FORMAL HECKE ALGEBRAS AND ALGEBRAIC ORIENTED COHOMOLOGY THEORIES

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Abstract. In the present paper we generalize the construction of the nil Hecke ring of Kostant-Kumar to the context of an arbitrary algebraic oriented cohomology theory of Levine-Morel and Panin-Smirnov, e.g. to Chow groups, Grothendieck's K_0, connective K-theory, elliptic cohomology, and algebraic cobordism. The resulting object, which we call a *formal (affine) Demazure algebra*, is parameterized by a one-dimensional commutative formal group law and has the following important property: specialization to the additive and multiplicative periodic formal group laws yields completions of the nil Hecke and the 0-Hecke rings respectively. We also introduce a deformed version of the formal (affine) Demazure algebra, which we call a *formal (affine) Hecke algebra*. We show that the specialization of the formal (affine) Hecke algebra to the additive and multiplicative periodic formal group laws gives completions of the degenerate (affine) Hecke algebra and the usual (affine) Hecke algebra respectively. We show that all formal affine Demazure algebras (and all formal affine Hecke algebras) become isomorphic over certain coefficient rings, proving an analogue of a result of Lusztig.

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Introduction

Geometric realizations of representations of algebras such as quantized enveloping algebras of Lie algebras and Hecke-type algebras have proved to be an exceptionally interesting and

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useful tool in both representation theory and geometry. In particular, the field of geometric representation theory has produced such results as the proof of the Kazhdan-Lusztig conjecture and the construction of canonical bases in quantized enveloping algebras. Geometric realizations are also often a precursor to *categorification*, a current topic of great interest.

Two fundamental constructions in geometric representation theory are of particular relevance to the current paper. The first arises from so-called *push-pull operators* (coming from the projection from the flag variety G/B to the quotient G/P of G by a minimal parabolic) on the singular cohomology or K-theory (i.e. Grothendieck's K_0) of the flag variety. If one works with singular cohomology, these operators generate the nil Hecke algebra. (When we use the term "nil Hecke algebra" here, we do not include the polynomial part.) If one works instead with K-theory, the push-pull operators generate the 0-Hecke algebra (the specialization of the Hecke algebra at $q = 0$). Adding the operators corresponding to multiplication by elements of the singular cohomology or K-theory, one obtains the affine analogues of the algebras above.

The above-mentioned algebras can also be realized in a more algebraic way. Let W be the Weyl group of a reduced root system, acting on the weight lattive Λ. In [KK86], Kostant and Kumar introduced a *twisted group algebra* Q_W, which is the smash product of the group ring $\mathbb{Z}[W]$ and the field of fractions Q of the polynomial ring S in Λ. Then they defined a subring R of Q_W generated by *Demazure elements* and elements of S and showed that R is similar to the 0-Hecke algebra: it satisfies the classical braid relation but a nilpotence relation instead of an idempotence one. For this reason, they called R the *nil Hecke algebra*. Following this approach, Evens and Bressler in [EB87] introduced the notion of a *generalized Hecke ring* (where the nilpotence/idempotence relation is replaced by a general quadratic one) which includes both 0-Hecke and nil Hecke algebras as examples. The Demazure elements play the role of a Hecke basis and have several geometric interpretations (as Demazure operators and push-pull operators) on the singular cohomology of the variety of Borel subgroups associated to the root system.

The second geometric construction relevant to the current paper is the realization of Hecke-type algebras via the geometry of the Steinberg variety. There is a natural structure of an algebra on the (co)homomology of the Steinberg variety via convolution. Again, the resulting algebra depends on the choice of (co)homology theory. Equivariant K-theory yields the affine Hecke algebra, equivariant singular cohomology yields the degenerate affine Hecke algebra, and top degree Borel-Moore homology yields the group algebra of the Weyl group. We refer the reader to [CG10, Gin] and the references therein for further details.

Another important ingredient of the current paper originates from algebraic topology and, especially, the cobordism theory of the 60s. The notion of an *algebraic oriented cohomology theory* was introduced by Levine-Morel [LM07] and by Panin-Smirnov [Pan03]. Roughly speaking, it is a cohomological-type functor $\mathtt{h}$ from the category of smooth algebraic varieties over a field to the category of commutative rings endowed with push-forward maps and characteristic classes (see §4). Basic examples of such functors are the Chow ring of algebraic cycles modulo rational equivalence, the Grothendieck group K_0, and the algebraic cobordism of Levine-Morel (see [LM07, §1.1] and [Pan03, §§2.1, 2.5, 3.8] for further examples). The theory of *formal group laws* has been used extensively in topology, especially in cobordism theory and, more generally, for studying topological oriented theories. The link between oriented cohomology and formal group laws is given by the Quillen formula expressing the first characteristic class $c_1^{\mathtt{h}}$ of a tensor product of two line bundles, $c_1^{\mathtt{h}}(L_1 \otimes L_2) = F(c_1^{\mathtt{h}}(L_1), c_1^{\mathtt{h}}(L_2))$ (see [LM07, Lem. 1.1.3]), where F is the one-dimensional commutative formal group law associated to $\mathtt{h}$.

In [CPZ], the authors generalized the notions of Demazure operators and push-pull operators to the context of an arbitrary algebraic oriented cohomology theory $\mathtt{h}$ and the associated formal group law F, e.g. replacing singular cohomology by connective K-theory, elliptic cohomology or algebraic cobordism. It is thus natural to ask if one can extend the Kostant-Kumar construction of the nil Hecke algebra and the convolution construction of the affine Hecke algebra to the setting of an arbitrary algebraic oriented cohomology theory or formal group law. In the present paper we provide an affirmative answer to the first question. We also define algebras that we believe should be related to the more general convolution algebras of the second question.

Given a formal group law F (corresponding to some algebraic oriented cohomology theory), we introduce the notion of a twisted formal group algebra Q_W^F. To do this, we replace the polynomial ring S of [KK86, §4] by the formal group algebra associated to F. We then define the *formal Demazure element* to be the expression in Q_W^F corresponding to the formal Demazure operator. One of our key objects is the algebra generated by the formal Demazure elements and the elements of the formal group algebra. We call this the *formal affine Demazure algebra* and denote it $\mathbf{D}_F$. The subalgebra generated by only the formal Demazure elements is called the *formal Demazure algebra*. Next, we "deform" these algebras by introducing an infinite cyclic group. Geometrically, this corresponds to introducing $\mathbb{C}^*$-actions on the relevant varieties. We call the deformed algebra the *formal (affine) Hecke algebra* associated to the formal group law. Specializing to the additive and multiplicative periodic formal group laws, which correspond to (equivariant) singular cohomology and K-theory respectively, we recover (completions of) all of the algebras mentioned above. This is summarized in the following table.

	Additive FGL	Multiplicative FGL
Alg. Oriented Cohom. Theory	(Equiv.) singular cohomology	(Equiv.) K-theory
Formal Demazure alg.	Nil Hecke alg.	0-Hecke alg.
Formal affine Demazure alg.	Affine nil Hecke alg.	Affine 0-Hecke alg.
Formal Hecke alg.	Group alg. of the Weyl Group	Hecke alg.
Formal affine Hecke alg.	Degenerate affine Hecke alg.	Affine Hecke alg.

Therefore, the algebra $\mathbf{D}_F$ can be viewed as a deformation space between the generalized Hecke rings studied by Bressler and Evens. We see that $\mathbf{D}_F$ shares many properties with affine Hecke algebras. However, it does *not* always satisfy the braid relations. In general, the braid relations are satisfied only up to lower order terms (see Proposition 5.7). This reflects the fact that formal Demazure operators for a general algebraic oriented cohomology theory depend on a choice of reduced decomposition of an element of the Weyl goup.

Our construction provides two things. First, it gives a uniform presentation of the fundamental algebras appearing in both the push-pull and Steinberg variety constructions. Second, it generalizes to other formal groups laws and algebraic oriented cohomology theories, yielding new algebras in the process. These new algebras should be thought of as natural generalizations of the Hecke-type algebras appearing in the table above. Given the representation theoretic importance of these Hecke-type algebras, we expect the new algebras defined here to be of interest to both geometers and representation theorists. For example, Hecke-type algebras have played crucial roles in the categorification of quantum groups and related algebras. Thus, it is natural to ask if the generalizations defined in the current paper can be used as building blocks in more general categorifications.

This paper is organized as follows. In the first four sections we recall basic definitions and facts used in the rest of the paper. We review the definition of a formal group law and the exponential map in Section 1. In Section 2, we recall the definition and basic properties of formal group rings/algebras following [CPZ, §2]. In Section 3, following [CPZ, §3], we recall the definition and basic properties of formal Demazure operators. Section 4 is devoted to algebraic oriented cohomology theories. We define the formal (affine) Demazure algebras and prove various facts about them in Sections 5 and 6. In particular, we describe them in terms of generators and relations in the case that the root system is simply-laced. We also show that they are all isomorphic over certain coefficient rings. In Section 7, we define the formal (affine) Hecke algebras and describe them in terms of generators and relations. We prove various properties about them in Section 8. In particular, we show that they are all isomorphic over certain coefficient rings (an analogue of a result of Lusztig ([Lus89, Thm. 9.3])).

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1. Formal group laws

In the present section we recall the definition and properties of formal group laws (see [Frö68, Ch. 1, §3, Ch. III, §1] and [LM07, Ch. 1 and 2] for details).

Definition 1.1 (Formal group law)**.** A one-dimensional commutative *formal group law* (FGL) is a pair (R, F), where R is a commutative ring, called the *coefficient ring*, and $F = F(u, v) \in R[\![u, v]\!]$ is a power series satisfying the following axioms:

(FG1) $F(u, 0) = F(0, u) = u \in R[\![u]\!]$,
(FG2) $F(u, v) = F(v, u)$, and
(FG3) $F(u, F(v, w)) = F(F(u, v), w) \in R[\![u, v, w]\!]$.

Note that axioms (FG1) and (FG2) imply that

$$F(u, v) = u + v + \textstyle\sum_{i,j\geq 1} a_{ij}u^i v^j, \quad \text{where } a_{ij} = a_{ji} \in R. \tag{1.1}$$

Given an integer $m \geq 1$ we use the notation

$$u +_F v := F(u, v), \quad m \cdot_F u := \underbrace{u +_F \cdots +_F u}_{m \text{ times}}, \quad \text{and } (-m) \cdot_F u := -_F(m \cdot_F u),$$

where $-_F u$ denotes the *formal inverse* of u, i.e. the unique power series in $R[\![u]\!]$ such that $u +_F (-_F u) = (-_F u) +_F u = 0$ (see [Frö68, Ch. 1, §3, Prop. 1]). We define

$$\mu_F(u) := \frac{-_F u}{-u} = 1 - a_{11}u + a_{11}^2u^2 - (a_{11}^3 + a_{12}a_{11} - a_{22} + 2a_{13})u^3 + \cdots \tag{1.2}$$

(see [LM07, (2.7)]). Note that $\mu_F(u)$ has a multiplicative inverse since its constant term is invertible.

Throughout the current paper, whenever a particular FGL is denoted using a subscript (e.g. F_A, F_M, F_L, F_U), we will use the same subscript to denote various quantities associated to that FGL. Thus, we will write $-_A u$ for $-_{F_A} u$, μ_L for μ_{F_L}, etc.

Example 1.2. (a) For the *additive* FGL $(\mathbb{Z}, F_A(u,v) = u+v)$ we have (see [LM07, Example 1.1.4])

$$-_A u = -u \quad \text{and} \quad \mu_A(u) = 1.$$

(b) For the *multiplicative* FGL $(R, F_M(u,v) = u+v-\beta uv)$, $\beta \in R$, $\beta \neq 0$, we have (see [LM07, Example 1.1.5])

$$-_M u = -u \textstyle\sum_{i\geq 0} \beta^i u^i \quad \text{and} \quad \mu_M(u) = \textstyle\sum_{i\geq 0} \beta^i u^i.$$

Observe that $(1-\beta u)\mu_M(u) = 1$, so $\mu_M(u)^{-1} = 1-\beta u$ in $R[\![u]\!]$. If $\beta \in R^\times$, where $R^\times$ denotes the group of invertible elements of R, we say that the FGL is *multiplicative periodic*.

(c) The *Lorentz* FGL (R, F_L) is given by

$$F_L(u,v) = \tfrac{u+v}{1+\beta uv} = (u+v)\textstyle\sum_{i\geq 0}(-\beta uv)^i, \quad \beta \in R,\ \beta \neq 0.$$

We have $-_L u = -u$ and $\mu_L(u) = 1$. Note that for $\beta = 1/c^2$, where c is the speed of light, the expression $F_L(u,v)$ corresponds to the addition of relativistic parallel velocities.

(d) Let E be the elliptic curve defined by the Tate model ([Tat74, §3]):

$$E: \quad v = u^3 + a_1 uv + a_2 u^2 v + a_3 v^2 + a_4 uv^2 + a_6 v^3. \tag{1.3}$$

Here the coefficient ring is $R = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$. The group law on E induces an *elliptic* FGL (R, F_E) with

$$F_E(u,v) = u + v - a_1 uv - a_2(u^2 v + uv^2) - 2a_3(u^3 v + uv^3) + (a_1 a_2 - 3a_3)u^2 v^2 + O(5)$$

(see [Lan87, Appendix 1, (3.6)]). We have

$$-_E u = \tfrac{-u}{1-a_1 u - a_3 v(u)}, \quad \mu_E(u) = \tfrac{1}{1-a_1 u - a_3 v(u)}$$

(see [Sil09, §IV.1,p. 120]), where $v(u)$ is considered as an element in $R[\![u]\!]$ after a recursive procedure in the Tate model.

(e) We define the *Lazard ring* $\mathbb{L}$ to be the commutative ring with generators a_{ij}, $i,j \in \mathbb{N}_+$, and subject to the relations that are forced by the axioms for formal group laws. The corresponding FGL $(\mathbb{L}, F_U(u,v) = u+v+\sum_{i,j\geq 1} a_{ij} u^i v^j)$ is then called the *universal* FGL (see [LM07, §1.1]). The series $\mu_U(u)$ is given by (1.2).

Let (R,F) and (R,F') be formal group laws. A *morphism of formal group laws* $f\colon (R,F) \to (R,F')$ is a formal power series $f \in R[\![u]\!]$ such that $f(u +_F v) = f(u) +_{F'} f(v)$. Given a FGL F over R, there is an isomorphism of FGLs after tensoring with $\mathbb{Q}$,

$$e_F\colon (R_{\mathbb{Q}}, F_A) \to (R_{\mathbb{Q}}, F),\ R_{\mathbb{Q}} = R \otimes_{\mathbb{Z}} \mathbb{Q},$$

given by the *exponential* series $e_F(u) \in R_{\mathbb{Q}}[\![u]\!]$ which satisfies the property $e_F(u+v) = e_F(u) +_F e_F(v)$ (see [Frö68, Ch. IV, §1]).

Example 1.3. (a) For a general FGL $F(u,v) = u + v + a_{11}uv + a_{12}(u^2 v + uv^2) + O(4)$ we have

$$e_F(u) = u + \tfrac{a_{11}}{2!}u^2 + \tfrac{a_{11}^2 + 2a_{12}}{3!}u^3 + O(4).$$

(b) For the multiplicative FGL we have
$$e_M(u) = \sum_{i\geq 1}(-\beta)^{i-1}\tfrac{u^i}{i!}, \quad \text{so that} \quad \beta e_M(u) = 1 - \exp(-\beta u).$$
(c) For the Lorentz FGL we have $e_L(u) = \frac{e^{2u}-1}{e^{2u}+1}$.
(d) For the elliptic FGL we have
$$e_E(u) = u - \tfrac{a_1}{2!}u^2 + \left(\tfrac{3a_1^2-2(a_1^2+a_2)}{3!}\right)u^3 + O(4).$$

2. Formal group algebras

Following [CPZ, §2], we recall the definition and basic properties of formal group algebras. These will play a fundamental role in our definition of formal (affine) Demazure and Hecke algebras.

Definition 2.1 (Formal group algebra)**.** Suppose (R,F) is a FGL and Λ is an abelian group. Let $R[x_\Lambda] := R[\{x_\lambda \mid \lambda \in \Lambda\}]$ denote the polynomial ring over R with variables indexed by Λ. Let $\varepsilon\colon R[x_\Lambda] \to R$ be the augmentation homomorphism which maps all x_λ, $\lambda \in \Lambda$, to 0 and consider the $(\ker\varepsilon)$-adic topology on $R[x_\Lambda]$. We define $R[\![x_\Lambda]\!]$ to be the $(\ker\varepsilon)$-adic completion of the polynomial ring $R[x_\Lambda]$. In particular, if Λ is finite of order n, then the ring $R[\![x_\Lambda]\!]$ is the usual ring of power series in n variables.

Let J_F be the closure of the ideal generated by the elements x_0 and $x_{\lambda_1+\lambda_2} - (x_{\lambda_1} +_F x_{\lambda_2})$ for all $\lambda_1, \lambda_2 \in \Lambda$. We define the *formal group algebra* (or *formal group ring*) to be the quotient (see [CPZ, Def. 2.4])
$$R[\![\Lambda]\!]_F := R[\![x_\Lambda]\!]/J_F.$$
The class of x_λ in $R[\![\Lambda]\!]_F$ will be denoted by the same letter. By definition, $R[\![\Lambda]\!]_F$ is a complete Hausdorff R-algebra with respect to the $(\ker\varepsilon)$-adic topology, where $\varepsilon\colon R[\![\Lambda]\!]_F \to R$ is the induced augmentation map. We define the augmentation ideal $\mathcal{I}_F := \ker\varepsilon$ to be the kernel of this induced map.

The assignment of the formal group algebra $R[\![\Lambda]\!]_F$ to the data (R,F,Λ) is functorial in the following ways (see [CPZ, Lem. 2.6]).

(a) Given a morphism $f\colon (R,F) \to (R,F')$ of FGLs, there is an induced continuous ring homomorphism $f^\star\colon R[\![\Lambda]\!]_{F'} \to R[\![\Lambda]\!]_F$, $x_\lambda \mapsto f(x_\lambda)$. If $f'\colon (R,F') \to (R,F'')$ is another morphism of FGLs, then $(f'f)^\star = f^\star(f')^\star$.
(b) Given a group homomorphism $f\colon \Lambda \to \Lambda'$, there is an induced continuous ring homomorphism $\hat{f}\colon R[\![\Lambda]\!]_F \to R[\![\Lambda']\!]_F$, $x_\lambda \mapsto x_{f(\lambda)}$. If $f'\colon \Lambda' \to \Lambda''$ is another group homomorphism, then $\widehat{f'f} = \widehat{f'}\hat{f}$.

Note that maps of the type $\hat{f}$ commute with maps of the type $f^\star$.

Example 2.2. The map $x_m \mapsto m \cdot_F x$, $m \in \mathbb{Z}$, defines R-algebra isomorphisms
$$R[\![\mathbb{Z}]\!]_F \cong R[\![x]\!] \quad \text{and} \quad R[\![\mathbb{Z}/n\mathbb{Z}]\!]_F \cong R[\![x]\!]/(n \cdot_F x).$$
More generally, there is a (non-canonical) R-algebra isomorphism (see [CPZ, Cor. 2.12])
$$R[\![\mathbb{Z}^n]\!]_F \cong R[\![x_1,\dots,x_n]\!],$$
where the right hand side is independent of F. This implies that if R is a domain, then so is $R[\![\mathbb{Z}^n]\!]_F$.

It follows from (1.1) that $n \cdot_F x = nx + x^2 p(x)$ for some $p(x) \in R[\![x]\!]$. Thus, if $n \in R^\times$, then $n \cdot_F x$ is the product of x and a unit in $R[\![x]\!]$, so $(x) = (n \cdot_F x)$ and $R[\![\mathbb{Z}/n\mathbb{Z}]\!]_F \cong R$.

Lemma 2.3. *Given a FGL (R, F), we have $\mu_F(x_\lambda)^{-1} = \mu_F(x_{-\lambda})$, for all $\lambda \in \Lambda$.*

Proof. This follows immediately from the fact that $-_F x_\lambda = x_{-\lambda}$ in $R[\![\Lambda]\!]_F$. ☐

We now consider what happens at a finite (truncated) level in $R[\![\Lambda]\!]_F$. Let $R[\Lambda]_F$ denote the subalgebra of $R[\![\Lambda]\!]_F$ equal to the image of $R[x_\Lambda]$ under the composition $R[x_\Lambda] \hookrightarrow R[\![x_\Lambda]\!] \twoheadrightarrow R[\![\Lambda]\!]_F$. Then $R[\![\Lambda]\!]_F$ is the completion of $R[\Lambda]_F$ at the ideal $(\ker \varepsilon) \cap R[\Lambda]_F$. As before, the assignment $(R, F, \Lambda) \mapsto R[\Lambda]_F$ is functorial with respect to group homomorphisms.

Example 2.4. (a) Suppose Λ is a free abelian group. Then for the additive FGL $F_A(u, v) = u + v$ over R we have ring isomorphisms (cf. [CPZ, Example 2.19])

$$R[\![\Lambda]\!]_A \cong S_R^*(\Lambda)^\wedge := \prod_{i=0}^{\infty} S_R^i(\Lambda) \quad \text{and} \quad R[\Lambda]_A \cong S_R^*(\Lambda) := \bigoplus_{i=0}^{\infty} S_R^i(\Lambda),$$

where $S_R^i(\Lambda)$ is the i-th symmetric power of Λ over R, and the isomorphisms are induced by sending x_λ to $\lambda \in S_R^1(\Lambda)$.

(b) Consider the group ring

$$R[\Lambda] := \left\{ \textstyle\sum_j r_j e^{\lambda_j} \mid r_j \in R,\ \lambda_j \in \Lambda \right\}.$$

Let $\varepsilon \colon R[\Lambda] \to R$ be the augmentation map, i.e. the R-linear map sending all e^λ, $\lambda \in \Lambda$, to 1. Let $R[\Lambda]^\wedge$ be the completion of $R[\Lambda]$ at $\ker \varepsilon$.

Assume that $\beta \in R^\times$. Then for the multiplicative periodic FGL $F_M(u, v) = u + v - \beta uv$ over R, we have R-algebra isomorphisms (cf. [CPZ, Example 2.20])

$$R[\![\Lambda]\!]_M \cong R[\Lambda]^\wedge \quad \text{and} \quad R[\Lambda]_M \cong R[\Lambda]$$

induced by $x_\lambda \mapsto \beta^{-1}(1 - e^{-\lambda})$ and $e^\lambda \mapsto (1 - \beta x_{-\lambda}) = (1 - \beta x_\lambda)^{-1}$ respectively. Using this identification, along with Example 1.2(b) and Lemma 2.3, we obtain

$$\mu_M(x_\lambda)\mu_M(x_{\lambda'}) = (1 - \beta x_{-\lambda})(1 - \beta x_{-\lambda'}) = e^{\lambda+\lambda'} = 1 - \beta x_{-\lambda-\lambda'} = \mu_M(x_{\lambda+\lambda'}).$$

Example 2.5. Fix a generator γ of $\mathbb{Z}$ and let $t = e^\gamma$ be the corresponding element in the group ring $R[\mathbb{Z}]$. According to the previous examples we have R-algebra isomorphisms

$$R[\![\mathbb{Z}]\!]_M \cong R[t, t^{-1}]^\wedge \quad \text{and} \quad R[\![\mathbb{Z}]\!]_A \cong R[\![\gamma]\!],$$

where $R[t, t^{-1}]^\wedge$ denotes the completion of $R[t, t^{-1}]$ at the ideal generated by $t - 1$. At the truncated levels, we have

$$R[\mathbb{Z}]_M \cong R[t, t^{-1}] \quad \text{and} \quad R[\mathbb{Z}]_A \cong R[\gamma],$$

given by $x_{n\gamma} \mapsto \beta^{-1}(1 - t^{-n})$ (with inverse map given by $t \mapsto 1 - \beta x_{-\gamma}$) and $x_{n\gamma} \mapsto n\gamma$ respectively.

3. Formal Demazure operators

In the present section we introduce, following [CPZ, §3], the notion of formal Demazure operators. We also state some of their properties that will be needed in our constructions. For the remainder of the paper, we assume that R is a commutative domain.

Consider a reduced root system (Λ, Φ, ϱ) as in [Dem73, §1], i.e. a free $\mathbb{Z}$-module Λ of finite rank (the *weight lattice*), a finite subset Φ of Λ whose elements are called *roots*, and a map $\varrho\colon \Lambda \to \Lambda^\vee := \operatorname{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ associating a *coroot* $\alpha^\vee \in \Lambda^\vee$ to every root α, satisfying certain axioms. The *reflection* map $\lambda \mapsto \lambda - \langle \alpha^\vee, \lambda\rangle\alpha$ is denoted by s_α. Here $\langle \cdot, \cdot \rangle$ denotes the natural pairing between $\Lambda^\vee$ and Λ.

The *Weyl group* W associated to a reduced root system is the subgroup of linear automorphisms of Λ generated by the reflections s_α. We fix sets of simple roots $\{\alpha_i\}_{i\in I}$ and fundamental weights $\{\omega_i\}_{i\in I}$. That is, $\omega_i \in \Lambda^\vee$ satisfies $\langle \omega_i, \alpha_j\rangle = \delta_{ij}$ for all $i, j \in I$. Let $\{s_i = s_{\alpha_i}\}_{i\in I}$ denote the corresponding set of simple reflections in W and let ℓ denote the usual length function on W. We say the root system is *simply laced* if $\langle \alpha_i^\vee, \alpha_j\rangle \in \{0, -1\}$ for all $i, j \in I$, $i \neq j$. For instance, the roots systems of type ADE are simply laced.

Fix a FGL (R, F). Since the Weyl group acts linearly on Λ, it acts by R-algebra automorphisms on $R[\![\Lambda]\!]_F$ via the functoriality in Λ of $R[\![\Lambda]\!]_F$ (see Section 2), i.e. we have

$$w(x_\lambda) = x_{w(\lambda)}, \text{ for all } w \in W,\ \lambda \in \Lambda.$$

Definition 3.1 (Formal Demazure operator Δ_α^F)**.** By [CPZ, Cor. 3.4], for any $\varphi \in R[\![\Lambda]\!]_F$ and root $\alpha \in \Phi$, the element $\varphi - s_\alpha(\varphi)$ is uniquely divisible by x_α. We define an R-linear operator Δ_α^F on $R[\![\Lambda]\!]_F$ (see [CPZ, Def. 3.5]), called the *formal Demazure operator*, by

$$\Delta_\alpha^F(\varphi) := \tfrac{\varphi - s_\alpha(\varphi)}{x_\alpha}, \quad \varphi \in R[\![\Lambda]\!]_F.$$

Observe that if F is the additive or multiplicative FGL, then Δ_α^F is the classical Demazure operator of [Dem73, §3 and §9]. We will often omit the superscript F when the FGL is understood.

Definition 3.2 (g^F, κ_α^F and C_α^F)**.** Consider the power series $g^F(u, v)$ defined by $u +_F v = u + v - uvg^F(u, v)$ and, for $\alpha \in \Phi$, let

$$\kappa_\alpha^F := g^F(x_\alpha, x_{-\alpha}) = \tfrac{1}{x_\alpha} + \tfrac{1}{x_{-\alpha}} \in R[\![\Lambda]\!]_F.$$

We define an R-linear operator C_α^F on $R[\![\Lambda]\!]_F$ (see [CPZ, Def. 3.11]) by

$$C_\alpha^F(\varphi) := \kappa_\alpha^F \varphi - \Delta_\alpha^F(\varphi), \quad \varphi \in R[\![\Lambda]\!]_F.$$

We will often omit the superscript F when the FGL is understood.

Lemma 3.3. *The following statements are equivalent.*

(a) $F(u, v) = (u + v)h(u, v)$ *for some* $h(u, v) \in R[\![u, v]\!]$.
(b) $\kappa_\alpha^F = 0$ *for all* $\alpha \in \Phi$.
(c) $\kappa_\alpha^F = 0$ *for some* $\alpha \in \Phi$.
(d) $\mu_F(u) = 1$.

If these equivalent conditions are satisfied, we write $\kappa^F = 0$. *If they are not satisfied, we write* $\kappa^F \neq 0$.

Proof. First suppose that $F(u,v)=(u+v)h(u,v)$ for some $h(u,v)\in R[\![u,v]\!]$. Then, for any $\alpha\in\Phi$, we have $F(x_\alpha,-x_\alpha)=0$. By the uniqueness of the formal inverse, this implies that $x_{-\alpha}=-_F x_\alpha=-x_\alpha$. Thus $\kappa_\alpha^F=0$ and so (a) implies (b). Clearly (b) implies (c).

Now suppose that $\kappa_\alpha^F=0$ for some $\alpha\in\Phi$. Then $x_\alpha\in\mathcal{I}_F\setminus\{0\}$ and $F(x_\alpha,-x_\alpha)=0$ in $R[\![\Lambda]\!]_F$. By the definition of a FGL, we have

$$F(x_\alpha,-x_\alpha)=x_\alpha^2\textstyle\sum_{i,j\ge 1}(-1)^j a_{ij}x_\alpha^{i+j-2}=x_\alpha^2\sum_{n\ge 0}b_n x_\alpha^n\in\mathcal{I}_F^2,$$
$$\text{where } b_n=\textstyle\sum_{i+j=n+2}(-1)^j a_{ij}.$$

We claim that $b_n=0$ for all $n\ge 0$. Indeed, let n_0 be the smallest n such that $b_{n_0}\neq 0$. Then

$$0=F(x_\alpha,-x_\alpha)=x_\alpha^2\textstyle\sum_{n\ge n_0}b_n x_\alpha^n=x_\alpha^{2+n_0}\sum_{n\ge n_0}b_n x_\alpha^{n-n_0}.$$

Since $x_\alpha\neq 0$ (this follows from [CPZ, Lem. 4.2]) and $R[\![\Lambda]\!]_F$ is a domain, we have

$$\textstyle\sum_{n\ge 0}b_n x_\alpha^{n-n_0}=0.$$

Applying the augmentation map, we obtain $b_{n_0}=0$, contradicting our choice of n_0.

Now let $F_i(u,v)$ be the i-th homogeneous component of F, $i\ge 2$. Since $F_i(u,-u)=b_{i-2}u^i=0$, $F_i(u,v)$ is divisible by $(u+v)$ for every $i\ge 2$. Since the homogeneous components of degree zero and one for any FGL are 0 and $u+v$, this implies that $F(u,v)=(u+v)h(u,v)$ for some $h(u,v)\in R[\![u,v]\!]$. Thus (c) implies (a).

Now, if $\mu_F(u)=1$, then $-x_\alpha=-_F x_\alpha=x_{-\alpha}$ and so $\kappa_\alpha^F=0$ for all $\alpha\in\Phi$. Thus (d) implies (b).

Finally, if $F(u,v)=(u+v)h(u,v)$ for some $h(u,v)\in R[\![u,v]\!]$, then $-_F u=-u$ by the uniqueness of the formal inverse (as above). Thus $\mu_F(u)=1$. Hence (a) implies (d). ☐

As in the case of the usual Demazure operators, the operators Δ_α^F and C_α^F satisfy Leibniz-type properties (see [CPZ, Props. 3.8 and 3.12]).

4. Algebraic oriented cohomology theories and characteristic maps

We now recall several facts concerning algebraic oriented cohomology theories. We refer the reader to [LM07] and [Pan03] for further details and examples.

An *algebraic oriented cohomology theory* (AOCT) is a contravariant functor $\mathtt{h}$ from the category of smooth projective varieties over a field k to the category of commutative unital rings which satisfies certain properties (see [LM07, §1.1]). Given a morphism $f\colon X\to Y$ of varieties, the map $\mathtt{h}(f)$ will be denoted f^* and called the *pullback* of f. One of the characterizing properties of $\mathtt{h}$ is that, for any proper map $f\colon X\to Y$, there is an induced map $f_*\colon \mathtt{h}(X)\to\mathtt{h}(Y)$ of $\mathtt{h}(Y)$-modules called the *push-forward* (here $\mathtt{h}(X)$ is an $\mathtt{h}(Y)$-module via f^*). A *morphism* of AOCTs is a natural transformation of functors that also commutes with push-forwards. Basic examples of AOCTs are Chow groups CH and Grothendieck's K_0 (see [Pan03, §§2.1, 2.5, 3.8] for further examples).

The connection between algebraic oriented cohomology theories and FGLs is as follows. Given two line bundles L_1 and L_2 over X, we have (see [LM07, Lem. 1.1.3])

$$c_1^{\mathtt{h}}(L_1\otimes L_2)=c_1^{\mathtt{h}}(L_1)+_F c_1^{\mathtt{h}}(L_2),$$

where $c_1^{\mathtt{h}}$ is the first characteristic class with values in $\mathtt{h}$ and F is a one-dimensional commutative FGL over the coefficient ring $R=\mathtt{h}(\operatorname{Spec}k)$ associated to $\mathtt{h}$.

There is an AOCT Ω defined over a field of characteristic zero, called *algebraic cobordism* (see [LM07, §1.2]), that is universal in the following sense: Given any AOCT $\mathtt{h}$ there is a unique morphism $\Omega \to \mathtt{h}$ of AOCTs. The FGL associated to Ω is the universal FGL F_U.

Moreover, given a FGL F over a ring R together with a morphism $\mathbb{L} \to R$, we define a functor $X \mapsto \mathtt{h}(X) := \Omega(X) \otimes_{\mathbb{L}} R$. If F satisfies certain conditions, the functor $\mathtt{h}$ gives an AOCT.

Example 4.1. In the above terms, the additive FGL corresponds to the theory of Chow groups. The multiplicative periodic FGL with $\beta \in R^\times$ corresponds to Grothendieck's K_0. The multiplicative FGL with $\beta \notin R^\times$ corresponds to connective K-theory.

Let G be a split simple simply connected linear algebraic group over a field k corresponding to the root system (Λ, Φ, ϱ). Fix a split maximal torus T and a Borel subgroup B so that $T \subseteq B \subseteq G$. Let G/B be the variety of Borel subgroups of G and let F be the FGL over R associated to an AOCT $\mathtt{h}$ satisfying the assumptions of [CPZ, Thm. 13.12]. Consider the formal group algebra $R[\![\Lambda]\!]_F$. Then there is a ring homomorphism, called the *characteristic map* (see [CPZ, §6]),

$$\mathfrak{c}_F\colon R[\![\Lambda]\!]_F \to \mathtt{h}(G/B), \quad x_\lambda \mapsto c_1^{\mathtt{h}}(L(\lambda)),$$

where $L(\lambda)$ is the line bundle associated to $\lambda \in \Lambda$. Note that this map is neither injective nor surjective in general. Its kernel contains the ideal generated by W-invariant elements, and $\mathtt{h}(G/B)$ modulo the ideal generated by the image of $\mathfrak{c}_F$ is isomorphic to $\mathtt{h}(G)$ (see [GZ12, Prop. 5.1]).

Example 4.2. (a) The characteristic map for the theory of Chow groups, i.e. corresponding to the additive FGL, is given by

$$\mathfrak{c}_A\colon \mathbb{Z}[\![\Lambda]\!]_A \to \mathrm{CH}(G/B), \quad x_\lambda \mapsto c_1\left(L(\lambda)\right),$$

which recovers the usual characteristic map for Chow groups (see [Dem74, §1.5]).

(b) The characteristic map for Grothendieck's K_0, i.e. corresponding to the multiplicative periodic FGL, is given by

$$\mathfrak{c}_M\colon \mathbb{Z}[\![\Lambda]\!]_M \to K_0(G/B), \quad x_\lambda \mapsto 1 - [L(\lambda)^\vee].$$

Restricting to the integral group ring $\mathbb{Z}[\Lambda]$ and using the identification of Example 2.4(b), we recover the usual characteristic map for K_0 ([Dem74, §1.6]) which maps e^λ to $[L(\lambda)]$.

(c) Algebraic cobordism Ω defined over a field of characteristic 0 satisfies the assumptions of [CPZ, Thm.13.12]. Therefore, we have the characteristic map

$$\mathfrak{c}_U\colon \mathbb{L}[\![\Lambda]\!]_U \to \Omega(G/B), \quad x_\lambda \mapsto c_1^{\Omega}(L(\lambda)).$$

Let G/P_i be the projective homogeneous variety, where P_i is the minimal parabolic subgroup of G corresponding to the simple root α_i, $i \in I$. Then

$$p\colon G/B = \mathbb{P}_{G/P_i}(1 \oplus L(\omega_i)) \to G/P_i$$

is the projective bundle associated to the vector bundle $1 \oplus L(\omega_i)$, there 1 denotes the trivial bundle of rank one (see, for example, [CPZ, §10.3]). Then the operators C_α^F introduced in Definition 3.2 have the following geometric interpretation in terms of *push-pull operators* (generalizing [PR99, Prop.]).

Proposition 4.3 ([CPZ, Prop. 10.10(4)]). *We have*

$$p^*p_*(\mathfrak{c}_F(\chi)) = \mathfrak{c}_F(C^F_\alpha(\chi)), \quad \textit{for all } \chi \in R[\![\Lambda]\!]_F.$$

5. Formal (affine) Demazure algebras: definitions

In the present section we introduce the notion of a twisted formal group algebra and a particular subalgebra, called the formal (affine) Demazure algebra, which is one of our main objects of interest. Our method is inspired by the approach of [KK86, §4.1].

Definition 5.1 (Twisted formal group algebra). Let $Q^F = Q^{(R,F)}$ denote the field of fractions of $R[\![\Lambda]\!]_F$. The action of the Weyl group W on $R[\![\Lambda]\!]_F$ induces an action by automorphisms on Q^F. We define the *twisted formal group algebra* to be the smash product $Q^F_W := R[W] \ltimes_R Q^F$. (This is sometimes denoted by $R[W]\#Q^F$.) In other words, Q^F_W is equal to $R[W] \otimes_R Q^F$ as an R-module, with multiplication given by

$$(\delta_{w'}\psi')(\delta_w\psi) = \delta_{w'w} w^{-1}(\psi')\psi \ \text{ for all } w, w' \in W, \ \psi, \psi' \in Q^F$$

(extended by linearity), where δ_w denotes the element in $R[W]$ corresponding to w (so we have $\delta_{w'}\delta_w = \delta_{w'w}$ for $w, w' \in W$).

Observe that Q^F_W is a free right Q^F-module (via right multiplication) with basis $\{\delta_w\}_{w\in W}$. Note that Q^F_W is not a Q^F-algebra (but only an R-algebra) since $\delta_e Q^F = Q^F\delta_e$ is not central in Q^F_W. We denote δ_e (the unit element of Q^F_W) by 1.

Definition 5.2 (Formal Demazure element). For each root $\alpha \in \Phi$, we define the corresponding *formal Demazure element*

$$\Delta^F_\alpha := \tfrac{1}{x_\alpha}(1 - \delta_{s_\alpha}) = \tfrac{1}{x_\alpha} - \delta_{s_\alpha}\tfrac{1}{x_{-\alpha}} \in Q^F_W$$

(cf. [KK86, (I_{24})]). We will omit the superscript F when the FGL is clear from the context.

We can now define our first main objects of study.

Definition 5.3 (Formal (affine) Demazure algebra). The *formal Demazure algebra* D_F is the R-subalgebra of Q^F_W generated by the formal Demazure elements Δ^F_i. The *formal affine Demazure algebra* $\mathbf{D}_F$ is the R-subalgebra of Q^F_W generated by D_F and $R[\![\Lambda]\!]_F$. When we wish to specify the coefficient ring, we write $D_{R,F}$ (resp. $\mathbf{D}_{R,F}$) for D_F (resp. $\mathbf{D}_F$).

Remark 5.4. Suppose $\mathtt{h}$ is an algebraic oriented cohomology theory satisfying the assumptions of [CPZ, Thm. 13.12] and with FGL F (see Section 4). By Proposition 4.3, we see that, under the characteristic map $\mathfrak{c}_F$, the affine Demazure algebra $\mathbf{D}_F$ corresponds to the algebra of operators on $\mathtt{h}(G/B)$ generated by left multiplication (by elements of $\mathtt{h}(G/B)$) and the push-pull operators p^*p_*.

Lemma 5.5 (cf. [KK86, Prop. 4.2]). *For all $\psi \in Q$ and $\alpha \in \Phi$, we have*

$$\psi\Delta_\alpha = \Delta_\alpha s_\alpha(\psi) + \Delta_\alpha(\psi),$$

where $\Delta_\alpha(\psi) = \frac{\psi - s_\alpha(\psi)}{x_\alpha} \in Q$ is the formal Demazure operator applied to ψ (see Definition 3.1).

Proof. We have

$$\psi\Delta_\alpha = \psi\left(\tfrac{1}{x_\alpha} - \delta_{s_\alpha}\tfrac{1}{x_{-\alpha}}\right) = \tfrac{\psi - s_\alpha(\psi)}{x_\alpha} + \left(\tfrac{1}{x_\alpha} - \delta_{s_\alpha}\tfrac{1}{x_{-\alpha}}\right)s_\alpha(\psi) = \Delta_\alpha(\psi) + \Delta_\alpha s_\alpha(\psi). \qquad \square$$

Proceeding from Lemma 5.5 by induction, we obtain the following general formula (cf. [KK86, (I_{26})])

$$\psi \varDelta_{\beta_1}\varDelta_{\beta_2}\cdots\varDelta_{\beta_s} = \sum_{(1\le i_1<\cdots<i_r\le s)} \varDelta_{\beta_{i_1}}\varDelta_{\beta_{i_2}}\cdots\varDelta_{\beta_{i_r}}\phi(i_1,\ldots,i_r),$$

where $\beta_1,\beta_2,\ldots,\beta_s\in\Phi$ are roots and $\phi(i_1,\ldots,i_r)\in Q^F$ is defined to be the composition $\Delta_{\beta_s}\circ\Delta_{\beta_{s-1}}\circ\cdots\circ\Delta_{\beta_1}$ applied to ψ, where the Demazure operators at the places $i_1,\ldots,i_r$ are replaced by the respective reflections.

Recall the elements κ_α, $\alpha\in\Phi$, from Definition 3.2. It is easy to verify that $\kappa_\alpha\delta_{s_\alpha}=\delta_{s_\alpha}\kappa_\alpha$, $\kappa_\alpha\varDelta_\alpha=\varDelta_\alpha\kappa_\alpha$, and

$$\varDelta_\alpha^2=\varDelta_\alpha\kappa_\alpha. \tag{5.1}$$

Example 5.6. (a) For the additive and Lorentz FGLs we obtain the *nilpotence relation* $\varDelta_\alpha^2=0$ since $\kappa_\alpha^A=\kappa_\alpha^L=0$.

(b) For the multiplicative FGL we obtain the relation $\varDelta_\alpha^2=\beta\varDelta_\alpha$, since $\kappa_\alpha^M=\beta$. In particular, if $\beta=1$ we obtain the *idempotence relation* $\varDelta_\alpha^2=\varDelta_\alpha$.

(c) For the elliptic FGL we have, in the notation of Example 1.2(d),

$$\varDelta_\alpha^2=\tfrac{a_1x_\alpha+a_3v(x_\alpha)}{x_\alpha}\varDelta_\alpha.$$

For example, if $a_3=0$, then $\varDelta_\alpha^2=a_1\varDelta_\alpha$.

To simplify notation in what follows, for $i,j,i_1,\ldots,i_k\in I$, we set

$$x_{\pm i}=x_{\pm\alpha_i},\ x_{\pm i\pm j}=x_{\pm\alpha_i\pm\alpha_j},\ \delta_{i_1i_2\ldots i_k}=\delta_{s_{i_1}s_{i_2}\cdots s_{i_k}},\ \Delta_i=\Delta_{\alpha_i},\ \kappa_i=\kappa_{\alpha_i}. \tag{5.2}$$

Furthermore, when we write an expression such as $\frac{\delta_w}{\varphi}$ for $w\in W$, $\varphi\in R[\![\Lambda]\!]_F$, we interpret this as being equal to $\delta_w\frac{1}{\varphi}$. That is, we consider the numerators of rational expressions to be to the left of their denominators.

Proposition 5.7. *Suppose $i,j\in I$ and let m_{ij} be the order of s_is_j in W. Then*

$$\underbrace{\varDelta_j\varDelta_i\varDelta_j\cdots}_{m_{ij}\text{ terms}}-\underbrace{\varDelta_i\varDelta_j\varDelta_i\cdots}_{m_{ij}\text{ terms}}=\sum_{w\in W,\,\ell(w)<m_{ij}}\varDelta_w\eta_{ij}^w \tag{5.3}$$

for some $\eta_{ij}^w\in Q^F$. In particular, we have the following:

(a) If $\langle\alpha_i,\alpha_j\rangle=0$, so that $m_{ij}=2$, then $\varDelta_i\varDelta_j=\varDelta_j\varDelta_i$.

(b) If $\langle\alpha_i,\alpha_j\rangle=-1$, so that $m_{ij}=3$, then

$$\varDelta_j\varDelta_i\varDelta_j-\varDelta_i\varDelta_j\varDelta_i=\varDelta_i\kappa_{ij}-\varDelta_j\kappa_{ji},$$

where

$$\kappa_{ij}=\tfrac{1}{x_{i+j}}\Big(\tfrac{1}{x_j}-\tfrac{1}{x_{-i}}\Big)-\tfrac{1}{x_ix_j}\in R[\![\Lambda]\!]_F. \tag{5.4}$$

Proof. We have

$$\underbrace{\varDelta_j\varDelta_i\varDelta_j\cdots}_{m_{ij}\text{ terms}}=\tfrac{1}{x_j}(1-\delta_j)\tfrac{1}{x_i}(1-\delta_i)\tfrac{1}{x_j}(1-\delta_j)\cdots.$$

Since the δ_w, $w \in W$, form a basis of Q_W as a right Q-module, this can be written as a sum of (right) Q-multiples of δ_w. The leading term (with respect to the length of w) is

$$(-1)^{m_{ij}}\delta_{\underbrace{s_j s_i s_j \cdots s_n s_k}_{m_{ij}\text{ terms}}}(x_{-\underbrace{s_k s_n \cdots s_j s_i}_{m_{ij}-1\text{ terms}}(\alpha_j)} \cdots x_{-s_k(\alpha_n)}x_{-\alpha_k})^{-1},$$

where $n = i$ and $k = j$ (resp. $n = j$ and $k = i$) if m_{ij} is odd (resp. even). Now, by [Bou81, Cor. 2 de la Prop. 17],

$$\alpha_k,\ s_k(\alpha_n), \ldots,\ \underbrace{s_k s_n \cdots s_j s_i}_{m_{ij-1}\text{ terms}}(\alpha_j)$$

are precisely the positive roots mapped to negative roots by $\delta_{s_j s_i s_j \cdots}$ (m_{ij} reflections in the subscript). Since $\delta_{s_j s_i s_j \cdots} = \delta_{s_i s_j s_i}$ (m_{ij} reflections in each subscript), we see that the highest order terms in $\Delta_j\Delta_i\Delta_j \cdots - \Delta_i\Delta_j\Delta_i \cdots$ cancel, proving the first part of the proposition.

Under the assumptions of (a), we have $s_i(\alpha_j) = \alpha_j$ and

$$\Delta_i\Delta_j = \left(\tfrac{1}{x_i} - \tfrac{\delta_i}{x_{-i}}\right)\left(\tfrac{1}{x_j} - \tfrac{\delta_j}{x_{-j}}\right) = \tfrac{1}{x_i x_j} - \tfrac{\delta_i}{x_{-i}x_j} - \tfrac{1}{x_i}\tfrac{\delta_j}{x_{-j}} + \tfrac{\delta_i}{x_{-i}}\tfrac{\delta_j}{x_{-j}} = \tfrac{1}{x_i x_j} - \tfrac{\delta_i}{x_{-i}x_j} - \tfrac{\delta_j}{x_i x_{-j}} + \tfrac{\delta_{ij}}{x_{-i}x_{-j}}.$$

Since the final expression is symmetric in i and j, we have $\Delta_i\Delta_j = \Delta_j\Delta_i$.

It remains to prove (b). We have $s_i(\alpha_j) = s_j(\alpha_i) = \alpha_i + \alpha_j$ and $s_i s_j(\alpha_i) = \alpha_j$. Thus

$$\begin{aligned}
\Delta_j\Delta_i\Delta_j &= \left(\tfrac{1}{x_j} - \tfrac{\delta_j}{x_{-j}}\right)\left(\tfrac{1}{x_i} - \tfrac{\delta_i}{x_{-i}}\right)\left(\tfrac{1}{x_j} - \tfrac{\delta_j}{x_{-j}}\right)\\
&= \tfrac{1}{x_i x_j^2} - \tfrac{1}{x_j x_i}\tfrac{\delta_j}{x_{-j}} - \tfrac{1}{x_j}\tfrac{\delta_i}{x_{-i}x_j} - \tfrac{\delta_j}{x_{-j}x_i x_j} + \tfrac{1}{x_j}\tfrac{\delta_i}{x_{-i}}\tfrac{\delta_j}{x_{-j}} + \tfrac{\delta_j}{x_{-j}x_i}\tfrac{\delta_j}{x_{-j}} + \tfrac{\delta_j}{x_{-j}}\tfrac{\delta_i}{x_{-i}x_j} - \tfrac{\delta_j}{x_{-j}}\tfrac{\delta_i}{x_{-i}}\tfrac{\delta_j}{x_{-j}}\\
&= \tfrac{1}{x_i x_j^2} - \tfrac{\delta_j}{x_{i+j}x_{-j}^2} - \tfrac{\delta_i}{x_{i+j}x_{-i}x_j} - \tfrac{\delta_j}{x_{-j}x_i x_j} + \tfrac{\delta_{ij}}{x_i x_{-i-j}x_{-j}} + \tfrac{1}{x_{i+j}x_{-j}x_j} + \tfrac{\delta_{ji}}{x_{-i-j}x_{-i}x_j} - \tfrac{\delta_{jij}}{x_{-i-j}x_{-i}x_{-j}}\\
&= \Delta_j\left(\tfrac{1}{x_i x_j} + \tfrac{1}{x_{i+j}x_{-j}}\right) + \Delta_i\tfrac{1}{x_{i+j}x_j} + \tfrac{\delta_{ij}}{x_i x_{-i-j}x_{-j}} + \tfrac{\delta_{ji}}{x_{-i-j}x_{-i}x_j} - \tfrac{\delta_{jij}}{x_{-i-j}x_{-i}x_{-j}} - \tfrac{1}{x_i x_{i+j}x_j}.
\end{aligned}$$

Using the fact that $s_i s_j s_i = s_j s_i s_j$, we obtain

$$\Delta_j\Delta_i\Delta_j - \Delta_i\Delta_j\Delta_i = \Delta_i\left(\tfrac{1}{x_{i+j}x_j} - \tfrac{1}{x_{i+j}x_{-i}} - \tfrac{1}{x_i x_j}\right) - \Delta_j\left(\tfrac{1}{x_{i+j}x_i} - \tfrac{1}{x_{i+j}x_{-j}} - \tfrac{1}{x_i x_j}\right). \tag{5.5}$$

It remains to prove that $\kappa_{ij} \in R[\![\Lambda]\!]_F$. We have

$$g(x_{i+j}, x_{-i}) = \tfrac{x_{i+j}+x_{-i}-x_j}{x_{i+j}x_{-i}} = \tfrac{1}{x_{-i}} + \tfrac{1}{x_{i+j}} - \tfrac{x_j}{x_{i+j}x_{-i}} \in R[\![\Lambda]\!]_F.$$

and, hence,

$$\kappa_{ij} = \tfrac{g(x_{i+j},x_{-i}) - g(x_i,x_{-i})}{x_j},$$

where $g(x_i, x_{-i}) = \frac{1}{x_i} + \frac{1}{x_{-i}} \in R[\![\Lambda]\!]_F$. Therefore, it suffices to show that $g(x_{i+j}, x_{-i}) - g(x_i, x_{-i})$ is divisible by x_j. The latter follows (taking $u_1 = x_i$, $u_2 = x_j$ and $v = x_{-i}$) from the congruence $u_1 +_F u_2 \equiv u_1 \pmod{u_2}$, which implies that $g(u_1 +_F u_2, v) \equiv g(u_1, v) \pmod{u_2}$. □

Remark 5.8. Note that

$$\kappa_{ij} = \tfrac{x_i(x_{-i}-x_j) - x_{i+j}x_{-i}}{x_i x_{-i}x_j x_{i+j}}.$$

By [BE90, p. 809], the numerator of the above expression equals zero if and only if $F(u, v) = u + v + a_{11}uv$ for some $a_{11} \in R$ (i.e. if and only if F is the additive or multiplicative FGL). Therefore, contrary to the situation for the additive and multiplicative FGLs, the formal Demazure elements do *not* satisfy the braid relations in general (cf. [BE90, Thm. 3.7]).

Remark 5.9. Observe that the key difference between our setting and the setting of [BE90] is that we deal with *algebraic theories* for which the groups $\mathtt{h}(BT)$ and their properties, which are used extensively in [BE90], are not well-defined or remain unknown. Instead, we rely on the formal group algebra $R[\![\Lambda]\!]_F$ (as a replacement for $\mathtt{h}(BT)$) and techniques introduced in [CPZ].

For each $w \in W$, fix a reduced decomposition $w = s_{i_1} \cdots s_{i_k}$ and set

$$\Delta_w = \Delta_{i_1} \cdots \Delta_{i_k}. \tag{5.6}$$

Note that, in general, Δ_w depends on the choice of reduced decomposition.

Definition 5.10 ($\tilde{R}$ and $R[\![\Lambda]\!]_F^{\sim}$)**.** Let $\tilde{R}$ be the subalgebra of Q^F defined by

$$\tilde{R} := R[W] \cdot R[\eta_{ij}^w \mid i, j \in I,\ w \in W,\ \ell(w) < m_{ij}], \tag{5.7}$$

where $R[W]\cdot$ denotes the natural action of the group algebra $R[W]$ of W on Q^F. Similarly, define

$$R[\![\Lambda]\!]_F^{\sim} := R[W] \cdot R[\![\Lambda]\!]_F[\eta_{ij}^w \mid i, j \in I,\ w \in W,\ \ell(w) < m_{ij}]. \tag{5.8}$$

Note that $R[\![\Lambda]\!]_F^{\sim} = R[\![\Lambda]\!]_F$ if the root system is simply laced (since $\kappa_{ij} \in R[\![\Lambda]\!]_F$).

The following lemma is an easy generalization of [KK86, Thm. 4.6] (which considers the case of the additive FGL).

Lemma 5.11. *The set $\{\Delta_w \mid w \in W\}$ forms a basis of $D_F \otimes_R \tilde{R}$ as a right (or left) $\tilde{R}$-module and a basis of $\mathbf{D}_F \otimes_{R[\![\Lambda]\!]_F} R[\![\Lambda]\!]_F^{\sim}$ as a right (or left) $R[\![\Lambda]\!]_F^{\sim}$-module.*

Proof. Since R is a domain, so are $\tilde{R}$ and $R[\![\Lambda]\!]_F^{\sim}$. By (5.1) and Proposition 5.7, we can write any product of formal Demazure elements as a $\tilde{R}$-linear combination of the elements Δ_w, $w \in W$. Combined with Lemma 5.5, we can write any product of formal Demazure elements and elements of $R[\![\Lambda]\!]_F^{\sim}$ as an $R[\![\Lambda]\!]_F^{\sim}$-linear combination of the elements Δ_w, $w \in W$. Thus $\{\Delta_w \mid w \in W\}$ is a spanning set of the modules in the statement of the lemma. Now, it is easy to see from the definition of the formal Demazure elements (Definition 5.2) that, for all $w \in W$,

$$\Delta_w = \textstyle\sum_{v\,:\,\ell(v) \leq \ell(w)} \delta_v a_w,$$

where the sum is over elements $v \in W$ with length less than or equal to the length of w, $a_v \in Q^F$ for all v, and $a_w \neq 0$. Thus, since $\{\delta_w \mid w \in W\}$ is a basis for Q_W^F as a right (or left) Q^F-module, we see that $\{\Delta_w \mid w \in W\}$ is also a basis for this module. In particular, the set $\{\Delta_w \mid w \in W\}$ is linearly independent over Q^F and hence over $\tilde{R}$ or $R[\![\Lambda]\!]_F^{\sim}$. ☐

Theorem 5.12. *Given a formal group law (R, F), the formal affine Demazure algebra $\mathbf{D}_F$ is generated as an R-algebra by $R[\![\Lambda]\!]_F$ and the formal Demazure elements Δ_i, $i \in I$, and satisfies the following relations:*

(a) $\varphi \Delta_i = \Delta_i s_i(\varphi) + \Delta_{\alpha_i}(\varphi)$ for all $i \in I$ and $\varphi \in R[\![\Lambda]\!]_F$;
(b) $\Delta_i^2 = \Delta_i \kappa_i$ for all $i \in I$, where $\kappa_i = \frac{1}{x_i} + \frac{1}{x_{-i}} \in R[\![\Lambda]\!]_F$;
(c) $\Delta_i \Delta_j = \Delta_j \Delta_i$ for all $i, j \in I$ such that $\langle \alpha_i, \alpha_j^\vee \rangle = 0$;

(d) $\Delta_j\Delta_i\Delta_j - \Delta_i\Delta_j\Delta_i = \Delta_i\kappa_{ij} - \Delta_j\kappa_{ji}$ *for all* $i, j \in I$ *such that* $\langle \alpha_i, \alpha_j^\vee \rangle = -1$, *where*

$$\kappa_{ij} = \tfrac{1}{x_{i+j}}\left(\tfrac{1}{x_j} - \tfrac{1}{x_{-i}}\right) - \tfrac{1}{x_i x_j} \in R[\![\Lambda]\!]_F$$

(note that, in general, $\kappa_{ij} \neq \kappa_{ji}$*);*

(e) *relation* (5.3) *for all* $i, j \in I$ *such that* $\langle \alpha_i, \alpha_j^\vee \rangle \leq -2$.

Furthermore, if the root system is simply laced, then (a)–(d) *form a complete set of relations. In arbitrary type,* (a)–(e) *form a complete set of relations (over* $R[\![\Lambda]\!]_F^\sim$*) for* $\mathbf{D}_F \otimes_{R[\![\Lambda]\!]_F} R[\![\Lambda]\!]_F^\sim$.

Proof. The first part of the theorem follows immediately from (5.1), Lemma 5.5 and Proposition 5.7.

Since R is a domain, so is $R[\![\Lambda]\!]_F^\sim$. Let $\tilde{\mathbf{D}}_F$ be the R-algebra generated by $R[\![\Lambda[\![_F^\sim$ and elements Δ'_i, $i \in I$, subject to the relations given in the theorem. Then we have a surjective ring homomorphism $\rho : \tilde{\mathbf{D}}_F \to \mathbf{D}_F \otimes_{R[\![\Lambda]\!]_F} R[\![\Lambda[\![_F^\sim$ which is the identity on $R[\![\Lambda[\![_F^\sim$ and maps Δ'_i to Δ_i. We wish to show that this map is an isomorphism.

For $w \in W$, define Δ'_w as in (5.6). The relations among the Δ'_w allow us to write any element of $\tilde{\mathbf{D}}_F$ in the form

$$\textstyle\sum_{w \in W} \Delta'_w a_w, \quad a_w \in R[\![\Lambda[\![_F^\sim.$$

Suppose such an element is in the kernel of ρ. Then

$$0 = \rho\left(\textstyle\sum_{w \in W} \Delta'_w a_w\right) = \textstyle\sum_{w \in W} \Delta_w a_w.$$

By Lemma 5.11, this implies that $a_w = 0$ for all w. Thus ρ is injective and hence an isomorphism. This completes the proof of the proposition once we recall that $R[\![\Lambda[\![_F^\sim \cong R[\![\Lambda]\!]_F$ in simply laced type. □

6. Formal (affine) Demazure algebras: examples and further properties

In this section we specialize the definition of the formal (affine) Demazure algebra to various FGLs. We then prove several important facts about these algebras in general. The first proposition demonstrates that our definition recovers classical objects in the additive and multiplicative cases.

Proposition 6.1. *(a) For the additive FGL over* $R = \mathbb{Z}$*, the formal affine Demazure algebra* $\mathbf{D}_A$ *is isomorphic to the completion of the* nil Hecke ring *of* [KK86, Def. 4.12]. *In this case, all the relations among the* Δ_i *are given by the braid relations and the nilpotence relations* $\Delta_i^2 = 0$ *(even in non-simply laced type).*

(b) For the additive FGL over $R = \mathbb{C}$*, the formal Demazure algebra* D_A *is isomorphic to the nil-Hecke ring of* [EB87, Def. 3].

(c) For the multiplicative FGL over $R = \mathbb{C}$ *with* $\beta = 1$*, the formal Demazure algebra* D_M *is the completion of the* 0-Hecke algebra, *which is the classical Hecke algebra specialized at* $q = 0$*. In this case all the relations among the* Δ_i *are given by the braid relations and the idempotence relations (even in non-simply laced type).*

Proof. For the additive FGL over $R = \mathbb{Z}$, Q is the field of fractions of the ring $S^*(\Lambda)^\wedge \cong R[\![\Lambda]\!]_A$ and Δ_i is the $-x_i$ of [KK86, (I_{24})]. This proves part (a). Similarly, if $R = \mathbb{C}$, then our Δ_i corresponds to the X_i of [EB87], proving part (b).

For the multiplicative FGL over $R = \mathbb{C}$ with $\beta = 1$, we have that Q is the field of fractions of $\mathbb{C}[\Lambda]^\wedge$ and $-\Delta_{\alpha_i}$ is the B_i of [EB87, §2] if we identify the simple root α_i of [EB87] with our $-\alpha_i$. This proves part (c). □

We now consider some other FGLs, where our definition gives new algebras.

Example 6.2 (Lorentz affine Demazure algebra)**.** Consider the Lorentz FGL (R, F_L). Then $x_{i+j} = \frac{x_i+x_j}{1+\beta x_i x_j}$, $\beta \in R$. Since $x_{-\lambda} = -x_\lambda$ for all $\lambda \in \Lambda$, we have

$$\kappa_{ij} = \tfrac{1+\beta x_i x_j}{x_i+x_j} \cdot \tfrac{x_i+x_j}{x_i x_j} - \tfrac{1}{x_i x_j} = \beta \tag{6.1}$$

for $i, j \in I$ with $\langle \alpha_i, \alpha_j \rangle = -1$. Therefore, relation (d) of Theorem 5.12 becomes

$$\Delta_j \Delta_i \Delta_j - \Delta_i \Delta_j \Delta_i = \beta(\Delta_i - \Delta_j) \ \text{ for all } i, j \text{ such that } \langle \alpha_i, \alpha_j \rangle = -1.$$

Example 6.3 (Elliptic affine Demazure algebra)**.** Consider the elliptic FGL (R, F_E). Set $\mu_i = \mu_E(x_i)$ and $g^E_{ij} = g^E(x_i, x_j)$. Then, for $i, j \in I$ with $\langle \alpha_i, \alpha_j \rangle = -1$, we have

$$\begin{aligned}\kappa_{ij} &= \tfrac{x_i(x_{-i}-x_j)-x_{i+j}x_{-i}}{x_i x_{-i} x_j x_{i+j}} = \tfrac{x_{-i}-x_j+x_{i+j}\mu_i}{x_{-i}x_j x_{i+j}} = \tfrac{x_{-i}-x_j+\mu_i(x_i+x_j-x_i x_j g^E_{ij})}{x_{-i}x_j x_{i+j}} = \tfrac{-x_j+\mu_i x_j-\mu_i x_i x_j g^E_{ij}}{x_{-i}x_j x_{i+j}} \\ &= \tfrac{\mu_i-1-\mu_i x_i g^E_{ij}}{x_{-i}x_{i+j}} = \tfrac{\mu_{-i}^{-1}-1-\mu_i x_i g^E_{ij}}{x_{-i}x_{i+j}} = \tfrac{-a_1 x_{-i}-a_3 v(x_{-i})+x_{-i}g^E_{ij}}{x_{i+j}x_{-i}} = \tfrac{g^E_{ij}-a_1}{x_{i+j}} - \tfrac{a_3 v(x_{-i})}{x_{i+j}x_{-i}}.\end{aligned}$$

Theorem 6.4. *For any two formal group laws (R, F) and (R, F') over the same ring R, we have $\mathbf{D}_{R_\mathbb{Q},F} \cong \mathbf{D}_{R_\mathbb{Q},F'}$ as algebras, where $R_\mathbb{Q} = R \otimes_\mathbb{Z} \mathbb{Q}$.*

Proof. It suffices to prove the result for the special case where $F' = F_A$. There is an isomorphism of FGLs

$$e_F : (R_\mathbb{Q}, F_A) \to (R_\mathbb{Q}, F)$$

given by the exponential series $e_F(u) \in R_\mathbb{Q}[\![u]\!]$ (see Section 1). This induces an isomorphism of formal group algebras

$$e_F^\star : R_\mathbb{Q}[\![\Lambda]\!]_F \to R_\mathbb{Q}[\![\Lambda]\!]_A.$$

This map commutes with the action of W and thus we have an induced isomorphism of twisted formal group algebras

$$e_F^\star : Q_W^{(R_\mathbb{Q},F)} \to Q_W^{(R_\mathbb{Q},A)}.$$

Thus $\mathbf{D}_{R_\mathbb{Q},F}$ is isomorphic to its image $D' := e_F^\star(\mathbf{D}_{R_\mathbb{Q},F})$ under this map. Now, D' is generated over $R_\mathbb{Q}[\![\Lambda]\!]_A$ by the elements

$$e_F^\star(\Delta_i^F) = \tfrac{1}{e_F(x_i)}(1-\delta_i) = \tfrac{x_i}{e_F(x_i)}\Delta_i^A, \quad i \in I.$$

Since $e_F(x_i)/x_i \in R_\mathbb{Q}[\![\Lambda]\!]_A$ is invertible in $R_\mathbb{Q}[\![\Lambda]\!]_A$ (because its constant term is invertible in $R_\mathbb{Q}$ – see Example 1.3(a)), D' is also generated over $R_\mathbb{Q}[\![\Lambda]\!]_A$ by Δ_i^A, $i \in I$, and thus isomorphic to $\mathbf{D}_{R_\mathbb{Q},A}$. □

Remark 6.5. Note that while Theorem 6.4 shows that all affine Demazure algebras are isomorphic when the coefficient ring is $R_\mathbb{Q}$, the isomorphism is *not* the naive one sending Δ_i^F to $\Delta_i^{F'}$. Furthermore, the completion (with respect to the augmentation map) is crucial. No assertion is made regarding an isomorphism (even over $R_\mathbb{Q}$) of *truncated* versions.

7. Formal (affine) Hecke algebras: definitions

In the present section we define deformed versions of the formal (affine) Demazure algebras. Our goal is to construct generalizations of the usual (affine) Hecke algebra and its degenerate analogue. These two examples correspond to the multiplicative periodic and additive FGL cases of our more general construction. We begin by reminding the reader of the definition of these classical objects.

Definition 7.1 (Hecke algebra)**.** The (classical) *Hecke algebra* H associated to the Weyl group W is the $\mathbb{Z}[t,t^{-1}]$-algebra with 1 generated (as a $\mathbb{Z}[t,t^{-1}]$-algebra) by elements $T_i := T_{s_i}$, $i \in I$, modulo

(a) the quadratic relations $(T_i + t^{-1})(T_i - t) = 0$ for all $i \in I$, and
(b) the braid relations $T_iT_jT_i \cdots = T_jT_iT_j \cdots$ (m_{ij} factors in both products) for any $i \neq j$ in I with s_is_j of order m_{ij} in W.

Definition 7.2 (Affine Hecke algebra)**.** The (classical) *affine Hecke algebra* $\mathbf{H}$ is $H \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Z}[t,t^{-1}][\Lambda]$, where the factors H and $\mathbb{Z}[t,t^{-1}][\Lambda]$ are subalgebras and the relations between the two factors are given by

$$e^{\lambda}T_i - T_ie^{s_i(\lambda)} = (t - t^{-1})\tfrac{e^{\lambda} - e^{s_i(\lambda)}}{1 - e^{-\alpha_i}}, \quad \lambda \in \Lambda,\ i \in I.$$

Remark 7.3. In Defintions 7.1 and 7.2, we have followed the conventions found, for instance, in [CMHL02, pp. 71–72] (except that we use t in place of v and T_i in place of $\tilde{T}_i$). These conventions differ somewhat from those found in other places in the literature. For instance, H as defined above is isomorphic to $H' \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Z}[q^{1/2}, q^{-1/2}]$, where H' is the Hecke algebra as defined in [Hum90, §7.4] or [CG10, Def. 7.1.1]. The T_{s_i} appearing in [Hum90, CG10] correspond to tT_{s_i} in our notation, where t corresponds to $q^{1/2}$.

Definition 7.4 (Degenerate affine Hecke algebra)**.** Let ϵ be an indeterminate. The *degenerate affine Hecke algebra* $\mathbf{H}_{\deg}$ is the unital associative $\mathbb{Z}[\epsilon]$-algebra that is $\mathbb{Z}[W] \otimes_{\mathbb{Z}} S^*_{\mathbb{Z}[\epsilon]}(\Lambda)$ as a $\mathbb{Z}[\epsilon]$-module and such that the subspaces $\mathbb{Z}[W]$ and $S^*_{\mathbb{Z}[\epsilon]}(\Lambda)$ are subalgebras and the following relations hold:

$$\delta_i\lambda - s_i(\lambda)\delta_i = -\epsilon\langle \alpha_i^\vee, \lambda\rangle, \quad i \in I,\ \lambda \in \Lambda.$$

Fix a free abelian group Γ of rank 1 with generator γ. Denote by R_F the formal group algebra $R[\![\Gamma]\!]_F$. For instance, $\mathbb{Z}_M \cong \mathbb{Z}[t,t^{-1}]^\wedge$ and $\mathbb{Z}_A \cong \mathbb{Z}[\![\gamma]\!]$ (see Example 2.5). Let $Q' := Q^{(R_F,F)}$ denote the fraction field of $R_F[\![\Lambda]\!]_F$ and let $Q'_W \cong R_F[W] \ltimes_R Q'$ be the respective twisted formal group algebra over R_F (see Definition 5.1). We will continue to use the shorthand (5.2). We are now ready to define our second main objects of study.

Definition 7.5 (Formal (affine) Hecke algebra)**.** The *formal Hecke algebra* H_F is the R_F-subalgebra of Q'_W generated by the elements

$$T_i^F := \begin{cases} \Delta_i^F \frac{\Theta_F}{\kappa_i^F} + \delta_i\mu_F(x_\gamma) & \text{if } \kappa^F \neq 0, \\ 2\Delta_i^F x_\gamma + \delta_i & \text{if } \kappa^F = 0, \end{cases} \tag{7.1}$$

for all $i \in I$, where $\Theta_F := \mu_F(x_\gamma) - \mu_F(x_{-\gamma}) \in R_F$. The *formal affine Hecke algebra* $\mathbf{H}_F$ is the R_F-subalgebra of Q'_W generated by H_F and

$$R_F[\![\Lambda]\!]_F^\kappa := \begin{cases} R_F[\![\Lambda]\!]_F[(\kappa_\alpha^F)^{-1} \mid \alpha \in \Phi] & \text{if } \kappa^F \neq 0, \\ R_F[\![\Lambda]\!]_F & \text{if } \kappa^F = 0. \end{cases}$$

We sometimes write T_i when the FGL is understood from the context. When we wish to specify the coefficient ring, we write $H_{R,F}$ (resp. $\mathbf{H}_{R,F}$) for H_F (resp. $\mathbf{H}_F$).

Remark 7.6. (a) If a_{11} (in the notation of (1.1)) is invertible in R, then κ_α^F is invertible in $R_F[\![\Lambda]\!]_F$ for all $\alpha \in \Phi$. Thus, $R_F[\![\Lambda]\!]_F^\kappa = R_F[\![\Lambda]\!]_F$.

(b) The coefficients $\frac{\Theta_F}{\kappa_i}$, $\mu_F(x_\gamma)$, and $x_\gamma - x_{-\gamma}$ appearing in Definition 7.5 are all invariant under the action of s_i.

(c) In the multiplicative case we have

$$\tfrac{\Theta_F}{\kappa_i} = \tfrac{\beta(x_\gamma - x_{-\gamma})}{\beta} = x_\gamma - x_{-\gamma}.$$

Since the additive FGL is the $\beta \to 0$ limit of the multiplicative, this motivates the choice of coefficient of $\mathit{\Delta}_i^F$ in the case $\kappa^F = 0$. More generally, one can show that when $\frac{\Theta_F}{\kappa_i^F}$ is expanded as a power series in x_i and x_γ, the leading term is equal to $2x_\gamma$. Similarly, when $\kappa^F = 0$, we have $x_{-\gamma} = -x_\gamma$ and so $\mu_F(x_\gamma) = 1$, the coefficient of δ_i.

Lemma 7.7. *For all $\psi \in Q'$ and $i \in I$, we have*

$$\psi T_i - T_i s_i(\psi) = \begin{cases} \frac{\Theta_F}{\kappa_i} \Delta_{\alpha_i}^F(\psi) & \text{if } \kappa^F \neq 0, \\ 2x_\gamma \Delta_{\alpha_i}^F(\psi) & \text{if } \kappa^F = 0. \end{cases} \tag{7.2}$$

In particular, $\varphi T_i - T_i s_i(\varphi) \in R_F[\![\Lambda]\!]_F^\kappa$ for all $\varphi \in R_F[\![\Lambda]\!]_F^\kappa$.

Proof. Let a and b be the coefficients of $\mathit{\Delta}_i$ and δ_i in (7.1), so that $T_i = \mathit{\Delta}_i a + \delta_i b$. By Lemma 5.5, we have

$$\psi T_i = \psi(\mathit{\Delta}_i a + \delta_{s_i} b) = (\mathit{\Delta}_i s_i(\psi) + \Delta_{\alpha_i}(\psi))a + \delta_{s_i} s_i(\psi) b = T_i s_i(\psi) + a\Delta_{\alpha_i}(\psi).$$

The last statement is an easy verification left to the reader. □

Lemma 7.8. *The elements T_i, $i \in I$, satisfy the quadratic relation*

$$T_i^2 = T_i \Theta_F + 1. \tag{7.3}$$

Thus T_i is invertible and $T_i^{-1} = T_i - \Theta_F$. Furthermore

$$(T_i + \mu_F(x_{-\gamma}))(T_i - \mu_F(x_\gamma)) = 0. \tag{7.4}$$

Proof. Let a and b be the coefficients of $\mathit{\Delta}_i$ and δ_i in (7.1), so that $T_i = \mathit{\Delta}_i a + \delta_i b$. Using (5.1) and the fact that $\mathit{\Delta}_i \delta_i + \delta_i \mathit{\Delta}_i = (\delta_i - 1)\kappa_i^F$, we have

$$T_i^2 = \mathit{\Delta}_i^2 a^2 + (\mathit{\Delta}_i \delta_i + \delta_i \mathit{\Delta}_i) ab + b^2 = \mathit{\Delta}_i \kappa_i^F a^2 + (\delta_i - 1)\kappa_i^F ab + b^2 = T_i(\kappa_i^F a) + b(b - \kappa_i^F a).$$

One readily verifies that in both cases in (7.1), we have $\kappa_i^F a = \Theta_F$ and $b(b - \kappa_i^F a) = 1$, completing the proof of the first statement in the lemma. The second two statements follow easily. □

Remark 7.9. Because of (7.4) and the fact that $\mu_F(x_{-\gamma}) = \mu_F(x_\gamma)^{-1}$, one may think of the power series $\mu_F(x_\gamma)$ as a generalization of the deformation parameter t of the classical Hecke algebra (see Definition 7.1(a)).

Proposition 7.10. *Suppose $i, j \in I$ and let m_{ij} be the order of $s_i s_j$ in W. Then*

$$\underbrace{T_j T_i T_j \cdots}_{m_{ij} \text{ terms}} - \underbrace{T_i T_j T_i \cdots}_{m_{ij} \text{ terms}} = \sum_{w \in W,\, \ell(w) < m_{ij}} T_w \tau_{ij}^w \tag{7.5}$$

for some $\tau_{ij}^w \in Q'$. In particular, we have the following.

(a) If $\langle \alpha_i, \alpha_j \rangle = 0$, so that $m_{ij} = 2$, then $T_i T_j = T_j T_i$.
(b) If $\langle \alpha_i, \alpha_j \rangle = -1$, so that $m_{ij} = 3$, then

$$T_j T_i T_j - T_i T_j T_i = (T_i - T_j)\sigma_{ij}, \quad \sigma_{ij} = \chi_{i+j}(\chi_j - \chi_{-i}) - \chi_i \chi_j, \tag{7.6}$$

where

$$\chi_\alpha = \begin{cases} \frac{\Theta_F}{x_\alpha \kappa_\alpha} & \text{if } \kappa^F \neq 0, \\ \frac{2x_\gamma}{x_\alpha} & \text{if } \kappa^F = 0, \end{cases} \qquad \alpha \in \Phi.$$

(We use the usual convention that $\chi_{\pm i} = \chi_{\pm \alpha_i}$ and $\chi_{\pm i \pm j} = \chi_{\pm \alpha_i \pm \alpha_j}$.) Moreover, $\sigma_{ij} = \sigma_{ji}$ commutes with δ_i and δ_j (and hence with T_i and T_j). If $\kappa^F = 0$, then $\sigma_{ij} = 4x_\gamma^2 \kappa_{ij} \in R_F[\![\Lambda]\!]_F$.

Proof. Set $\mu = \mu_F(x_\gamma)$. (Thus $\mu = 1$ iff $\kappa^F = 0$ by Lemma 3.3.) In both cases (i.e. $\kappa^F \neq 0$ or $\kappa^F = 0$), we have $T_j = \chi_j + (\mu - \chi_j)\delta_j$. Now

$$\underbrace{T_j T_i T_j \cdots}_{m_{ij} \text{ terms}} = (\chi_j + (\mu - \chi_j)\delta_j)(\chi_i + (\mu - \chi_i)\delta_i)(\chi_j + (\mu - \chi_j)\delta_j) \cdots .$$

Since the δ_w, $w \in W$, form a basis of Q'_W as a right Q'-module, this can be written as a sum of (right) Q'-multiplies of δ_w. The leading term (with respect to the length of w) is

$$\delta_{\underbrace{s_j s_i s_j \cdots s_n s_k}_{m_{ij} \text{ terms}}} (\mu - \chi_{-\underbrace{s_k s_n \cdots s_j s_i}_{m_{ij}-1 \text{ terms}}(\alpha_j)}) \cdots (\mu - \chi_{-s_k(\alpha_n)})(\mu - \chi_{-\alpha_k}),$$

where $n = i$ and $k = j$ (resp. $n = j$ and $k = i$) if m_{ij} is odd (resp. even). As in the proof of Proposition 5.7, we see that the highest order terms in $T_j T_i T_j \cdots - T_i T_j T_i \cdots$ cancel, proving the first part of the proposition.

Part (a) follows immediately from Proposition 5.7(a) and the facts that, under the assumptions, $\delta_i \delta_j = \delta_j \delta_i$, $\Delta_i \delta_j = \delta_j \Delta_i$, and $\Delta_j \delta_i = \delta_i \Delta_j$.

It remains to prove (b). We have

$$\begin{aligned} T_j T_i T_j &= \big(\chi_j + (\mu - \chi_j)\delta_j\big)\big(\chi_i + (\mu - \chi_i)\delta_i\big)\big(\chi_j + (\mu - \chi_j)\delta_j\big) \\ &= \chi_i \chi_j^2 + (\mu - \chi_j)(\mu - \chi_{-j})\chi_{i+j} + \big((\mu - \chi_j)\chi_{i+j}\chi_{-j} + (\mu - \chi_j)\chi_i\chi_j\big)\delta_j \\ &\quad + (\mu - \chi_i)\chi_j\chi_{i+j}\delta_i + (\mu - \chi_j)(\mu - \chi_{i+j})\chi_i\delta_j\delta_i + (\mu - \chi_i)(\mu - \chi_{i+j})\chi_j\delta_i\delta_j \\ &\quad + (\mu - \chi_j)(\mu - \chi_i)(\mu - \chi_{i+j})\delta_j\delta_i\delta_j. \end{aligned}$$

Using the fact that $\delta_j\delta_i\delta_j = \delta_j\delta_i\delta_j$, we see that

$$\begin{aligned} T_jT_iT_j - T_iT_jT_i &= \chi_i\chi_j^2 - \chi_i^2\chi_j + (\mu-\chi_j)(\mu-\chi_{-j})\chi_{i+j} - (\mu-\chi_i)(\mu-\chi_{-i})\chi_{i+j} \\ &\quad - \sigma_{ji}(\mu-\chi_j)\delta_j + \sigma_{ij}(\mu-\chi_i)\delta_i \\ &= \sigma_{ij}\chi_i - \sigma_{ji}\chi_j - \sigma_{ji}(\mu-\chi_j)\delta_j + \sigma_{ij}(\mu-\chi_i)\delta_i + \mu\chi_{i+j}(\chi_i+\chi_{-i}-\chi_j-\chi_{-j}) \\ &= \sigma_{ij}T_i - \sigma_{ji}T_j + \mu\chi_{i+j}(\chi_i+\chi_{-i}-\chi_j-\chi_{-j}) \\ &= \sigma_{ij}T_i - \sigma_{ji}T_j + \mu(\sigma_{ji}-\sigma_{ij}) \\ &= \sigma_{ij}(T_i-\mu) - \sigma_{ji}(T_j-\mu). \end{aligned}$$

If $\kappa^F = 0$, then clearly $\sigma_{ij} = 4x_\gamma^2\kappa_{ij}$. Since, in this case, $\kappa_{ij} = \kappa_{ji}$ (use the fact that $x_{-i} = -x_i$ in (5.4)), we have $\sigma_{ij} = \sigma_{ji}$. If $\kappa^F \neq 0$, we have

$$\begin{aligned} \sigma_{ij} &= \Theta_F^2 \tfrac{x_ix_{-i}\kappa_i - x_ix_j\kappa_j - x_{-i}x_{i+j}\kappa_{i+j}}{x_ix_{-i}x_jx_{i+j}\kappa_i\kappa_j\kappa_{i+j}} \\ &= \Theta_F^2 \tfrac{(x_i+x_{-i})x_{-j}x_{-i-j} - (x_j+x_{-j})x_ix_{-i-j} - (x_{i+j}+x_{-i-j})x_{-i}x_{-j}}{(x_i+x_{-i})(x_j+x_{-j})(x_{i+j}+x_{-i-j})} \\ &= -\Theta_F^2 \tfrac{x_ix_jx_{-i-j} + x_{-i}x_{-j}x_{i+j}}{(x_i+x_{-i})(x_j+x_{-j})(x_{i+j}+x_{-i-j})} = -\Theta_F^2 \left(\tfrac{1}{x_ix_jx_{-i-j}} + \tfrac{1}{x_{-i}x_{-j}x_{i+j}}\right)\tfrac{1}{\kappa_i\kappa_j\kappa_{i+j}}, \end{aligned}$$

which implies that $\sigma_{ij} = \sigma_{ji}$. Thus we have

$$T_jT_iT_j - T_iT_jT_i = \sigma_{ij}(T_i - T_j)$$

The fact that σ_{ij} commutes with δ_i and δ_j is an easy verification left to the reader. □

For each $w \in W$, fix a reduced decomposition $w = s_{i_1}\cdots s_{i_k}$ and set

$$T_w = T_{i_1}\cdots T_{i_k}. \tag{7.7}$$

Note that, in general, T_w depends on the choice of reduced decomposition.

Definition 7.11 ($\tilde{R}_F$ and $R_F[\![\Lambda]\!]_F^{\sim}$)**.** Let $\tilde{R}_F$ be the subalgebra of Q' defined by

$$\tilde{R}_F := R_F[W]\cdot R_F[\tau_{ij}^w \mid i,j\in I,\ w\in W,\ \ell(w) < m_{ij}]. \tag{7.8}$$

where $R_F[W]\cdot$ denotes the natural action of the group algebra $R_F[W]$ of W on Q'. Similarly, define

$$R_F[\![\Lambda]\!]_F^{\sim} := R_F[W]\cdot R_F[\![\Lambda]\!]_F^{\kappa}[\tau_{ij}^w \mid i,j\in I,\ w\in W,\ \ell(w) < m_{ij}]. \tag{7.9}$$

Note that $R_F[\![\Lambda]\!]_F^{\sim} = R_F[\![\Lambda]\!]_F$ if the root system is simply laced and $\kappa^F = 0$ (since $\sigma_{ij} = 4x_\gamma^2\kappa_{ij} \in R_F[\![\Lambda]\!]_F$ in that case).

Lemma 7.12. *The set $\{T_w \mid w\in W\}$ forms a basis of $H_F \otimes_{R_F} \tilde{R}_F$ as a right (or left) $\tilde{R}_F$-module and a basis of $\mathbf{H}_F \otimes_{R_F[\![\Lambda]\!]_F} R_F[\![\Lambda]\!]_F^{\sim}$ as a right (or left) $R_F[\![\Lambda]\!]_F^{\sim}$-module.*

Proof. The proof is analogous to that of Lemma 5.11 and will be omitted. □

Theorem 7.13. *Given a formal group law (R,F), the formal affine Hecke algebra $\mathbf{H}_F$ is generated as an R_F-algebra by $R_F[\![\Lambda]\!]_F^{\kappa}$ and the elements T_i, $i\in I$, and satisfies*

(a) relation (7.2) for all $i\in I$ and $\varphi \in R_F[\![\Lambda]\!]_F^{\kappa}$,
(b) $(T_i + \mu_F(x_{-\gamma}))(T_i - \mu_F(x_\gamma)) = 0$ for all $i \in I$,
(c) $T_iT_j = T_jT_i$ for all $i,j\in I$ such that $\langle \alpha_i, \alpha_j^\vee\rangle = 0$,
(d) relation (7.6) for all $i,j\in I$ such that $\langle \alpha_i, \alpha_j^\vee\rangle = -1$, and

(e) *relation* (7.5) *for all* $i, j \in I$ *such that* $\langle \alpha_i, \alpha_j^\vee \rangle \leq -2$.

Furthermore, (a)–(e) *form a complete set of relations (over* $R_F[\![\Lambda]\!]_F^{\sim}$*) for* $\mathbf{H}_F \otimes_{R_F[\![\Lambda]\!]_F} R_F[\![\Lambda]\!]_F^{\sim}$.

Proof. The first part of the theorem follows immediately from Lemmas 7.7 and 7.8 and Proposition 7.10. The second part is analogous to the proof of Theorem 5.12 and will be omitted. □

8. Formal (affine) Hecke algebras: examples and further properties

In this final section we specialize the definition of the formal (affine) Hecke algebra to various FGLs, yielding classical algebras as well as new ones. We then prove several important facts about these algebras in general.

As in Section 5, we have a map $Q'_W \to \operatorname{End}_{R_F} Q'$. Since the operators T_i^F preserve $R_F[\![\Lambda]\!]_F^\kappa$, we have an induced map $\mathbf{H}_F \to \operatorname{End}_{R_F} R_F[\![\Lambda]\!]_F^\kappa$ of R_F-algebras. Recall that if a_{11} (in the notation of (1.1)) is invertible in R, then κ_α^F is invertible for all $\alpha \in \Phi$, and so $R_F[\![\Lambda]\!]_F^\kappa = R_F[\![\Lambda]\!]_F$.

Proposition 8.1. *If* a_{11} *is invertible in* R*, then the map* $\mathbf{H}_F \to \operatorname{End}_{R_F} R_F[\![\Lambda]\!]_F$ *described above is injective. In other words, the natural action of* $\mathbf{H}_F$ *on* $R_F[\![\Lambda]\!]_F$ *is faithful.*

Proof. Suppose, contrary to the statement of the proposition, that the given map is not injective. Let $a \in \mathbf{H}_F$ be in the kernel of this map, with $a \neq 0$. In other words, a acts by zero on $R_F[\![\Lambda]\!]_F$ under the associated action. By Lemma 7.12, we may write

$$a = \textstyle\sum_{w \in W} T_w a_w, \quad a_w \in R_F[\![\Lambda]\!]_F^{\sim}.$$

Now, clearly $a\varphi$ also acts by zero on $R_F[\![\Lambda]\!]_F$ for all $\varphi \in R_F[\![\Lambda]\!]_F$. Choosing φ to be a common denominator of all the a_w, we see that we may assume that $a_w \in R_F[\![\Lambda]\!]_F$ for all $w \in W$.

For $\varphi \in R_F[\![\Lambda]\!]_F$, define the *degree* of φ to be

$$\deg \varphi := \max\{m \in \mathbb{Z}_{\geq 0} \mid \varphi \in \mathcal{I}_F^m\},$$

where $\mathcal{I}_F$ is the kernel of the augmentation map $\varepsilon \colon R_F[\![\Lambda]\!]_F \to R_F$ (i.e. the element x_γ is *not* mapped to zero). We adopt the convention that $\deg 0 = -\infty$. Then the formal Demazure operators lower degree and the coefficients $\mu_F(x_\gamma)$, $\frac{\Theta_F}{\kappa_i^F}$ and x_γ appearing in Definition (7.1) of T_i preserve degree. Thus, if $\deg \varphi = m$, we have

$$T_i(\varphi) = \mu_F(x_\gamma) s_i(\varphi) + (\text{terms of degree} < m).$$

Furthermore, $\deg(\varphi\varphi') = \deg \varphi + \deg \varphi'$ for $\varphi, \varphi' \in R_F[\![\Lambda]\!]_F$. Indeed, it follows by definition that $\deg(\varphi\varphi') \geq \deg \varphi + \deg \varphi'$. If $\deg(\varphi\varphi') > \deg \varphi + \deg \varphi'$, then in the associate graded algebra we have $\varphi\varphi' = 0$, where $\varphi \neq 0$ and $\varphi' \neq 0$. Identifying the associated graded algebra with the polynomial algebra (by [CPZ, Lem. 4.2]) we obtain a contradiction as the polynomial algebra is a domain.

Let m be the maximum degree of the a_w, $w \in W$, and set $W' = \{w \in W \mid \deg a_w = m\}$. Then, for all $\varphi \in R_F[\![\Lambda]\!]_F$, we have

$$\begin{aligned} 0 = a(\varphi) &= \textstyle\sum_{w \in W'} T_w(a_w \varphi) + \sum_{w \in W \setminus W'} T_w(a_w \varphi) \\ &= \textstyle\sum_{w \in W'} \mu_F(x_\gamma)^{\ell(w)} s_w(a_w \varphi) + b, \end{aligned}$$

where the last summation lies in $\mathcal{I}_F^{m+\deg\varphi}$ and $b \notin \mathcal{I}_F^{m+\deg\varphi}$. It follows that

$$\textstyle\sum_{w\in W'} \mu_F(x_\gamma)^{\ell(w)} s_w(a_w\varphi) = \sum_{w\in W'} s_w\left(\mu_F(x_\gamma)^{\ell(w)} a_w\varphi\right) = 0 \quad \text{for all } \varphi \in R_F[\![\Lambda]\!]_F.$$

The above sum is therefore also equal to zero in $\mathcal{I}_F^{m+\deg\varphi}/\mathcal{I}_F^{m+\deg\varphi+1}$. But $\bigoplus_n \mathcal{I}_F^n/\mathcal{I}_F^{n+1} \cong S^*_{R_F}(\Lambda)$, by [CPZ, Lem. 4.2]. Since the action of $R_F[\![W]\!] \ltimes S^*_{R_F}(\Lambda)$ on $S^*_{R_F}(\Lambda)$ is faithful (see, for example, the argument in [Kle05, Second Proof of Thm. 3.2.2]), we have that $\mu_F(x_\gamma)^{\ell(w)}a_w = 0$ (hence $a_w = 0$) for all $w \in W'$. But this contradicts the choice of m. ☐

Remark 8.2. In the additive and multiplicative cases, Proposition 8.1 reduces to known embeddings of the (degenerate) affine Hecke algebra into endomorphism rings. See the proof of Proposition 8.3.

The following proposition demonstrates that our definition of the formal (affine) Hecke algebra recovers classical objects in the additive and multiplicative cases.

Proposition 8.3. *Suppose $R = \mathbb{Z}$.*

(a) *For the additive FGL, we have the following isomorphisms of algebras:*

$$\mathbf{H}_A \cong \mathbf{H}_{\deg}^\wedge := \mathbf{H}_{\deg} \otimes_{\mathbb{Z}[\epsilon]} \mathbb{Z}[\![\gamma]\!], \quad H_A \cong \mathbb{Z}_A[W] \cong \mathbb{Z}[W] \otimes_{\mathbb{Z}} \mathbb{Z}[\![\gamma]\!],$$

where $\epsilon = -2\gamma$.

(b) *For the multiplicative periodic FGL, we have the following isomorphisms of algebras:*

$$\mathbf{H}_M \cong \mathbf{H} \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Z}[t,t^{-1}]^\wedge, \quad H_M \cong H \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Z}[t,t^{-1}]^\wedge.$$

Proof. It is easy to see that for the additive and multiplicative FGLs in simply laced type, the relations of Theorem 7.13 become the relations of the respective algebras in the statement of the proposition. However, we provide a proof that remains valid in all types (i.e. not necessarily simply laced). Note that in the both the additive case (where $\kappa^F = 0$) and multiplicative periodic case (where $a_{11} = \beta$ is invertible and hence all κ_α, $\alpha \in \Phi$, are invertible), we have $R_F[\![\Lambda]\!]_F^\kappa = R_F[\![\Lambda]\!]_F$.

Consider first the additive FGL. Recall the identification $\mathbb{Z}_A \cong \mathbb{Z}[\![\gamma]\!]$ of Example 2.5. The injective map $\mathbf{H}_A \hookrightarrow \operatorname{End}_{\mathbb{Z}_A} \mathbb{Z}_A[\![\Lambda]\!]_A$ is given on the T_i by

$$T_i = 2\gamma\Delta_i^A + \delta_i \mapsto s_i + 2\gamma\tfrac{1}{\alpha}(s_i - 1).$$

Thus $\mathbf{H}_A$ is isomorphic to the subalgebra $\mathbf{H}'_A$ of $\operatorname{End}_{\mathbb{Z}_A} \mathbb{Z}_A[\![\Lambda]\!]_A$ generated by multiplication by elements of $\mathbb{Z}_A[\![\Lambda]\!]_A$ and the operators $s_i + 2\gamma\frac{1}{\alpha}(s_i - 1)$.

Observe that, in the notation of [Gin, §12], the algebra $S^*_{\mathbb{Z}_A}(\Lambda)^\wedge \otimes_{\mathbb{Z}} \mathbb{C} = S^*_{\mathbb{Z}}(\Lambda)^\wedge \otimes_{\mathbb{Z}} \mathbb{C}[\![\gamma]\!]$ can be identified with the completion of the algebra $\mathbb{C}[\mathfrak{h},\gamma]$ of polynomial functions on $\mathfrak{h}$ with coefficients in $\mathbb{C}[\gamma]$. If we let $\epsilon = -2\gamma$, then we see that $\mathbf{H}'_A$ is precisely the completion of the image of $\mathbf{H}_{\deg}$ under the faithful action on $\mathbb{C}[\mathfrak{h},\epsilon]$ given by Demazure-Lusztig type operators (see [Gin, Prop. 12.2] or [Kle05, Second Proof of Thm. 3.2.2]). This proves the first isomorphism of part (a). The second follows by considering the subalgebra generated by the T_i.

Now consider the multiplicative periodic FGL $F_M(u,v) = u + v - \beta uv$, $\beta \in \mathbb{Z}^\times$. We have (see Example 1.2(b))

$$\mu_M(x_\gamma) = 1 - \beta x_{-\gamma} = t \text{ and } \Theta_M = \beta(x_\gamma - x_{-\gamma}) = t - t^{-1} \in \mathbb{Z}[t,t^{-1}]^\wedge,$$

under the identifications of Example 2.5. Using the above and the identifications of Example 2.4(b), the injective map $\mathbf{H}_M \hookrightarrow \operatorname{End}_{\mathbb{Z}_M} \mathbb{Z}_M[\![\Lambda]\!]_M$ is given on the T_i by

$$T_i = \Delta_i^M \tfrac{\Theta_M}{\kappa_i^M} + \delta_i \mu_M(x_\gamma) = \tfrac{t-t^{-1}}{1-e^{-\alpha_i}}(1-\delta_i) + t\delta_i \mapsto t^{-1}\tfrac{1-s_i}{e^{-\alpha_i}-1} - t\tfrac{1-e^{-\alpha_i}s_i}{e^{-\alpha_i}-1}. \tag{8.1}$$

We identify $\mathbb{Z}_M[\![\Lambda]\!]_M$ with $\mathbb{Z}[q,q^{-1}][P] \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Z}[t,t^{-1}]^\wedge$ in the notation of [Lus85] (where the P and q of [Lus85] are our Λ and t^2, respectively) via the map $e^\lambda \mapsto -\lambda$ (see Example 2.4(b)). (The negative sign in front of the α_i arises from the twisting of the action of $\mathbb{Z}[q,q^{-1}][P]$ on itself by a sign in [Lus85, (8.2)].) Under this identification, the right hand side of (8.1) corresponds to the Demazure-Lusztig operator [Lus85, (8.1)], where the T_s of [Lus85] corresponds to our tT_i, where $s = s_i$ (see Remark 7.3). Therefore, the actions of $\mathbf{H}_M$ and $\mathbf{H}$ on $\mathbb{Z}_M[\![\Lambda]\!]_M \cong \mathbb{Z}[q,q^{-1}][P] \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Z}[t,t^{-1}]^\wedge$ coincide. The action of $\mathbf{H}_M$ is faithful by Proposition 8.1 and the action of $\mathbf{H}$ is also known to be faithful (see, for example, [Gin, Prop. 12.2(i)] or note that the action of $\mathbf{H}$ specializes to the standard action of $\mathbb{Z}[W] \ltimes \mathbb{Z}[\Lambda]$ on $\mathbb{Z}[\Lambda]$ when $q = 1$). Thus we have the first isomorphism of part (b). The second follows by considering the subalgebra generated by the T_i. □

For other FGLs our definition gives new algebras as the following examples indicate.

Example 8.4 (Lorentz case)**.** For the Lorentz FGL F_L, we have $\mu_L(u) = 1$, $\Theta_L = 0$, and $\kappa = 0$. Since $\kappa_{ij} = \beta$ (see (6.1)), we have $\sigma_{ij} = 4\beta x_\gamma^2$. Thus the relations (a)–(d) of Theorem 7.13 become

(a) $\varphi T_i - T_i s_i(\varphi) = 2x_\gamma \Delta^L_{\alpha_i}(\varphi)$ for all $\varphi \in R_F[\![\Lambda]\!]_F,\ i \in I$.
(b) $T_i^2 = 1$ for all $i \in I$,
(c) $T_iT_j = T_jT_i$ for all $i, j \in I$ such that $\langle \alpha_i, \alpha_j^\vee \rangle = 0$,
(d) $T_iT_jT_i - T_jT_iT_j = 4\beta x_\gamma^2(T_i - T_j)$ for all $i, j \in I$ such that $\langle \alpha_i, \alpha_j \rangle = -1$.

These form a complete set of relations in the simply laced case.

Example 8.5 (Elliptic case)**.** For the elliptic FGL F_E, we have

$$\mu_E(u) = \tfrac{1}{1-a_1u-a_3v(u)}, \quad \Theta_E = \tfrac{2\psi-\psi^2}{1-\psi} = \tfrac{\psi}{1-\psi} + \psi,$$

where $\psi = a_1x_\gamma + a_3v(x_\gamma)$ (see Example 1.2(d)). If, for example, $a_3 = 0$, then

$$\mu_E(u) = \tfrac{1}{1-a_1u}, \quad \Theta_E = \tfrac{2a_1x_\gamma - a_1^2x_\gamma^2}{1-a_1x_\gamma}, \quad \kappa_i = a_1 \quad \text{for all } i \in I,$$

and so

$$T_i = \Delta_i^E \tfrac{2x_\gamma - a_1x_\gamma^2}{1-a_1x_\gamma} + \delta_i \tfrac{1}{1-a_1u} \quad \text{for all } i \in I.$$

Furthermore, when $a_3 = 0$, we have

$$\chi_i = \tfrac{\Theta_E}{x_i a_1}$$

and so

$$\sigma_{ij} = \tfrac{\Theta_E}{x_ia_1}\tfrac{\Theta_E}{x_ja_1} + \tfrac{\Theta_E}{x_{i+j}a_1}\left(\tfrac{\Theta_E}{x_{-i}a_1} - \tfrac{\Theta_E}{x_ja_1}\right) = -\tfrac{\Theta_E^2}{a_1^2}\kappa_{ij}.$$

Example 8.6 (Universal formal Hecke algebra)**.** We call the formal Hecke algebra H_U corresponding to the universal FGL F_U the *universal formal Hecke algebra*. Observe that H_U is an algebra over $\mathbb{L}_U$, where $\mathbb{L}$ is the Lazard ring. Note that in this case we have

$$\begin{aligned}\Theta_U &= -a_{11}(x_\gamma - x_{-\gamma}) + a_{11}^2(x_\gamma^2 - x_{-\gamma}^2) - (a_{11}^3 + a_{12}a_{11} - a_{22} + 2a_{13})(x_\gamma^3 - x_{-\gamma}^3) + \cdots \\ &= -2a_{11}x_\gamma - 2(a_{11}^3 + a_{11}a_{12} - a_{22} + 2a_{13})x_\gamma^3 + \cdots .\end{aligned}$$

Theorem 8.7. *Suppose (R,F) and (R,F') are FGLs over the same ring R, with either $\kappa^F = 0$ or a_{11} invertible in R (in the notation of (1.1)). Then*

$$\mathbf{H}_F \otimes_{R_F[\![\Lambda]\!]_F} R'_F[\![\Lambda]\!]_F \cong \mathbf{H}_{F'} \otimes_{R_{F'}[\![\Lambda]\!]_{F'}} R'_{F'}[\![\Lambda]\!]_{F'}$$

as algebras, where $R'_F = (R \otimes_{\mathbb{Z}} \mathbb{Q})_F \otimes_{\mathbb{Q}} \mathbb{Q}[x_\gamma^{-1}]$ (and similarly, with F replaced by F').

Proof. It suffices to prove the result when $F' = F_A$. Let $R_{\mathbb{Q}} = R \otimes_{\mathbb{Z}} \mathbb{Q}$. As in the proof of Theorem 6.4, we have an isomorphism of twisted formal group algebras

$$e_F^\star : Q_W^{(R'_F,F)} \to Q_W^{(R'_A,A)}.$$

Since

$$e_F^\star(x_\gamma^{-1}) = \tfrac{1}{e_F(x_\gamma)} = \tfrac{x_\gamma}{e_F(x_\gamma)} x_\gamma^{-1}$$

and $\frac{x_\gamma}{e_F(x_\gamma)} \in (R_{\mathbb{Q}})_A$ is invertible in $(R_{\mathbb{Q}})_A$, we see that $e_F^\star(R'_F) = R'_A$ and so $e_F^\star(R'_F[\![\Lambda]\!]_F) = R'_A[\![\Lambda]\!]_A$. The algebra $\mathbf{H}_F \otimes_{R_F[\![\Lambda]\!]_F} R'_F[\![\Lambda]\!]_F$ is isomorphic to its image $\mathbf{H}' := e_F^\star(\mathbf{H}_F \otimes_{R_F[\![\Lambda]\!]_F} R'_F[\![\Lambda]\!]_F)$ under $e_F^\star$.

We first consider the case where $\kappa^F = 0$. Then $\mathbf{H}_F \otimes_{R_F[\![\Lambda]\!]_F} R'_F[\![\Lambda]\!]_F$ is generated over $R'_F[\![\Lambda]\!]_F$ by (the element 1 and) the elements

$$T_i^F - 1 = 2\Delta_i^F x_\gamma + \delta_i - 1 = \Xi_i^F(\delta_i - 1), \quad i \in I,$$

where

$$\Xi_i^F = 1 - \tfrac{2x_\gamma}{x_i} \in Q_W^{(R_F,F)}.$$

We see that

$$e_F^\star(T_i^F - 1) = e_F^\star(\Xi_i^F)(\delta_i - 1) = \tfrac{e_F^\star(\Xi_i^F)}{\Xi_i^A}(T_i^A - 1).$$

Thus it suffices to show that $e_F^\star(\Xi_i^F)/\Xi_i^A$ lies in $R'_A[\![\Lambda]\!]_A$ and is invertible in $R'_A[\![\Lambda]\!]_A$ (i.e. has invertible constant term). Now,

$$\begin{aligned} \tfrac{e_F^\star(\Xi_i^F)}{\Xi_i^A} &= \left(1 - \tfrac{2e_F(x_\gamma)}{e_F(x_i)}\right)\left(1 - \tfrac{2x_\gamma}{x_i}\right)^{-1} \\ &= \left(\tfrac{x_i}{e_F(x_i)}\tfrac{e_F(x_\gamma)}{x_\gamma} - \tfrac{x_i}{2x_\gamma}\right)\left(1 - \tfrac{x_i}{2x_\gamma}\right)^{-1}. \end{aligned}$$

Note that $e_F(x_i)/x_i, e_F(x_\gamma)/x_\gamma \in R'_A[\![\Lambda]\!]_A$ are invertible in $R'_A[\![\Lambda]\!]_A$ (with constant term one). Since $1 - x_i/(2x_\gamma) \in R'_A[\![\Lambda]\!]_A$ is also clearly invertible in $R'_A[\![\Lambda]\!]_A$, we are done.

Now consider the case where $\kappa^F \neq 0$ and a_{11} is invertible in R (hence in R'_A). The elements

$$T_i^F - \mu_F(x_\gamma) = \rho_i(\delta_i - 1), \quad i \in I,$$

where

$$\rho_i = \mu_F(x_\gamma) - \tfrac{\Theta_F}{x_i \kappa_i^F} \in Q_W^{(R_F,F)}$$

generate $\mathbf{H}_F \otimes_{R_F[\![\Lambda]\!]_F} R'_F[\![\Lambda]\!]_F$ over $R'_F[\![\Lambda]\!]_F$ (along with the element 1). Since

$$e_F^\star(T_i^F - \mu_F(x_\gamma)) = \tfrac{e_F^\star(\rho_i)}{\Xi_i^A}(T_i^A - 1),$$

it follows as in the $\kappa^F = 0$ case that it suffices to show that $e_F^\star(\rho_i)/\Xi_i^A$ lies in $R'_A[\![\Lambda]\!]_A$ and is invertible in $R'_A[\![\Lambda]\!]_A$ (i.e. has invertible constant term).

For any $x \in R'_A[\![\Lambda]\!]_A$ we set

$$\psi(x) = \tfrac{1 - \mu_F(e_F(x))}{x} = a_{11} + O(1) \in R'_A[\![\Lambda]\!]_A$$

so that $\mu_F(e_F(x)) = 1 - x\psi(x)$. Then (as $x_{-\lambda} = -x_\lambda$ in $R_A[\![\Lambda]\!]_A$)

$$\begin{aligned}\frac{e_F^\star(\rho_i)}{\Xi_i^A} &= \left(\mu_F(e_F(x_\gamma)) - \frac{\mu_F(e_F(x_\gamma))-\mu_F(e_F(-x_\gamma))}{1-\mu_F(e_F(x_{-i}))}\right)\left(1-\frac{2x_\gamma}{x_i}\right)^{-1}\\ &= \left(1 - x_\gamma\psi(x_\gamma) - \frac{x_\gamma}{x_i}\cdot\frac{\psi(-x_\gamma)+\psi(x_\gamma)}{\psi(-x_i)}\right)\left(1-\frac{2x_\gamma}{x_i}\right)^{-1}\\ &= \left(\frac{\psi(-x_\gamma)+\psi(x_\gamma)}{\psi(-x_i)} + x_i\psi(x_\gamma) - \frac{x_i}{x_\gamma}\right)\left(2-\frac{x_i}{x_\gamma}\right)^{-1}.\end{aligned}$$

Since a_{11} is invertible, we have

$$\frac{\psi(-x_\gamma)+\psi(x_\gamma)}{\psi(-x_i)} = \frac{2a_{11}+O(1)}{a_{11}+O(1)} = 2 + O(1).$$

Combining all of the above computations, we see that

$$\frac{e_F^\star(\rho_i)}{\Xi_i^A} = 1 + O(1) \in R'_A[\![\Lambda]\!]_A$$

is invertible in $R'_A[\![\Lambda]\!]_A$ as desired. □

Remark 8.8. (a) It is known that certain localizations or completions of the affine Hecke algebra and degenerate affine Hecke algebra are isomorphic (see [Lus89, Thm. 9.3] and [Rou, §3.1.7]). Theorem 8.7 can be seen as an analogue of these results.

(b) Note that while Theorem 8.7 shows that all affine Hecke algebras (satisfying the hypotheses of the proposition) become isomorphic over appropriate rings, the isomorphism is *not* the naive one sending T_i^F to $T_i^{F'}$. Furthermore, the completion (with respect to the augmentation map) is crucial. No assertion is made regarding an isomorphism (even over $\mathbb{Q}$) of *truncated* versions. See Remark 6.5.

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