Involutions of a Clifford algebra induced by involutions of orthogonal group in characteristic 2

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Abstract

Among the involutions of a Clifford algebra, those induced by the involutions of the orthogonal group are the most natural ones. In this work, several basic properties of these involutions, such as the relations between their invariants, their occurrences and their decompositions are investigated.

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1 Introduction

The main object of this work is to study the involutions of a Clifford algebra induced by the involutions of the orthogonal group in characteristic two. For the case of characteristic $\neq 2$, these involutions were studied in [8] in connection with the periodicity of real Clifford algebras with involution. Also they were considered in [15] in connection with the Pfister Factor Conjecture, which was finally settled in [1]. In characteristic $\neq 2$, some properties of these involutions were also investigated in [9].

Our starting point is a general analysis of involutions of orthogonal group in characteristic 2. For the case of characteristic different from 2, the situation is much more straightforward as it is known that if τ is an involution in the orthogonal group $O(V, q)$ then there exists a decomposition $V = U \perp U^{\perp}$ with $\tau|_U = \mathrm{id}$ and $\tau|_{U^{\perp}} = -\mathrm{id}$. For the case where the characteristic of the base field is 2, the situation is quite more subtle. Wiitala in [17, Thm. 2] provides a characterization of involutions of orthogonal groups in characteristic 2. For every involution τ in $O(V, q)$, he proves that there exists a decomposition $V = W \perp V_1 \perp V_2 \perp \cdots$, where the restriction of τ to W is trivial and exactly one of the following is true: (1) each V_i is a two-dimensional subspace of V and the restriction of τ to V_i is a reflection; (2) each V_i is a four-dimensional subspace of V invariant under τ such that the fixed space of $\tau|_{V_i}$ is a totally isotropic space of dimension 2. In §3, several geometric interpretations of Wiitala's result are obtained and his characterization is complemented.

An initial observation for the study of the involutions is that every involution of a Clifford algebra $C(V)$ which leaves V invariant, is induced by an involution in the orthogonal group $O(V,q)$. Let $\tau \in O(V,q)$ be an involution and let J_τ be the involution induced by τ on $C(V)$. In §4, elementary investigations about these involutions such as the calculation of their type (that is if J_τ is of orthogonal or symplectic type) are done. Also some relations between invariants of J_τ (e.g., the type and the discriminant (in the sense of [7])) and invariants of τ (e.g., the Dickson invariant and the spinor norm) are obtained. Finally, in the case of the nontriviality of the Arf invariant, these questions for the restriction of J_τ to the even Clifford algebra $C_0(V)$ are investigated.

In §5, involutorial versions of some relations between the Clifford algebra and the even Clifford algebra (needed in the sequel) obtained by Mammone, Tignol and Wadsworth in [10] are presented. These relations link an orthogonal sum of a quadratic plane (i.e., a two-dimensional quadratic space) over F and an n-dimensional quadratic space over F to the Clifford algebra of an $(n+2)$-dimensional quadratic space.

In §6, analogous to [9], a geometric characterization of an involution of a quaternion algebra is obtained. It turns out that if (Q,σ) is a quaternion algebra with involution then there exists a quadratic plane $(\mathbb{E},\varphi)$ and an involution $\tau \in O(\mathbb{E},\varphi)$ such that $(Q,\sigma) \simeq (C(\mathbb{E}),J_\tau)$. Also σ is of symplectic (resp. orthogonal) type if and only if $\tau = \mathrm{id}$ (resp. τ is a reflection), see (6.1). In view of this result, it is deduced that as an algebra with involution, (Q,σ) is characterized by the spinor norm of τ. Then we show that a multiquaternion algebra with involution $(A,\sigma) := (Q_1,\sigma_1)\otimes\cdots\otimes(Q_n,\sigma_n)$ is isomorphic to the Clifford algebra with involution $(C(V),J_\tau)$ where $\tau \in O(V,q)$ is a suitable involution of a certain quadratic space (V,q). Also a more precise description of τ, depending on the type of σ, is given. In particular it is shown that if σ is of symplectic type then the involution $\tau \in O(V,q)$ can be chosen to be identity and if σ is of orthogonal type then τ can be chosen to be an orthogonal sum of reflections (see (6.3)).

Let (Q,σ) be a quaternion algebra with an involution of the second kind. In contrary to what happens in characteristic different from 2 (see [9, (6.10)]), (Q,σ) is not isomorphic to a Clifford algebra with involution $(C(V),J_\tau)$ for any (V,q) and τ (6.9). Instead of that it is shown that (Q,σ) is isomorphic to an even Clifford algebra with involution $(C_0(V),J_\tau)$ where (V,q) is a quadratic space of dimension 4 over the fixed field of σ restricted to the center of Q. More generally if K/F is a separable quadratic extension with nontrivial automorphism ρ, it is shown that the multiquaternion algebra $(A,\sigma) := (K,\rho)\otimes(Q_1,\sigma_1)\otimes\cdots\otimes(Q_n,\sigma_n)$ is isomorphic to the even Clifford algebra with involution $(C_0(V),J_\tau)$ for a suitable quadratic space (V,q) over F and an involution τ in $O(V,q)$. Furthermore it is shown that if $n \equiv 0 \mod 2$ then τ can be chosen to be identity and if $n \equiv 1 \mod 2$ then τ can be chosen to be a reflection (see (6.7)). In should be mentioned that in low dimensions (e.g., $n \leqslant 2$), these results, for arbitrary characteristic, are already included in [5].

The main object of §7 is to investigate the behavior of the involution J_τ on $C_0(V)$ in the case where (V,q) has trivial Arf invariant. Note that as in this case $C_0(V)$ is not central simple, one may wonder how does J_τ act on simple components of $C_0(V)$. Also one may want to see what is the relation between the spinor norm of τ and the discriminant of the restriction of J_τ to simple components of $C_0(V)$. In this section these questions and some other related questions similar to those investigated in the previous sections, are studied.

It is relevant to mention that in characteristic $\neq 2$, the behavior of J_τ on simple components of $C_0(V)$, even for the simplest case $\tau = \mathrm{id}$, has been of importance in many applications of Clifford algebras in the literature (see, e.g., [3], [16], [11], [13]).

In characteristic 2, when τ is arbitrary we are able to give a precise description of the action of J_τ on simple components of $C_0(V)$. It turns out that, depending on τ, the involution J_τ can also be a *switch* involution. See (7.8) and (7.11).

In a final appendix, a universal mapping property for the even Clifford algebra, analogous to [9, (3.3)], which is used to shorten some proofs in the paper is presented.

2 Preliminaries and terminology

Let F be a field of characteristic 2 and let V be a finite dimensional vector space over F. A *quadratic form* over F is a map $q : V \to F$ such that $q(\lambda u + \mu v) = \lambda^2 q(u) + \mu^2 q(v) + \lambda\mu B(u, v)$ for every $\lambda, \mu \in F$ and $u, v \in V$, where $B : V \times V \to F$ is a bilinear form over V. We have $B(u, v) = q(u) + q(v) + q(u + v)$ and so the bilinear form B is uniquely determined by q. In particular $B(v, v) = 0$ for every $v \in V$, i.e., B is an *alternating form.* Two vectors $u, v \in V$ are called *orthogonal* if $B(u, v) = 0$. The *radical* of V is defined as $\operatorname{rad} V = \{x \in V \mid B(x, y) = 0 \text{ for every } y \in V\}$. We say that q is *nondefective* if $\ker(q|_{\operatorname{rad} V}) = 0$. If the bilinear form B is nondegenerate, (V, q) is called *regular.* We call (V, q) or (V, q, B) a *quadratic space* if q is regular. Any quadratic space (V, q) is necessarily even dimensional and has a basis $\mathcal{B} = \{u_1, v_1, \cdots, u_n, v_n\}$ such that $B(u_i, v_i) = 1$ for each i and all other distinct pairs of $\mathcal{B}$ are orthogonal (see [14, p. 339]). Such a basis is called a *symplectic basis* of V. We denote by $[a, b]$ the two dimensional quadratic space (V, q) where $V = Fu + Fv$, $q(u) = a$, $q(v) = b$ and $B(u, v) = 1$. Also we call $\{u, v\}$ a *standard symplectic basis* of (V, q).

We say that q *represents* $a \in F$, if there is a nonzero vector $v \in V$ such that $q(v) = a$. The set of values in $F^\times = F \setminus \{0\}$ represented by q is denoted by $D_F(q)$. The quadratic space (V, q) is called *universal* if $D_F(q) = F^\times$. If (V, q, B) is a quadratic space and $a \in F^\times$, the *scaled* quadratic space $(V, a \cdot q, a \cdot B)$ is defined by $a \cdot q(v) = aq(v)$ for every $v \in V$.

An *isometry* between two quadratic spaces such as $(V_1, q_1) \simeq (V_2, q_2)$ is an isomorphism of vector spaces $\tau : V_1 \to V_2$ such that $q_2(\tau(v)) = q_1(v)$ for every $v \in V_1$. Similarly, an isometry between two bilinear spaces such as $(V_1, B_1) \simeq (V_2, B_2)$ is an isomorphism of vector spaces $\tau : V_1 \to V_2$ such that $B_2(\tau(u), \tau(v)) = B_1(u, v)$ for every $u, v \in V_1$. Note that $(V_1, q_1) \simeq (V_2, q_2)$ implies that $(V_1, B_1) \simeq (V_2, B_2)$ but the converse is not always true. The group of isometries of (V, q) is denoted by $O(V, q)$ and is called the *orthogonal group* of (V, q).

Let (V_1, q_1) and (V_2, q_2) be two quadratic spaces over F and define the quadratic form $q_1 \perp q_2$ over $V_1 \oplus V_2$ via $(q_1 \perp q_2)(v_1 + v_2) = q_1(v_1) + q_2(v_2)$. We denote this quadratic space by $(V_1 \perp V_2, q_1 \perp q_2)$ and we call it the *orthogonal* *sum* of (V_1, q_1) and (V_2, q_2). If (V_1, q_1) and (V_2, q_2) are quadratic spaces over F and τ_1 and τ_2 are isometries of (V_1, q_1) and (V_2, q_2), respectively, the *orthogonal sum* of τ_1 and τ_2 which is denoted by $\tau_1 \perp \tau_2$ is an isometry of $(V_1 \perp V_2, q_1 \perp q_2)$ defined by $(\tau_1 \perp \tau_2)(v_1 + v_2) = \tau_1(v_1) + \tau_2(v_2)$.

All F-algebras considered in this work are implicitly supposed to be associative. The center of an F-algebra A is denoted by $Z(A)$ and the group of invertible elements of A is denoted by A^*. If $u \in A$ is an invertible element, we use the notation $\mathrm{Int}(u)$ for the inner automorphism of A induced by u:

$$\mathrm{Int}(u) : A \to A, \qquad \mathrm{Int}(u)(x) = uxu^{-1}.$$

Let (V, q) be a quadratic space over a field F of characteristic 2 and let A be an F-algebra. A linear map $f : V \to A$ is called *compatible* with q if for every $x \in V$, we have $f(x)^2 = q(x) \cdot 1$. In particular if $\{v_1, \cdots, v_n\}$ is a basis of V, then the linear map f is compatible with q if and only if for every $1 \leqslant i \leqslant n$, we have $f(v_i)^2 = q(v_i) \cdot 1$ and for every $1 \leqslant i, j \leqslant n$, we have $f(v_i)f(v_j) + f(v_j)f(v_i) = B(v_i, v_j) \cdot 1$.

The Clifford algebra of a quadratic space (V, q) is denoted by $C(V, q)$, or $C(V)$ if no confusion arises, and we consider V as a subspace of $C(V)$. We have $uv = vu + B(u, v) \cdot 1$ for every $u, v \in V$. The *even Clifford algebra* of (V, q) is denoted by $C_0(V, q)$ or $C_0(V)$.

If τ is an isometry of (V, q), then τ induces an algebra automorphism I_τ of $C_0(V)$. The *Dickson invariant*

$$\mathrm{D} : O(V, q) \to \mathbb{Z}/2\mathbb{Z}$$

is defined by $\mathrm{D}(\tau) = 0 \in \mathbb{Z}/2\mathbb{Z}$, if $I_\tau|_{Z(C_0(V))} = \mathrm{id}$ and $\mathrm{D}(\tau) = 1 \in \mathbb{Z}/2\mathbb{Z}$ otherwise.

Consider the additive subgroup $\wp(F) = \{x + x^2 \,|x \in F\}$ of $(F, +)$. Let $\mathcal{B} = \{u_1, v_1, \cdots, u_n, v_n\}$ be a symplectic basis of (V, q). It can be shown that the class of $\sum_{i=1}^n q(u_i)q(v_i)$ in the quotient group $F/\wp(F)$, which is called the *Arf invariant* of (V, q) and is denoted by $\Delta(V, q)$ or $\Delta(q)$, is independent of the choice of $\mathcal{B}$ (see [14, Ch. 9, (4.2)]). In particular the Arf invariant of the form $[a, b]$ is $ab + \wp(F)$.

If (V, q) is a quadratic space, a nonzero vector $v \in V$ is called *isotropic* if $q(v) = 0$ and *anisotropic* otherwise. A subspace W of V is called *isotropic* if there is an isotropic vector in W, otherwise W is called *anisotropic*. Every isotropic quadratic space is universal, see [12, Ch. I, (4.5)]. The subspace W is *totally isotropic* if $q(w) = 0$ for all $w \in W$. If W and W' are two maximal totally isotropic subspaces of V, then one has $\dim W = \dim W'$ (see [4, (12.11)]). The dimension of a maximal totally isotropic subspace of V is called the *Witt index* of (V, q). By a *quadratic plane* we mean a 2-dimensional quadratic space $(\mathbb{E}, \varphi)$. Every quadratic plane has a basis $\{u, v\}$ such that $B(u, v) = 1$. For such a basis we have $(uv)^2 = uv + q(u)q(v) \cdot 1$, where this equality is considered in the Clifford algebra of $(\mathbb{E}, \varphi)$. Two quadratic planes $(\mathbb{E}_1, \varphi_1)$ and $(\mathbb{E}_2, \varphi_2)$ are isometric as bilinear spaces but not necessarily isometric as quadratic spaces. It is known that every quadratic space is an orthogonal sum of quadratic planes (see [12, Ch. I, (4.3)]). Up to isometry, there is a unique isotropic quadratic plane $(\mathbb{E}, \varphi)$, which is called a *hyperbolic plane*. For every hyperbolic plane $(\mathbb{E}, \varphi)$ we have $\varphi \simeq [1, 0]$. We denote a hyperbolic plane by $\mathbb{H}$. A quadratic space (V, q) is called *hyperbolic* if it is an orthogonal sum of hyperbolic planes. Every hyperbolic space has trivial Arf invariant.

An isometry τ of (V, q) is called an *involution* in $O(V, q)$, if $\tau^2 = \mathrm{id}$. Let $u \in V$ be an anisotropic vector and define a linear transformation $\tau_u : V \to V$ via $\tau_u(v) = v + \frac{B(v,u)}{q(u)}u$. It is easy to see that τ_u is an involution in $O(V, q)$. Note that if $B(u, v) = 0$ then $\tau_u(v) = v$, also $\tau_u(u) = u$. We call τ_u the (*orthogonal*) *reflection along* u. In the literature, these isometries are also called *orthogonal transvections*. It is easy to see that if $u \in V$ is an anisotropic vector, then the restriction of the inner automorphism $\mathrm{Int}(u)$ of $C(V)$ to V is just the reflection τ_u. If (V, q) is a quadratic space, then by a theorem due to Dieudonné, every isometry σ of (V, q) is a product of reflections except for the case that $\dim V = 4$, $|F| = 2$ and the Witt index of (V, q) is 2 (see [4, (14.16)]). If $\sigma = \tau_{u_1} \cdots \tau_{u_k} \in O(V, q)$, then it can be shown that the class of $\Pi_{i=1}^{k} q(u_i)$ in the quotient group $F^\times/F^{\times 2}$, which is called the *spinor norm* of σ and is denoted by $\theta(\sigma)$, is independent of the choice of the reflections τ_{u_i}'s, see [4, (14.17)]. In particular if $\sigma = \tau_u$ is a reflection, then the spinor norm of τ_u is the class of $q(u)$ in the quotient group $F^\times/F^{\times 2}$.

For an F-central simple algebra A the integer $\sqrt{\dim_F A}$ is called the *degree* of A. An *involution* on A is an anti-automorphism σ of A such that $\sigma^2 = \mathrm{id}$. Let (A, σ) be an F-central simple algebra with involution. Consider the subfield K of F consisting of the elements a of F such that $\sigma(a) = a$. It is known that either $F = K$, or F/K is a separable field extension of degree 2. In the first case we say σ is of *the first kind* and in the second case we say σ is of *the second kind* (or of *unitary type*).

Let V be a finite dimensional vector space over a field F and let B be a nondegenerate bilinear form over V. The unique involution σ_B on $\mathrm{End}_F(V)$, characterized by $B(x, f(y)) = B(\sigma_B(f)(x), y)$ for every $x, y \in V$ and $f \in \mathrm{End}_F(V)$, is called the *adjoint involution* of $\mathrm{End}_F(V)$ with respect to B.

Let (A, σ) be an F-central simple algebra of degree n with involution of the first kind. If L is a splitting field of A and V is an n-dimensional vector space over L, then there is a nondegenerate symmetric or antisymmetric bilinear form B over V such that $(A_L, \sigma_L) \simeq (\mathrm{End}_L(V), \sigma_B)$ (see [6, (2.1)]). An involution σ of the first kind is said to be of *symplectic type* if for any splitting field L of A and any isomorphism $(A_L, \sigma_L) \simeq (\mathrm{End}_L(V), \sigma_B)$, the bilinear form B is alternating. Otherwise we say that σ is of *orthogonal type*. For short, we say that the central simple algebra with involution (A, σ) is of orthogonal (resp. symplectic) type if σ is of orthogonal (resp. symplectic) type.

A quaternion algebra over a field F of characteristic 2, is a 4-dimensional F-algebra Q with a basis $\{1, i, j, k\}$ subject to the relations $i^2 + i = \alpha \in F$, $j^2 = \beta \in F^\times$, $ij = k$ and $ij + ji = j$. Note that these relations imply that $k^2 = (ij)^2 = \alpha\beta$. This algebra is denoted by $[\alpha, \beta)$ and we call $\{1, i, j, k\}$ a *standard basis* of Q. The map $\gamma : Q \to Q$ defined by $\gamma(a + bi + cj + dk) = a + b(i + 1) + cj + dk$ is an involution of the first kind on Q which is called *the canonical involution of* Q. The canonical involution γ is the unique involution on Q such that $\gamma(x)x \in F$ for every $x \in Q$, see [14, Ch.8, (11.2)].

If (A, σ) is an F-central simple algebra with involution, the set of alternating elements in A is defined as follows:

$$\mathrm{Alt}(A, \sigma) = \{a - \sigma(a) | a \in A\}.$$

If A is of even degree $n = 2m$ over F and σ is of orthogonal type, the *determinant* of σ is the square class of the reduced norm of any alternating unit:

$$\det \sigma = \mathrm{Nrd}_A(a)F^{\times 2} \in F^\times/F^{\times 2} \quad \text{for } a \in \mathrm{Alt}(A,\sigma) \cap A^*,$$

and the *discriminant* of σ is the signed determinant:

$$\mathrm{disc}\,\sigma = (-1)^m \det \sigma \in F^\times/F^{\times 2}.$$

Note that by [7, Lem. 2.1] or [6, (7.1)], the determinant and the discriminant are well-defined. Also if $\mathrm{char}\,F = 2$ the discriminant and the determinant coincide.

3 Wiitala's characterization of involutions of orthogonal groups in characteristic 2

Definition 3.1. Let (V, q, B) be a quadratic space over a field F of characteristic 2 and let τ be an involution in $O(V, q)$. We call the subspace $\mathrm{Fix}(V, \tau) = \{v \in V \mid \tau v = v\}$, *the fixed subspace of* τ.

We state here the following definition from [17].

Definition 3.2. Let $(\mathbb{A}, q)$ be a quadratic space of dimension 4 over a field F of characteristic 2. An involution $\tau \in O(\mathbb{A}, q)$ is called an *interchange isometry* if the fixed subspace $\mathrm{Fix}(\mathbb{A}, \tau)$ is a two dimensional totally isotropic space.

Remark 3.3. In [17], such maps are called "basic null involutions". In this work have preferred to use the term interchange isometry. Also in view of (3.6) below, this naming is not unreasonable.

Lemma 3.4. *Let $(\mathbb{A}, q, B)$ be a quadratic space of dimension 4 over a field F of characteristic 2. If τ is an isometry of $(\mathbb{A}, q)$, then τ is an interchange isometry if and only if there is a basis $\{w, x, y, z\}$ of $\mathbb{A}$ such that $\{x, w\}$ is a basis of $\mathrm{Fix}(\mathbb{A}, \tau)$, $q(w) = q(x) = 0$, $\tau(y) = x + y$, $\tau(z) = w + z$, $B(x, z) = B(w, y) = 1$ and all other pairs of vectors from this basis are orthogonal. In particular if τ is an interchange isometry of $(\mathbb{A}, q)$, then $(\mathbb{A}, q)$ is necessarily hyperbolic.*

Proof. See [17, Lem. 2]. ∎

Remark 3.5. If $(\mathbb{A}, q, B)$ is a 4-dimensional quadratic space over a field F of characteristic 2 and τ is an interchange isometry of $(\mathbb{A}, q)$, the basis $\{w, x, y, z\}$ of $\mathbb{A}$ with the properties stated in (3.4), can be chosen with the additional property that $q(y) = q(z) = 0$; in fact set $\alpha = q(y)$, $\beta = q(z)$, $y' = y + \alpha w$ and $z' = z + \beta x$. It is easy to see that the basis $\{w, x, y', z'\}$ has the desired property.

Proposition 3.6. *Let $(\mathbb{A}, q, B)$ be a quadratic space of dimension 4 over a field F of characteristic 2 and let τ be an isometry of $(\mathbb{A}, q)$. The following statements are equivalent:* (1) *τ is an interchange isometry.* (2) *There exist hyperbolic planes $\mathbb{H}_1$ and $\mathbb{H}_2$ in $\mathbb{A}$ such that $\mathbb{A} = \mathbb{H}_1 \perp \mathbb{H}_2$, $\tau(\mathbb{H}_1) = \mathbb{H}_2$ and $\tau(\mathbb{H}_2) = \mathbb{H}_1$.* (3) *There exist quadratic planes $\mathbb{E}_1$ and $\mathbb{E}_2$ in $\mathbb{A}$ such that $\mathbb{A} = \mathbb{E}_1 \perp \mathbb{E}_2$, $\tau(\mathbb{E}_1) = \mathbb{E}_2$ and $\tau(\mathbb{E}_2) = \mathbb{E}_1$.*

Proof. (1) ⇒ (2): Let $\{w, x, y, z\}$ be the basis of $\mathbb{A}$ with the properties stated in (3.5) and set $u_1 = y$, $u_2 = x + y = \tau(u_1)$, $v_1 = w + z$ and $v_2 = z = \tau(v_1)$. Then

$$\begin{aligned} B(u_1, u_2) &= B(y, x + y) = 0, \\ B(u_1, v_1) &= B(y, w + z) = B(y, w) + B(y, z) = 1 + 0 = 1, \\ B(u_1, v_2) &= B(y, z) = 0, \\ B(v_1, u_2) &= B(\tau(v_2), \tau(u_1)) = B(v_2, u_1) = 0, \\ B(v_1, v_2) &= B(w + z, z) = 0, \\ B(u_2, v_2) &= B(\tau(u_1), \tau(v_1)) = B(u_1, v_1) = 1. \end{aligned}$$

As $q(u_1) = q(y) = 0$ and $q(v_2) = q(z) = 0$, the subspaces $Fu_1 + Fv_1$ and $Fu_2 + Fv_2$ of $\mathbb{A}$ are hyperbolic. Let $\mathbb{H}_1 = Fu_1 + Fv_1$ and $\mathbb{H}_2 = \tau(\mathbb{H}_1) = Fu_2 + Fv_2$. Then $\mathbb{H}_1 = \mathbb{H}_2^{\perp}$. Finally as $\tau^2 = \mathrm{id}$, we obtain $\tau(\mathbb{H}_2) = \mathbb{H}_1$.

(2) ⇒ (3) is immediate.

(3) ⇒ (1): Set $u_2 = \tau(u_1)$ and $v_2 = \tau(v_1)$. Then $\mathbb{E}_2 = Fu_2 + Fv_2$. Let $v = au_1 + bu_2 + cv_1 + dv_2 \in \mathrm{Fix}(\mathbb{A}, \tau)$, where $a, b, c, d \in F$. Then $au_2 + bu_1 + cv_2 + dv_1 = au_1 + bu_2 + cv_1 + dv_2$. Hence $a = b$ and $c = d$, i.e., $v = a(u_1 + u_2) + c(v_1 + v_2)$. On the other hand the subspace $F(u_1 + u_2) + F(v_1 + v_2)$ of $\mathbb{A}$ is fixed by τ. So $\mathrm{Fix}(\mathbb{A}, \tau) = F(u_1 + u_2) + F(v_1 + v_2)$. Finally we have $q(u_1 + u_2) = q(v_1 + v_2) = B(u_1 + u_2, v_1 + v_2) = 0$, i.e., $\mathrm{Fix}(\mathbb{A}, \tau)$ is a 2-dimensional totally isotropic space. So τ is an interchange isometry. □

Theorem 3.7. *Let (V, q, B) be a quadratic space over a field F of characteristic 2 and let τ be an isometry of (V, q). Then τ is an involution in $O(V, q)$ if and only if there is a decomposition of V into regular subspaces $V = W \perp \mathbb{E}_1 \perp \cdots \perp \mathbb{E}_r \perp \mathbb{A}_1 \perp \cdots \perp \mathbb{A}_s$ such that:*

(*a*) *r and s are nonnegative integers and for every i,* $\dim \mathbb{E}_i = 2$ *and* $\dim \mathbb{A}_i = 4$.

(*b*) *τ maps each subspace W, $\mathbb{E}_i$ and $\mathbb{A}_i$ into itself.*

(*c*) $\tau|_W = \mathrm{id}$.

(*d*) *$\tau|_{\mathbb{E}_i}$ is a reflection.*

(*e*) *$\tau|_{\mathbb{A}_i}$ is an interchange isometry.*

Proof. See [17, Thm. 1]. □

Definition 3.8. Let (V, q, B) be a quadratic space over a field F of characteristic 2 and let τ be an involution in $O(V, q)$. We call the decomposition $V = W \perp \mathbb{E}_1 \perp \cdots \perp \mathbb{E}_r \perp \mathbb{A}_1 \perp \cdots \perp \mathbb{A}_s$ in (3.7), a *Wiitala decomposition* of (V, τ) and we call the subspace W, a *maximal fixed orthogonal summand* of (V, τ); in view of (3.9) below, this naming is justified. For every Wiitala decomposition $V = W \perp \mathbb{E}_1 \perp \cdots \perp \mathbb{E}_r \perp \mathbb{A}_1 \perp \cdots \perp \mathbb{A}_s$ of (V, τ), we have a decomposition $\tau = \mathrm{id}_W \perp \tau_1 \perp \cdots \perp \tau_r \perp \tau'_1 \perp \cdots \perp \tau'_s$ which is called a *Wiitala decomposition* of τ where τ_i is a reflection of $\mathbb{E}_i$ for $i = 1, \cdots, r$ and τ'_i is an interchange isometry of $\mathbb{A}_i$ for $i = 1, \cdots, s$ (i.e., the integers r and s are the respective number of reflections and interchange isometries in the above Wiitala decomposition of τ).

Proposition 3.9. *Let (V,q,B) be a quadratic space over a field F of characteristic 2 and let τ be an involution in $O(V,q)$. Let $U\subseteq V$ be an orthogonal summand of V such that $\tau|_U=\mathrm{id}$. Then there exists a maximal fixed orthogonal summand W of (V,τ) such that $U\subseteq W$. In particular $\dim W$ is uniquely determined by τ.*

Proof. Let U' be an orthogonal complement of U, i.e., $V=U\perp U'$. Then $\tau|_{U'}$ is an involution in $O(U',q|_{U'})$. Let $U'=W_1\perp\mathbb{E}_1\perp\cdots\perp\mathbb{E}_r\perp\mathbb{A}_1\perp\cdots\perp\mathbb{A}_s$ be a Wiitala decomposition of $(U',\tau|_{U'})$ and set $W=U\perp W_1$. Then $V=W\perp\mathbb{E}_1\perp\cdots\perp\mathbb{E}_r\perp\mathbb{A}_1\perp\cdots\perp\mathbb{A}_s$ is a Wiitala decomposition of (V,τ). So W is a maximal fixed orthogonal summand of (V,τ) and $U\subseteq W$. $\square$

Remark 3.10. Let (V,q,B) be a quadratic space over a field F of characteristic 2 and let τ be an involution in $O(V,q)$. Let W be a maximal fixed orthogonal summand of (V,τ). Then W is not necessarily unique up to isometry. For example let $(\mathbb{E}_i,\varphi_i)\simeq[\alpha,\beta]$, $i=1,2$, be an anisotropic quadratic plane over F with standard symplectic basis $\{u_i,v_i\}$. Set $(V,q)=(\mathbb{E}_1,\varphi_1)\perp(\mathbb{E}_2,\varphi_2)$ and $\tau=\mathrm{id}_{\mathbb{E}_1}\perp\tau_{u_2}$ where τ_{u_2} is the reflection of $\mathbb{E}_2$ along u_2. Then τ is an involution in $O(V,q)$ and $W_1=\mathbb{E}_1$ is a maximal fixed orthogonal summand of (V,τ). Now set $W=F(u_1+u_2)+Fv_1$ and $\mathbb{E}=Fu_2+F(v_1+v_2)$. Then $V=W\perp\mathbb{E}$ and $\tau|_W=\mathrm{id}$. Also $\tau|_{\mathbb{E}}$ is the reflection along u_2, because $\tau(u_2)=u_2=\tau_{u_2}(u_2)$ and $\tau(v_1+v_2)=v_1+(v_2+\frac{1}{q(u_2)}u_2)=(v_1+v_2)+\frac{B(v_1+v_2,u_2)}{q(u_2)}u_2=\tau_{u_2}(v_2)$. So $V=W\perp\mathbb{E}$ is a Wiitala decomposition of (V,τ) and W is a maximal fixed orthogonal summand of (V,τ). Now we have $q(u_1+u_2)=\alpha+\alpha=0$, so W is isotropic and $W_1\not\simeq W$.

Definition 3.11. Let (V,q,B) be a quadratic space over a field F of characteristic 2 and let τ be an involution in $O(V,q)$. Let $V=W\perp\mathbb{E}_1\perp\cdots\perp\mathbb{E}_r\perp\mathbb{A}_1\perp\cdots\perp\mathbb{A}_s$ be a Wiitala decomposition of (V,τ). We call a symplectic basis

$$\{u_1,v_1,\cdots,u_t,v_t,u'_1,v'_1,\cdots,u'_r,v'_r,u''_1,v''_1,\cdots,u''_{2s},v''_{2s}\}$$

of V a *τ-symplectic basis* of (V,τ) if it satisfies the following conditions:

(a) The set $\{u_1,v_1,\cdots,u_t,v_t\}$ is a symplectic basis of the subspace W (so $t=\frac{1}{2}\dim W$).

(b) For $1\leqslant i\leqslant r$, $\{u'_i,v'_i\}$ is a symplectic basis of the subspace $\mathbb{E}_i$ such that $q(u'_i)\neq 0$, $\tau(u'_i)=u'_i$ and $\tau(v'_i)=v'_i+\frac{1}{q(u'_i)}u'_i$ (i.e., $\tau|_{\mathbb{E}_i}=\tau_{u'_i}$).

(c) For $1\leqslant i\leqslant s$, $\{u''_{2i-1},v''_{2i-1},u''_{2i},v''_{2i}\}$ is a basis of the subspace $\mathbb{A}_i$ such that the subspace $Fu''_{2i-1}+Fv''_{2i-1}$ is perpendicular to the subspace $Fu''_{2i}+Fv''_{2i}$, $\tau(u''_{2i-1})=u''_{2i}$, $\tau(u''_{2i})=u''_{2i-1}$, $\tau(v''_{2i-1})=v''_{2i}$ and $\tau(v''_{2i})=v''_{2i-1}$ (i.e., $\tau|_{\mathbb{A}_i}$ is an interchange isometry).

For simplicity, we denote the above τ-symplectic basis of V by $\{u_1,v_1,\cdots,u_n,v_n\}$ where $n=t+r+2s$ and for $t+1\leqslant i\leqslant t+r$, we have $u_i=u'_{i-t}$ and $v_i=v'_{i-t}$ and for $t+r+1\leqslant i\leqslant t+r+2s=n$, we have $u_i=u''_{i-t-r}$ and $v_i=v''_{i-t-r}$.

Remark 3.12. Let $(\mathbb{E},\varphi)$ be a quadratic plane over a field F of characteristic 2 and let $\tau\neq\mathrm{id}$ be an involution in $O(\mathbb{E},\varphi)$. Then τ is a reflection (see [17, Lem. 1]). In particular for every quadratic plane $(\mathbb{E},\varphi)$ and every involution $\tau\in O(\mathbb{E},\varphi)$, there exists an anisotropic vector $u\in\mathbb{E}$ such that $\tau(u)=u$.

Lemma 3.13. *Let $(\mathbb{E}, \varphi)$ be a quadratic plane over a field F of characteristic 2 and let τ be an involution in $O(\mathbb{E}, \varphi)$. Let $t = \dim \mathrm{Fix}(\mathbb{E}, \tau) - \frac{1}{2}\dim \mathbb{E}$ and let $u \in \mathbb{E}$ be an anisotropic vector such that $\tau(u) = u$. Then for every $x \in \mathbb{E}$, we have $\tau(x) = x + (t+1)B(x,u)u^{-1}$.*

Proof. If $\tau = \mathrm{id}$, then $t = 1$ and $\tau(x) = x = x + (t+1)B(x,u)u^{-1}$. Otherwise by (3.12), we have $\tau = \tau_u$, $t = 0$ and $\tau(x) = x + B(x,u)u^{-1} = x + (t+1)B(x,u)u^{-1}$. □

Lemma 3.14. *Let (V, q, B) be a quadratic space over a field F of characteristic 2 and let τ be an involution in $O(V, q)$.*

- (a) *If $\dim V = 2$ then we have $\tau = \mathrm{id}$ if and only if for every $v \in V$ we have $B(v, \tau(v)) = 0$.*
- (b) *If $\dim V = 4$ and $\tau \neq \mathrm{id}$, then τ is an interchange isometry if and only if for every $v \in V$, we have $B(v, \tau(v)) = 0$.*
- (c) *We have $B(v, \tau(v)) = 0$ for every $v \in V$ if and only if there is no reflection in any Wiitala decomposition of τ.*

Proof. (a) If $\tau = \mathrm{id}$, the conclusion is evident. Conversely suppose that $\tau \neq \mathrm{id}$. By (3.12) there exists an anisotropic vector $u \in V$ such that τ is the reflection along u, i.e., $\tau = \tau_u$. Let $v \in V$ with $B(u, v) = 1$. Then $B(v, \tau(v)) = B(v, v + \frac{1}{q(u)}u) = \frac{1}{q(u)} \neq 0$.

(b) Let τ be an interchange isometry. Choose a basis $\{w, x, y, z\}$ of V with the properties stated in (3.4). Let $v \in V$ and write $v = aw + bx + cy + dz$ where $a, b, c, d \in F$. We have

$$\begin{aligned} B(v, \tau(v)) &= B(aw + bx + cy + dz, aw + bx + c(y + x) + d(z + w)) \\ &= B(aw + bx + cy + dz, (a + d)w + (b + c)x + cy + dz) \\ &= ac + bd + c(a + d) + d(b + c) = 0, \end{aligned}$$

and we are done. Conversely suppose that $B(v, \tau(v)) = 0$ for every $v \in V$. As $\tau \neq \mathrm{id}$, if τ is not an interchange isometry then by (3.7) there is necessarily a quadratic plane $\mathbb{E} \subseteq V$ such that $\tau|_{\mathbb{E}}$ is a reflection. But (a) implies that there exists $v \in \mathbb{E}$ such that $B(v, \tau(v)) \neq 0$, this contradicts the hypothesis.

(c) If there is a reflection in a Wiitala decomposition of τ, then there exists $v \in V$ such that $B(v, \tau(v)) \neq 0$. Conversely suppose that there is no reflection in any Wiitala decomposition of τ. By (3.7), we can write $V = W \perp \mathbb{A}_1 \perp \cdots \perp \mathbb{A}_s$, where $\tau|_W = \mathrm{id}$ and $\tau|_{\mathbb{A}_i}$ is an interchange isometry. For every $v \in V$ we can write $v = v_0 + v_1 + \cdots + v_s$ where $v_0 \in W$ and $v_i \in \mathbb{A}_i$, $i = 1, \ldots, s$. Then using (b) we obtain

$$\begin{aligned} B(v, \tau(v)) &= B(v_0, \tau(v_0)) + B(v_1, \tau(v_1)) + \cdots + B(v_s, \tau(v_s)) \\ &= 0 + 0 + \cdots + 0 = 0. \end{aligned}$$

□

Proposition 3.15. *Let (V, q, B) be a quadratic space of dimension 6 over a field F of characteristic 2 and let τ be an involution in $O(V, q)$. Suppose that V has a decomposition of the form $V = \mathbb{E} \perp \mathbb{A}$ where $\mathbb{E}$ is a quadratic plane* *in V, $\tau|_{\mathbb{E}}$ is a reflection and $\tau|_{\mathbb{A}}$ is an interchange isometry. Then there is a decomposition of V of the form $V = \mathbb{E}_1 \perp \mathbb{E}_2 \perp \mathbb{E}_3$ such that $\mathbb{E}_1$, $\mathbb{E}_2$ and $\mathbb{E}_3$ are quadratic planes in V and for every $i = 1, 2, 3$, $\tau|_{\mathbb{E}_i}$ is a reflection.*

Proof. See [17, Thm. 2]. □

Theorem 3.16. *Let (V, q, B) be a quadratic space over a field F of characteristic 2 and let τ be an involution in $O(V, q)$. Then there is a decomposition of (V, τ) in exactly one of the following forms:*

- (a) *$V = W \perp \mathbb{E}_1 \perp \cdots \perp \mathbb{E}_r$ where $\tau|_W = \mathrm{id}$, $\mathbb{E}_i$ is a quadratic subplane of V and $\tau|_{\mathbb{E}_i}$ is a reflection for every $1 \leqslant i \leqslant r$.*
- (b) *$V = W \perp \mathbb{A}_1 \perp \cdots \perp \mathbb{A}_s$ where $\tau|_W = \mathrm{id}$, $\dim \mathbb{A}_i = 4$ and $\tau|_{\mathbb{A}_i}$ is an interchange isometry for every $1 \leqslant i \leqslant s$.*

Furthermore the type of decomposition occurring and the numbers r and s are uniquely determined by τ.

Proof. See [17, Thm. 2]. □

Definition 3.17. Let (V, q, B) be a quadratic space over a field F of characteristic 2 and let τ be an involution in $O(V, q)$. We say that τ is of *reflectional* (resp. *interchanging*) *kind* if V has a decomposition of the form (a) (resp. (b)) in (3.16).

The following result is a restatement of (3.14 (c)):

Corollary 3.18. *Let (V, q, B) be a quadratic space over a field F of characteristic 2 and let τ be an involution in $O(V, q)$. Then τ is of interchanging kind if and only if $B(v, \tau(v)) = 0$ for every $v \in V$.*

Proposition 3.19. *Let (V, q) be a quadratic space over a field F of characteristic 2 and let τ be an involution in $O(V, q)$. Let r be the number of reflections in any Wiitala decomposition of (V, τ). Then $r \equiv \dim \mathrm{Fix}(V, \tau) \mod 2$ and $\dim \mathrm{Fix}(V, \tau) = \frac{1}{2} \dim W + \frac{1}{2} \dim V$ where W is a maximal fixed orthogonal summand of (V, τ). In particular $\dim \mathrm{Fix}(V, \tau) \geqslant \frac{1}{2} \dim V$. Also we have $\dim \mathrm{Fix}(V, \tau) = \frac{1}{2} \dim V$ if and only if $W = 0$.*

Proof. Let $V = W \perp \mathbb{E}_1 \perp \cdots \perp \mathbb{E}_r \perp \mathbb{A}_1 \perp \cdots \perp \mathbb{A}_s$ be a Wiitala decomposition of (V, τ). Then for $1 \leqslant i \leqslant r$, $\tau|_{\mathbb{E}_i}$ is a reflection, so the fixed subspace of $\tau|_{\mathbb{E}_i}$ is one dimensional. Similarly for $1 \leqslant i \leqslant s$, $\tau|_{\mathbb{A}_i}$ is an interchange isometry, so the fixed subspace of $\tau|_{\mathbb{A}_i}$ is two dimensional. So the fixed subspace of the restriction of τ to $\mathbb{E}_1 \perp \cdots \perp \mathbb{E}_r \perp \mathbb{A}_1 \perp \cdots \perp \mathbb{A}_s$ has dimension $r + 2s$ and we obtain $\dim \mathrm{Fix}(V, \tau) = r + 2s + \dim W = (r + 2s + \frac{1}{2} \dim W) + \frac{1}{2} \dim W = \frac{1}{2} \dim V + \frac{1}{2} \dim W$. Finally since W is a regular subspace of v, $\dim W$ is even, so $r \equiv \dim \mathrm{Fix}(V, \tau) \mod 2$. □

Remark 3.20. Let (V, q, B) be a quadratic space over a field F of characteristic 2 and let τ be an involution in $O(V, q)$. If $\dim \mathrm{Fix}(V, \tau) = \frac{1}{2} \dim V$ and τ is of interchanging kind, then by (3.19), (V, τ) has trivial maximal fixed orthogonal summand. So by (3.16), $V = \mathbb{A}_1 \perp \cdots \perp \mathbb{A}_s$. By (3.6), V is a hyperbolic space. In particular the Arf invariant of q is trivial. If τ is of reflectional kind, depending on (V, q), $\Delta(q)$ can be trivial or nontrivial. For example if $V = \mathbb{H}$ and τ is a reflection of V, then $\Delta(q)$ is trivial, while if $V = \mathbb{E}$ is the quadratic plane $[1, \delta]$ with $\delta \in F \backslash \wp(F)$ and τ is a reflection of V, then $\Delta(q) = \delta + \wp(F) \in F/\wp(F)$ is nontrivial.

4 Generalities on some natural involutions of a Clifford algebra

Remark 4.1. Let (V, q) be a quadratic space over a field F of characteristic 2 and let $C(V)$ be the Clifford algebra of (V, q). If $\tau : V \to V$ is an involution in $O(V, q)$, then τ induces an involution J^q_τ on $C(V)$ such that $J^q_\tau(v) = \tau(v)$ for every $v \in V$. Also if σ is an involution on $C(V)$ such that $\sigma(V) = V$, then $\tau = \sigma|_V$ is involution in $O(V, q)$. We call J^q_τ the involution induced by τ on $C(V)$ and we usually abbreviate J^q_τ by J_τ.

Remark 4.2. Let $(\mathbb{E}, \varphi)$ be a quadratic plane over a field F of characteristic 2 and let $\tau = \tau_u$ be a reflection along some anisotropic vector $u \in \mathbb{E}$. Extend $\{u\}$ to a symplectic basis $\{u, v\}$ of $\mathbb{E}$. Consider the involution J_τ on $C(\mathbb{E})$. We have $J_\tau(u) = u$, $J_\tau(v) = v + \frac{1}{q(u)}u$ and $J_\tau(uv) = \tau(v)\tau(u) = (v + \frac{1}{q(u)}u)u = vu + 1 = uv$. So if $x = a + bu + cv + duv \in C(\mathbb{E})$ where $a, b, c, d \in F$, then we get $x - J_\tau(x) = \frac{c}{q(u)}u$, i.e., $\mathrm{Alt}(C(\mathbb{E}), J_\tau) = Fu$. In particular we have $x - J_\tau(x) \in F$ if and only if $x = J_\tau(x)$.

Proposition 4.3. *Let $(\mathbb{E}, \varphi, B)$ be a quadratic plane over a field F of characteristic 2 and let τ be an involution in $O(\mathbb{E}, \varphi)$. The involution J_τ on $C(\mathbb{E})$ is of symplectic type if and only if $\tau = \mathrm{id}$ if and only if $\mathrm{D}(\tau) = 0 \in \mathbb{Z}/2\mathbb{Z}$. In other words J_τ is of orthogonal type if and only if τ is a reflection.*

Proof. Suppose that $\tau = \mathrm{id}$ and $\{x, y\}$ is a basis of $\mathbb{E}$ such that $B(x, y) = 1$. Then $J_\tau(xy) = \tau(y)\tau(x) = yx = xy + 1$, so $1 \in \mathrm{Alt}(C(\mathbb{E}), J_\tau)$ and by [6, (2.6)], J_τ is of symplectic type.

Conversely suppose that $\tau \neq \mathrm{id}$. Then by (3.12), τ is a reflection along some anisotropic vector $u \in \mathbb{E}$, i.e., $\tau = \tau_u$. By (4.2), $\mathrm{Alt}(C(\mathbb{E}), J_\tau) = Fu$, hence $1 \notin \mathrm{Alt}(C(\mathbb{E}), J_\tau)$. So by [6, (2.6)], J_τ is of orthogonal type.

This proves the first part of the proposition. For the second part, if $\tau \neq \mathrm{id}$ is an involution in $O(\mathbb{E}, \varphi)$, then τ is a reflection and for any reflection τ we have $\mathrm{D}(\tau) = 1 \in \mathbb{Z}/2\mathbb{Z}$ (see [14, Ch. 9, (4.11)]). □

Lemma 4.4. *Let (V_1, q_1) and (V_2, q_2) be quadratic spaces over a field F of characteristic 2. If τ_1 and τ_2 are involutions in $O(V_1, q_1)$ and $O(V_2, q_2)$, respectively, then $(C(V_1), J_{\tau_1}) \otimes (C(V_2), J_{\tau_2}) \simeq (C(V_1 \perp V_2), J_{\tau_1 \perp \tau_2})$.*

Proof. It is easy to see that the linear map $\varphi : V_1 \perp V_2 \to C(V_1) \otimes C(V_2)$ defined by $\varphi(v_1 + v_2) = v_1 \otimes 1 + 1 \otimes v_2$ where $v_1 \in V_1$ and $v_2 \in V_2$ can be extended to an isomorphism $\varphi : C(V_1 \perp V_2) \simeq C(V_1) \otimes C(V_2)$ (cf. [4, p. 120]). So it remains to show that $\varphi \circ J_{\tau_1 \perp \tau_2} = (J_{\tau_1} \otimes J_{\tau_2}) \circ \varphi$. It is enough to check that $\varphi \circ J_{\tau_1 \perp \tau_2}(w) = (J_{\tau_1} \otimes J_{\tau_2}) \circ \varphi(w)$ for every $w \in V_1 \perp V_2$. Write $w = v_1 + v_2$ where $v_1 \in V_1$ and $v_2 \in V_2$, then

$$\begin{aligned}
\varphi \circ J_{\tau_1 \perp \tau_2}(w) &= \varphi \circ J_{\tau_1 \perp \tau_2}(v_1 + v_2) = \varphi(\tau_1(v_1) + \tau_2(v_2)) \\
&= \tau_1(v_1) \otimes 1 + 1 \otimes \tau_2(v_2) = (J_{\tau_1} \otimes J_{\tau_2})(v_1 \otimes 1 + 1 \otimes v_2) \\
&= (J_{\tau_1} \otimes J_{\tau_2}) \circ \varphi(v_1 + v_2) = (J_{\tau_1} \otimes J_{\tau_2}) \circ \varphi(w).
\end{aligned}$$

□

Lemma 4.5. *Let (V, q) be a quadratic space over a field F of characteristic 2 and let τ be an involution in $O(V, q)$. Then for every $a \in F^\times$ we have $(C_0(V, q), J^q_\tau) \cong (C_0(V, a \cdot q), J^{a \cdot q}_\tau)$.*

Proof. Define a bilinear map $\psi : V \times V \to C_0(V, a \cdot q)$ via $\psi(x,y) = a^{-1} \cdot xy$ for every $x, y \in V$. For every $x, y, z \in V$ we have $\psi(x,y)\psi(y,z) = (a^{-1} \cdot xy)(a^{-1} \cdot yz) = a^{-2}aq(y) \cdot xz = a^{-1}q(y) \cdot xz = q(y) \cdot \psi(x,z)$ and $\psi(x,x) = a^{-1} \cdot x^2 = a^{-1}a \cdot q(x) = q(x) \cdot 1$. So ψ is an even Clifford map and by (A.5), there exists an F-algebra homomorphism $\Psi : C_0(V,q) \to C_0(V, a \cdot q)$ induced by $\Psi(xy) = a^{-1} \cdot xy$ for every $x, y \in V$. The map ψ is clearly an isomorphism.

Let $\{u_1, v_1, \cdots, u_n, v_n\}$ be a symplectic basis of (V,q). For every $1 \leqslant i, j \leqslant n$ we have

$$\Psi \circ J_\tau^q(u_iu_j) = \Psi(\tau(u_j)\tau(u_i)) = a^{-1}\tau(u_j)\tau(u_i) = J_\tau^{a\cdot q}(a^{-1}u_iu_j) = J_\tau^{a\cdot q} \circ \Psi(u_iu_j).$$

Similarly we have the following relations

$$\Psi \circ J_\tau^q(v_iv_j) = J_\tau^{a\cdot q} \circ \Psi(v_iv_j),$$
$$\Psi \circ J_\tau^q(u_iv_j) = J_\tau^{a\cdot q} \circ \Psi(u_iv_j).$$

So $\Psi \circ J_\tau^q(x) = J_\tau^{a\cdot q} \circ \Psi(x)$ for every $x \in C_0(V,q)$. This completes the proof. $\square$

Proposition 4.6. *Let $(\mathbb{A}, q)$ be a quadratic space of dimension 4 over a field F of characteristic 2 and let τ be an interchange isometry of $(\mathbb{A}, q)$. Then the involution J_τ on $C(\mathbb{A})$ is of orthogonal type.*

Proof. Let $(\mathbb{E}, \varphi)$ be a quadratic plane over F and let ρ be a reflection of $(\mathbb{E}, \varphi)$. Set $V = \mathbb{E} \perp \mathbb{A}$. Then $\tau \perp \rho$ is an involution in $O(V, \varphi \perp q)$. By (3.15), there is a decomposition $V = \mathbb{E}_1 \perp \mathbb{E}_2 \perp \mathbb{E}_3$ such that $\tau_i := (\tau \perp \rho)|_{\mathbb{E}_i}$ is a reflection for every $i = 1, 2, 3$. Hence by (4.3), J_{τ_i} is an involution of orthogonal type on $C(\mathbb{E}_i)$. Using (4.4) we obtain $(C(V), J_{\tau\perp\rho}) \simeq (C(\mathbb{E}_1), J_{\tau_1}) \otimes (C(\mathbb{E}_2), J_{\tau_2}) \otimes (C(\mathbb{E}_3), J_{\tau_3})$, and by [6, (2.23)], $J_{\tau\perp\rho}$ is of orthogonal type.

On the other hand we have $(C(V), J_{\tau\perp\rho}) \simeq (C(\mathbb{A}), J_\tau) \otimes (C(\mathbb{E}), J_\rho)$. Since $(C(V), J_{\tau\perp\rho})$ is of orthogonal type, again by [6, (2.23)], J_τ is of orthogonal type on $C(\mathbb{A})$ and we are done. $\square$

Theorem 4.7. *Let (V, q) be a quadratic space over a field F of characteristic 2 and let τ be an involution in $O(V,q)$. Then the involution J_τ on $C(V)$ is of orthogonal type if and only if (V, τ) has trivial maximal fixed orthogonal summand if and only if $\dim \mathrm{Fix}(V, \tau) = \frac{1}{2} \dim V$.*

Proof. Let $V = W \perp \mathbb{E}_1 \perp \cdots \perp \mathbb{E}_r \perp \mathbb{A}_1 \perp \cdots \perp \mathbb{A}_s$ be a Wiitala decomposition of (V, τ). We have $(C(V), J_\tau) \simeq (C(W), J_{\mathrm{id}}) \otimes (C(W'), J_{\tau|_{W'}})$ where $W' = \mathbb{E}_1 \perp \cdots \perp \mathbb{E}_r \perp \mathbb{A}_1 \perp \cdots \perp \mathbb{A}_s$. The conclusion follows from (4.3), (4.4), (4.6) and [6, (2.23)]. The equivalence of the last two statements follows from (3.19). $\square$

Corollary 4.8. *Let (V, q) be a quadratic space over a field F of characteristic 2.*

(*a*) *The involution J_{id} on $C(V)$ is of symplectic type.*

(*b*) *Let τ be a reflection of V. If $\dim V = 2$, then the involution J_τ on $C(V)$ is of orthogonal type. If $\dim V \geqslant 4$, then J_τ is of symplectic type.*

Proof. (*a*) We have $\dim \mathrm{Fix}(V, \mathrm{id}) = \dim V > \frac{1}{2} \dim V$. So by (4.7), J_{id} is of symplectic type.

(*b*) If $\dim V = 2$, then by (4.3), J_τ is of orthogonal type. If $\dim V \geqslant 4$, then $\dim \mathrm{Fix}(V, \tau) = \dim V - 1 > \frac{1}{2} \dim V$ and by (4.7), J_τ is of symplectic type. $\square$

Proposition 4.9. *Let (V, q) be a quadratic space over a field F of characteristic 2 and let τ be an involution in $O(V, q)$. Then $\mathrm{D}(\tau)$ is equal to the class of $\dim \mathrm{Fix}(V, \tau)$ in $\mathbb{Z}/2\mathbb{Z}$.*

Proof. Let $V = W \perp \mathbb{E}_1 \perp \cdots \perp \mathbb{E}_r \perp \mathbb{A}_1 \perp \cdots \perp \mathbb{A}_s$ be a Wiitala decomposition of (V, τ) and let $\{u_1, v_1, \cdots, u_t, v_t, u'_1, v'_1, \cdots, u'_r, v'_r, u''_1, v''_1, \cdots, u''_{2s}, v''_{2s}\}$ be a τ-symplectic basis with respect to this decomposition and set $z = u_1v_1 + \cdots + u_tv_t + u'_1v'_1 + \cdots + u'_rv'_r + u''_1v''_1 + \cdots + u''_{2s}v''_{2s}$. Let I_τ be the automorphism of $C_0(V)$ induced by τ. Then $I_\tau|_W = \mathrm{id}$. Also for $1 \leqslant i \leqslant r$, we have $I_\tau(u'_iv'_i) = u'_i(v'_i + \frac{1}{q(u'_i)}u'_i) = u'_iv'_i + 1$. Finally for $1 \leqslant i \leqslant s$, we have $I_\tau(u''_{2i-1}v''_{2i-1} + u''_{2i}v''_{2i}) = u''_{2i}v''_{2i} + u''_{2i-1}v''_{2i-1}$. Hence $I_\tau(z) = z + r \cdot 1$, where r is the number of reflections in the above Wiitala decomposition of τ. Note that by (3.19) the number of reflections modulo two is independent of the decomposition of τ. Since $Z(C_0(V))$ is generated by 1 and z over F (see [4, (13.12)]), the Dickson invariant $\mathrm{D}(\tau)$ is equal to the class of r in $\mathbb{Z}/2\mathbb{Z}$. On the other hand by (3.19), $r \equiv \dim \mathrm{Fix}(V, \tau) \mod 2$ and therefore, $\mathrm{D}(\tau)$ is equal to the class of $\dim \mathrm{Fix}(V, \tau)$ in $\mathbb{Z}/2\mathbb{Z}$. □

Lemma 4.10. *Let F be a field of characteristic 2 and let A a be central simple algebra over F. If $[A : F] = n^2$ is even and $x \in A$ is an element such that $x^2 = \alpha \in F$, then $\mathrm{Nrd}(x) = \alpha^{\frac{n}{2}}$.*

Proof. We have $\mathrm{Nrd}(x)^2 = \mathrm{Nrd}(\alpha) = \alpha^n$. Since n is even and $\mathrm{char}\, F = 2$, we get $\mathrm{Nrd}(x) = \alpha^{\frac{n}{2}}$. □

Proposition 4.11. *Let F be a field of characteristic 2 and let (V, q) be a quadratic space over F. Let τ be an involution in $O(V, q)$ such that J_τ is of orthogonal type (in other words $\dim \mathrm{Fix}(V, \tau) = \frac{1}{2} \dim V$) on $C(V)$.*

(*a*) *If $\dim V = 2$, then $\mathrm{disc}\, J_\tau = \theta(\tau)$.*

(*b*) *If $\dim V \geqslant 4$, then $\mathrm{disc}\, J_\tau = 1 \in F^\times/F^{\times 2}$.*

Proof. (*a*) Since J_τ is of orthogonal type, by (4.3), we have $\tau = \tau_u$ where $u \in V$ is an anisotropic vector. By (4.2), we have $u \in \mathrm{Alt}(C(V), J_{\tau_u})$. On the other hand since $u^2 = q(u) \neq 0$, the vector u is invertible in $C(V)$ and by (4.10), $\mathrm{Nrd}(u) = q(u)$. So $\mathrm{disc}\, J_{\tau_u} = \mathrm{Nrd}(u)F^{\times 2} = q(u)F^{\times 2} = \theta(\tau_u)$.

(*b*) By (4.7), (V, τ) has trivial maximal fixed orthogonal summand, so (V, τ) has a Wiitala decomposition $V = \mathbb{E}_1 \perp \cdots \perp \mathbb{E}_r \perp \mathbb{A}_1 \perp \cdots \perp \mathbb{A}_s$. If $\dim V = 4$, then either $\tau = \tau_1 \perp \tau_2$ where τ_i, $i = 1, 2$, is a reflection of $\mathbb{E}_i$ or τ is an interchange isometry of V. In the first case we have $(C(V), J_\tau) \simeq (C(\mathbb{E}_1), J_{\tau_1}) \otimes (C(\mathbb{E}_2), J_{\tau_2})$. Using [6, (7.3)], we have $\mathrm{disc}\, J_\tau = \mathrm{disc}(J_{\tau_1} \otimes J_{\tau_2}) = 1$. Now consider the case where τ is an interchange isometry. Let $\{w, x, y, z\}$ be the basis of V stated in (3.4). Then $J_\tau(xz) = \tau(z)\tau(x) = (w + z)x = wx + zx = wx + xz + 1$, i.e., $wx + 1 \in \mathrm{Alt}(C(V), J_\tau)$. Since $B(x, w) = 0$ we have $wx = xw$. We therefore obtain $(wx + 1)^2 = wxwx + 1 = w^2x^2 + 1 = q(w)q(x) + 1 = 1$. So $wx+1 \in C(V)^*$ and by (4.10), $\mathrm{Nrd}(wx+1) = 1$. So $\mathrm{disc}\, J_\tau = \mathrm{Nrd}(wx+1)F^{\times 2} = 1 \in F^\times/F^{\times 2}$.

If $\dim V > 4$, then using the Wiitala decomposition of (V, τ), there exist two nonzero subspaces V_1 and V_2 of V such that $V = V_1 \perp V_2$ and V_1 and V_2 are stable under τ. So $(C(V), J_\tau) \simeq (C(V_1), J_{\tau_1}) \otimes (C(V_2), J_{\tau_2})$, where $\tau_i = \tau|_{V_i}$, $i = 1, 2$, and by [6, (7.3)], $\mathrm{disc}\, J_\tau = \mathrm{disc}(J_{\tau_1} \otimes J_{\tau_2}) = 1$. □

Lemma 4.12. *Let (V,q) be a quadratic space over a field F of characteristic 2 and let τ be an involution in $O(V,q)$. Let $\{u_1,v_1,\cdots,u_t,v_t,u'_1,v'_1,\cdots,u'_r,v'_r,$ $u''_1,v''_1,\cdots,u''_{2s},v''_{2s}\}$ be a τ-symplectic basis of V. Set $z=u_1v_1+\cdots+u_tv_t+u'_1v'_1+\cdots+u'_rv'_r+u''_1v''_1+\cdots+u''_{2s}v''_{2s}\in C_0(V)$ and $\delta=q(u_1)q(v_1)+\cdots+q(u_t)q(v_t)+q(u'_1)q(v'_1)+\cdots+q(u'_r)q(v'_r)+q(u''_1)q(v''_1)+\cdots+q(u''_{2s})q(v''_{2s})\in F$. Then $z^2=z+\delta\cdot 1$ and $J_\tau(z)=z+t\cdot 1$. (Note that according to* (3.19), $t=\dim\mathrm{Fix}(V,\tau)-\frac{1}{2}\dim V$*).*

Proof. By [4, p. 123], we have $z^2=z+\delta\cdot 1$. Let $V=W\perp\mathbb{E}_1\perp\cdots\perp\mathbb{E}_r\perp\mathbb{A}_1\perp\cdots\perp\mathbb{A}_s$ be a Wiitala decomposition of (V,τ) such that $\{u_1,v_1,\cdots,u_t,v_t,u'_1,v'_1,\cdots,u'_r,v'_r,\ u''_1,v''_1,\cdots,u''_{2s},v''_{2s}\}$ is a τ-symplectic basis with respect to this decomposition and $t=\dim\mathrm{Fix}(V,\tau)-\frac{1}{2}\dim V$. For $1\leqslant i\leqslant t$, we have $J_\tau(u_iv_i)=\tau(v_i)\tau(u_i)=v_iu_i=u_iv_i+1$. Similarly for $1\leqslant i\leqslant r$, we have $J_\tau(u'_iv'_i)=\tau(v'_i)\tau(u'_i)=(v'_i+\frac{1}{q(u'_i)}u'_i)u'_i=v'_iu'_i+1=u'_iv'_i$. Finally for $1\leqslant i\leqslant s$, we have $J_\tau(u''_{2i-1}v''_{2i-1}+u''_{2i}v''_{2i})=v''_{2i}u''_{2i}+v''_{2i-1}u''_{2i-1}=u''_{2i-1}v''_{2i-1}+u''_{2i}v''_{2i}$. We therefore obtain $J_\tau(z)=z+t\cdot 1$ and the result is proved. □

Theorem 4.13. *Let (V,q) be a quadratic space over a field F of characteristic 2 and let τ be an involution in $O(V,q)$. Let $t=\dim\mathrm{Fix}(V,\tau)-\frac{1}{2}\dim V$. Suppose that $\Delta(q)$ is nontrivial. Consider the involution J_τ on $C_0(V)$:*

(*a*) *If $t\equiv 0 \mod 2$ then J_τ is of the first kind. Furthermore if $t=0$ then J_τ is of orthogonal type and if $t\neq 0$ then J_τ is of symplectic type.*

(*b*) *If $t\equiv 1 \mod 2$ then J_τ is of the second kind.*

Proof. Let $\{u_1,v_1,\cdots,u_n,v_n\}$ be a τ-symplectic basis of V and let W be a maximal fixed orthogonal summand of (V,τ). Set $z=u_1v_1+\cdots+u_nv_n$. Then by (4.12), we have $J_\tau(z)=t\cdot 1+z$ where $t=\frac{1}{2}\dim W=\dim\mathrm{Fix}(V,\tau)-\frac{1}{2}\dim V$. By [4, (13.12)], $Z(C_0(V))$ is generated as an F-algebra by 1 and z over F. If $t\equiv 0 \mod 2$ then $J_\tau(z)=z$, i.e., $J_\tau|_{Z(C_0(V))}=\mathrm{id}$. Hence J_τ is of the first kind. Now if $t=0$, then $W=0$ and by (4.7), J_τ is of orthogonal type on $C(V)$, so by [6, (2.6)] $1\notin\mathrm{Alt}(C(V),J_\tau)$. So $1\notin\mathrm{Alt}(C_0(V),J_\tau)$ and by [6, (2.6)], J_τ is of orthogonal type on $C_0(V)$. Similarly if $t\neq 0$, then $t\geqslant 2$, so $u_1,v_1\in W$ and $J_\tau(u_1v_1)=v_1u_1=u_1v_1+1$. Hence $1\in\mathrm{Alt}(C_0(V),J_\tau)$ and again by [6, (2.6)], J_τ is of symplectic type.

If $t\equiv 1 \mod 2$ then $J_\tau(z)=z+1$, so J_τ is of the second kind. □

Remark 4.14. Let (V,q) be a quadratic space over a field F of characteristic 2 and let τ be an involution in $O(V,q)$. If $\Delta(q)$ is nontrivial, then $C_0(V)$ is a central simple algebra over $Z=Z(C_0(V))$ and Z/F is a quadratic field extension. More precisely if $\{u_1,v_1,\cdots,u_n,v_n\}$ is a symplectic basis of V and $z=u_1v_1+\cdots+u_nv_n$, then $Z=F(z)$. If $x=\alpha+\beta z\in Z$ for $\alpha,\beta\in F$ and $x^2\in F$, then by (4.12) we have $\alpha^2+\beta^2(z+\delta)\in F$ where $\delta=\sum_{i=1}^n q(u_i)q(v_i)$. So $\beta=0$ and $x\in F$. This shows that $Z^{\times 2}\cap F^\times=F^{\times 2}$. Therefore, there exists a natural injection $F^\times/F^{\times 2}\hookrightarrow Z^\times/Z^{\times 2}$ and we may consider $F^\times/F^{\times 2}$ as a subgroup of $Z^\times/Z^{\times 2}$.

Proposition 4.15. *Let (V,q) be a quadratic space over a field F of characteristic 2 and let τ be an involution in $O(V,q)$. Suppose that $\Delta(q)$ is nontrivial and J_τ is an involution of orthogonal type (in other words $\dim\mathrm{Fix}(V,\tau)=\frac{1}{2}\dim V$) on $C_0(V)$. Let $i: F^\times/F^{\times 2}\hookrightarrow Z^\times/Z^{\times 2}$ be the natural injection discussed in* (4.14)*.*

(*a*) *If* $\dim V = 4$, *then* $\operatorname{disc} J_\tau = i(\theta(\tau))$.

(*b*) *If* $\dim V \geqslant 6$, *then* $\operatorname{disc} J_\tau$ *is trivial.*

Proof. Since $\Delta(q)$ is nontrivial, by [4, (13.12)], $C_0(V)$ is a central simple algebra. Let $\{u_1, v_1, \cdots, u_n, v_n\}$ be a τ-symplectic basis of V. By (3.19), (V, τ) has trivial maximal fixed orthogonal summand, so (V, τ) has a Wiitala decomposition $V = \mathbb{E}_1 \perp \cdots \perp \mathbb{E}_r \perp \mathbb{A}_1 \perp \cdots \perp \mathbb{A}_s$. If τ is of interchanging kind, then by (3.20), $\Delta(q)$ is trivial which contradicts the assumption. So τ is of reflectional kind and we may assume that τ is an orthogonal sum of reflections $\tau_i = \tau_{u_i}$ of $\mathbb{E}_i$. We have $J_\tau(u_1v_2) = \tau_2(v_2)\tau_1(u_1) = (v_2 + \frac{1}{q(u_2)}u_2)u_1 = v_2u_1 + \frac{1}{q(u_2)}u_2u_1 = u_1v_2 + \frac{1}{q(u_2)}u_1u_2$, i.e., $u_1u_2 \in \operatorname{Alt}(C_0(V), J_\tau)$. We also have $(u_1u_2)^2 = q(u_1)q(u_2) \neq 0$, so $u_1u_2 \in C_0(V)^*$.

(*a*) By (4.10), $\operatorname{Nrd}(u_1u_2) = q(u_1)q(u_2)$. Therefore, $\operatorname{disc} J_\tau = \operatorname{Nrd}(u_1u_2)Z^{\times 2} = q(u_1)q(u_2)Z^{\times 2}$. Finally note that we have $\tau = \tau_{u_1}\tau_{u_2}$ where we consider τ_{u_i} as a reflection of V and therefore, $\operatorname{disc} J_\tau = i(\theta(\tau)) = q(u_1)q(u_2)Z^{\times 2}$.

(*b*) As $\dim V \geqslant 6$, we obtain $\deg(C_0(V)) \equiv 0 \mod 4$. Since $(u_1u_2)^2 = q(u_1)q(u_2)$, by (4.10), we have $\operatorname{Nrd}(u_1u_2) = 1$, so $\operatorname{disc} J_\tau = \operatorname{Nrd}(u_1u_2)Z^{\times 2}$ is trivial. □

Remark 4.16. The assumption that $\Delta(q)$ is nontrivial in (4.15) is necessary, because by [4, (13.12)], if $\Delta(q)$ is trivial, then $C_0(V)$ is not central simple algebra and the reduced norm is not defined. Also we have assumed that $\dim V \geqslant 4$, because if $\dim V = 2$ then J_τ is identity and $\operatorname{disc} J_\tau$ is not defined.

5 On a result of Mammone, Tignol and Wadsworth

In [10] it has been shown that if $(\mathbb{E}, \varphi)$ is the quadratic plane $[a, b]$ where $a \neq 0$, then for any quadratic space (V, q) we have $C(\mathbb{E} \perp V, \varphi \perp q) \simeq C(\mathbb{E}', \varphi') \otimes C(V, a \cdot q)$ and $C_0(\mathbb{E} \perp V, \varphi \perp q) \simeq C_0(\mathbb{E}'', \varphi'') \otimes C(V, a \cdot q)$, where $(\mathbb{E}', \varphi')$ and $(\mathbb{E}'', \varphi'')$ are quadratic planes $[a, b + a^{-1}\delta_1]$ and $[1, \delta_1 + \delta_2]$, respectively, here $\delta_1 \in F$ is a representative of the class $\Delta(q) \in F/\wp(F)$ and $\delta_2 \in F$ is a representative of the class $\Delta(\varphi) \in F/\wp(F)$, i.e., $\Delta(q) = \delta_1 + \wp(F) \in F/\wp(F)$ and $\Delta(\varphi) = \delta_2 + \wp(F) \in F/\wp(F)$. We complement these results by showing that:

Proposition 5.1. *Let F be a field of characteristic 2 and let $(\mathbb{E}, \varphi)$ and (V, q) be two quadratic spaces over F such that $\dim \mathbb{E} = 2$.*
(i) If $\varphi \simeq [a, b]$ for $a, b \in F$ and $a \neq 0$, then $C(\mathbb{E} \perp V, \varphi \perp q) \simeq C(\mathbb{E}', \varphi') \otimes C(V, a \cdot q) \simeq C(\mathbb{E}' \perp V, \varphi' \perp a \cdot q)$ where $(\mathbb{E}', \varphi')$ is the quadratic plane $[a, b + a^{-1}\delta]$ and $\delta \in F$ is a representative of the class $\Delta(q) \in F/\wp(F)$. ([10, Prop. 5])
(ii) Let σ and τ be involutions in $O(\mathbb{E}, \varphi)$ and $O(V, q)$, respectively. Then there exist $a, b \in F$ with $a \neq 0$ such that $\varphi \simeq [a, b]$ and $(C(\mathbb{E} \perp V, \varphi \perp q), J^{\varphi \perp q}_{\sigma \perp \tau}) \simeq (C(\mathbb{E}' \perp V, \varphi' \perp a \cdot q), J^{\varphi' \perp a \cdot q}_{\sigma' \perp \tau})$ where $(\mathbb{E}', \varphi')$ is the quadratic plane $[a, b + a^{-1}\delta]$, $\delta \in F$ is a representative of the class $\Delta(q) \in F/\wp(F)$ and σ' is a suitable involution in $O(\mathbb{E}', \varphi')$. More precisely suppose that $\{x, y\}$ is a standard symplectic basis of $[a, b + a^{-1}\delta]$ and $t = \dim \operatorname{Fix}(\mathbb{E} \perp V, \sigma \perp \tau) - \frac{1}{2}\dim(\mathbb{E} \perp V)$. If $t \equiv 0 \mod 2$ then $\sigma' = \tau_x$ and if $t \equiv 1 \mod 2$ then $\sigma' = \mathrm{id}_{\mathbb{E}'}$. In other words $\sigma'(x) = x$ and $\sigma'(y) = y + (t+1)x^{-1}$.

Proof. (*i*) This part has been proved in [10, Prop. 5], but as in the proof of part (*ii*) we need an explicit form of this isomorphism, we recall its construction here. Let $\{u, v\}$ be a standard symplectic basis of $(\mathbb{E}, \varphi = [a, b])$, i.e., $\varphi(u) = a$, $\varphi(v) = b$ and $B_\varphi(u, v) = 1$. Let $\{u_1, v_1, \cdots, u_n, v_n\}$ be a symplectic basis of (V, q) and set $\delta = \sum_{i=1}^{n} q(u_i)q(v_i) \in F$. Let $\{x, y\}$ be a standard symplectic basis of $(\mathbb{E}', \varphi' = [a, b + a^{-1}\delta])$. Define the linear map $f : (\mathbb{E} \perp V, \varphi \perp q) \to C(\mathbb{E}' \perp V, \varphi' \perp a \cdot q)$ via $f(u) = x$, $f(v) = y + a^{-1}x^{-1}(u_1v_1 + \cdots + u_nv_n)$, $f(u_i) = x^{-1}u_i$ and $f(v_i) = x^{-1}v_i$, $1 \leqslant i \leqslant n$. Let $B = B_{\varphi\perp q}$ be the bilinear form associated to the quadratic form $\varphi \perp q$. We have the following relations:

$$\begin{aligned}
f(u)^2 &= x^2 = \varphi'(x) = a = \varphi(u),\\
f(u_i)^2 &= (x^{-1}u_i)^2 = x^{-2}u_i^2 = (a^{-1})(a \cdot q(u_i)) = q(u_i),\\
f(v_i)^2 &= (x^{-1}v_i)^2 = x^{-2}v_i^2 = (a^{-1})(a \cdot q(v_i)) = q(v_i).
\end{aligned}$$

Since $\{u_1, v_1, \cdots, u_n, v_n\}$ is a symplectic basis of (V, q), the set $\{u_1, a^{-1}v_1, \cdots, u_n, a^{-1}v_n\}$ is also a symplectic basis of $(V, a \cdot q)$. We have $a \cdot q(u_i) = aq(u_i)$ and $a \cdot q(a^{-1}v_i) = a^{-1}q(v_i)$. So by (4.12) we have (the calculations are performed in $C(V, a \cdot q)$):

$$\begin{aligned}
(a^{-1}u_1v_1 + \cdots + a^{-1}u_nv_n)^2 &= (a^{-1}u_1v_1 + \cdots + a^{-1}u_nv_n)\\
&\quad + aq(u_1)a^{-1}q(v_1) + \cdots + aq(u_n)a^{-1}q(v_n)\\
&= a^{-1}(u_1v_1 + \cdots + u_nv_n) + \delta,
\end{aligned}$$

So $(u_1v_1 + \cdots + u_nv_n)^2 = a(u_1v_1 + \cdots + u_nv_n) + a^2\delta$ in $C(V, a \cdot q)$ and we obtain:

$$\begin{aligned}
f(v)^2 &= (y + a^{-1}x^{-1}(u_1v_1 + \cdots + u_nv_n))^2\\
&= \varphi'(y) + a^{-2}x^{-2}(a(u_1v_1 + \cdots + u_nv_n) + a^2\delta)\\
&\quad + a^{-1}(x^{-1}y + yx^{-1})(u_1v_1 + \cdots + u_nv_n)\\
&= (b + a^{-1}\delta) + a^{-3}(a(u_1v_1 + \cdots + u_nv_n) + a^2\delta)\\
&\quad + a^{-2}(xy + yx)(u_1v_1 + \cdots + u_nv_n)\\
&= b + a^{-1}\delta + a^{-2}(u_1v_1 + \cdots + u_nv_n) + a^{-1}\delta\\
&\quad + a^{-2}(u_1v_1 + \cdots + u_nv_n) = b = \varphi(v).
\end{aligned}$$

Now clearly for every $1 \leqslant i \leqslant n$, we have $f(u)f(u_i) + f(u_i)f(u) = 0 = B(u, u_i)$ and $f(u)f(v_i) + f(v_i)f(u) = 0 = B(u, v_i)$. Similarly for every $1 \leqslant i \neq j \leqslant n$, we have $f(u_i)f(u_j) + f(u_j)f(u_i) = 0 = B(u_i, u_j)$, $f(v_i)f(v_j) + f(v_j)f(v_i) = 0 = B(v_i, v_j)$ and $f(u_i)f(v_j) + f(v_j)f(u_i) = 0 = B(u_i, v_j)$. For every $1 \leqslant i \leqslant n$, we have

$$\begin{aligned}
f(u_i)f(v_i) + f(v_i)f(u_i) &= x^{-1}u_ix^{-1}v_i + x^{-1}v_ix^{-1}u_i = x^{-2}(u_iv_i + v_iu_i)\\
&= a^{-1}(a \cdot B_q(u_i, v_i)) = B(u_i, v_i).
\end{aligned}$$

We also have

$$\begin{aligned}
f(u)f(v) + f(v)f(u) &= x(y + a^{-1}x^{-1}(u_1v_1 + \cdots + u_nv_n))\\
&\quad + (y + a^{-1}x^{-1}(u_1v_1 + \cdots + u_nv_n))x\\
&= xy + yx = 1 = B(u, v).
\end{aligned}$$

For every $1 \leqslant i \leqslant n$, we have

$$\begin{aligned}
f(u_i)f(v)+f(v)f(u_i) &= x^{-1}u_i(y+a^{-1}x^{-1}(u_1v_1+\cdots+u_nv_n)) \\
&\quad + (y+a^{-1}x^{-1}(u_1v_1+\cdots+u_nv_n))x^{-1}u_i \\
&= x^{-1}u_iy+yx^{-1}u_i \\
&\quad + x^{-1}u_i(a^{-1}x^{-1}u_iv_i)+(a^{-1}x^{-1}u_iv_i)x^{-1}u_i \\
&= (a^{-1}x)u_iy+y(a^{-1}x)u_i \\
&\quad + a^{-1}x^{-2}u_i(u_iv_i)+a^{-1}x^{-2}(u_iv_i)u_i \\
&= a^{-1}u_i(xy+yx)+a^{-2}(u_iu_iv_i+u_i(u_iv_i+a)) \\
&= a^{-1}u_i+a^{-1}u_i=0=B(u_i,v).
\end{aligned}$$

Similarly for every $1 \leqslant i \leqslant n$, we have

$$f(v_i)f(v)+f(v)f(v_i)=B(v_i,v).$$

So the map f is compatible with $\varphi \perp q$ and it can be extended to an isomorphism $f: C(\mathbb{E} \perp V, \varphi \perp q) \to C(\mathbb{E}' \perp V, \varphi' \perp a \cdot q)$.

(ii) By (3.12), there is an anisotropic vector $u \in \mathbb{E}$ such that $\sigma(u)=u$. (So if $\sigma \neq \mathrm{id}_{\mathbb{E}}$, then $\sigma=\tau_u$). Extend $\{u\}$ to a symplectic basis $\{u,v\}$ of $\mathbb{E}$. Set $a=\varphi(u)$ and $b=\varphi(v)$. Then $\varphi \simeq [a,b]$. By *(i)*, we have $C(\mathbb{E} \perp V, \varphi \perp q) \simeq C(\mathbb{E}' \perp V, \varphi' \perp a \cdot q)$ where $(\mathbb{E}',\varphi')$ is the quadratic plane $[a, b+a^{-1}\delta]$ and $\delta \in F$ is a representative of the class $\Delta(q) \in F/\wp(F)$. We claim that $(C(\mathbb{E} \perp V, \varphi \perp q), J_{\sigma\perp\tau}^{\varphi\perp q}) \simeq (C(\mathbb{E}' \perp V, \varphi' \perp a \cdot q), J_{\sigma'\perp\tau}^{\varphi'\perp a\cdot q})$. We must show that for every $w \in C(\mathbb{E} \perp V)$ we have $(f \circ J_{\sigma\perp\tau}^{\varphi\perp q})(w) = (J_{\sigma'\perp\tau}^{\varphi'\perp a\cdot q} \circ f)(w)$. It is enough to check this for $w=u,v,u_i,v_i$, $i=1,\cdots,n$. Note that in both cases $\sigma'=\mathrm{id}_{\mathbb{E}'}$ and $\sigma'=\tau_x$ we have $\sigma'(x)=x$. We have

$$\begin{aligned}
(f \circ J_{\sigma\perp\tau}^{\varphi\perp q})(u) &= f(u)=x=\sigma'(x)=J_{\sigma'\perp\tau}^{\varphi'\perp a\cdot q}(x) \\
&= (J_{\sigma'\perp\tau}^{\varphi'\perp a\cdot q} \circ f)(u).
\end{aligned}$$

Since for every $1 \leqslant i \leqslant n$, $\tau(u_i) \in V$, we get $f(\tau(u_i))=x^{-1}\tau(u_i)$ and similarly $f(\tau(v_i))=x^{-1}\tau(v_i)$. So for every $1 \leqslant i \leqslant n$ we have

$$\begin{aligned}
(f \circ J_{\sigma\perp\tau}^{\varphi\perp q})(u_i) &= f(\tau(u_i))=x^{-1}\tau(u_i)=\sigma'(x^{-1})\tau(u_i) \\
&= J_{\sigma'\perp\tau}^{\varphi'\perp a\cdot q}(x^{-1}u_i)=(J_{\sigma'\perp\tau}^{\varphi'\perp a\cdot q} \circ f)(u_i).
\end{aligned}$$

Similarly for every $1 \leqslant i \leqslant n$ we have

$$\begin{aligned}
(f \circ J_{\sigma\perp\tau}^{\varphi\perp q})(v_i) &= f(\tau(v_i))=x^{-1}\tau(v_i)=\sigma'(x^{-1})\tau(v_i) \\
&= J_{\sigma'\perp\tau}^{\varphi'\perp a\cdot q}(x^{-1}v_i)=(J_{\sigma'\perp\tau}^{\varphi'\perp a\cdot q} \circ f)(v_i).
\end{aligned}$$

Set $t_1=\dim \mathrm{Fix}(\mathbb{E},\sigma)-\frac{1}{2}\dim \mathbb{E}$ and $t_2=\dim \mathrm{Fix}(V,\tau)-\frac{1}{2}\dim V$. Then $t=t_1+t_2$. Note that τ is also an involution in $O(V,a\cdot q,a\cdot B_q)$. Since $\{u_1,v_1,\cdots,u_n,v_n\}$ is a τ-symplectic basis of (V,q), the set $\{u_1,a^{-1}v_1,\cdots,u_n,a^{-1}v_n\}$ is also a τ-symplectic basis of $(V,a\cdot q)$. By (4.12), we have $J_\tau^{a\cdot q}(u_1a^{-1}v_1+\cdots+u_na^{-1}v_n)=t_2\cdot 1+u_1(a^{-1}v_1)+\cdots+u_n(a^{-1}v_n)$ in $C(V,a\cdot q)$. So $\tau(v_1)\tau(u_1)+\cdots+\tau(v_n)\tau(u_n)=t_2\cdot a+u_1v_1+\cdots+u_nv_n$ in $C(V,a\cdot q)$. We have $\sigma'(y)=y+(t+1)x^{-1}$.

So using (3.13) we obtain

$$\begin{aligned}(J^{\varphi'\perp a\cdot q}_{\sigma'\perp\tau}\circ f)(v) &= J^{\varphi'\perp a\cdot q}_{\sigma'\perp\tau}(y+a^{-1}x^{-1}(u_1v_1+\cdots+u_nv_n))\\ &= \sigma'(y)+a^{-1}(\tau(v_1)\tau(u_1)+\cdots+\tau(v_n)\tau(u_n))\sigma'(x^{-1})\\ &= y+(t+1)x^{-1}+a^{-1}(t_2\cdot a+u_1v_1+\cdots+u_nv_n)x^{-1}\\ &= y+(t+t_2+1)x^{-1}+a^{-1}(u_1v_1+\cdots+u_nv_n)x^{-1}\\ &= y+a^{-1}(u_1v_1+\cdots+u_nv_n)x^{-1}+(t_1+1)x^{-1}\\ &= f(v+(t_1+1)u^{-1})=f(\sigma(v))=(f\circ J^{\varphi\perp q}_{\sigma\perp\tau})(v),\end{aligned}$$

which completes the proof. □

Corollary 5.2. *Let (V,q) be a $2n$-dimensional quadratic space over a field F of characteristic 2 and let τ be a reflection of V. If $n\geqslant 2$, then there exists a $2n$-dimensional quadratic space (V',q') over F such that $(C(V),J_\tau)\simeq(C(V'),J_{\mathrm{id}})$.*

Proof. Let $u_1\in V$ be an anisotropic vector such that $\tau=\tau_{u_1}$. Extend $\{u_1\}$ to a symplectic basis $\{u_1,v_1,\cdots,u_n,v_n\}$ of V. Set $\mathbb{E}_1=Fu_1+Fv_1$ and $V_1=Fu_2+Fv_2+\cdots+Fu_n+Fv_n$. Then $V=\mathbb{E}_1\perp V_1$, $\tau|_{\mathbb{E}_1}$ is a reflection and $\tau|_{V_1}=\mathrm{id}$. Set $t:=\dim\mathrm{Fix}(V,\tau)-\frac{1}{2}\dim V=(2n-1)-n=n-1$. If $n\equiv 0 \mod 2$, then by (5.1 (ii)), we have $(C(V),J_\tau)\simeq(C(\mathbb{E}_1\perp V_1),J_{\tau|_{\mathbb{E}_1}\perp\mathrm{id}})\simeq(C(V'),J_{\mathrm{id}})$ where (V',q') is a $2n$-dimensional quadratic space. If $n\equiv 1 \mod 2$, we have $n\geqslant 3$, so $\dim V_1\geqslant 4$. As (V_1,q) is regular we can write $V_1=\mathbb{E}_2\perp V_2$ where $\mathbb{E}_2$ is a quadratic plane. We have $\dim\mathrm{Fix}(\mathbb{E}_1\perp V_2,\tau|_{\mathbb{E}_1}\perp\mathrm{id})-\frac{1}{2}\dim(\mathbb{E}_1\perp V_2)=(2n-3)-(n-1)=n-2\equiv 1 \mod 2$. By (4.4) and (5.1 (ii)), we have

$$\begin{aligned}(C(V),J_\tau) &\simeq (C(\mathbb{E}_1),J_{\tau|_{\mathbb{E}_1}})\otimes(C(V_1),J_{\mathrm{id}})\\ &\simeq ((C(\mathbb{E}_1),J_{\tau|_{\mathbb{E}_1}})\otimes(C(V_2),J_{\mathrm{id}}))\otimes(C(\mathbb{E}_2),J_{\mathrm{id}})\\ &\simeq (C(V_3),J_{\mathrm{id}})\otimes(C(\mathbb{E}_2),J_{\mathrm{id}})\simeq(C(V'),J_{\mathrm{id}}).\end{aligned}$$

Here V_3 is a $(2n-2)$-dimensional quadratic space which is determined by (5.1 (ii)) and $V'=V_3\perp\mathbb{E}_3$. □

Corollary 5.3. *Let F be a field of characteristic 2 and let $(\mathbb{E},\varphi)$ and (V,q) be quadratic spaces over F such that $\dim\mathbb{E}=2$. Let $(\mathbb{E}',\varphi')$ be the quadratic plane $[1,\delta+\delta']$ where $\delta\in F$ is a representative of the class $\Delta(\varphi)\in F/\wp(F)$ and $\delta'\in F$ is a representative of the class $\Delta(q)\in F/\wp(F)$. Let σ and τ be involutions in $O(\mathbb{E},\varphi)$ and $O(V,q)$, respectively.*
(i) We have $C_0(\mathbb{E}\perp V,\varphi\perp q)\simeq C_0(\mathbb{E}',\varphi')\otimes C(V,a\cdot q)$. ([10, Prop. 5])
(ii) We have $(C_0(\mathbb{E}),J_\sigma)\otimes(C(V),J_\tau)\simeq(C_0(\mathbb{E}'\perp V),J_{\sigma'\perp\tau})$ where σ' is a suitable involution in $O(\mathbb{E}',\varphi')$.
(iii) There exists an element $a\in D_F(\varphi)$ such that $(C_0(\mathbb{E}\perp V),J_{\sigma\perp\tau})\simeq(C_0(\mathbb{E}'),J_{\sigma'})\otimes(C(V,a\cdot q),J^{a\cdot q}_\tau)$ where σ' is a suitable involution in $O(\mathbb{E}',\varphi')$.

More precisely if $\{x,y\}$ is a standard symplectic basis of $[1,\delta+\delta']$ and $t=\dim\mathrm{Fix}(\mathbb{E}\perp V,\sigma\perp\tau)-\frac{1}{2}\dim(\mathbb{E}\perp V)$, the involution $\sigma'\in O(\mathbb{E}',\varphi')$ in (ii) and (iii) can be described as follows: if $t\equiv 0 \mod 2$ then $\sigma'=\tau_x$ and if $t\equiv 1 \mod 2$ then $\sigma'=\mathrm{id}_{\mathbb{E}'}$.

Proof. By (3.12), there is an anisotropic vector $u\in\mathbb{E}$ such that $\sigma(u)=u$ (so if $\sigma\neq\mathrm{id}_{\mathbb{E}}$, then $\sigma=\tau_u$). Extend $\{u\}$ to a symplectic basis $\{u,v\}$ of $\mathbb{E}$. Set $a=$

$\varphi(u)$ and $b = \varphi(v)$. Then $\varphi \simeq [a, b]$. Let $\{u_1, v_1, \cdots, u_n, v_n\}$ be a τ-symplectic basis of V. Let $\delta = \varphi(u)\varphi(v) = ab \in F$ and $\delta' = \sum_{i=1}^n q(u_i)q(v_i) \in F$.

(ii) By (4.5), $(C_0(\mathbb{E}, \varphi), J_\sigma) \simeq (C_0(\mathbb{E}, a\cdot\varphi), J_\sigma)$, so we may assume that $a = 1$, $b = \delta$ and $\varphi = [1, \delta]$. By (5.1 (ii)), there exists an isomorphism $(C(\mathbb{E} \perp V, \varphi \perp q), J_{\sigma\perp\tau}) \simeq (C(\mathbb{E}' \perp V, \varphi' \perp q), J_{\sigma'\perp\tau})$, such that if $t \equiv 0 \mod 2$ then $\sigma' = \tau_x$ and if $t \equiv 1 \mod 2$ then $\sigma' = \mathrm{id}_{\mathbb{E}'}$. So $\sigma'(x) = x$ and if $\sigma' \neq \mathrm{id}$, then $\sigma' = \tau_x$. Using this isomorphism and (4.4), the map $\psi : (C(\mathbb{E}, \varphi), J_\sigma) \otimes (C(V, q), J_\tau) \simeq (C(\mathbb{E} \perp V, \varphi \perp q), J_{\sigma\perp\tau}) \simeq (C(\mathbb{E}' \perp V), J_{\sigma'\perp\tau})$ induced by $\psi(u \otimes 1) = x$, $\psi(v \otimes 1) = y + x^{-1}(u_1v_1 + \cdots + u_nv_n)$, $\psi(1 \otimes u_i) = x^{-1}u_i$ and $\psi(1 \otimes v_i) = x^{-1}v_i$ is an isomorphism. The restriction of ψ to $(C_0(\mathbb{E}), J_\sigma) \otimes (C(V), J_\tau)$ maps $(C_0(\mathbb{E}), J_\sigma) \otimes (C(V), J_\tau)$ to $(C_0(\mathbb{E}' \perp V), J_{\sigma'\perp\tau})$ and the result is proved.

(iii) By (4.5), we have

$$(C_0(\mathbb{E} \perp V), J^{\varphi\perp q}_{\sigma\perp\tau}) \simeq (C_0(\mathbb{E} \perp V), J^{a\cdot\varphi\perp a\cdot q}_{\sigma\perp\tau}), \tag{1}$$

where $(\mathbb{E}, a \cdot \varphi) \simeq [1, \delta]$. Note that $\{a^{-1}u, v\}$ is a standard symplectic basis of $(\mathbb{E}, a \cdot \varphi)$. Let $(\mathbb{E}'', \varphi'')$ be the quadratic plane $[1, \delta + \delta' + \delta'] \simeq [1, \delta] \simeq (\mathbb{E}, a \cdot \varphi)$ with a standard symplectic basis $\{x', y'\}$. Set $t_1 = \dim \mathrm{Fix}(\mathbb{E}, \sigma) - \frac{1}{2} \dim \mathbb{E}$, $t_2 = \dim \mathrm{Fix}(V, \tau) - \frac{1}{2} \dim V$ and $t'_1 = \dim \mathrm{Fix}(\mathbb{E}', \sigma') - \frac{1}{2} \dim \mathbb{E}'$. We have $t = t_1 + t_2$. By replacing $\mathbb{E}$ with $\mathbb{E}'$ and σ with σ' in (ii), we obtain an isomorphism

$$(C_0(\mathbb{E}'), J_{\sigma'}) \otimes (C(V), J^{a\cdot q}_\tau) \simeq (C_0(\mathbb{E}'' \perp V), J^{\varphi''\perp a\cdot q}_{\sigma''\perp\tau}), \tag{2}$$

where σ'' is an involution in $O(\mathbb{E}'', \varphi'')$. Furthermore if $t'_1 + t_2 \equiv 0 \mod 2$ then $\sigma'' = \tau_{x'}$ and if $t'_1 + t_2 \equiv 1 \mod 2$ then $\sigma'' = \mathrm{id}_{\mathbb{E}''}$.

Now the linear map $f : \mathbb{E} \to \mathbb{E}''$ defined by $f(a^{-1}u) = x'$ and $f(v) = y'$ induces an isomorphism $C(\mathbb{E}, a \cdot \varphi) \simeq C(\mathbb{E}'', \varphi'')$. Under this isomorphism, τ_u corresponds to $\tau_{x'}$ and $\mathrm{id}_{\mathbb{E}}$ corresponds to $\mathrm{id}_{\mathbb{E}''}$, i.e., σ corresponds to σ''. So we can rewrite (2) as follows:

$$(C_0(\mathbb{E}'), J_{\sigma'}) \otimes (C(V), J^{a\cdot q}_\tau) \simeq (C_0(\mathbb{E} \perp V), J^{a\cdot\varphi\perp a\cdot q}_{\sigma\perp\tau}), \tag{3}$$

Furthermore if $t'_1 + t_2 \equiv 0 \mod 2$ then $\sigma = \tau_u$ (i.e., $t_1 = 0$) and if $t'_1 + t_2 \equiv 1 \mod 2$ then $\sigma = \mathrm{id}_{\mathbb{E}}$ (i.e., $t_1 = 1$). So we have $t'_1 + t_2 \equiv t_1 \mod 2$ which implies that $t = t_1 + t_2 \equiv t'_1 \mod 2$.

Using (1) and (3) we obtain the desired isomorphism:

$$(C_0(\mathbb{E} \perp V), J^{\varphi\perp q}_{\sigma\perp\tau}) \simeq (C_0(\mathbb{E}'), J_{\sigma'}) \otimes (C(V), J^{a\cdot q}_\tau).$$

Finally as $t \equiv t'_1 \mod 2$, if $t \equiv 0 \mod 2$ then $t'_1 = 0$ (i.e., $\sigma' = \tau_x$) and if $t \equiv 1 \mod 2$ then $t'_1 = 1$ (i.e., $\sigma' = \mathrm{id}_{\mathbb{E}'}$). □

6 Multiquaternion algebras with involution

Let (Q, σ) be a quaternion algebra with involution of the first kind. In [9] it has been shown that if $\mathrm{char}\, F \neq 2$, then there exists a 2-dimensional quadratic space (V, q) and an involution τ in $O(V, q)$ such that $(Q, \sigma) \simeq (C(V), J_\tau)$ (see [9, (6.2)]). Here we state this result for a field F of characteristic 2.

Theorem 6.1. *Let (Q, σ) be a quaternion algebra with an involution of the first kind over a field F of characteristic 2. Then there exists a quadratic plane*

$(\mathbb{E},\varphi)$ *over* F *with an involution* $\tau \in O(\mathbb{E},\varphi)$ *such that* $(Q,\sigma) \simeq (C(\mathbb{E}), J_\tau)$. *Furthermore if* σ *is of symplectic type, then* τ *is necessarily equal to* id *and if* σ *is of orthogonal type, then* $\tau \neq \mathrm{id}$ *is necessarily a reflection.*

Proof. Let $Q = [\alpha,\beta)$ for $\alpha \in F, \beta \in F^\times$ and consider a standard basis $\{1,i,j,k\}$ of Q. Let γ be the canonical involution of Q. Then by [14, Ch. 8, (7.4)], there exists some $x \in Q^*$ with $\gamma(x) = x$ such that $\sigma = \mathrm{Int}(x) \circ \gamma$. So if we write $x = a + bi + cj + dk$ for some $a,b,c,d \in F$, then $b = 0$ and we obtain $x = a + cj + dk$. If $c = d = 0$, then $\sigma = \gamma$. Let $\mathbb{E}$ be the quadratic plane $[\beta, \alpha\beta^{-1}]$ over F. By [14, Ch. 9, (4.6)], we have $Q \simeq C(\mathbb{E})$. Using (4.3), the involution J_{id} on $C(\mathbb{E})$ is of symplectic type. Since γ is the unique involution of symplectic type on Q, we have $(Q,\gamma) \simeq (C(\mathbb{E}), J_{\mathrm{id}})$.

Now suppose that c or d is nonzero, then there exists $s,t \in F$ such that $cs + dt = a$. Let $(\mathbb{E},\varphi)$ be a quadratic plane over F with a basis $\{u,v\}$ and the relations $\varphi(u) = t^2 + \alpha\beta$, $\varphi(v) = s^2 + \beta$ and $B(u,v) = \beta \neq 0$. Define the linear map $f : \mathbb{E} \to Q$ by $f(u) = t + k$ and $f(v) = s + j$. We have $f(u)^2 = (t+k)^2 = t^2 + \alpha\beta = \varphi(u)$ and $f(v)^2 = (s+j)^2 = s^2 + \beta = \varphi(v)$. We also have

$$\begin{aligned} f(u)f(v) + f(v)f(u) &= (t+k)(s+j) + (s+j)(t+k) = kj + jk \\ &= ijj + jij = (ij + ji)j = j^2 = \beta = B(u,v). \end{aligned}$$

So the map f is compatible with φ and it can be extended to an isomorphism $\psi : C(\mathbb{E}) \simeq Q$. Let $I = \psi^{-1} \circ \gamma \circ \psi$. Then I is an involution on $C(\mathbb{E})$ and $(Q,\gamma) \simeq (C(\mathbb{E}), I)$. We have the following relations:

$$I(u) = \psi^{-1} \circ \gamma \circ \psi(u) = \psi^{-1} \circ \gamma(t+k) = \psi^{-1}(t+k) = u,$$

$$I(v) = \psi^{-1} \circ \gamma \circ \psi(v) = \psi^{-1} \circ \gamma(s+j) = \psi^{-1}(s+j) = v.$$

So $I = J_{\mathrm{id}_{\mathbb{E}}}$. Now let $T = \psi^{-1} \circ \sigma \circ \psi$. Then T is an involution on $C(\mathbb{E})$ and $(Q,\sigma) \simeq (C(\mathbb{E}), T)$. We have $\psi(du + cv) = d(t+k) + c(s+j) = a + cj + dk = x$, so $\psi^{-1}(x) = du + cv \in \mathbb{E}$. Since $\sigma = \mathrm{Int}(x) \circ \gamma$, we obtain $T = \psi^{-1} \circ \sigma \circ \psi = \psi^{-1} \circ \mathrm{Int}(x) \circ \gamma \circ \psi = (\psi^{-1} \circ \mathrm{Int}(x) \circ \psi) \circ (\psi^{-1} \circ \gamma \circ \psi) = \mathrm{Int}(\psi^{-1}(x)) \circ J_{\mathrm{id}_{\mathbb{E}}}$. By (4.1), in order to show that $T = J_\tau$ for some involution in $O(\mathbb{E},\varphi)$, it is enough to show that $T(\mathbb{E}) = \mathbb{E}$ or equivalently $T(w) = \mathrm{Int}(\psi^{-1}(x)) \circ J_{\mathrm{id}_{\mathbb{E}}}(w) = \mathrm{Int}(\psi^{-1}(x))(w) \in \mathbb{E}$ for every $w \in \mathbb{E}$. But since $x \in Q^*$, we get $x^2 \neq 0$, so $\varphi(\psi^{-1}(x)) = \psi^{-1}(x)^2 \neq 0$, i.e., $\psi^{-1}(x)$ is anisotropic. On the other hand we have $\psi^{-1}(x) \in \mathbb{E}$, so $\mathrm{Int}(\psi^{-1}(x))|_{\mathbb{E}}$ is a reflection and $T(\mathbb{E}) = \mathbb{E}$. So $\sigma = J_\tau$ for some isometry τ of $(\mathbb{E},\varphi)$.

This proves the first part of the result. The second part follows from (4.3). □

Corollary 6.2. *Let* $(\mathbb{E},\varphi,B)$ *and* $(\mathbb{E}',\varphi',B')$ *be two quadratic planes over a field* F *of characteristic* 2. *If* τ *and* τ' *are reflections of* $(\mathbb{E},\varphi)$ *and* $(\mathbb{E}',\varphi')$, *respectively, then the following statements are equivalent:* (1) $(C(\mathbb{E}), J_\tau) \simeq (C(\mathbb{E}'), J_{\tau'})$. (2) $C(\mathbb{E}) \simeq C(\mathbb{E}')$ *and* $\theta(\tau) = \theta(\tau')$.

Proof. According to (4.11), we have $\operatorname{disc} J_\tau = \theta(\tau)$ and $\operatorname{disc} J_{\tau'} = \theta(\tau')$. The implication (1) $\Rightarrow$ (2) is immediate and (2) $\Rightarrow$ (1) follows from [6, (7.4)]. □

The next result is analogous to [9, (6.3)].

Theorem 6.3. *Let F be a field of characteristic 2 and let $(Q_1,\sigma_1),\cdots,(Q_n,\sigma_n)$ be quaternion algebras over F with involutions of the first kind. Let $(A,\sigma)\simeq(Q_1,\sigma_1)\otimes\cdots\otimes(Q_n,\sigma_n)$. Then there is a quadratic space (V,q) of dimension $2n$ over F and an involution τ in $O(V,q)$ such that $(A,\sigma)\simeq(C(V),J_\tau)$. Furthermore if σ is of orthogonal type then $\dim \mathrm{Fix}(V,\tau)=\frac{1}{2}\dim V$ and τ can be chosen to be an orthogonal sum of reflections and if σ is of symplectic type then τ can be chosen to be identity.*

Proof. By (6.1), for each $1\leqslant i\leqslant n$ there is a quadratic plane $(\mathbb{E}_i,\varphi_i)$ with an involution $\tau_i\in O(\mathbb{E}_i,\varphi_i)$ such that $(Q_i,\sigma_i)\simeq(C(\mathbb{E}_i),J_{\tau_i})$ and by (4.4), $(A,\sigma)\simeq(C(V),J_\tau)$ where $(V,q)=(\mathbb{E}_1,\varphi_1)\perp\cdots\perp(\mathbb{E}_n,\varphi_n)$ and $\tau=\tau_1\perp\cdots\perp\tau_n$. If σ is of orthogonal type then by [6, (2.23)], each σ_i is of orthogonal type and by (6.1), τ_i is a reflection for each i, so τ is an orthogonal sum of n reflections and $\dim\mathrm{Fix}(V,\tau)=\frac{1}{2}\dim V$. If σ is of symplectic type then by [6, (2.23)], at least one of the σ_i's is of symplectic type, say σ_1. Then by (6.1), $(Q_1,\sigma_1)\simeq(C(\mathbb{E}_1),J_{\mathrm{id}})$. Now if σ_2 is of symplectic type then $(Q_1,\sigma_1)\otimes(Q_2,\sigma_2)\simeq(C(\mathbb{E}_1),J_{\mathrm{id}})\otimes(C(\mathbb{E}_2),J_{\mathrm{id}})\simeq(C(\mathbb{E}_1\perp\mathbb{E}_2),J_{\mathrm{id}})$. If σ_2 is of orthogonal type, one can write $(Q_2,\sigma_2)\simeq(C(\mathbb{E}_2),J_{\tau_2})$ where τ_2 is a reflection of $\mathbb{E}_2$. So $(Q_1,\sigma_1)\otimes(Q_2,\sigma_2)\simeq(C(\mathbb{E}_1),J_{\mathrm{id}})\otimes(C(\mathbb{E}_2),J_{\tau_2})\simeq(C(\mathbb{E}_1\perp\mathbb{E}_2),J_{\mathrm{id}\perp\tau_2})$. By (5.2) this is isomorphic to $(C(V),J_{\mathrm{id}})$ for a suitable 4-dimensional quadratic space (V,q) over F. Now induction on n completes the proof. □

Lemma 6.4. *Let K/F be a separable quadratic extension of fields of characteristic 2 with the nontrivial automorphism ρ. Then there is a quadratic plane $(\mathbb{E},\varphi)$ over F such that $(K,\rho)\simeq(C_0(\mathbb{E}),J_{\mathrm{id}})$. Furthermore if $(\mathbb{E},\varphi)$ is a quadratic plane over F and τ is an involution in $O(\mathbb{E},\varphi)$ such that $(K,\rho)\simeq(C_0(\mathbb{E}),J_\tau)$, then τ is necessarily identity.*

Proof. Since K/F is separable we may assume $K=F(x)$ where $x^2+x=\delta\in F^\times$. Let $(\mathbb{E},\varphi)$ be the quadratic space $[1,\delta]$ with a standard symplectic basis $\{u,v\}$. Define $f:K\to C_0(\mathbb{E})$ via $f(\lambda+\mu x)=\lambda+\mu uv$ for every $\lambda,\mu\in F$. We have $f(x)^2=(uv)^2=uv+\delta=f(x+\delta)=f(x^2)$. So for every $\lambda_1,\mu_1,\lambda_2,\mu_2\in F$ we have $f((\lambda_1+\mu_1x)(\lambda_2+\mu_2x))=f((\lambda_1+\mu_1uv)(\lambda_2+\mu_2uv))$. Hence f is an F-algebra homomorphism and in fact an isomorphism, because it is obviously surjective. We have

$$J_{\mathrm{id}_{\mathbb{E}}}\circ f(x)=J_{\mathrm{id}_{\mathbb{E}}}(uv)=vu=uv+1=f(x+1)=f\circ\rho(x).$$

So $(K,\rho)\simeq(C_0(\mathbb{E}),J_{\mathrm{id}})$.

Now suppose that $(K,\rho)\simeq(C_0(\mathbb{E}),J_\tau)$ for a quadratic plane $(\mathbb{E},\varphi)$ and an involution $\tau\in O(\mathbb{E},\varphi)$. As $\rho(x)=x+1$, there is $y\in C_0(\mathbb{E})\subseteq C(\mathbb{E})$ such that $J_\tau(y)=y+1$, i.e., $1\in\mathrm{Alt}(C(\mathbb{E}),J_\tau)$, so by [6, (2.6)], $(C(\mathbb{E}),J_\tau)$ is of symplectic type and hence by (4.3), $\tau=\mathrm{id}$. □

Remark 6.5. In contrast with the case where $\operatorname{char}F\neq 2$, if (V,q) is a $2n$-dimensional quadratic space over a field F of characteristic 2, then there is no quadratic form $q':V'\to F$ with $\dim V'=2n-1$ such that $C_0(V)\cong C(V')$. In fact suppose that there is an F-algebra isomorphism $f:C(V')\simeq C_0(V)$. Since $\dim V'$ is odd, (V',q') is not regular and there is an element $u\in V'$ such that $B(u,v)=0$ for all $v\in V'$ and hence $u\in Z(C(V'))$. So we must have $f(u)\in Z(C_0(V))$. Therefore, we can write $f(u)=\lambda+\mu(u_1v_1+\cdots+u_nv_n)$ where $\{u_1,v_1,\cdots,u_n,v_n\}$ is a symplectic basis of (V,q). Since f is an

F-isomorphism, $\mu \neq 0$. On the other hand we have $u^2 = q'(u) \in F$, but $f(u)^2 = \lambda^2 + \mu^2(q(u_1)q(v_1) + \cdots + q(u_n)q(v_n) + u_1v_1 + \cdots + u_nv_n) \notin F$ which is a contradiction.

Remark 6.6. Let K/F be a separable quadratic extension of fields with the nontrivial automorphism ρ. If $\operatorname{char} F \neq 2$, then there is a 1-dimensional quadratic space over F such that $K \simeq C(V)$. But if $\operatorname{char} F = 2$, then (6.4) and (6.5) show that there is no 1-dimensional quadratic space (V, q) over F such that $K \simeq C(V)$.

Theorem 6.7. *Let F be a field of characteristic 2 and let $(Q_1, \sigma_1), \cdots, (Q_n, \sigma_n)$ be quaternion algebras over F with involutions of the first kind. Let K/F be a separable quadratic extension with the nontrivial automorphism ρ. If $(A, \sigma) \simeq (Q_1, \sigma_1) \otimes \cdots \otimes (Q_n, \sigma_n) \otimes (K, \rho)$, then there is a quadratic space (V, q) of dimension $2n+2$ over F and an involution τ in $O(V, q)$ such that $(A, \sigma) \simeq (C_0(V), J_\tau)$. Furthermore if $n \equiv 0 \mod 2$, then τ can be chosen to be identity and if $n \equiv 1 \mod 2$, then τ can be chosen to be a reflection.*

Proof. By (6.3), there is a quadratic space (V', q') of dimension $2n$ over F and an involution $\tau' \in O(V', q')$ such that $(Q_1, \sigma_1) \otimes \cdots \otimes (Q_n, \sigma_n) \simeq (C(V'), J_{\tau'})$. Also by (6.4), there is a quadratic plane $(\mathbb{E}, \varphi)$ over F such that $(K, \rho) \simeq (C_0(\mathbb{E}), J_{\mathrm{id}})$. By (6.3), we can choose (V', τ') such that either $\tau' = \mathrm{id}_{V'}$ or $\tau' = \tau'_1 \perp \cdots \perp \tau'_n$ where τ'_i, $i = 1, \cdots, n$, is a reflection of some quadratic subplane $\mathbb{E}'_i$ of V'. Set $t = \dim \operatorname{Fix}(V' \perp \mathbb{E}, \tau' \perp \mathrm{id}) - \frac{1}{2}\dim(V' \perp \mathbb{E})$.

First suppose that $\tau' = \mathrm{id}_{V'}$. We have $t = n + 1$. If $n \equiv 0 \mod 2$, then by (5.3 (*ii*)), there exists a quadratic plane $(\mathbb{E}', \varphi')$ over F such that $(A, \sigma) \simeq (C(V'), J_{\mathrm{id}}) \otimes (C_0(\mathbb{E}), J_{\mathrm{id}}) \simeq (C_0(V' \perp \mathbb{E}'), J_{\mathrm{id} \perp \mathrm{id}})$. So $(A, \sigma) \simeq (C_0(V), J_{\mathrm{id}})$ for $V = V' \perp \mathbb{E}'$. If $n \equiv 1 \mod 2$, then by (5.3 (*ii*)), there exists a quadratic space $(\mathbb{E}', \varphi')$ over F with a reflection τ'' such that $(A, \sigma) \simeq (C(V'), J_{\mathrm{id}}) \otimes (C_0(\mathbb{E}), J_{\mathrm{id}}) \simeq (C_0(V' \perp \mathbb{E}'), J_{\mathrm{id} \perp \tau''})$. So $(A, \sigma) \simeq (C_0(V), J_\tau)$ for $V = V' \perp \mathbb{E}'$ and the reflection $\tau = \mathrm{id}_{V'} \perp \tau''$.

Now suppose that $\tau' = \tau'_1 \perp \cdots \perp \tau'_n$. We have $t = 1$, so by (5.3 (*ii*)), there exists a quadratic space $(\mathbb{E}', \varphi')$ over F such that $(A, \sigma) \simeq (C(V'), J_{\tau'}) \otimes (C_0(\mathbb{E}), J_{\mathrm{id}}) \simeq (C_0(V' \perp \mathbb{E}'), J_{\tau' \perp \mathrm{id}_{\mathbb{E}'}})$. If $\dim V' = 2$, then $(A, \sigma) \simeq (C_0(V), J_\tau)$ where $V = V' \perp \mathbb{E}'$ and $\tau = \tau' \perp \mathrm{id}_{\mathbb{E}'}$ is a reflection of V. So suppose that $\dim V' \geqslant 4$. We have

$$(A, \sigma) \simeq (C_0(V' \perp \mathbb{E}'), J_{\tau' \perp \mathrm{id}_{\mathbb{E}'}}) \simeq (C_0(\mathbb{E}'_1 \perp W), J_{\tau'_1 \perp \rho}),$$

where $W = \mathbb{E}'_2 \perp \cdots \perp \mathbb{E}'_n \perp \mathbb{E}'$ and $\rho = \tau'_2 \perp \cdots \perp \tau'_n \perp \mathrm{id}_{\mathbb{E}'}$. We have $\dim \operatorname{Fix}(\mathbb{E}'_1 \perp W, \tau'_1 \perp \rho) - \frac{1}{2}\dim(\mathbb{E}'_1 \perp W) = 1$, so by (5.3 (*iii*)), $(C_0(\mathbb{E}'_1 \perp W), J_{\tau'_1 \perp \rho}) \simeq (C_0(\mathbb{E}''), J_{\mathrm{id}}) \otimes (C(W, a \cdot q_1), J_\rho)$ where $(\mathbb{E}'', \varphi'')$ is a suitable quadratic space over F, q_1 is the restriction of $q' \perp \varphi'$ to W and $a \in D_F(q|_{\mathbb{E}'_1})$. Since $\dim \operatorname{Fix}(W, \rho) = n + 1 > n = \frac{1}{2}\dim W$, by (4.7), the involution J_ρ on $C(W)$ is of symplectic type. So by (6.3), there exists a quadratic space (V'', q'') over F such that $(C(W, a \cdot q_1), J_\rho) \simeq (C(V''), J_{\mathrm{id}})$. So we have $(A, \sigma) \simeq (C_0(\mathbb{E}''), J_{\mathrm{id}}) \otimes (C(V''), J_{\mathrm{id}})$. Now by an argument similar to the case $\tau' = \mathrm{id}_{V'}$, we have $(A, \sigma) \simeq (C_0(V), J_{\mathrm{id}})$ if $n \equiv 0 \mod 2$, and $(A, \sigma) \simeq (C_0(V), J_\tau)$ for some reflection τ if $n \equiv 1 \mod 2$ and the result is proved. ∎

Corollary 6.8. *Let (Q, σ) be a quaternion algebra with involution of the second kind over a field K of characteristic 2. Let $F \subseteq K$ be the fixed subfield of $\sigma|_K$. Then there exists a 4-dimensional quadratic space $(\mathbb{A}, q)$ over F such that $(Q, \sigma) \simeq (C_0(\mathbb{A}), J_\tau)$ where τ is a suitable reflection of $(\mathbb{A}, q)$.* ([5]).

Proof. The result follows from [14, Ch. 8, (11.2)] and (6.7). □

Remark 6.9. With the notation of (6.7), if $n \equiv 1 \mod 2$ then the involution $\tau \in O(V, q)$ can not be chosen to be identity; because in this case $\dim V = 2n + 2 \equiv 0 \mod 4$ and J_{id} is an involution on $C(V)$ of the first kind, i.e., the involution σ is of the first kind which contradicts the assumption. In particular the involution τ of (6.8) can not be chosen to be identity. Also there is no $(2n-1)$-dimensional quadratic space (V, q) over F such that $A \simeq C(V)$. This fact is an easy consequence of (6.5) and (6.7).

Proposition 6.10. *Let $(\mathbb{A}, q)$ be a 4-dimensional quadratic space over a field F of characteristic 2 and let τ be an interchange isometry of $(\mathbb{A}, q)$. Then $(C(\mathbb{A}), J_\tau) \simeq (C(\mathbb{H}_1), J_{\tau_{u_1}}) \otimes (C(\mathbb{H}_2), J_{\tau_{u_2}})$ where $(\mathbb{H}_1, \varphi_1)$ and $(\mathbb{H}_2, \varphi_2)$ are isometric to the hyperbolic plane $[1, 0]$ over F with respective standard symplectic bases $\{u_1, v_1\}$ and $\{u_2, v_2\}$ and τ_{u_i}, $i = 1, 2$, is the reflection along the anisotropic vector $u_i \in \mathbb{H}_i$. In particular $\theta(\tau_{u_1}) = \theta(\tau_{u_2}) = 1 \in F^\times/F^{\times 2}$. Furthermore if $\{w, x, y, z\}$ is a basis of $\mathbb{A}$ with the properties stated in (3.5), then $(C(\mathbb{H}_1), J_{\tau_{u_1}})$ (resp. $(C(\mathbb{H}_2), J_{\tau_{u_2}})$) is isomorphic, as an F-algebra with involution, to the F-quaternion algebra generated by $\{1 + x + w + xw, y + xz + xzw\}$ (resp. $\{1 + x + xw, y + xz + xzw + z + zw\}$), which are invariant under J_τ.*

Proof. Let $\{w, x, y, z\}$ be a basis of $\mathbb{A}$ with the properties stated in (3.5), i.e., $q(w) = q(x) = q(y) = q(z) = 0$, $\tau(w) = w$, $\tau(x) = x$, $\tau(y) = x + y$, $\tau(z) = w + z$, $B(w, y) = B(x, z) = 1$ and all other pairs of vectors from this basis are orthogonal. Set $(\mathbb{A}', q', B') = (\mathbb{H}_1, \varphi_1) \perp (\mathbb{H}_2, \varphi_2)$. Then the set $\mathcal{B} = \{u_1, v_1, u_2, v_2\}$ is a symplectic basis of $\mathbb{A}'$. Define the linear map $f : \mathbb{A}' \to C(\mathbb{A})$ by $f(u_1) = 1 + x + w + xw$, $f(v_1) = y + xz + xzw$, $f(u_2) = 1 + x + xw$ and $f(v_2) = y + xz + xzw + z + zw = f(v_1) + z + zw$. Using the above relations and straightforward calculations in Clifford algebra we obtain that for every $x, y \in \mathcal{B}$, we have $f(x)^2 = q'(x)$ and $f(x)f(y) + f(y)f(x) = B'(x, y)$. So the map f is compatible with q' and it can be extended to an isomorphism $\psi : C(\mathbb{A}') \simeq C(\mathbb{A})$. We claim that for every $v \in C(\mathbb{A}')$, $J_\tau \circ \psi(v) = \psi \circ (J_{\tau_{u_1}} \perp J_{\tau_{u_2}})(v)$. It is enough to check this for $v = u_1$, $v = v_1$, $v = u_2$ and $v = v_2$. We have the following relations:

$$\begin{aligned} J_\tau \circ \psi(u_1) &= J_\tau(1 + x + w + xw) = 1 + x + w + wx \\ &= \psi(u_1) = \psi \circ (J_{\tau_{u_1}} \perp J_{\tau_{u_2}})(u_1), \end{aligned}$$

$$J_\tau \circ \psi(u_2) = J_\tau(1 + x + xw) = 1 + x + wx = \psi(u_2) = \psi \circ (J_{\tau_{u_1}} \perp J_{\tau_{u_2}})(u_2),$$

$$\begin{aligned} J_\tau \circ \psi(v_1) &= J_\tau(y + xz + xzw) = (x + y) + (w + z)x + w(w + z)x \\ &= (x + y) + (wx + zx) + (q(w)x + wzx) \\ &= (x + y) + (xw + xz + 1) + (0 + xzw + w) \\ &= (y + xz + xzw) + (1 + x + w + xw) = \psi(v_1 + u_1) \\ &= \psi(v_1 + \frac{1}{q(u_1)} u_1) = \psi \circ (J_{\tau_{u_1}} \perp J_{\tau_{u_2}})(v_1), \end{aligned}$$

$$
\begin{aligned}
J_\tau \circ \psi(v_2) &= J_\tau(y + xz + xzw + z + zw) \\
&= (x+y) + (w+z)x + w(w+z)x + (w+z) + w(w+z) \\
&= (x+y) + (wx + zx) + (q(w)x + wzx) + (w+z) + (q(w) + wz) \\
&= (x+y) + (xw + xz + 1) + (0 + xzw + w) + (w+z) + (0 + zw) \\
&= (y + xz + xzw + z + zw) + (1 + x + xw) = \psi(v_2 + u_2) \\
&= \psi(v_2 + \frac{1}{q(u_2)} u_2) = \psi \circ (J_{\tau_{u_1}} \perp J_{\tau_{u_2}})(v_2).
\end{aligned}
$$

Finally note that $\theta(\tau_{u_1}) = q(u_1)F^{\times 2} = 1 \in F^\times/F^{\times 2}$ and $\theta(\tau_{u_2}) = q(u_2)F^{\times 2} = 1 \in F^\times/F^{\times 2}$ and the result is proved. $\square$

Remark 6.11. Using (6.10), one can give a shorter proof for (4.6) and (4.11 (b)).

7 Involutions of an even Clifford algebra with trivial Arf invariant

Remark 7.1. Let F be a field and let (A, σ) and (B, τ) be two F-algebras with involutions. Let $A \times B$ be the direct product of A and B. The involution on $A \times B$ defined by $(a, b) \mapsto (\sigma(a), \tau(b))$ is denoted by $\sigma \times \tau$.

The proof of the next result follows from the standard properties of tensor and direct product and we left it to the reader.

Lemma 7.2. *Let F be a field and let (A, σ), (B, τ) and (C, ρ) be F-algebras with involution. Then there exists an isomorphism of F-algebras with involution $((A \times B) \otimes_F C, (\sigma \times \tau) \otimes \rho) \simeq ((A \otimes_F C) \times (B \otimes_F C), (\sigma \otimes \rho) \times (\tau \otimes \rho))$.*

Remark 7.3. Let (V, q) be a quadratic space with trivial Arf invariant and let τ be an involution in $O(V, q)$. We claim that there is a τ-symplectic basis $\{u_1, v_1, \cdots, u_n, v_n\}$ of V such that $\sum_{i=1}^n q(u_i)q(v_i) = 0$. If τ is an orthogonal sum of interchange isometries, then by (3.6), $V = \mathbb{H}_1 \perp \cdots \cdots \mathbb{H}_{2s}$. Choose a standard symplectic basis $\{u_i, v_i\}$ of $\mathbb{H}_i \simeq [1, 0]$, $i = 1, \cdots, 2s$. Then $\{u_1, v_1, \cdots, u_n, v_n\}$ is the desired basis. Otherwise there exists either an identity map or a reflection in the Wiitala decomposition of τ. So there is a τ-symplectic basis $\{u_1, v_1, \cdots, u_n, v_n\}$ of V such that $q(u_1) \neq 0$. We have $\sum_{i=1}^n q(u_i)q(v_i) = a^2 + a \in \wp(F)$, where $a \in F$. If $a \neq 0$, by replacing v_1 with $v_1' = v_1 + aq(u_1)^{-1}u_1$, we have $\sum_{i=1}^n q(u_i)q(v_i) = 0$.

Lemma 7.4. *For $n \geqslant 2$, let $(\mathbb{A}_1, q_1), \cdots, (\mathbb{A}_n, q_n)$ be 4-dimensional hyperbolic spaces over a field F of characteristic 2. Then we have $C_0(\mathbb{A}_1 \perp \cdots \perp \mathbb{A}_n) \simeq C_0(\mathbb{A}_1) \otimes C(\mathbb{A}_2 \perp \cdots \perp \mathbb{A}_n)$.*

Proof. For every $1 \leqslant i \leqslant n$, there exist hyperbolic planes $\mathbb{H}_{2i-1}$ and $\mathbb{H}_{2i}$ such that $\mathbb{A}_i = \mathbb{H}_{2i-1} \perp \mathbb{H}_{2i}$. Let $\{u_i, v_i\}$, $i = 1, \cdots, 2n$, be a standard symplectic basis of $\mathbb{H}_i \simeq [1, 0]$. Set $(W_1, q_1) = \mathbb{H}_1 \perp \mathbb{H}_3 \perp \cdots \perp \mathbb{H}_{2n-1}$ and $(W_2, q_2) = \mathbb{H}_2 \perp \mathbb{H}_4 \perp \cdots \perp \mathbb{H}_{2n}$.

Define the linear map $f_1 : W_1 \to C(W_1)$ via $f_1(u_1) = u_1$, $f_1(v_1) = v_1 + u_1(u_3v_3 + \cdots + u_{2n-1}v_{2n-1})$, $f_1(u_{2i-1}) = u_1u_{2i-1}$ and $f_1(v_{2i-1}) = u_1v_{2i-1}$ for $i = 2, \cdots, n$.

Similarly define the linear map $f_2 : W_2 \to C(W_2)$ via $f_2(u_2) = u_2$, $f_2(v_2) = v_2 + u_2(u_4v_4 + \cdots + u_{2n}v_{2n})$, $f_2(u_{2i}) = u_2u_{2i}$ and $f_2(v_{2i}) = u_2v_{2i}$ for $i = 2, \cdots, n$.

By the proof of (5.1), the maps f_1 and f_2 are compatible with q_1 and q_2, respectively; for example for the map f_1, considering the notation of the proof of (5.1) we have $\mathbb{E}' = \mathbb{E} = \mathbb{H}_1$, $V = \mathbb{H}_3 \perp \cdots \perp \mathbb{H}_{2n-1}$, $u = x = u_1$, $v = y = v_1$, $a = q(u_1) = 1$, $x^{-1} = u_1$ and $\delta = q(u_3)q(v_3) + \cdots + q(u_{2n-1})q(v_{2n-1}) = 0$. Now define the linear map $f_1' : W_1 \to C(W_1)$ the same as f_1 except that $f_1'(v_1) = v_1 + u_1(u_3v_3 + \cdots + u_{2n-1}v_{2n-1} + (n-1)\cdot 1) = f_1(v_1) + (n-1)\cdot u_1$. We claim that f_1' is also compatible with q_1. It is enough to show that for every $x, y \in \{u_1, v_1, \cdots, u_{2n-1}, v_{2n-1}\}$, we have $f_1'(x)^2 = q_1(x)$ and $f_1'(x)f_1'(y) + f_1'(y)f_1'(x) = B_{q_1}(x, y)$. It is clear that $(n-1)\cdot u_1 = f_1'(v_1) - f_1(v_1)$ commutes with $f_1(u_1)$, $f_1(u_{2i-1})$ and $f_1(v_{2i-1})$ for $i = 2, \cdots, n$. Since f_1 is compatible with q_1, in order to see that f_1' is compatible with q_1, it is enough to show that $f_1'(v_1)^2 = q_1(v_1)$. We have

$$\begin{aligned}
f_1'(v_1)^2 &= (f_1(v_1) + (n-1)\cdot u_1)^2 \\
&= f_1(v_1)^2 + (n-1)^2 q_1(u_1)\cdot 1 \\
&\quad + (n-1)\cdot u_1(v_1 + u_1(u_3v_3 + \cdots + u_{2n-1}v_{2n-1})) \\
&\quad + (n-1)\cdot (v_1 + u_1(u_3v_3 + \cdots + u_{2n-1}v_{2n-1}))u_1 \\
&= q_1(v_1)\cdot 1 + (n-1)^2\cdot 1 + (n-1)\cdot (u_1v_1 + v_1u_1) \\
&= q_1(v_1)\cdot 1 + (n-1)\cdot 1 + (n-1)\cdot 1 = q_1(v_1)\cdot 1.
\end{aligned}$$

So the claim is proved. Since the image of f_1' commutes with that of f_2, the map $f = f_1' \oplus f_2$ on $W_1 \perp W_2$ is compatible with $q = q_1 \perp q_2$ and it can be extended to an isomorphism $\bar{f} : C(L) = C(W_1 \perp W_2) \simeq C(L) = C(\mathbb{A}_1 \perp L')$ where $L = \mathbb{A}_1 \perp \cdots \perp \mathbb{A}_n$ and $L' = \mathbb{A}_2 \perp \cdots \perp \mathbb{A}_n$. By (4.4), there exists an isomorphism $g : C(\mathbb{A}_1 \perp L') \simeq C(\mathbb{A}_1) \otimes C(L')$. So there exists an isomorphism $\psi = g \circ \bar{f} : C(L) \simeq C(\mathbb{A}_1) \otimes C(L')$. The restriction of ψ to $C_0(L)$ maps $C_0(L)$ to $C_0(\mathbb{A}_1) \otimes C(L')$. So we have an isomorphism $\psi : C_0(L) \simeq C_0(\mathbb{A}_1) \otimes C(L')$. $\square$

Proposition 7.5. *For $n \geqslant 2$, let $(\mathbb{A}_1, q_1), \cdots, (\mathbb{A}_n, q_n)$ be 4-dimensional quadratic spaces over a field F of characteristic 2. For $i = 1, \cdots, n$, let τ_i be an interchange isometry of $\mathbb{A}_i$. We have $(C_0(\mathbb{A}_1 \perp \cdots \perp \mathbb{A}_n), J_{\tau_1 \perp \cdots \perp \tau_n}) \simeq (C_0(\mathbb{A}_1), J_{\tau_1}) \otimes (C(\mathbb{A}_2 \perp \cdots \perp \mathbb{A}_n), J_{\tau_2 \perp \cdots \perp \tau_n})$.*

Proof. By (3.6), for every $1 \leqslant i \leqslant n$, there exist quadratic planes $\mathbb{H}_{2i-1}$ and $\mathbb{H}_{2i}$ such that $\mathbb{A}_i = \mathbb{H}_{2i-1} \perp \mathbb{H}_{2i}$, $\tau_i(\mathbb{H}_{2i-1}) = \mathbb{H}_{2i}$ and $\tau_i(\mathbb{H}_{2i}) = \mathbb{H}_{2i-1}$. Let $\{u_i, v_i\}$, $i = 1, \cdots, 2n$, be a standard symplectic basis of $\mathbb{H}_i \simeq [1, 0]$ such that for $1 \leqslant i \leqslant n$, $\tau(u_{2i-1}) = u_{2i}$, $\tau(u_{2i}) = u_{2i-1}$, $\tau(v_{2i-1}) = v_{2i}$ and $\tau(v_{2i}) = v_{2i-1}$. Set $L = \mathbb{A}_1 \perp \cdots \perp \mathbb{A}_n$, $L' = \mathbb{A}_2 \perp \cdots \perp \mathbb{A}_n$, $\tau = \tau_1 \perp \cdots \perp \tau_n$ and $\tau' = \tau_2 \perp \cdots \perp \tau_n$. Let $\psi : C_0(L) \simeq C_0(\mathbb{A}_1) \otimes C(L')$ be the isomorphism discussed in (7.4). We want to show that $(J_{\tau_1} \otimes J_{\tau'}) \circ \psi = \psi \circ J_\tau$. We have

$$(J_{\tau_1} \otimes J_{\tau'}) \circ \psi(u_1) = (J_{\tau_1} \otimes J_{\tau'})(u_1 \otimes 1) = u_2 \otimes 1 = \psi(u_2) = \psi \circ J_\tau(u_1),$$

similarly

$$(J_{\tau_1} \otimes J_{\tau'}) \circ \psi(u_2) = \psi \circ J_\tau(u_2).$$

We also have

$$\begin{aligned}(J_{\tau_1} \otimes J_{\tau'}) \circ \psi(v_1) &= (J_{\tau_1} \otimes J_{\tau'})((n-1) \cdot u_1 \otimes 1 + v_1 \otimes 1 \\ &\quad + u_1 \otimes (u_3v_3 + \cdots + u_{2n-1}v_{2n-1})) \\ &= (n-1) \cdot u_2 \otimes 1 + v_2 \otimes 1 + u_2 \otimes (v_4u_4 + \cdots + v_{2n}u_{2n}) \\ &= (n-1) \cdot u_2 \otimes 1 + v_2 \otimes 1 \\ &\quad + u_2 \otimes (u_4v_4 + \cdots + u_{2n}v_{2n} + (n-1) \cdot 1) \\ &= v_2 \otimes 1 + u_2 \otimes (u_4v_4 + \cdots + u_{2n}v_{2n}) = \psi(v_2) = \psi \circ J_\tau(v_1),\end{aligned}$$

similarly

$$(J_{\tau_1} \otimes J_{\tau'}) \circ \psi(v_2) = \psi \circ J_\tau(v_2).$$

For every $2 \leqslant i \leqslant n$, we have

$$\begin{aligned}(J_{\tau_1} \otimes J_{\tau'}) \circ \psi(u_{2i-1}) &= (J_{\tau_1} \otimes J_{\tau'})(u_1 \otimes u_{2i-1}) = u_2 \otimes u_{2i} \\ &= \psi(u_{2i}) = \psi \circ J_\tau(u_{2i-1}),\end{aligned}$$

similarly

$$(J_{\tau_1} \otimes J_{\tau'}) \circ \psi(v_{2i-1}) = \psi \circ J_\tau(v_{2i-1}).$$

Finally for every $2 \leqslant i \leqslant n$, we have

$$\begin{aligned}(J_{\tau_1} \otimes J_{\tau'}) \circ \psi(u_{2i}) &= (J_{\tau_1} \otimes J_{\tau'})(u_2 \otimes u_{2i}) = u_1 \otimes u_{2i-1} \\ &= \psi(u_{2i-1}) = \psi \circ J_\tau(u_{2i}),\end{aligned}$$

and similarly

$$(J_{\tau_1} \otimes J_{\tau'}) \circ \psi(v_{2i}) = \psi \circ J_\tau(v_{2i}).$$

□

Remark 7.6. Let (V, q) be a quadratic space over a field F of characteristic 2 and let $C(V)$ be the Clifford algebra of (V, q). If τ is an involution in $O(V, q)$, then τ induces an involution J'^q_τ on $C(V) \times C(V)$ such that $J'^q_\tau(v, 0) = (0, \tau(v))$ and $J'^q_\tau(0, v) = (\tau(v), 0)$ for every $v \in V$. We usually denote J'^q_τ by J'_τ.

We need the following straightforward result.

Lemma 7.7. *Let (V, q) and (V', q') be two quadratic spaces over a field F of characteristic 2 and let τ and τ' be involutions in $O(V, q)$ and $O(V', q')$, respectively. If $(C(V), J_\tau) \simeq (C(V'), J_{\tau'})$, then $(C(V) \times C(V), J'_\tau) \simeq (C(V') \times C(V'), J'_{\tau'})$.*

Proposition 7.8. *For $n \geqslant 2$, let (V, q) be a $2n$-dimensional quadratic space over a field F of characteristic 2 and let τ be an involution in $O(V, q)$. Suppose that $\Delta(q)$ is trivial and $V = \mathbb{E} \perp V'$ where $\mathbb{E}$ is a quadratic subplane of V such that $\tau(\mathbb{E}) = \mathbb{E}$ and $\tau(V') = V'$. Let $t = \dim \mathrm{Fix}(V, \tau) - \frac{1}{2} \dim V$, $\tau' = \tau|_{V'}$ and $u_1 \in \mathbb{E}$ be an anisotropic vector such that $\tau(u_1) = u_1$. Set $a = q(u_1)$.*

(a) *If $t \equiv 0 \mod 2$, then $(C_0(V, q), J_\tau) \simeq (C(V', a{\cdot}q) \times C(V', a{\cdot}q), J^{a\cdot q}_{\tau'} \times J^{a\cdot q}_{\tau'})$.*

(b) *If* $t \equiv 1 \mod 2$*, then* $(C_0(V,q), J_\tau) \simeq (C(V', a\cdot q) \times C(V', a\cdot q), J'^{a\cdot q}_{\tau'})$.

Also $C(V', a\cdot q)$ *is Brauer equivalent to* $C(V,q)$*. By abuse of notation, we have denoted the form* $a\cdot q|_{V'}$ *by* $a\cdot q$*.*

Proof. Let $\{u_2, v_2, \cdots, u_n, v_n\}$ be a symplectic basis of V'. By (7.3), we can extend $\{u_1\}$ to a symplectic basis $\{u_1, v_1\}$ of $\mathbb{E}$ such that $q(u_1)q(v_1) + \cdots + q(u_n)q(v_n) = 0$. Let $z = u_1v_1 + \cdots + u_nv_n$. Then by [4, pp. 122-124], the map $\psi : C(V', a\cdot q) \times C(V', a\cdot q) \to C_0(V,q)$ induced by $\psi(u_i, 0) = u_1u_iz$, $\psi(v_i, 0) = u_1v_iz$, $\psi(0, u_i) = u_1u_i(z+1)$ and $\psi(0, v_i) = u_1v_i(z+1)$, $i = 1, \cdots, n$, is an F-algebra isomorphism. Set $\sigma = \psi^{-1}\circ J_\tau \circ \psi$. Then σ is an involution on $C(V', a\cdot q) \times C(V', a\cdot q)$.

(a) We must show that $J_\tau \circ \psi = \psi \circ (J^{a\cdot q}_{\tau'} \times J^{a\cdot q}_{\tau'})$ or equivalently $\sigma = J^{a\cdot q}_{\tau'} \times J^{a\cdot q}_{\tau'}$. For this it suffices to verify that for every $w, w' \in C(V')$ we have $\sigma(w, w') = (\tau'(w),\ \tau'(w'))$. It is enough to check this for $(w, w') = (x, 0)$ and $(w, w') = (0, x)$ where $x \in V'$. By (4.12), we have $\tau(z) = z$. As $z \in Z(C_0(V))$, we have $zw = wz$ for every $w \in C_0(V)$. Also $u_1x = xu_1$ for every $x \in V'$. By the hypothesis, for every $x \in V'$ we have $\tau(x) \in V'$ and

$$\begin{aligned}\sigma(x,0) &= \psi^{-1}\circ J_\tau \circ \psi(x,0) = \psi^{-1}\circ J_\tau(u_1xz) = \psi^{-1}(z\tau(x)u_1)\\ &= \psi^{-1}(u_1\tau(x)z) = (\tau(x), 0) = (\tau'(x), 0).\end{aligned}$$

Similarly

$$\begin{aligned}\sigma(0,x) &= \psi^{-1}\circ J_\tau \circ \psi(0,x) = \psi^{-1}\circ J_\tau(u_1x(z+1)) = \psi^{-1}((z+1)\tau(x)u_1)\\ &= \psi^{-1}(u_1\tau(x)(z+1)) = (0, \tau(x)) = (0, \tau'(x)).\end{aligned}$$

So $(C_0(V,q), J_\tau) \simeq (C(V', a\cdot q) \times C(V', a\cdot q), J^{a\cdot q}_{\tau'} \times J^{a\cdot q}_{\tau'})$.

(b) Like in the previous case, it suffices to verify that for every $w, w' \in C(V')$ we have $\sigma(w, w') = (\tau'(w'),\ \tau'(w))$. It is enough to check this for $(w, w') = (x, 0)$ and $(w, w') = (0, x)$ where $x \in V'$. By (4.12), we have $\tau(z) = z + 1$. Again by the hypothesis, for every $x \in V'$ we have $\tau(x) \in V'$ and

$$\begin{aligned}\sigma(x,0) &= \psi^{-1}\circ J_\tau \circ \psi(x,0) = \psi^{-1}\circ J_\tau(u_1xz) = \psi^{-1}((z+1)\tau(x)u_1)\\ &= \psi^{-1}(u_1\tau(x)(z+1)) = (0, \tau(x)) = (0, \tau'(x)).\end{aligned}$$

Similarly

$$\begin{aligned}\sigma(0,x) &= \psi^{-1}\circ J_\tau \circ \psi(0,x) = \psi^{-1}\circ J_\tau(u_1x(z+1)) = \psi^{-1}(z\tau(x)u_1)\\ &= \psi^{-1}(u_1\tau(x)z) = (\tau(x), 0) = (\tau'(x), 0).\end{aligned}$$

So $(C_0(V,q), J_\tau) \simeq (C(V', a\cdot q) \times C(V', a\cdot q), J'^{a\cdot q}_{\tau'})$.

The last statement follows from (5.1 (i)) and the triviality of the Arf invariant. □

Proposition 7.9. *Let* (V,q) *be a* $2n$*-dimensional quadratic space over a field* F *of characteristic* 2 *and let* τ *be an involution in* $O(V,q)$*. Then there exists a quadratic space* (V', q') *such that* $(C(V) \times C(V), J'_\tau) \simeq (C(V') \times C(V'), J'_{\mathrm{id}})$*.*

Proof. If J_τ is of symplectic type, then by (6.3), there exists a quadratic space (V', q') over F such that $(C(V), J_\tau) \simeq (C(V'), J_{\mathrm{id}})$, so by (7.7), we have $(C(V)\times$

$C(V), J'_\tau) \simeq (C(V') \times C(V'), J'_{\mathrm{id}})$. So consider the case where J_τ is of orthogonal type, thus by (4.7), τ is either an orthogonal sum of reflections or an orthogonal sum of interchange isometries. By (6.10), the Clifford algebra of a 4-dimensional quadratic space with an involution induced by an interchange isometry is isomorphic to the Clifford algebra of a 4-dimensional quadratic space with an involution induced by an orthogonal sum of two reflections, so in order to prove the result we may consider the case where $\tau = \tau_1 \perp \cdots \perp \tau_n$ is an orthogonal sum of reflections. Let $V = \mathbb{E}_1 \perp \cdots \perp \mathbb{E}_n$ be a Wiitala decomposition of (V, τ) such that $\tau_i = \tau|_{\mathbb{E}_i}$. Let $\{u_1, v_1, \cdots, u_n, v_n\}$ be a τ-symplectic basis of V and $\delta = q(u_1)q(v_1) + \cdots + q(u_n)q(v_n) \in F$. Let $(\mathbb{E}_{n+1}, \varphi)$ be the quadratic plane $[1, \delta]$ over F with a standard symplectic basis $\{u_{n+1}, v_{n+1}\}$ and set $z = u_1v_1 + \cdots + u_{n+1}v_{n+1} \in C(V \perp \mathbb{E}_{n+1})$. Consider the involution $\rho = \tau \perp \mathrm{id}_{\mathbb{E}_{n+1}} \in O(V \perp \mathbb{E}_{n+1}, q \perp \varphi)$. We have $\dim \mathrm{Fix}(V \perp \mathbb{E}_{n+1}, \rho) - \frac{1}{2}\dim(V \perp \mathbb{E}_{n+1}) = (n+2) - (n+1) = 1$ and $q(u_1)q(v_1) + \cdots + q(u_{n+1})q(v_{n+1}) = 0$, so by (7.8 (b)), we have $(C_0(V \perp \mathbb{E}_{n+1}), J_\rho) \simeq (C(V) \times C(V), J'_\tau)$; note that $a = \varphi(u_{n+1}) = 1$. On the other hand $V \perp \mathbb{E}_{n+1} = \mathbb{E}_1 \perp V_1$ where $V_1 = \mathbb{E}_2 \perp \cdots \perp \mathbb{E}_{n+1}$. Again by (7.8 (b)), we have $(C_0(V \perp \mathbb{E}_{n+1}), J_\rho) \simeq (C_0(\mathbb{E}_1 \perp V_1), J_{\tau_1 \perp \rho_1}) \simeq (C(V_1, a \cdot q \perp a \cdot \varphi) \times C(V_1, a \cdot q \perp a \cdot \varphi), J'_{\rho_1})$ where $\rho_1 = \tau_2 \perp \cdots \perp \tau_n \perp \mathrm{id}_{\mathbb{E}_{n+1}}$ and $a = q(u_1) \neq 0$. By (4.7), the involution J_{ρ_1} on $C(V_1, a \cdot q \perp a \cdot \varphi)$ is of symplectic type, so by (6.3), we may write $(C(V_1, a \cdot q \perp a \cdot \varphi), J_{\rho_1}) \simeq (C(V', q'), J_{\mathrm{id}})$ for a suitable quadratic space (V', q'). So by (7.7), $(C(V_1, a \cdot q \perp a \cdot \varphi) \times C(V_1, a \cdot q \perp a \cdot \varphi), J'_{\rho_1}) \simeq (C(V') \times C(V'), J'_{\mathrm{id}})$. Therefore, $(C(V) \times C(V), J'_\tau) \simeq (C(V') \times C(V'), J'_{\mathrm{id}})$ and the result is proved. □

Lemma 7.10. *Let $(\mathbb{A}, q)$ be a 4-dimensional quadratic space over a field F of characteristic 2 and let τ be an interchange isometry of $(\mathbb{A}, q)$. Then $(C_0(\mathbb{A}), J_\tau) \simeq (C(\mathbb{H}), J_{\tau_u}) \times (C(\mathbb{H}), J_{\mathrm{id}})$ where $\mathbb{H}$ is a hyperbolic subplane of $\mathbb{A}$ and u is a suitable anisotropic element of $\mathbb{H}$.*

Proof. By (3.6), there exist a decomposition $\mathbb{A} = \mathbb{H}_1 \perp \mathbb{H}_2$ of $\mathbb{A}$ into hyperbolic planes $\mathbb{H}_1$ and $\mathbb{H}_2$ over F and standard symplectic bases $\{u_i, v_i\}$ of $\mathbb{H}_i \simeq [1, 0]$, $i = 1, 2$, such that $\tau(u_1) = u_2$, $\tau(u_2) = u_1$, $\tau(v_1) = v_2$ and $\tau(v_2) = v_1$. We have $q(u_1)q(v_1) + q(u_2)q(v_2) = 0$. Set $z = u_1v_1 + u_2v_2 \in C_0(\mathbb{A})$. By (4.12), we have $z^2 = z$ and $\tau(z) = z$. Also by (7.8), the map $\psi : C(\mathbb{H}_2) \times C(\mathbb{H}_2) \to C_0(\mathbb{A})$ induced by $\psi(u_2, 0) = u_1u_2z$, $\psi(v_2, 0) = u_1v_2z$, $\psi(0, u_2) = u_1u_2(z+1)$ and $\psi(0, v_2) = u_1v_2(z+1)$ is an isomorphism. Set $\sigma = \psi^{-1} \circ J_\tau \circ \psi$. Then σ is an involution on $C(\mathbb{H}_2) \times C(\mathbb{H}_2)$. We claim that $\sigma(x, y) = (J_{\tau_{u_2}}(x), J_{\mathrm{id}}(y))$ for every $x, y \in C(\mathbb{H}_2)$. It is enough to check this equality for $(x, y) = (u_2, 0)$, $(v_2, 0)$, $(0, u_2)$ and $(0, v_2)$. We have

$$\begin{aligned}
\sigma(u_2, 0) &= \psi^{-1} \circ J_\tau \circ \psi(u_2, 0) = \psi^{-1} \circ J_\tau(u_1u_2z) = \psi^{-1}(zu_1u_2) \\
&= \psi^{-1}(u_1u_2z) = (u_2, 0) = (J_{\tau_{u_2}}(u_2), 0),
\end{aligned}$$

Note that we have $u_1z = u_1(u_1v_1 + u_2v_2) = v_1 + u_1u_2v_2$, so $v_1 = u_1z + u_1u_2v_2$ and we obtain

$$\begin{aligned}
\sigma(v_2, 0) &= \psi^{-1} \circ J_\tau \circ \psi(v_2, 0) = \psi^{-1} \circ J_\tau(u_1v_2z) = \psi^{-1}(zv_1u_2) \\
&= \psi^{-1}(u_2v_1z) = \psi^{-1}(u_2(u_1z + u_1u_2v_2)z) = \psi^{-1}(u_1u_2z + u_1v_2z) \\
&= (u_2 + v_2, 0) = (v_2 + \frac{B(u_2, v_2)}{q(u_2)}u_2, 0) = (J_{\tau_{u_2}}(v_2), 0).
\end{aligned}$$

Similarly we have

$$\begin{aligned}\sigma(0,u_2) &= \psi^{-1}\circ J_\tau \circ \psi(0,u_2) = \psi^{-1}\circ J_\tau(u_1u_2(z+1))\\ &= \psi^{-1}((z+1)u_1u_2) = \psi^{-1}(u_1u_2(z+1)) = (0,u_2) = (0, J_{\mathrm{id}}(u_2)),\end{aligned}$$

Finally

$$\begin{aligned}\sigma(0,v_2) &= \psi^{-1}\circ J_\tau \circ \psi(0,v_2) = \psi^{-1}\circ J_\tau(u_1v_2(z+1)) = \psi^{-1}((z+1)v_1u_2)\\ &= \psi^{-1}(u_2v_1(z+1)) = \psi^{-1}(u_2(u_1z+u_1u_2v_2)(z+1))\\ &= \psi^{-1}(0+u_1v_2(z+1)) = (0,v_2) = (0, J_{\mathrm{id}}(v_2)).\end{aligned}$$

□

Theorem 7.11. *Let (V,q) be a $2n$-dimensional quadratic space over a field F of characteristic 2 and let τ be an involution in $O(V,q)$. Suppose that $\Delta(q)$ is trivial and let $t = \dim \mathrm{Fix}(V,\tau) - \frac{1}{2}\dim V$.*

(*a*) *If $t = 0$ and τ is of reflectional kind, then there exists a $(2n-2)$-dimensional quadratic space (V',q') over F such that $(C_0(V), J_\tau) \simeq (C(V')\times C(V'), J_{\tau'}\times J_{\tau'})$ where τ' is an involution in $O(V',q')$ which is an orthogonal sum of $n-1$ reflections. More precisely (V',q') is similar to a subform of (V,q) which can be explicitly described by (7.8).*

(*b*) *If $t = 0$ and τ is of interchanging kind, then there exist $(2n-2)$-dimensional quadratic spaces (V',q') and (V'',q'') over F such that $(C_0(V), J_\tau) \simeq (C(V')\times C(V''), J_{\tau'}\times J_{\mathrm{id}})$ where τ' is an involution in $O(V',q')$ which is an orthogonal sum of $n-1$ reflections. Furthermore we have $C(V') \simeq C(V'')$ and $C(V')$ is isomorphic to the Clifford algebra of a hyperbolic space over F.*

(*c*) *If $0 \neq t \equiv 0 \mod 2$, then there exists a $(2n-2)$-dimensional quadratic space (V',q') over F such that $(C_0(V), J_\tau) \simeq (C(V'), J_{\mathrm{id}}) \times (C(V'), J_{\mathrm{id}})$.*

(*d*) *If $t \equiv 1 \mod 2$, then there exists a $(2n-2)$-dimensional quadratic space (V',q') over F such that $(C_0(V), J_\tau) \simeq (C(V')\times C(V'), J'_{\mathrm{id}})$.*

Proof. Let $V = W \perp \mathbb{E}_1 \perp \cdots \perp \mathbb{E}_r \perp \mathbb{A}_1 \perp \cdots \perp \mathbb{A}_s$ be a Wiitala decomposition of (V,τ) where W is a maximal fixed orthogonal summand of (V,τ). Let $\{u_1, v_1, \cdots, u_n, v_n\}$ be a τ-symplectic basis of V with respect to this decomposition. By (3.19), we have $t = \frac{1}{2}\dim W$. Set $z = u_1v_1 + \cdots + u_nv_n \in C_0(V)$. By (4.12), $J_\tau(z) = z + t\cdot 1$.

(*a*) We have $W = 0$ and $J_\tau(z) = z$. As τ is of reflectional kind, we may assume that $s = 0$ and $r = n$, so one can write $\tau = \tau_{u_1} \perp \cdots \perp \tau_{u_n}$ where τ_{u_i} is the reflection along the anisotropic vector $u_i \in \mathbb{E}_i$. Let $V' = \mathbb{E}_2 \perp \cdots \perp \mathbb{E}_n$ and $a = q(u_1)$. By (7.8 (a)), $(C_0(V), J_\tau) \simeq (C(V', a\cdot q)\times C(V', a\cdot q), J_{\tau'}\times J_{\tau'})$ where $\tau' = \tau_{u_2} \perp \cdots \perp \tau_{u_n}$.

(*b*) We have $W = 0$ and $J_\tau(z) = z$. As τ is of interchanging kind, we may assume that $r = 0$, $n = 2s$ and $\tau = \tau_1 \perp \cdots \perp \tau_s$ where τ_i is an interchange isometry of $\mathbb{A}_i$, $i = 1, \cdots, s$. If $n = 2$, by (7.10), we have $(C_0(V), J_\tau) \simeq (C(\mathbb{H})\times C(\mathbb{H}), J_{\tau'}\times J_{\mathrm{id}})$ where $\mathbb{H}$ is a hyperbolic subplane of V and τ' is a reflection of

$\mathbb{H}$. If $n \geqslant 3$, by (7.5), we have $(C_0(V), J_{\tau_1 \perp \cdots \perp \tau_s}) \simeq (C_0(\mathbb{A}_1), J_{\tau_1}) \otimes (C(\mathbb{A}_2 \perp \cdots \perp \mathbb{A}_s), J_{\tau_2 \perp \cdots \perp \tau_s})$. Using (7.5), (7.10) and (7.2) we obtain

$$\begin{aligned}
(C_0(V), J_{\tau_1 \perp \cdots \perp \tau_s}) &\simeq (C_0(\mathbb{A}_1), J_{\tau_1}) \otimes (C(\mathbb{A}_2 \perp \cdots \perp \mathbb{A}_s), J_{\tau_2 \perp \cdots \perp \tau_s}) \\
&\simeq ((C(\mathbb{H}_1), J_{\tau_1'}) \times (C(\mathbb{H}_1), J_{\mathrm{id}})) \\
&\quad \otimes (C(\mathbb{A}_2 \perp \cdots \perp \mathbb{A}_s), J_{\tau_2 \perp \cdots \perp \tau_s}) \\
&\simeq ((C(\mathbb{H}_1), J_{\tau_1'}) \otimes (C(\mathbb{A}_2 \perp \cdots \perp \mathbb{A}_s), J_{\tau_2 \perp \cdots \perp \tau_s})) \\
&\quad \times ((C(\mathbb{H}_1), J_{\mathrm{id}}) \otimes (C(\mathbb{A}_2 \perp \cdots \perp \mathbb{A}_s), J_{\tau_2 \perp \cdots \perp \tau_s})),
\end{aligned}$$

where $\mathbb{H}_1$ is a hyperbolic subplane of $\mathbb{A}_1$ and τ_1' is a reflection of $\mathbb{H}_1$. By (6.10), we have $(C(\mathbb{A}_2 \perp \cdots \perp \mathbb{A}_s), J_{\tau_2 \perp \cdots \perp \tau_s}) \simeq (C(L), J_\rho)$, where $L = \mathbb{H}_2 \perp \cdots \perp \mathbb{H}_{2s-1}$ is an orthogonal sum of $2s-2 = n-2$ hyperbolic planes $\mathbb{H}_i$ over F and $\rho = \tau_2' \perp \cdots \perp \tau_{2s-1}'$ is an orthogonal sum of $2s-2$ reflections τ_i' of $\mathbb{H}_i$. So we obtain

$$\begin{aligned}
(C_0(V), J_{\tau_1 \perp \cdots \perp \tau_s}) &\simeq ((C(\mathbb{H}_1), J_{\tau_1'}) \otimes (C(L), J_\rho)) \times ((C(\mathbb{H}_1), J_{\mathrm{id}}) \otimes (C(L), J_\rho)) \\
&\simeq (C(\mathbb{H}_1 \perp L), J_{\tau_1' \perp \rho}) \times (C(\mathbb{H}_1 \perp L), J_{\mathrm{id} \perp \rho}). \qquad (4)
\end{aligned}$$

By (4.7), $(C(\mathbb{H}_1 \perp L), J_{\mathrm{id}\perp\rho})$ is of symplectic type, so by (6.3), there exists a quadratic space (V'', q'') over F such that $(C(\mathbb{E}_1' \perp L), J_{\mathrm{id}\perp\rho}) \simeq (C(V''), J_{\mathrm{id}})$, so (4) leads to $(C_0(V), J_\tau) \simeq (C(V') \times C(V''), J_{\tau'} \times J_{\mathrm{id}})$, where $V' = \mathbb{H}_1 \perp L$ and $\tau' = \tau_1' \perp \rho = \tau_1' \perp \cdots \perp \tau_{2s-1}'$. Finally note that as L is a hyperbolic space, V' is also a hyperbolic space, so $C(V') \simeq C(V'')$ is isomorphic to the Clifford algebra of a hyperbolic space over F.

(*c*) We have $\dim W \geqslant 4$ and $J_\tau(z) = z$. So we may assume that for the vector space $W_1 = Fu_1 + Fv_1 + Fu_2 + Fv_2$ we have $\tau|_{W_1} = \mathrm{id}_{W_1}$ where $a = q(u_1) \neq 0$. Let $\mathbb{E} = Fu_1 + Fv_1$, $V_1 = Fu_2 + Fv_2 + \cdots + Fu_n + Fv_n$ and $\tau' = \tau|_{V_1}$. By (7.8 (a)), $(C_0(V), J_\tau) \simeq (C(V_1, a\cdot q) \times C(V_1, a \cdot q), J_{\tau'} \times J_{\tau'})$. On the other hand as $\dim \mathrm{Fix}(V_1, \tau|_{V_1}) - \frac{1}{2}\dim V_1 = t - 1 \geqslant 1$, by (4.7), the involution $J_{\tau'}$ is of symplectic type, so by (6.3), we have $(C(V_1), J_{\tau'}) \simeq (C(V'), J_{\mathrm{id}})$ for some quadratic space (V', q') over F, so $(C_0(V), J_\tau) \simeq (C(V') \times C(V'), J_{\mathrm{id}} \times J_{\mathrm{id}})$.

(*d*) If $t = 1$, then $\dim W = 2$ and we have $V = W \perp W'$ where $W' = W^\perp$. Let $u \in W$ be an anisotropic vector and $a = q(u)$. By (7.8 (b)) we have $(C_0(V, q), J_\tau) \simeq (C_0(W \perp W'), J_{\mathrm{id}_W \perp \tau'}) \simeq (C(W', a\cdot q) \times C(W', a \cdot q), J'_{\tau'})$ where $\tau' = \tau|_{W'}$ and by (6.3) and (7.7), $(C(W', a\cdot q) \times C(W', a\cdot q), J'_{\tau'}) \simeq (C(V') \times C(V'), J'_{\mathrm{id}_{V'}})$ for some quadratic space (V', q'). So $(C_0(V,q), J_\tau) \simeq (C(V') \times C(V'), J'_{\mathrm{id}_{V'}})$. □

Lemma 7.12. *For $i = 1, 2$, let (A_i, σ_i) and (B_i, τ_i) be central simple algebras with involution over a field F. If $(A_1 \times A_2, \sigma_1 \times \sigma_2) \simeq (B_1 \times B_2, \tau_1 \times \tau_2)$, then either $(A_1, \sigma_1) \simeq (B_1, \tau_1)$ and $(A_2, \sigma_2) \simeq (B_2, \tau_2)$ or $(A_1, \sigma_1) \simeq (B_2, \tau_2)$ and $(A_2, \sigma_2) \simeq (B_1, \tau_1)$.*

Proof. Let $\psi : (A_1 \times A_2, \sigma_1 \times \sigma_2) \simeq (B_1 \times B_2, \tau_1 \times \tau_2)$ be an isomorphism. The set $B_1 \times \{0\}$ is a two sided ideal of $B_1 \times B_2$, so $\psi^{-1}(B_1 \times \{0\})$ is a two sided ideal of $A_1 \times A_2$. As A_1 and A_2 are simple rings and ψ is an isomorphism, we have either $\psi^{-1}(B_1 \times \{0\}) = A_1 \times \{0\}$ or $\psi^{-1}(B_1 \times \{0\}) = \{0\} \times A_2$. Suppose that the first case occurs. Define a map $f : A_1 \to B_1$ as follows: for every $a_1 \in A_1$, there exists an element $b_1 \in B_1$ such that $\psi(a_1, 0) = (b_1, 0)$. Set $f(a_1) = b_1$, i.e., for every $a_1 \in A_1$ we have $\psi(a_1, 0) = (f(a_1), 0)$. As $\psi : A_1 \times \{0\} \to B_1 \times \{0\}$ is an isomorphism, the map f is an isomorphism of F-algebras. For every $a_1 \in A_1$ we have $(\tau_1 \times \tau_2) \circ \psi(a_1, 0) = \psi \circ (\sigma_1 \times \sigma_2)(a_1, 0)$, so $(\tau_1 \times \tau_2)(f(a_1), 0) = \psi(\sigma_1(a_1), 0)$. From this we obtain $(\tau_1(f(a_1)), 0) = (f(\sigma_1(a_1)), 0)$ which means that $\tau_1 \circ f = f \circ \sigma_1$. So we have proved that $(A_1, \sigma_1) \simeq (B_1, \tau_1)$. Similarly $(A_2, \sigma_2) \simeq (B_2, \tau_2)$.

If the second case occurs, a similar argument shows that $(A_1, \sigma_1) \simeq (B_2, \tau_2)$ and $(A_2, \sigma_2) \simeq (B_1, \tau_1)$ which completes the proof. □

Corollary 7.13. *For $i = 1, \cdots, n$, let (A_i, σ_i) and (B_i, τ_i) be central simple algebras with involution over a field F. If $(A_1 \times \cdots \times A_n, \sigma_1 \times \cdots \times \sigma_n) \simeq (B_1 \times \cdots \times B_n, \tau_1 \times \cdots \times \tau_n)$, then there exists a permutation $\pi \in S_n$ such that for every $1 \leqslant i \leqslant n$, $(A_i, \sigma_i) \simeq (B_{\pi(i)}, \tau_{\pi(i)})$.*

Remark 7.14. The isomorphisms of (7.11 (a)) and (7.11 (b)) are the best possible, in the sense that the involution $\tau' \in O(V', q')$ in (7.11 (a)) and (7.11 (b)) can not be chosen to be identity; because by (4.7), $(C(V'), J_{\tau'})$ is of orthogonal type and $(C(V'), J_{\mathrm{id}})$ is an algebra with involution of symplectic type. Now by (7.12), the involution $J_{\tau'}$ must be of orthogonal type, so τ' can not be chosen to be identity.

In (7.11), we have shown that if $\tau \in O(V, q)$ is an involution and $\Delta(q) = 0$, then $(C_0(V), J_\tau)$ can be decomposed as a direct product of two isomorphic Clifford algebras with involution. Conversely in (7.15) we will express a direct product of two isomorphic Clifford algebras with involution in the form of $(C_0(V), J_\tau)$ for a suitable quadratic space (V, q) with trivial Arf invariant and an involution $\tau \in O(V, q)$. The proof is very similar to that of (7.11).

Remark 7.15. For $i = 1, 2$, let (V_i, q_i) be a $2n$-dimensional quadratic space over a field F of characteristic 2. For $i = 1, 2$, let τ_i be an involution in $O(V_i, q_i)$ and set $t_i = \dim \mathrm{Fix}(V_i, \tau_i) - \frac{1}{2} \dim V_i$. Then there exists a $(2n+2)$-dimensional quadratic space (V, q) over F with trivial Arf invariant such that $(C(V_1) \times C(V_2), J_{\tau_1} \times J_{\tau_2}) \simeq (C_0(V), J_\tau)$ if and only if one of the following conditions is satisfied:

- (a) $t_1 = t_2 = 0$ and $(C(V_1), J_{\tau_1}) \simeq (C(V_2), J_{\tau_2})$. In this case τ is necessarily an orthogonal sum of reflections. In particular $\dim \mathrm{Fix}(V, \tau) = \frac{1}{2} \dim V$, so by (4.7), J_τ is necessarily of orthogonal type.
- (b) $t_1 = 0$ and $0 \neq t_2 \equiv 0 \mod 2$ (resp. $t_2 = 0$ and $0 \neq t_1 \equiv 0 \mod 2$), $n \equiv 1 \mod 2$, $C(V_1) \simeq C(V_2)$ and for every $u \in \mathrm{Fix}(V_1, \tau_1)$, we have $q(u) \in F^2$. In this case τ is necessarily an orthogonal sum of interchange isometries. In particular $\dim \mathrm{Fix}(V, \tau) = \frac{1}{2} \dim V$, so by (4.7), J_τ is necessarily of orthogonal type.
- (c) $0 \neq t_i \equiv 0 \mod 2$, $i = 1, 2$, and $(C(V_1), J_{\tau_1}) \simeq (C(V_2), J_{\tau_2})$. In this case τ has necessarily nontrivial maximal fixed orthogonal summand, i.e., $\dim \mathrm{Fix}(V, \tau) > \frac{1}{2} \dim V$, so by (4.7), J_τ is necessarily of symplectic type.

Lemma 7.16. *Let $(\mathbb{E}_1, \varphi_1)$ and $(\mathbb{E}_2, \varphi_2)$ be two quadratic planes over a field F of characteristic 2. Let τ_i be a reflection of $\mathbb{E}_i$ and let τ_i' be an involution in $O(\mathbb{E}_i, \varphi_i)$ where either $(\tau_1', \tau_2') = (\tau_1, \tau_2)$ or $(\tau_1', \tau_2') = (\mathrm{id}, \mathrm{id})$. If $(C(\mathbb{E}_1) \times C(\mathbb{E}_1), J_{\tau_1} \times J_{\tau_1'}) \simeq (C(\mathbb{E}_2) \times C(\mathbb{E}_2), J_{\tau_2} \times J_{\tau_2'})$, then $\theta(\tau_1) = \theta(\tau_2)$.*

Proof. By the hypothesis we have $(C(\mathbb{E}_1) \times C(\mathbb{E}_1), J_{\tau_1} \times J_{\tau_1}) \simeq (C(\mathbb{E}_2) \times C(\mathbb{E}_2), J_{\tau_2} \times J_{\tau_2})$ or $(C(\mathbb{E}_1) \times C(\mathbb{E}_1), J_{\tau_1} \times J_{\mathrm{id}}) \simeq (C(\mathbb{E}_2) \times C(\mathbb{E}_2), J_{\tau_2} \times J_{\mathrm{id}})$. In the first case by (7.12), we have $(C(\mathbb{E}_1), J_{\tau_1}) \simeq (C(\mathbb{E}_2), J_{\tau_2})$ which implies that $\theta(\tau_1) = \theta(\tau_2)$. So consider the second case. By (4.8 (b)), $(C(\mathbb{E}_1), J_{\tau_1})$ is of orthogonal type and by (4.8 (a)), $(C(\mathbb{E}_2), J_{\mathrm{id}})$ is of symplectic type, so $(C(\mathbb{E}_1), J_{\tau_1}) \not\simeq (C(\mathbb{E}_2), J_{\mathrm{id}})$. Again by (7.12), we have $(C(\mathbb{E}_1), J_{\tau_1}) \simeq (C(\mathbb{E}_2), J_{\tau_2})$, in particular $\theta(\tau_1) = \theta(\tau_2)$. $\square$

Proposition 7.17. *For $n \geqslant 2$, let (V, q) be a $2n$-dimensional quadratic space over a field F of characteristic 2 and let τ be an involution in $O(V, q)$. Suppose that $\Delta(q)$ is trivial and* $\dim \mathrm{Fix}(V, \tau) = \frac{1}{2} \dim V$.
(i) If τ is of reflectional kind and $(C_0(V), J_\tau) \simeq (C(V') \times C(V'), J_{\tau'} \times J_{\tau'})$ is the isomorphism in (7.11 (a))*, where (V', q') is a $(2n-2)$-dimensional quadratic space over F and τ' is an involution in $O(V', q')$ which is an orthogonal sum of reflections (in particular by* (4.7)*, $(C(V'), J_{\tau'})$ is of orthogonal type), then*

- (a) *If $n = 2$ then* $\mathrm{disc}\, J_{\tau'} = \theta(\tau)$.
- (b) *If $n \geqslant 3$ then* $\mathrm{disc}\, J_{\tau'} = 1 \in F^\times / F^{\times 2}$.

(ii) If τ is of interchanging kind and $(C_0(V), J_\tau) \simeq (C(V') \times C(V''), J_{\tau'} \times J_{\mathrm{id}})$ is the isomorphism of (7.11 (b))*, where (V', q') and (V'', q'') are $(2n-2)$-dimensional quadratic space over F and τ' is an involution in $O(V', q')$ which is an orthogonal sum of reflections (in particular by* (4.7)*, $(C(V'), J_{\tau'})$ is of orthogonal type), then* $\mathrm{disc}\, J_{\tau'} = 1 \in F^\times / F^{\times 2}$.

Proof. (i) (a) Note that by (4.11 (a)), $\mathrm{disc}\, J_{\tau'} = \theta(\tau')$ and by (7.16), $\theta(\tau')$ is independent of the choice of V'. As $\dim \mathrm{Fix}(V, \tau) = \frac{1}{2} \dim V$, (V, τ) has trivial maximal fixed orthogonal summand. Since τ is of reflectional kind, (V, τ) has a Wiitala decomposition $V = \mathbb{E}_1 \perp \mathbb{E}_2$ where $\tau_i = \tau|_{\mathbb{E}_i}$ is a reflection along some anisotropic vector $u_i \in \mathbb{E}_i$, $i = 1, 2$. By (7.8), we can write $(C_0(V), J_\tau) \simeq (C(\mathbb{E}_2, a \cdot q) \times C(\mathbb{E}_2, a \cdot q), J_{\tau_2} \times J_{\tau_2})$ where $a = q(u_1)$. By (4.11 (a)), $\mathrm{disc}\, J_{\tau_2} = \theta(\tau_2) = aq(u_2)F^{\times 2} = q(u_1)q(u_2)F^{\times 2} = \theta(\tau)$.

(b) We have $\dim V' \geqslant 4$, so by (4.11 (b)), $\mathrm{disc}\, J_{\tau'}$ is trivial.
(ii) If $n = 2$, by (7.10), we can choose $(V', q') = (V'', q'') = (\mathbb{H}, \varphi)$ where $(\mathbb{H}, \varphi)$ is a hyperbolic plane over F and $\tau' = \tau_u$ where $\varphi(u) = 1$. Also by (7.16), $\theta(\tau')$ is independent of the choice of $\mathbb{E}$. So by (4.11), $\mathrm{disc}\, J_{\tau'} = \varphi(u)F^{\times 2} = 1 \in F/F^{\times 2}$. If $n \geqslant 3$, we have $\dim V' \geqslant 4$, so by (4.11 (b)), $\mathrm{disc}\, J_{\tau'}$ is trivial. $\square$

A Appendix

Definition A.1. Let (V, q) be a quadratic space over a field F of characteristic 2 and let A be an F-algebra. A bilinear map $\psi : V \times V \to A$ is called an *even Clifford map* if for every $x, y, z \in V$ we have:

- (a) $\psi(x, y)\psi(y, z) = q(y) \cdot \psi(x, z)$.
- (b) $\psi(x, x) = q(x) \cdot 1_A$.

Remark A.2. Let (V, q, B) be a quadratic space over a field F of characteristic 2 and let A be an F-algebra. Let $\psi : V \times V \to A$ be an even Clifford map and $u, v \in V$. Then by computing $\psi(u + v, u + v)$, we have $\psi(u, v) + \psi(v, u) = B(u, v) \cdot 1_A$.

Lemma A.3. *Let (V, q, B) be a quadratic space over a field F of characteristic 2 and let A be an F-algebra. Let $\psi : V \times V \to A$ be an even Clifford map and $u, v, w, x \in V$. Then $\psi(u, v)\psi(w, x) = \psi(u, w)\psi(v, x) + B(v, w) \cdot \psi(u, x)$.*

Proof. Using (a) in (A.1) and the bilinearity of ψ, we have

$$q(v+w)\cdot\psi(u, x) = (q(v+w)+B(v, w))\cdot\psi(u, x)+\psi(u, v)\psi(w, x)+\psi(u, w)\psi(v, x).$$

So $\psi(u, v)\psi(w, x) = \psi(u, w)\psi(v, x) + B(v, w) \cdot \psi(u, x)$. $\square$

Corollary A.4. *Let (V, q, B) be a quadratic space over a field F of characteristic 2 and let A be an F-algebra. Let $\psi : V \times V \to A$ be an even Clifford map and $u, v \in V$. Then $\psi(u, v)\psi(u, v) = q(u)q(v) \cdot 1_A + B(u, v) \cdot \psi(u, v)$.*

Theorem A.5. *Let (V, q) be a quadratic space over a field F of characteristic 2. There exists an even Clifford map j from $V \times V$ to $C_0(V)$ which satisfies the following conditions:*

(a) *As an F-algebra $C_0(V)$ is generated by $1_{C_0(V)}$ and $\{j(x, y)|x, y \in V\}$.*

(b) *For every even Clifford map $\psi : V \times V \to A$, there exists a unique F-algebra homomorphism $\Psi : C_0(V) \to A$ such that $\psi = \Psi \circ j$, i.e., the following diagram commutes:*

$$\begin{tikzcd} V \times V \arrow[rr, "\psi"] \arrow[dr, "j"'] & & A \\ & C_0(V) \arrow[ur, "\Psi"'] & \end{tikzcd} \tag{5}$$

Proof. Define $j : V \times V \to C_0(V)$ via $j(u, v) = uv$. Then j is clearly an even Clifford map. Let $\psi : V \times V \to A$ be an even Clifford map. Let $u_1 \in V$ be an anisotropic vector and set $a = q(u_1)$. Extend $\{u_1\}$ to a symplectic basis $\{u_1, v_1, \cdots, u_n, v_n\}$ of V and set $z = u_1v_1 + \cdots u_nv_n$. By (4.12), we have $z^2 = z + \delta \cdot 1$ where $\delta = \sum_{i=1}^{n} q(u_i)q(v_i) \in F$ is a representative of the class $\Delta(q) \in F/\wp(F)$. By [4, (13.12)], the map $g : C_0(V) \to C(W, a \cdot q) \otimes (F + Fz)$ induced by $g(u_1w) = w \otimes 1$ and $g(z) = 1 \otimes z$ is an isomorphism, where W is the subspace of V generated by $u_2, v_2, \cdots, u_n, v_n$. Define the linear map $f_1 : W \to A$ via $f_1(w) = \psi(u_1, w)$. By (A.4), for every $w \in W$ we have $f_1(w)^2 = \psi(u_1, w)\psi(u_1, w) = q(u_1)q(w) \cdot 1_A + B(u_1, w) \cdot \psi(u_1, w) = a \cdot q(w)$. So the map f_1 is compatible with $a \cdot q$ and it can be extended to an F-algebra homomorphism $\bar{f}_1 : C(W, a \cdot q) \to A$. Now define the map $f_2 : F + Fz \to A$ via $f_2(\alpha + \beta z) = \alpha + \beta(\psi(u_1, v_1) + \cdots + \psi(u_n, v_n))$. By (A.4), we have $\psi(u_i, v_i)^2 = q(u_i)q(v_i) \cdot 1_A + \psi(u_i, v_i)$. Also for every $i \neq j$, by (A.2) and (A.3), we have

$$\psi(u_i, v_i)\psi(u_j, v_j) = \psi(u_i, u_j)\psi(v_i, v_j) = \psi(u_j, u_i)\psi(v_j, v_i) = \psi(u_j, v_j)\psi(u_i, v_i)$$

So

$$\begin{aligned} f_2(z)^2 &= (\psi(u_1, v_1) + \cdots + \psi(u_n, v_n))^2 = \psi(u_1, v_1)^2 + \cdots + \psi(u_n, v_n)^2 \\ &= q(u_1)q(v_1) \cdot 1_A + \psi(u_1, v_1) + \cdots + q(u_n)q(v_n) \cdot 1_A + \psi(u_n, v_n) \\ &= \delta \cdot 1_A + \psi(u_1, v_1) + \cdots + \psi(u_n, v_n) = f_2(\delta + z) = f_2(z^2). \end{aligned}$$

Therefore, $f_2((\alpha_1 + \beta_1 z)(\alpha_2 + \beta_2 z)) = f_2(\alpha_1 + \beta_1 z) f_2(\alpha_2 + \beta_2 z)$ for every $\alpha_1, \beta_1, \alpha_2, \beta_2 \in F$ and f_2 is an F-algebra homomorphism.

We claim that the image of $\bar{f}_1$ commutes with that of f_2. We must show that for every $w \in W$:

$$\psi(u_1, w)(\psi(u_1, v_1)+\cdots+\psi(u_n, v_n)) = (\psi(u_1, v_1) + \cdots + \psi(u_n, v_n))\psi(u_1, w). \quad (6)$$

It is enough to check this for $w = u_i$ and $w = v_i$, $i = 1, \cdots, n$. If $i = 1$, (6) is clearly true. So fix an index i with $2 \leqslant i \leqslant n$. If $j \neq 1$ and $j \neq i$, by (A.2) and (A.3), we have $\psi(u_1, u_i)\psi(u_j, v_j) = \psi(u_j, v_j)\psi(u_1, u_i)$. If $j = 1$ or $j = i$, we have

$$\begin{aligned}
&\psi(u_1, u_i)\psi(u_1, v_1) + \psi(u_1, u_i)\psi(u_i, v_i) \\
&\qquad = \psi(u_1, u_1)\psi(u_i, v_1) + B(u_i, u_1) \cdot \psi(u_1, v_1) + q(u_i) \cdot \psi(u_1, v_i) \\
&\qquad = q(u_1) \cdot \psi(u_i, v_1) + q(u_i) \cdot \psi(u_1, v_i).
\end{aligned}$$

Similarly

$$\psi(u_1, v_1)\psi(u_1, u_i) + \psi(u_i, v_i)\psi(u_1, u_i) = q(u_1) \cdot \psi(u_i, v_1) + q(u_i) \cdot \psi(u_1, v_i).$$

So $\psi(u_1, u_i)\psi(u_1, v_1)+\psi(u_1, u_i)\psi(u_i, v_i) = \psi(u_1, v_1)\psi(u_1, u_i)+\psi(u_i, v_i)\,\psi(u_1, u_i)$ and the equality (6) is achieved for $w = u_i$. Similarly (6) is true for $w = v_i$ and the claim is proved. By [2, Ch. 5, Thm. 2] one can consider the F-algebra homomorphism $f = \bar{f}_1 \otimes f_2 : C(W, a \cdot q) \otimes (F + Fz) \to A$.

Set $\Psi = f \circ g$. We claim that $\psi = \Psi \circ j$. For every $v \in V$ we can write $v = \alpha u_1 + \beta v_1 + w$, where $\alpha, \beta \in F$ and $w \in W$. Since $j(v, v') = vv'$ for every $v, v' \in V$, it is enough to show that $\psi(u_1, u_1) = \Psi(u_1 u_1)$, $\psi(v_1, v_1) = \Psi(v_1 v_1)$, $\psi(u_1, v_1) = \Psi(u_1 v_1)$, $\psi(u_1, w) = \Psi(u_1 w)$, $\psi(v_1, w) = \Psi(v_1 w)$ and $\psi(w_1, w_2) = \Psi(w_1 w_2)$ for every $w, w_1, w_2 \in W$. The first two relations follow from the F-linearity of Ψ and the property (b) of (A.1). For every $w \in W$ we have

$$\Psi(u_1 w) = f \circ g(u_1 w) = \bar{f}_1(w) = \psi(u_1, w),$$

Similarly

$$\begin{aligned}
\Psi(w_1 w_2) &= a^{-1} \cdot \Psi(u_1 w_1 u_1 w_2) = a^{-1} \cdot (f \circ g)(u_1 w_1)(f \circ g)(u_1 w_2) \\
&= a^{-1} \cdot \bar{f}_1(w_1)\bar{f}_1(w_2) = a^{-1} \cdot \psi(u_1, w_1)\psi(u_1, w_2) \\
&= a^{-1} \cdot \psi(w_1, u_1)\psi(u_1, w_2) = a^{-1} q(u_1) \cdot \psi(w_1, w_2) = \psi(w_1, w_2).
\end{aligned}$$

The relations $\Psi(u_1 v_1) = \psi(u_1, v_1)$ and $\Psi(w v_1) = \psi(w, v_1)$ are obtained with the same arguments. For the last one, we use $wv_1 = v_1 w = a^{-1}(wu_1 z + wu_1 u_2 v_2 + \cdots + wu_1 u_n v_n)$.

Finally, the uniqueness of Ψ follows from the facts that as an F-algebra, $C_0(V)$ is generated by $1_{C_0(V)}$ and the set $\{j(u, v) | u, v \in V\}$, and the relation $\Psi(uv) = \Psi \circ j(u, v) = \psi(u, v)$. □

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