A NOTE ON THE PYTHAGORAS NUMBER OF REAL FUNCTION FIELDS

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ABSTRACT. We show that the pythagoras number of a real function field is strictly larger than its transcendence degree. In the purely transcendental case, this is a well known consequence of the Cassels-Pfister Theorem.

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1. INTRODUCTION

A field F is called *real* if -1 is not a sum of squares in F, and *nonreal* otherwise. The *pythagoras number of F* is defined as

$$p(F) = \inf \{n \in \mathbb{N} \mid \text{every sum of squares in } F \text{ is a sum of } n \text{ squares}\}.$$

The main goal of this note is to prove the following lower estimate for the pythagoras number of real function fields.

Theorem 1.1. *Let F be real field that is finitely generated of transcendence degree $d \geq 0$ over a subfield. Then $p(F) \geq d + 1$.*

This result will be shown in Theorem 3.2. It is well known in the case where F is a rational function field[1] in d variables over some real field. In fact, a slightly better bound is known in this case:

Theorem 1.2 (Cassels, Ellison, Pfister)**.** *Let K be real. Then*

$$p(K(X_1, \dots, X_d)) \geq \begin{cases} d+2 & d \geq 2 \\ 2 & d = 1. \end{cases}$$

In case $d \geq 3$, we will reduce Theorem 1.1 to the case $d-1$ of Theorem 1.2. But before we go into the technical details of the proof of Theorem 1.1, let us put the result in some context.

The quantitative study of sums of squares in function fields over the field $\mathbb{R}$ of real numbers was fueled by Pfister's discovery [Pf1] that 2^d bounds the pythagoras number of any function field of transcendence degree d over $\mathbb{R}$. Optimality of this bound is unknown for $d \geq 3$. Obtaining good lower estimates for the pythagoras number of a real function fields $F/\mathbb{R}$ in terms of its transcendence degree is therefore of natural interest.

The search for lower estimates historically focused on the rational case $F = \mathbb{R}(X_1, \dots, X_d)$ for indeterminates $X_1, \dots, X_d$. This focus is explained by the

[1] also called a *purely transcendental field extension*

fact that the rational case allows to work with the Cassels-Pfister theorem [La, Chap. IX, Theorem 1.3], but also by a feeling that the set of pythagoras numbers of all function fields of fixed transcendence degree d should obtain its maximum for the rational function field. Theorem 1.2 yields the best known lower estimate for the rational case. It also yields optimality of Pfister's bound 2^d in the case of transcendence degree $d = 1, 2$.

In fact, it is known that 2 is not only the maximum of the set of all pythagoras numbers of function field of transcendence degree one over $\mathbb{R}$, but that this set equals $\{2\}$. This follows from the Diller-Dress theorem [La, Chap. VIII, 5.7], by which no finite extension of a rational function field can have pythagoras number one.

Consequently, it is known that $2 \leq p(F) \leq 4$ for any function field $F/\mathbb{R}$ of transcendence degree two. Theorem 1.1 shows that $3 \leq p(F) \leq 4$ when F is real (a closer analysis of our proof of Theorem 1.1 shows that the same also holds when F is nonreal and $\mathbb{R}$ is relatively algebraically closed in F).

A natural question is to ask whether there exists a real function field $F/\mathbb{R}$ of transcendence degree two with $p(F) = 3$. Our methods do not give an answer.

Let me now give a different perspective on Theorem 1.1: As with any field invariant, the behavior of the pythagoras number under field extensions is of natural interest. Not much is known on this behaviour for finite extensions L/F of real fields. By [Pf2, Chap. 7, Prop. 1.13] we have the theoretical bound $p(L) \leq [L : F] \cdot p(F)$ for a finite extension of real fields L/F (but to my knowledge, no example is even known where $p(L) > p(F) + 1$). Conversely, examples of quadratic real extensions L/F such that $p(F)$ is arbitrarily large and $p(L) = 2$ were constructed in [Pr]. However, these examples are far from being finitely generated over some 'natural' subfield F_0. Theorem 1.1 explains why such examples cannot be finitely generated of transcendence degree two or higher over any subfield.

We conclude the introduction by outlining the proof of Theorem 1.2. In the literature (see e.g. [Pf2, p. 96/97]), this proof is given in the case of real closed base fields but it easily extends to the more general situation of Theorem 1.2:

It is clear that $1+X^2$ is not a square in $K(X)$ for any real field K. This shows $p(K(X)) \geq 2$. More generally, a consequence ([Pf2, Chap. 1, Theorem 3.2]) of the Cassels-Pfister theorem implies that if $a \in K^\times$ is not a sum of m squares in K for some $m \in \mathbb{N}$ then $a + X^2$ is not a sum of $m+1$ squares in $K(X)$, which yields by iteration that $1 + X_1^2 + \cdots + X_d^2$ is not a sum of d squares in $K(X_1, \ldots, X_d)$. For $d \geq 2$, one can do better than this: Let us denote by $f(X, Y)$ the two-variate polynomial [2] given by the following sum of 4 squares

$$\left(1 + X^2 - 2X^2Y^2\right)^2 + \left(X(1 - X^2)Y^2\right)^2 + \left(X(1 - X^2)Y\right)^2 + \left(X^2(1 - X^2)Y\right)^2.$$

It was shown in [CEP] that $f(X, Y)$ is not a sum of 3 squares in $K(X, Y)$. Iterating the same consequence of the Cassels-Pfister Theorem as before, we obtain that $f(X_1, X_2)+X_3^2+\ldots+X_d^2$ is not a sum of $d+1$ squares $K(X_1, \ldots, X_d)$ when $d \geq 2$. This yields Theorem 1.2.

[2] historically, the so called 'Motzkin polynomial' was considered, which can be obtained from the given one by dividing out $(1 + X^2)^2$, and K was assumed to be real closed.

2. Valuations on real function fields

We will first fix some terminology. Let V be an variety defined over a real field K. We call $P \in V$ a *real* point if its residue field $\kappa(P) = \mathcal{O}_{V,P}/\mathfrak{m}_{V,P}$ is real. We will make use of the following fact, which is well known to real geometers.

Lemma 2.1. *Assume that V is irreducible. Then its function field $\kappa(V)$ is real if and only if V contains a real smooth closed point.*

Proof. We can assume that V is affine. Let us first assume that V contains a real smooth closed point P. In particular the local ring $\mathcal{O}_{V,P}$ is regular of Krull dimension $\dim V = d$. Any regular system $(x_1, \dots, x_d)$ of $\mathcal{O}_{V,P}$ yields a K-valuation v on $\kappa(V)$ with residue field $\kappa(P)$. This valuation v corresponds to the composition of places given by the localizations of $\mathcal{O}_{V,P}/(x_1, \dots, x_i)$ after the principal prime ideal generated by the residue class of x_{i+1} for $0 \leq i \leq d-1$.

Suppose that $\kappa(V)$ is nonreal. Then $0 = x_1^2 + \cdots + x_s^2$ for some $s \geq 1$ and $x_i \in \kappa(V)^\times$. Assume that $v(x_1) = \min\{v(x_1), \dots, v(x_s)\}$. After dividing the equality by x_1^2, we can assume that $x_1 = 1$ and $v(x_i) \geq 0$ for all $1 \leq i \leq s$. It follows that -1 is a sum of squares in the residue field $\kappa(P)$ of v, which yields the contradiction. Hence $\kappa(V)$ is real.

To show the converse, let us suppose that $\kappa(V)$ is real. In the case where K is real closed, [BCR, Proposition 7.6.4 (i)] yields the existence of a real smooth closed point in V. For arbitrary real K, let K' denote the relative algebraic closure of K in $\kappa(V)$. Considered over K' the variety V is geometrically irreducible. Now let R denote a real closure of K' with respect to an ordering on $\kappa(V)$. The base change V_R of V from base K' to base R is irreducible with function field $\kappa(V_{\mathbb{R}}) = \kappa(V) \otimes_{K'} R$ and the unique ordering of R extends to $\kappa(V_R)$. There exists a closed point $P_R \in V_R$ with $\kappa(P_R) = R$. Let P denote the image of P under the first projection of the base change. Then $\kappa(P) \subseteq \kappa(P_R) = R$ and hence $\kappa(P)$ is real. □

Proposition 2.2. *Let F be a real field that is finitely generated over some subfield K of transcendence degree $d \geq 1$. Then there exists a discrete K-valuation of rank one on F whose residue field is a rational function field in $d-1$ variables over a finite real extension of K.*

Proof. Let V be a variety over K with function field F. By Lemma 2.1, V contains a real smooth closed point P.

The exceptional fiber of the blowing-up $B\ell_{\{P\}} : V' \to V$ of V along $\{P\}$ is isomorphic over K to $\mathbb{P}^{n-1}_{\kappa(P)}$ (see [Li, Chapter 8, Theorem 1.19]). In particular, its generic point $\eta \in V'$ is of codimension one in V' with residue field $\kappa(\eta) = \kappa(P)(X_1, \dots, X_{d-1})$. As V' is regular, the local ring at η is a discrete valuation ring, hence yielding the claimed discrete K-valuation of rank one on F. □

We will present a relative version of Proposition 2.2 in the final section of this note.

Proposition 2.3. *Let F be a field and K a relatively algebraically closed real subfield of F. Assume furthermore that F is of transcendence degree at least two over K. Then there exists a discrete K-valuation of rank one on F with nonreal residue field E and K relatively algebraically closed in E.*

Proof. We can consider F as the function field of an affine geometrically irreducible K-variety V that is regular of codimension one. Consider V as a K-closed subset in an ambient affine space $\mathbb{A}^n$ for some $n \geq 2$. After replacing V by a K-birationally equivalent affine variety embedded in $n+1$-dimensional affine space, we will use intersections with suitable hyperplanes in the ambient space in order to obtain a geometrically irreducible closed K-subvarieties C of codimension one in V without closed real points. The local ring of the generic point of on such C will then give rise to a discrete valuation of rank one on F whose residue field E is the function field of C. The fact that C is geometrically irreducible will correspond to the property that K is relatively algebraically closed in E and the fact that C contains no closed real points will imply that E is nonreal (by Lemma 2.1).

Let $\mathbb{S}^n \subset \mathbb{A}^{n+1}$ denote the hypersurface defined by

$$X_1^2 + \cdots + X_{n+1}^2 = 1.$$

Let us consider $\mathbb{A}^n$ as a closed subset of $\mathbb{A}^{n+1}$ defined by $x_{n+1} = 0$ and consider the stereographic projection $\varphi : \mathbb{A}^n \to \mathbb{S}^n$ with respect to the point $(0, \ldots, 0, 1) \in \mathbb{S}^n$, which is the birational morphism that is given by mapping each point $P \in \mathbb{A}^n$ to the unique second intersection with $\mathbb{S}^n$ of the unique line passing through P and $(0, \ldots, 0, 1)$. By replacing V with the K-Zariski-closure of $\varphi(V)$, we can assume that V is a closed subset of $\mathbb{S}^n$.

For any point $P = (p_1, \ldots, p_{n+1}) \in \mathbb{A}^{n+1} \setminus \{(0, \ldots, 0)\}$ we associate the hyperplane $H(P) \subset \mathbb{A}^{n+1}$ defined by the linear equation

$$p_1 X_1 + \cdots + p_{n+1} X_{n+1} = p_1^2 + \cdots + p_{n+1}^2.$$

It follows from [J, Theorem 6.3], a variant of Bertini's Theorem[3] that

$$U = \{P \in \mathbb{A}^{n+1} \mid H(P) \cap V \text{ is of codimension one and geometrically irreducible}\}$$

contains a nonempty K-Zariski open subset of $\mathbb{A}^{n+1}$.

Let W denote the subset of $\mathbb{A}^{n+1}$ consisting of all points $P = (p_1, \ldots, p_{n+1}) \in \mathbb{A}^{n+1}(\mathbb{Q})$ with $p_1^2 + \cdots + p_{n+1}^2 > 1$. We show the hyperplane $H(P)$ corresponding to such a $\mathbb{Q}$-point P intersects $\mathbb{S}^n$ only in nonreal points.

Suppose otherwise that there exists a finite real extension L/K and a point $Q = (q_1, \ldots, q_{n+1}) \in \mathbb{A}^{n+1}(L)$ with $Q \in H(P) \cap \mathbb{S}^n$. Let '$\leq$' denote an arbitrary field ordering on L (which automatically extends the unique field ordering on $\mathbb{Q}$). Using the Cauchy-Schwartz inequality, we obtain that

$$\begin{aligned}
(p_1^2 + \ldots + p_{n+1}^2) &= (q_1^2 + \ldots + q_{n+1}^2)(p_1^2 + \ldots + p_{n+1}^2) \\
&\geq (q_1 p_1 + \ldots + q_{n+1} p_{n+1})^2 \\
&= (p_1^2 + \ldots + p_{n+1}^2)^2,
\end{aligned}$$

contradicting the choice of P such that $(p_1^2 + \ldots + p_{n+1}^2) > 1$.

The subset W of $\mathbb{A}^{n+1}$ is K-Zariski dense whereby $W \cap U$ is nonempty. Choosing any $P \in W \cap U$ and setting $C = H(P) \cap V$ finishes the proof. □

[3] for simplicity of the exposition, we define $H(P)$ slightly different than in [J, 3.6]. However, this does not compromise the validity of the argument.

3. An application to the pythagoras number

A proof of the following observation can be found in [BGV, Lemma 4.1].

Lemma 3.1. *Let $v : F \to \mathbb{Z} \cup \{\infty\}$ be a discrete valuation of rank one. Let κ_v denote its residue field. Then $p(F) \geq p(\kappa_v)$. Moreover $p(F) \geq 3$ if κ_v is nonreal and -1 is not a square in κ_v*

This can be applied together with our earlier observations to obtain lower estimates for the pythagoras number of function fields.

Theorem 3.2 (An absolute estimate)**.** *Let F be a real field that is finitely generated of transcendence degree d over a subfield of F. Then $p(F) \geq d + 1$.*

Proof. Let K denote a subfield of F over which F is finitely generated of transcendence degree n. By Proposition 2.2 there is a discrete K-valuation of rank one and residue field $L(X_1, \dots, X_{d-1})$ with L/K a finite real extension. Theorem 1.2 together with Lemma 3.1 yields that $p(F) \geq d - 1 + 2 = d + 1$ in the case where $d \geq 3$.

In the case $d = 2$, the fact that $p(F) \geq 3$ follows from Lemma 3.1 together with Proposition 2.3 which asserts the existence of a discrete K-valuation with nonreal residue field in which -1 is not a square.

Finally, the fact that $p(F) \geq 2$ for $d = 1$ follows from the observations that $1 + X^2$ is not a square in F in the case where $F = K(X)$, and the general case where $F/K(X)$ is a finite field extension follows from a reduction [La, VIII.5.7] to the case $F = K(X)$. ☐

We also show a relative bound for the pythagoras number of function fields. Its proof relies on an observation that will be obtained in Section 4.

Theorem 3.3 (A relative estimate)**.** *Let F and E be real fields that are finitely over a common subfield K. Assume that there exists a K-place $\varphi : F \to E \cup \{\infty\}$ and that $d := \mathrm{trdeg}_K F - \mathrm{trdeg}_K E \geq 2$. Then $p(F) \geq p(E(X_1, \dots X_{d-1}))$. In particular $p(F) \geq p(E) + d - 1$ and, moreover, $p(F) \geq p(E) + d$ if $d \geq 3$.*

Proof. This follows from Lemma 3.1, together with the same arguments as the proof of Theorem 1.2 as outlined in the introduction, as soon as we show that $E(X_1, \dots, X_{d-1})$ is the residue field of a discrete valuation of rank one. Proposition 4.2 of the following section will do just that. ☐

Remark 3.4. Note that the K-place $\varphi : F \to E \cup \{\infty\}$ in the statement of Theorem 3.3 is not assumed to be surjective nor is the corresponding valuation assumed to be discrete or of finite rank.

4. Valuations on function fields revisited

In this final section, we continue our observations of Section 2. More precisely, we want to obtain a relative version of Proposition 2.2.

A *catenary ring* is a ring such that all maximal strictly increasing chains of prime ideals connecting two fixed prime ideals have the same length. If every finitely generated algebra over a given ring is catenary then this ring is called *universally catenary.* By [E, Corollary 13.6] localizations and a finitely generated algebras over a universally catenary ring are again universally catenary, and fields are universally catenary.

Lemma 4.1. *Let A be a regular local universally catenary ring with field of fractions F and residue field K. Suppose that the Krull dimension d of A is at least two. For any simple field extension L/K, the ring A is dominated by a regular local universally catenary ring $B \subseteq F$ of Krull dimension $(d - \operatorname{trdeg}_K L)$ and with residue field L.*

Proof. By the regularity assumption, the maximal ideal $\mathfrak{m}$ of A is generated by some elements $y_0, \ldots, y_{d-1}$. Set

$$\tilde{A} \cong A[T_1, \ldots, T_{d-1}]/(y_0 T_i - y_i)_{1 \le i \le d-1}.$$

Note that $\tilde{A} \cong A[\frac{y_1}{y_0}, \ldots, \frac{y_{d-1}}{y_0}] \subset F$, and that $\tilde{A}$ is a again of dimension d. Note that $\mathfrak{m}A[T_1, \ldots, T_{d-1}]$ is a height one prime ideal with

$$A[T_1, \ldots, T_{d-1}]/\mathfrak{m}A[T_1, \ldots, T_{d-1}] \cong K[T_1, \ldots, T_{d-1}].$$

Hence, $\mathfrak{m}\tilde{A}$ is a prime ideal with $\tilde{A}/\mathfrak{m}\tilde{A} \cong K[T_1, \ldots, T_{d-1}]$. Let $\mathfrak{M}$ denote the preimage in $\tilde{A}$ of the kernel of a K-homomorphism $K[T_1, \ldots, T_{d-1}] \to L$ such that the field of fractions of the image is L. Let B denote the localization of $\tilde{A}$ with respect to $\mathfrak{M}$. Note that B is a regular local $(d - \operatorname{trdeg}_K L)$-dimensional universally catenary ring that dominates A. □

Proposition 4.2. *Let K be a field of characteristic zero. Let F/K and E/K be finitely generated field extensions such that $d := \operatorname{trdeg}_K(F) - \operatorname{trdeg}_K(E) \ge 2$. If E contains the residue field of a K-valuation on F, then $E(X_1, \ldots, X_{d-1})$ is the residue field of a discrete K-valuation of rank one.*

Proof. Let w denote the K-valuation on F whose residue field κ_w is contained in E. Let V denote a proper (e.g. projective) variety over K with function field F. The valuative criterion of properness ([Li, Chapter 3, Corollary 3.26]) yield the existence of a point $P \in V$ such that the valuation ring of w dominates the local ring $\mathcal{O}_{V,P}$ of V at P. As a localization of a finitely generated integral K-algebra, this ring is universally catenary. The residue field $\kappa(P)$ of $\mathcal{O}_{V,P}$ is thus contained in κ_w and the dimension of $\mathcal{O}_{V,P}$ is $\operatorname{trdeg}_K(F) - \operatorname{trdeg}_K(\kappa(P)) \ge 2$. We can assume that $\mathcal{O}_{V,P}$ is regular by resolution of singularities in characteristic zero [Hi]. Applying Lemma 4.1 iteratively, one obtains a regular local ring B of Krull-dimension one (i.e. B is a discrete valuation ring of F) with residue field $E(X_1, \ldots, X_{d-1})$. □

Remark 4.3. We could have deduced Proposition 2.2 from Proposition 4.2 by considering a place given by any real smooth closed point on a variety over K with function field F.

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