinzip für Involutionen und hermitesche Formen.* Diplomarbeit, Universität Konstanz, 2004.
Online: `http://kops.ub.uni-konstanz.de/handle/urn:nbn:de:bsz:352-opus-13143`

[3] I. Dejaiffe. Formes antihermitiennes devenant hyperboliques sur un corps de déploiement. *C. R. Acad. Sci., Paris, Sér. I, Math.* **332** (2001): 105–108.

[4] R.L. Graham, D.E. Knuth, O. Patashnik. *Concrete mathematics. A foundation for computer science.* Addison-Wesley Publishing Company, Advanced Book Program, Reading, MA, 1989.

[5] P. Gille and T. Szamuely. *Central simple algebras and Galois cohomology.* Cambridge University Press (2006).
[6] N. Jacobson. A note on hermitian forms. *Bull. Amer. Math. Soc.* **46** (1940): 264–268.
[7] N.A. Karpenko. Hyperbolicity of orthogonal involutions. *Documenta Math., Extra Volume Suslin* (2010): 371–389.
[8] M.-A. Knus, A.S. Merkurjev, M. Rost, J.-P. Tignol, *The Book of Involutions*, Coll. Pub. **44**, Amer. Math. Soc., Providence, RI (1998).
[9] T.Y. Lam, *Introduction to quadratic forms over fields*, Graduate Studies in Mathematics **67**, American Mathematical Society, Providence, RI (2005).
[10] T.Y. Lam. *Orderings, valuations and quadratic forms.* CBMS Regional Conf. Ser. Math., 52, published by AMS, 1983.
[11] D.W. Lewis. Witt rings as integral rings. *Invent. Math.* **90** (1987): 631–633.
[12] D.W. Lewis, J.-P. Tignol, On the signature of an involution, *Arch. Math.* **60** (1993): 128–135.
[13] D.W. Lewis, T. Unger. A local-global principle for algebras with involution and hermitian forms. *Math. Z.* **244** (2003): 469–477.
[14] R. Parimala, R. Sridharan, V. Suresh. Hermitian analogue of a theorem of Springer. *J. Algebra* **243** (2001): 780–789.
[15] A. Pfister. Quadratische Formen in beliebigen Körpern. *Invent. Math.* **1** (1966): 116–132.
[16] A. Quéguiner. Signature des involutions de deuxième espèce. *Arch. Math.* **65** (1995): 408–412.
[17] W. Scharlau. *Quadratic and Hermitian forms.* Grundlehren 270, Springer, Berlin, 1985.
[18] W. Scharlau. Induction theorems and the structure of the Witt group. *Invent. Math.* **4** (1970): 37–44.
[19] J.-P. Tignol. Hyperbolicity of symplectic and unitary involutions. Appendix to a paper of N. Karpenko. *Documenta Math., Extra Volume Suslin* (2010): 389–392.
[20] T. Unger, Quadratic Forms and Central Simple Algebras with Involution, Ph.D. Thesis, National University of Ireland, Dublin (2000).

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