Specialisation and good reduction for algebras with involution

Sofie Beke[a]

[a]*Ghent University, Department of Mathematics, Krijgslaan 281, 9000 Gent, Belgium*

Abstract

Given a place from one field to another, the isotropy behaviour of Azumaya algebras with involution over the valuation ring corresponding to the place is studied. In particular, it is shown that isotropic right ideals specialise in an appropriate way. This provides a natural analogue to the existing specialisation theory for symmetric bilinear spaces. We devote particular attention to the case of a Henselian valuation ring in which 2 is invertible, where the specialisation results can be strenghened. In turn, this allows us to show that isomorphism of Azumaya algebras with involution over the Henselian valuation ring can be detected rationally. We use this to define a notion of good reduction with respect to places for algebras with involution.

Keywords: Azumaya algebras with involution, central simple algebras with involution, Brauer group, (skew–)hermitian spaces, bilinear spaces, (Henselian) valuation rings, value functions.

1. Introduction

In the 1970's, M. Knebusch developed a specialisation theory for quadratic and symmetric bilinear forms over fields with respect to places (cf. [13]). A comprehensive treatment of this topic, and its applications in the generic splitting theory of quadratic forms, can be found in [14]. The present article strives for an analogous treatment of specialisation for involutions on central simple algebras over fields, which are closely related to quadratic and symmetric bilinear forms (cf. [16]). We will assume basic knowledge of valuation theory over fields, bilinear form theory, and the theory of central simple algebras and their involutions over fields. For valuation theory, we refer to [9], for the other topics to [25] and [16].

In the rest of the introduction, λ denotes a place from a field F to a field L, and $\mathcal{O}$ its associated valuation ring. A symmetric bilinear space (V,b) over F (by which we mean that (V,b) is non–singular) is said to have *good reduction with respect to* λ if it is obtained by scalar extension from a symmetric bilinear space over $\mathcal{O}$. The latter is called *a* λ*–unimodular representation for* (V,b). One can then associate a "residue" symmetric bilinear space over L to (V,b), by extending scalars of a λ–unimodular representation for (V,b) from $\mathcal{O}$ to L. The Witt class of

Email address: sbeke@cage.ugent.be

The author is a PhD Fellow of the Research Foundation - Flanders (FWO). Part of this work was carried out during a research stay at the University of Konstanz, whose hospitality is greatly acknowledged.

this bilinear space over L is independent of the choice of the λ–unimodular representation, by [14, Theorem 1.15]. If the characteristic of L is different from 2, this implies that its isometry class is independent of this choice. In that case, the residue bilinear space is denoted by $\lambda_*(V,b)$.

1.1 Theorem. *Suppose that* $\mathrm{char}(L) \neq 2$. *Let* (V,b) *be a symmetric bilinear space over* F *with good reduction with respect to* λ. *If* (V,b) *is isotropic, then* $\lambda_*(V,b)$ *is isotropic as well.*

Proof. In [14, Theorem 1.26], it is shown that if an orthogonal sum of symmetric bilinear spaces over F and one of its summands have good reduction with respect to λ, then the remaining summand also has good reduction with respect to λ. Using this combined with the fact that hyperbolic planes have good reduction with respect to λ yields the statement. □

It is well known that one can associate to a symmetric bilinear space over F its adjoint algebra with involution. If the space has good reduction with respect to λ, then its adjoint algebra with involution is obtained by scalar extension from an Azumaya algebra with involution with center $\mathcal{O}$. The isotropy behaviour of the bilinear space under λ then carries over to the adjoint algebra with involution. In this article, we consider, not only in the split case, algebras with involution over F that are obtained by scalar extension from Azumaya algebras with involution over $\mathcal{O}$, and we investigate their isotropy behaviour under λ.

Let $(\mathcal{A},\sigma)$ be an $\mathcal{O}$–*algebra with involution*, i.e. $\mathcal{A}$ is an Azumaya algebra with center $\mathcal{O}$ or a separable quadratic $\mathcal{O}$–algebra, and σ is an $\mathcal{O}$–linear involution of the first or second kind on $\mathcal{A}$. In Theorem 5.7, we show that an isotropic right ideal I of $(\mathcal{A},\sigma)_F$ of a certain dimension yields an isotropic right ideal of $(\mathcal{A},\sigma)_L$ of the same dimension. In particular, we have that isotropy (resp. metabolicity) of $(\mathcal{A},\sigma)_F$ yields isotropy (resp. metabolicity) of $(\mathcal{A},\sigma)_L$. The statement on ideals can be rephrased in terms of the index of an algebra with involution. In this formulation, Theorem 1.1 can be obtained as a corollary of Theorem 5.7. The main ingredient of the proof of Theorem 5.7 is to show that for a right ideal I of $\mathcal{A}_F$, we have that $I \cap \mathcal{A}$ is free as a $Z(\mathcal{A})$–module. This is done in a more general module theoretical setting in section 4, using the theory of value functions on F–vector spaces (developed in [23, 26]).
Suppose that $\mathcal{O}$ is Henselian and that 2 is invertible in $\mathcal{O}$. Then the result on the index can be strenghtened. Namely, in Theorem 6.7, we show that for an $\mathcal{O}$–algebra with involution, isotropy (resp. hyperbolicity) can be lifted from the residue field of $\mathcal{O}$ back to F. This is an analogue of a result for symmetric bilinear spaces over $\mathcal{O}$. An important ingredient of the proof of Theorem 6.7 is the existence of a special kind of value function, shown in Proposition 4.7. In the last section, we use the lifting result from Theorem 6.7 in order to show that $\mathcal{O}$–algebras with involution that become isomorphic over F are already isomorphic over $\mathcal{O}$. This allows us to define a notion of good reduction with respect to places for algebras with involution.

In order to prove the aforementioned lifting result for isotropy and hyperbolicity, and the isomorphism result, we need to jump back and forth between Azumaya algebras with involution and (skew–)hermitian spaces over Azumaya algebras with involution without zero divisors. The preliminary results on Azumaya algebras with involution over valuation rings and the Brauer group of a valuation ring, are given in section 2. (Skew–)hermitian spaces are treated in section 3. We expect that many of the results in those sections will look very natural to people who are familiar with these concepts, but since we could not find proofs of the statements in the specific setting of valuation rings, and to make the article more self–contained, we give explicit arguments.

We set some general notation for the rest of this article. A *ring* will always mean an associative ring with unit. Let C be a ring. We denote the set of invertible elements in C by $C^\times$. Let R be a commutative ring. Let M be a finitely generated, free R−module. Every R−basis for M has the same cardinality (see e.g. [17, (III.4.2)]). We will use the term *dimension*, denoted by $\dim_R(M)$, for this cardinality. (In the literature, one also uses the term *rank*). Let S be a commutative R−algebra. If $(e_i)_{i\in I}$ is an R−basis for M, then $(e_i \otimes 1)_{i\in I}$ is an S−basis for $M_S = M \otimes_R S$ by [17, (XVI.2.7)]. We will denote $e_i \otimes 1$ again by e_i.
Let A be an R−algebra. Define a new multiplication on A by $a * b = ba$, for all $a, b \in A$. The R−module A with the new operation $*$ as multiplication is also an R−algebra, called *the opposite algebra of* A, for which we use the standard notation A^{op}.

2. Azumaya algebras with involution over valuation rings and the Brauer group

We recall some basics on Azumaya algebras with involution over rings, with an emphasis on valuation rings. The general theory of Azumaya algebras and their involutions can be found in [15].

We fix a domain R, and denote its fraction field by F. We further fix a ring S, which is either equal to R or to $R[z]$, where z is an element that is not in R, and such that $z^2 = az + b$, with $a, b \in R$ such that $a^2 + 4b \in R^\times$. In the latter case, we set $f(x) = x^2 - ax - b \in R[x]$, and if S is a domain, we denote its fraction field by K. If $S = R[z]$, then S is in particular a *separable quadratic R−algebra*, in the sense of [15, (I.1.3.6), (I.7.3.3)]. Furthermore, in that case, there is a unique nontrivial R−automorphism ι of $R[z]$, namely the one induced by mapping z to $a - z$. If R is a local ring, then all separable quadratic R−algebras are of the form $R[z]$ as above.
An associative S−algebra $\mathcal{A}$ is called *an Azumaya algebra over S* if $\mathcal{A}$ is finitely generated as an S−module and for every maximal ideal $\mathfrak{m}$ of S, we have that $\mathcal{A}/\mathfrak{m}\mathcal{A}$ is a central simple $S/\mathfrak{m}$−algebra. An Azumaya algebra over S has center S by [15, (III.5.1.1)], and is projective as an S−module by [15, (III.5.1)]. If S is a field, then an Azumaya algebra over S is just a central simple S−algebra.

An *involution* on a ring is an anti–autmorphism of order at most 2. Let $\mathcal{A}$ be an Azumaya algebra over S and σ an R−linear involution on $\mathcal{A}$ such that, if $S \neq R$, then σ restricts to ι on S. If $S = R$, then σ is called *an involution of the first kind*; otherwise, it is called *an involution of the second kind*. We call the pair $(\mathcal{A}, \sigma)$ an *R−algebra with involution*. If S is not a domain, then we call $(\mathcal{A}, \sigma)$ *degenerate*.
Let R' be a domain that is an R−algebra. We write $(\mathcal{A}, \sigma)_{R'} = (\mathcal{A}_{R'}, \sigma_{R'}) = (\mathcal{A} \otimes_R R', \sigma \otimes_R \mathrm{id}_{R'})$.

2.1 Proposition. *Let $(\mathcal{A}, \sigma)$ be an R−algebra with involution and let R' be a domain that is also an R−algebra. Then $(\mathcal{A}, \sigma)_{R'}$ is an R'−algebra with involution.*

Proof. We have that $\mathcal{A}_{R'} \cong \mathcal{A} \otimes_S (S \otimes_R R')$ is an Azumaya algebra over $S \otimes_R R'$ by [15, (III.5.1.9)]. The statement is then clear if σ is of the first kind. Suppose that σ is of the second kind. Then $S \otimes_R R' \cong R'[z]$ and $a^2 + 4b \in R^\times \subset R'^\times$. Furthermore, $\sigma_{R'}$ is an R'−linear involution on $\mathcal{A}_{R'}$, which restricts to the nontrivial R'−automorphism of $S \otimes_R R'$. □

Let $(\mathcal{A}, \sigma)$ and $(\mathcal{A}', \sigma')$ be R−algebras with involution. By *an isomorphism of R−algebras with involution* $(\mathcal{A}, \sigma) \to (\mathcal{A}', \sigma')$, we mean an isomorphism $\varphi : \mathcal{A} \to \mathcal{A}'$ of R−algebras such that

$\varphi \circ \sigma = \sigma' \circ \varphi$. We call $(\mathcal{A}, \sigma)$ and $(\mathcal{A}', \sigma')$ R*–isomorphic* if there exists an isomorphism of R–algebras with involution $(\mathcal{A}\ \sigma) \to (\mathcal{A}', \sigma')$, and denote this by $(\mathcal{A}, \sigma) \cong_R (\mathcal{A}', \sigma')$. Suppose moreover that there exist R–isomorphisms $f : S \to Z(\mathcal{A})$ and $f' : S \to Z(\mathcal{A}')$, and an isomorphism of R–algebras with involution $(\mathcal{A}, \sigma) \to (\mathcal{A}', \sigma')$ that is S–linear with respect to f and f'. Then we say that $(\mathcal{A}, \sigma)$ and $(\mathcal{A}', \sigma')$ are S*–isomorphic* and we write $(\mathcal{A}, \sigma) \cong_S (\mathcal{A}', \sigma')$.

Let $\mathcal{B}$ be an Azumaya algebra over R. The map $\mathrm{sw}_{\mathcal{B}} : \mathcal{B} \times \mathcal{B}^{\mathrm{op}} \to \mathcal{B} \times \mathcal{B}^{\mathrm{op}} : (a, b) \mapsto (b, a)$ defines an involution of the second kind on $\mathcal{B} \times \mathcal{B}^{\mathrm{op}}$, called *the switch involution*.

2.2 Proposition. *Let $\mathcal{A}$ be an Azumaya algebra over $R \times R$. Then there exist Azumaya algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ over R such that $\mathcal{A} \cong \mathcal{A}_1 \times \mathcal{A}_2$. Furthermore, if σ is an involution of the second kind on $\mathcal{A}$, then $\mathcal{A}_2 \cong \mathcal{A}_1^{\mathrm{op}}$ and $(\mathcal{A}, \sigma) \cong_R (\mathcal{A}_1 \times \mathcal{A}_1^{\mathrm{op}}, \mathrm{sw}_{\mathcal{A}_1})$.*

Proof. Let $\mathcal{A}_1 = \mathcal{A}(1, 0)$ and $\mathcal{A}_2 = \mathcal{A}(0, 1)$. These are R–algebras and it is clear that $\mathcal{A} \cong \mathcal{A}_1 \times \mathcal{A}_2$. For each maximal ideal $\mathfrak{m}$ of R, we have that $\mathcal{A}/\mathcal{A}(\mathfrak{m} \times R)$ is a central simple algebra over $(R \times R)/(\mathfrak{m} \times R)$. Using the natural isomorphisms, this gives $\mathcal{A}_1/\mathfrak{m}\mathcal{A}_1$ the structure of a central simple algebra over $R/\mathfrak{m}$. Hence, $\mathcal{A}_1$ is an Azumaya algebra over R. Similarly, by considering $\mathcal{A}/\mathcal{A}(R \times \mathfrak{m})$, we get that $\mathcal{A}_2$ is an Azumaya algebra over R.

Let σ be an involution of the second kind on $\mathcal{A}$. Since σ restricts to the switch involution on $R \times R$, the map $g : \mathcal{A}_1^{\mathrm{op}} \to \mathcal{A}_2$ defined by $\sigma(x, 0) = (0, g(x))$ is an R–isomorphism. Under the induced R–isomorphism $\mathcal{A}_1 \times \mathcal{A}_2 \cong \mathcal{A}_1 \times \mathcal{A}_1^{\mathrm{op}}$, the involution σ corresponds to the switch involution $\mathrm{sw}_{\mathcal{A}_1}$. □

2.3 Proposition. *Suppose that R is integrally closed in F. Then S is the integral closure of R in $S \otimes_R F$. Furthermore, S is a domain if and only if $S \otimes_R F$ is a field, and the latter is then the fraction field of S. If S is not a domain, then $S \cong R \times R$.*

Proof. If S is not a domain, then $S \cong R \times R$ by [15, (III.4.4.3)], and $R \times R$ is the integral closure of R in $S \otimes_R F \cong F \times F$. Suppose that S is a domain. Since R is integrally closed in F, we have that $f(x)$ is irreducible in $R[x]$ if and only if it is irreducible in $F[x]$. Since $f(x)$ is monic, this implies that, if $f(x)$ is irreducible in $R[x]$, then it generates a prime ideal in $R[x]$. If this is the case, then $S \cong R[x]/(f(x))$ is a domain, and $S \otimes_R F$ is a field. Furthermore, [10, (6.1.2)] yields that S is the integral closure of R in $S \otimes_R F$, since the discriminant of $f(x)$ is in $R^\times$. □

We recall the following terminology. Two valuation rings of a field are called *incomparable* if none of the two is contained in the other. Let F'/F be a field extension. We say that a valuation ring $\mathcal{O}'$ of F' with maximal ideal $\mathcal{M}'$ is *lying over* R if $R \subset \mathcal{O}'$ and $\mathcal{M}' \cap R$ is a maximal ideal of R. If R is a valuation ring of F, then we also say that $\mathcal{O}'$ is an *extension of R to F'*.

2.4 Corollary. *Suppose that R is integrally closed in F. If S is a domain, then it is the intersection of the valuation rings of K lying over R.*

Proof. This follows immediately from Proposition 2.3 combined with [9, (3.1.3)]. □

In the rest of this section, we work with Azumaya algebras with involution over valuation rings.

2.5 Proposition. *Assume that R is a valuation ring. Denote its maximal ideal by $\mathfrak{m}$ and its residue field by κ. Suppose that S is a domain. Then one of the following cases occurs:*

(a) There is a unique valuation ring of K extending R. Then S is equal to this extension. Furthermore, S has the same value group as R, has maximal ideal $\mathfrak{m}S$ and its residue field is a separable quadratic extension of κ.

(b) There are two valuation rings of K extending R. These both have the same value group as R and residue field κ. Furthermore, S is equal to their intersection, and $S/\mathfrak{m}S \cong \kappa \times \kappa$.

Proof. By Corollary 2.4, S is the intersection of the valuation rings of K extending R. By [9, (3.2.9)], since K/F is a quadratic extension, S is either a valuation ring or the intersection of two (incomparable) valuation rings. Let $\bar{f}(x) = x^2 - \bar{a}x - \bar{b} \in \kappa[x]$. Note that the discriminant of $\bar{f}(x)$ is nonzero, since $a^2 + 4b \in R^\times$, and hence, $\bar{f}(x)$ is separable. It is easy to see that S is a valuation ring if and only if $\bar{f}(x)$ is irreducible. If this is the case, then $\mathfrak{m}S$ is the unique maximal ideal of S. Then the residue field of S is a separable quadratic extension of κ, and hence, by [9, (3.2.3)], R and S have the same value group.
Suppose that S is the intersection of two incomparable valuation rings. Both of these valuation rings have the same value group and residue field as R by [9, (3.3.4)]. We have that S has two maximal ideals by [9, (3.2.7)]. Now $\bar{f}(x)$ is reducible over κ, and hence, $S/\mathfrak{m}S \cong \kappa \times \kappa$, since $\bar{f}(x)$ is separable. □

2.6 Proposition. *Assume that R is a valuation ring. Let $\mathcal{A}$ be an Azumaya algebra over S. The following hold:*

(a) $\mathcal{A}$ is free as an R–module. If S is a domain, then $\mathcal{A}$ is also free as an S–module.

(b) Every S–automorphism of $\mathcal{A}$ is inner.

Proof. Since $\mathcal{A}$ is an Azumaya algebra over S, it is finitely generated, projective as an S–module by [15, (III.5.1)]. Since S is a finite–dimensional R–module, it follows that $\mathcal{A}$ is also finitely generated, projective as an R–module.
Suppose that S is a domain. Then S has at most two maximal ideals by Proposition 2.5 and [9, (3.2.7)]. The second part of (a) then follows from the main theorem of [11], which states that finitely generated, projective modules over a semilocal domain are free. Combining the latter with [1, (3.6)] then yields (b).
Suppose that $S \cong R \times R$. By Proposition 2.2, there exist Azumaya algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ over S such that $\mathcal{A} \cong \mathcal{A}_1 \times \mathcal{A}_2$. Let $\varphi \in \mathrm{Aut}_S(\mathcal{A})$. For $i = 1, 2$, one checks that the restriction of φ to $\mathcal{A}_i$ is an R–automorphism of $\mathcal{A}_i$. By the first part of the proof, it follows that there exist $u \in \mathcal{A}_1^\times, v \in \mathcal{A}_2^\times$ such that $\varphi|_{\mathcal{A}_1} = \mathrm{Int}(u)$ and $\varphi|_{\mathcal{A}_2} = \mathrm{Int}(v)$. Then $\varphi = \mathrm{Int}(u, v)$. □

2.7 Proposition. *Assume that R is a valuation ring. Let $\mathcal{A}$ be an Azumaya algebra over S and let σ and σ' be two R–linear involutions of the same kind on $\mathcal{A}$. Then there exists an element $s \in \mathcal{A}^\times$ such that $\sigma(s) = \varepsilon s$ and $\sigma' = \mathrm{Int}(s) \circ \sigma$, where $\varepsilon \in \{1, -1\}$ if σ and σ' are of the first kind, and $\varepsilon = 1$ if σ and σ' are of the second kind.*

Proof. Since $\sigma' \circ \sigma$ is an S–automorphism of $\mathcal{A}$, Proposition 2.6 (b) yields that there exists $s \in \mathcal{A}^\times$ such that $\sigma' = \mathrm{Int}(s) \circ \sigma$. It follows that $\mathrm{id}_{\mathcal{A}} = \sigma'^2 = \mathrm{Int}(s\sigma(s)^{-1})$. This implies that $s\sigma(s)^{-1} = \lambda^{-1}$, for some $\lambda \in S^\times$. In other words, $\sigma(s) = \lambda s$. It follows that $s = \sigma^2(s) = \sigma(\lambda)\lambda s$. Since $s \in \mathcal{A}^\times$, we get that $\sigma(\lambda)\lambda = 1$. If σ and σ' are of the first kind, this implies that $\lambda^2 = 1$ and hence, $\lambda = \pm 1$.
Suppose that σ and σ' are of the second kind. Then $\sigma(\lambda)\lambda = \iota(\lambda)\lambda$. It suffices to show a Hilbert

90 type statement for S, namely that there exists $\mu \in S^\times$ such that $\mu = \lambda\iota(\mu)$, since this would yield $\mathrm{Int}(s) = \mathrm{Int}(\mu s)$ and $\sigma(\mu s) = \sigma(s)\iota(\mu) = \lambda\iota(\mu)s = \mu s$. If S is not a domain, then it is clear that such μ exists. So, for the rest of the proof, we assume that S is a domain. Using the Hilbert 90 theorem for K/F, there exists $\tilde{\mu} \in K$ such that $\lambda\iota(\tilde{\mu}) = \tilde{\mu}$. By Proposition 2.5, S is either a valuation ring of K or the intersection of two valuation rings of K. Suppose first that S is a valuation ring. Since, by Proposition 2.5, the value groups of S and R are equal, there exists $a \in F$ such that $a\tilde{\mu} \in S^\times$, and furthermore $\lambda\iota(a\tilde{\mu}) = a\lambda\iota(\tilde{\mu}) = a\tilde{\mu}$. Suppose that $S = \mathcal{O}_1 \cap \mathcal{O}_2$, with $\mathcal{O}_1$ and $\mathcal{O}_2$ the valuation rings of K extending R. Then $\mathcal{O}_2 = \iota(\mathcal{O}_1)$ by [9, (3.2.14)]. By the previous case, we may assume that $\tilde{\mu} \in \mathcal{O}_1^\times$. Then $\iota(\tilde{\mu}) \in \mathcal{O}_2^\times$, and since $\lambda \in S^\times = \mathcal{O}_1^\times \cap \mathcal{O}_2^\times$, it follows that $\tilde{\mu} \in \mathcal{O}_2^\times$ as well. Hence, $\tilde{\mu} \in S^\times$ and we are done. □

The concept of the Brauer group of a field has been extended to commutative rings in [1]. Let T be a commutative ring. A T−module P is called *faithful* if whenever $t \in T$ is such that $tP = 0$, then $t = 0$. Let $\mathcal{A}$ and $\mathcal{A}'$ be Azumaya algebras over T. Then $\mathcal{A}$ and $\mathcal{A}'$ are called *Brauer equivalent*, denoted by $\mathcal{A} \sim \mathcal{A}'$, if there exist finitely generated, faithfully projective T−modules P and P' such that $\mathcal{A} \otimes_T \mathrm{End}_T(P) \cong \mathcal{A}' \otimes_T \mathrm{End}_T(P')$. The set of Brauer equivalence classes of Azumaya algebras over T forms a group, where the group structure is induced by the tensor product. This group is called *the Brauer group of* T, denoted by $\mathrm{Br}(T)$. It is well known that $\mathrm{Br}(T)$ is a torsion group. Given an Azumaya algebra $\mathcal{A}$ over T, we denote its Brauer class in $\mathrm{Br}(T)$ by $[\mathcal{A}]$, and refer to the order of $[\mathcal{A}]$ in $\mathrm{Br}(T)$ as *the exponent of* $\mathcal{A}$.

We say that T *has the Wedderburn property* if the following holds:

Let $\mathcal{A}$ be an Azumaya algebra over T. Then there exists an up to isomorphism unique Azumaya algebra Δ over T without zero divisors, and a finite–dimensional right Δ−module V such that $\mathcal{A} \cong \mathrm{End}_\Delta(V)$ as T−algebras.

We show below that valuation rings and intersections of two valuation rings have the Wedderburn property. In order to do this, we first show that Azumaya algebras with such rings as centers are so–called *left and right Bézout rings*, i.e. all finitely generated left or right ideals of the algebra are principal.

2.8 Proposition. *Let Δ be an Azumaya algebra with center a valuation ring or the intersection of two valuation rings. Then Δ is a left and right Bézout ring. Assume that Δ does not have zero divisors. Then every finitely generated, torsion–free left or right Δ−module is free.*

Proof. We denote $Z(\Delta)$ by T and its fraction field by F. We first show that Δ is a left and right Bézout ring. If T is a valuation ring, then this follows immediately from [18, (5.6)], since Δ is a so–called Dubrovin valuation ring of Δ_F, by [18, (7.13)]. So, suppose that T is the intersection of two incomparable valuation rings $\mathcal{O}_1$ and $\mathcal{O}_2$ of F. For $i = 1, 2$, let $\mathfrak{M}_i$ be the unique maximal ideal of $\mathcal{O}_i$. Then $\mathcal{M}_1 = \mathfrak{M}_1 \cap T$ and $\mathcal{M}_2 = \mathfrak{M}_2 \cap T$ are the maximal ideals of T by [9, (3.2.7)], and $\mathcal{O}_1 = T_{\mathcal{M}_1}$ and $\mathcal{O}_2 = T_{\mathcal{M}_2}$ by [9, (3.2.6)]. Let $\Delta_1 = \Delta \otimes_T \mathcal{O}_1$ and $\Delta_2 = \Delta \otimes_T \mathcal{O}_2$. Then Δ_i is an Azumaya algebra over $\mathcal{O}_i$ by [15, (III.5.1.9)], and therefore integral over $\mathcal{O}_i$. One easily checks that $\Delta = \Delta_1 \cap \Delta_2$.
By [18, (7.13)], Δ_1 and Δ_2 are Dubrovin valuation rings of Δ_F. If we show that Δ_1 and Δ_2 are pairwise incomparable, i.e. $\Delta_1 \subsetneq \Delta_2$ and $\Delta_2 \subsetneq \Delta_1$, then it follows from [18, (15.5) and (15.7)] that Δ is a left and right Bézout ring, using that Δ_i is integral over $\mathcal{O}_i$, and that $\Delta_i = \Delta\,\mathcal{O}_i$ inside Δ_F, for $i = 1, 2$. So, suppose for the sake of contradiction that $\Delta_1 \subset \Delta_2$. Then $\mathcal{O}_1 \subset \Delta_2$. Since

Δ_2 is an Azumaya algebra over $\mathcal{O}_2$, it is free over $\mathcal{O}_2$ by Proposition 2.6 (a), and an $\mathcal{O}_2$−basis of Δ_2 is an F−basis of Δ_F. It follows that $\mathcal{O}_2 = Z(\Delta_2) = Z(\Delta_F) \cap \Delta_2 = F \cap \Delta_2 \supset \mathcal{O}_1$, but this contradicts the fact that $\mathcal{O}_1$ and $\mathcal{O}_2$ are incomparable. Similarly, $\Delta_2 \subsetneq \Delta_1$. Hence, Δ is a left and right Bézout ring.

Assume that Δ does not have zero divisors. The fact that every finitely generated, torsion–free right Δ−module is free is shown in [6, (2.3.19)]. The statement for left modules follows in an analogous way. □

We thank M. Ojanguren for providing the main ideas for the proof of the following results.

2.9 Proposition. *Let T be a valuation ring and denote its fraction field by F. Let Δ be an Azumaya algebra over T without nontrivial idempotents. Then Δ does not have zero divisors, and $\Delta \otimes_T F$ is a division algebra.*

Proof. We have that $D = \Delta \otimes_T F$ is a central simple F−algebra by [15, (III.5.1.9)]. It is clear that if D is a division algebra, then Δ does not have zero divisors, and vice versa. In order to show that D is a division algebra, it suffices to show that D only has trivial idempotents. So, suppose for the sake of contradiction that D contains an idempotent $x \neq 0, 1$. Consider the right ideal xD of D and let $I = xD \cap \Delta$. This is a right ideal of Δ different from Δ itself. If we can show that Δ/I is projective as a Δ−module, then the exact sequence

$$0 \longrightarrow I \longrightarrow \Delta \longrightarrow \Delta/I \longrightarrow 0$$

splits, which implies that $\Delta \cong I \oplus \Delta/I$. The projection from Δ to I then yields a nontrivial idempotent in $\mathrm{End}_\Delta(I \oplus \Delta/I) \cong \Delta$, where the isomorphism $\Delta \to \mathrm{End}_\Delta(\Delta)$ is given by left multiplication. Clearly, Δ/I is finitely generated over Δ, and then also over T, and it is easily seen that Δ/I is torsion–free over T. Proposition 2.8 yields that Δ/I is free over T and so in particular projective. It follows from [15, (VII.8.2.6)] that Δ/I is projective as a Δ−module. □

2.10 Proposition. *Let F be a field and T a valuation ring of F or the intersection of two valuation rings of F. Then T has the Wedderburn property.*

Proof. In [7, Corollary 1], it is shown that any Azumaya algebra over T is isomorphic to a matrix algebra over an up to isomorphism uniquely determined Azumaya algebra over T without zero divisors. The statement now follows from Proposition 2.9. □

2.11 Proposition. *Let F be a field and T a valuation ring of F or the intersection of two valuation rings of F. Then the following hold:*

(a) *The natural map $\mathrm{Br}(T) \to \mathrm{Br}(F)$ is injective.*

(b) *Let $\mathcal{A}$ and $\mathcal{A}'$ be Azumaya algebras over T. If $\mathcal{A}_F \cong \mathcal{A}'_F$, then $\mathcal{A} \cong \mathcal{A}'$.*

Proof. We first show (a). Let $\mathcal{A}$ be an Azumaya algebra over T such that $\mathcal{A}_F$ is split, i.e. there exist a simple right $\mathcal{A}_F$−module V such that $\mathcal{A}_F \cong \mathrm{End}_F(V)$. Note that $\dim_F(V)^2 = \dim_F(\mathcal{A}_F) = \dim_T(\mathcal{A})$. Let $u \in V \setminus \{0\}$ and consider the right $\mathcal{A}$−module $M = u\mathcal{A}$. Since V is a simple right $\mathcal{A}_F$−module, it follows that $MF = V$. Since $\mathcal{A}$ is finitely generated as a T−module, M is finitely generated as a T−module as well, and furthermore torsion–free, since

V is free over T. Proposition 2.8 yields that M is free as a T–module. The T–homomorphism $\varphi : \mathcal{A} \to \mathrm{End}_T(M)$ mapping $a \in \mathcal{A}$ to right multiplication by a, induces an F–homomorphism $\varphi_F : \mathcal{A}_F \to \mathrm{End}_F(V)$, which is a isomorphism. Hence, φ is injective and therefore, $\varphi(\mathcal{A}) \cong \mathcal{A}$. By [1, (3.3)], there is an Azumaya algebra $\mathcal{B}$ over T such that $\mathrm{End}_T(M) \cong \varphi(A) \otimes_T \mathcal{B}$. Since $\dim_T(\mathcal{A}) = \dim_T(M)^2 = \dim_T(\mathrm{End}_T(M))$, it follows that $\mathcal{B} \cong T$, and hence $\mathcal{A} \cong \mathrm{End}_T(M)$.

Let $\mathcal{A}$ and $\mathcal{A}'$ be Azumaya algebras over T. By Proposition 2.10, there exist Azumaya algebras Δ and Δ' over T without zero divisors, and uniquely determined $n, n' \in \mathbb{N}$ such that $\mathcal{A} \cong M_n(\Delta)$ and $\mathcal{A}' \cong M_{n'}(\Delta')$. Suppose that $\mathcal{A}_F \cong \mathcal{A}'_F$. Then $[\Delta_F] = [\mathcal{A}_F] = [\mathcal{A}'_F] = [\Delta'_F] \in \mathrm{Br}(F)$. By (a), it follows that $[\Delta] = [\Delta'] \in \mathrm{Br}(T)$, and hence $\Delta \cong \Delta'$, by Proposition 2.10. Since $\mathcal{A}_F \cong \mathcal{A}'_F$, this implies that $n = n'$ and hence, we get that $\mathcal{A} \cong \mathcal{A}'$, proving (b). □

2.12 Corollary. *Let F be a field and T a valuation ring of F. Let $\mathcal{A}$ and $\mathcal{A}'$ be Azumaya algebras over $T \times T$. If $\mathcal{A}_{F\times F} \cong \mathcal{A}'_{F\times F}$, then $\mathcal{A} \cong \mathcal{A}'$.*

Proof. By Proposition 2.2, there exist Azumaya algebras $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}'_1, \mathcal{A}'_2$ over T such that $\mathcal{A} \cong \mathcal{A}_1 \times \mathcal{A}_2$ and $\mathcal{A}' \cong \mathcal{A}'_1 \times \mathcal{A}'_2$. Suppose that $(\mathcal{A}_1)_F \times (\mathcal{A}_2)_F \cong (\mathcal{A}'_1)_F \times (\mathcal{A}'_2)_F$. Since the simple components are unique up to isomorphism, we may assume that $(\mathcal{A}_1)_F \cong (\mathcal{A}'_1)_F$ and $(\mathcal{A}_2)_F \cong (\mathcal{A}'_2)_F$. Invoking Proposition 2.11, it follows that $\mathcal{A}_1 \cong \mathcal{A}'_1$ and $\mathcal{A}_2 \cong \mathcal{A}'_2$, and hence, $\mathcal{A} \cong \mathcal{A}'$. □

3. Hermitian and skew–hermitian spaces

We recall some preliminaries on (skew–)hermitian spaces over rings with involution, and more specifically over Azumaya algebras with involution over valuation rings. In the latter case, we prove a Witt decomposition result, which will be used in section 6. A standard reference for the theory of (skew–)hermitian spaces over rings with involution is [15].

3.1. Preliminaries

Let C be a (not necessarily commutative) ring and θ an involution on C. Let V be a finitely generated, projective right C–module. A *sesquilinear form on V (with respect to θ)* is a bi–additive map $h : V \times V \to C$ such that for all $x, y \in V$ and all $\alpha, \beta \in C$, we have that $h(x\alpha, y\beta) = \theta(\alpha)h(x,y)\beta$. Let $\varepsilon = \pm 1$. If in addition $h(y,x) = \varepsilon\theta(h(x,y))$ for all $x, y \in V$, then h is called *an ε–hermitian form* and (V, h) *an ε–hermitian module*. Furthermore, h is called *hermitian* if $\varepsilon = 1$ and *skew–hermitian* if $\varepsilon = -1$. If $\theta = \mathrm{id}_C$, then h is clearly a bilinear form.

Let (V, h) be an ε–hermitian module. Let $V^* = \mathrm{Hom}_C(V, C)$. This is a left C–module. Define the right C–module ${}^\theta V^*$ by ${}^\theta V^* = \{{}^\theta\varphi \mid \varphi \in V^*\}$ with the operations ${}^\theta\varphi + {}^\theta\psi = {}^\theta(\varphi + \psi)$, $({}^\theta\varphi)\alpha = {}^\theta(\theta(\alpha)\varphi)$ for all $\varphi, \psi \in V^*$ and all $\alpha \in C$. Then h is called *non–singular* if the adjoint transformation

$$\hat{h} : V \to {}^\theta V^* : x \mapsto {}^\theta\varphi, \quad \text{where } \varphi(y) = h(x,y) \text{ for all } y \in V,$$

is an isomorphism of right C–modules. We call (V, h) *an ε–hermitian space* if h is non–singular. If V is a free C–module, then $\hat{h}$ is an isomorphism if and only if the matrix of h with respect to any C–basis of V is invertible.

Let $\varphi : (C, \theta) \to (C', \theta')$ be a homomorphism of rings with involution, i.e. φ is a ring homomorphism from C to C' such that $\theta' \circ \varphi = \varphi \circ \theta$. Consider C' as a left C–module via φ. Let (V, h) be an ε–hermitian module over (C, θ). Then the map $h_{C'} : V_{C'} \times V_{C'} \to C'$ defined by

$$h_{C'}(x \otimes a', y \otimes b') = \theta'(a')\varphi(h(x,y))b' \quad \text{for all } x, y \in V \text{ and all } a', b' \in C',$$

is an ε–hermitian form on $V_{C'}$ with respect to θ'.

3.1 Notation. Let R be a domain. Let $(\mathcal{C}, \theta)$ be an R–algebra with involution and (V, h) an ε–hermitian module over $(\mathcal{C}, \theta)$. Let R' be a domain that is also an R–algebra. Then we denote the ε–hermitian module $(V_{\mathcal{C}_{R'}}, h_{\mathcal{C}_{R'}})$ by $(V, h)_{R'}$.

Let U be a C–submodule of V. The *orthogonal complement of* U, which is equal to $\{x \in V \mid h(x,y) = 0 \text{ for all } y \in U\}$, is denoted by $U^\perp$. The subspace U is called *totally isotropic* if $U \subset U^\perp$. The ε–hermitian module (V, h) is called *isotropic* if it contains a nonzero totally isotropic subspace U, and *anisotropic* otherwise. Equivalently, (V, h) is isotropic if there exists an element $0 \neq x \in V$ such that $h(x, x) = 0$. (V, h) is called *metabolic* if it contains a direct summand U such that $U^\perp = U$. There is also a notion of a *hyperbolic* ε–hermitian space (see [15, (I.3.5)]), and if $2 \in C^\times$, then these notions coincide (see [15, (I.3.7.3)]).

Let (V, h) and (V', h') be two ε–hermitian modules over (C, θ). They are called *isometric*, denoted by $(V, h) \simeq (V', h')$, if there is a C–linear bijection $\varphi : V \to V'$ such that $h(x, y) = h'(\varphi(x), \varphi(y))$ for all $x, y \in V$. They are called *similar* if there exists $a \in C$ such that $(V, h) \simeq (V', ah')$. The *orthogonal sum* of (V, h) and (V', h') is the ε–hermitian space $(V \oplus V', h \perp h')$, where $(h \perp h')(x + x', y + y') = h(x, y) + h'(x', y')$, for all $x, y \in V$ and all $x', y' \in V'$, denoted by $(V, h) \perp (V', h')$.

Let $\alpha_1, \ldots, \alpha_n \in C^\times$ be elements such that $\theta(\alpha_i) = \varepsilon\alpha_i$. Then the matrix $\operatorname{diag}(\alpha_1, \ldots, \alpha_n)$ defines a non–singular ε–hermitian form on C^n with respect to θ. We denote the corresponding ε–hermitian space by $\langle \alpha_1, \ldots, \alpha_n \rangle_\theta$.

3.2 Proposition. *Let $\varepsilon = \pm 1$. Let (V, h) be an ε–hermitian space over (C, θ) and let U be a C–submodule of V that is finitely generated and projective over C. If $h|_U$ is non–singular, then $(V, h) \simeq (U, h|_U) \perp (U^\perp, h|_{U^\perp})$, and $h|_{U^\perp}$ is also non–singular.*

Proof. See [15, (I.3.6.1), (I.3.6.2)]. □

In the rest of section 3, we work with (skew–)hermitian spaces over Azumaya algebras with involution without zero divisors. We fix a field F and a valuation ring $\mathcal{O}$ of F.

Let (Δ, θ) be an $\mathcal{O}$–algebra with involution without zero divisors. Let $\varepsilon = \pm 1$ and let (V, h) be an ε–hermitian space over (Δ, θ). Then V is free as a Δ–module by Proposition 2.8. Using this, it follows that hyperbolic ε–hermitian spaces of the same Δ–dimension are isometric.

3.3 Lemma. *Let Δ be an Azumaya algebra with center a valuation ring or the intersection of two valuation rings. Let $(a_1, \ldots, a_m)$ be a unimodular row over Δ, i.e. there exist $b_1, \ldots, b_m \in \Delta$ such that $\sum_{i=1}^m a_i b_i = 1$. Then there exists an invertible $(m \times m)$–matrix U over Δ having $(a_1, \ldots, a_m)$ as its first row, such that the inverse of U has $(b_1, \ldots, b_m)$ as its first column.*

Proof. The statement holds for Hermite rings by [6, (0.4.1)]. Left and right Bézout rings without zero divisors are Hermite rings by [6, (2.3.4), (2.3.17)]. So, the statement follows from Proposition 2.8. □

3.4 Proposition. *Let (Δ, θ) be an $\mathcal{O}$–algebra with involution without zero divisors. Let $\varepsilon = \pm 1$ and let (V, h) be an ε–hermitian space over (Δ, θ). Then (V, h) is the orthogonal sum of an anisotropic ε–hermitian space over (Δ, θ) and a metabolic ε–hermitian space over (Δ, θ).*

Proof. If h is anisotropic, there is nothing to prove. So, suppose that there exists $0 \neq x \in V$ such that $h(x, x) = 0$. Let $(e_1, \dots, e_r)$ be a Δ–basis for V. Then $x = \sum_{i=1}^r e_i x_i$, with $x_1, \dots, x_r \in \Delta$. By Proposition 2.8, Δ is a left Bézout ring. Hence, there exists $d \in \Delta$ such that $\Delta x_1 + \dots + \Delta x_r = \Delta d$. It follows that there exist $b_1, \dots, b_r, c_1, \dots, c_r \in \Delta$ such that $x_i = b_i d$ and $\sum_{i=1}^r c_i x_i = d$. Let $y = \sum_{i=1}^r e_i b_i$. Then $0 = h(x, x) = \theta(d) h(y, y) d$. Since Δ does not have zero divisors, it follows that $h(y, y) = 0$. Since $\sum_{i=1}^r c_i b_i = 1$, by Lemma 3.3, there exists an invertible matrix U over Δ with $(b_1, \dots, b_r)$ as its first column. Let $(f_1, \dots, f_r) = (e_1, \dots, e_r)U$. Then $f_1 = y$. Since U is invertible, this means that $(y, f_2, \dots, f_r)$ is a Δ–basis for V. It follows that there exists $\varphi \in \mathrm{Hom}_\Delta(V, \Delta)$ such that $\varphi(y) = 1$. Since h is non–singular, there exists $y' \in V$ such that $1 = \varphi(y) = h(y', y)$. Consider the right Δ–subspace $U = y\Delta + y'\Delta$ of V, and note that in fact $U = y\Delta \oplus y'\Delta$. The matrix of $h|_U$ is given by

$$\begin{pmatrix} 0 & \varepsilon \\ 1 & h(y', y') \end{pmatrix}.$$

This matrix is invertible over Δ. Hence, $h|_U$ is non–singular and isotropic. It is a so–called metabolic plane. Proposition 3.2 yields that $(V, h) \simeq (U, h|_U) \perp (U^\perp, h|_{U^\perp})$. Furthermore, by Proposition 3.2, $(U^\perp, h|_{U^\perp})$ is also an ε–hermitian space over (Δ, θ) and hence, $U^\perp$ is free over Δ. If $h|_{U^\perp}$ is anisotropic, we are done. If $h|_{U^\perp}$ is isotropic, then we can repeat the above procedure. Eventually we obtain a decomposition of the desired form. □

3.5 Remark. Note that Proposition 3.4 yields that an ε–hermitian space over (Δ, θ) is isotropic if and only if it contains a "unimodular" isotropic vector.

3.6 Proposition. *Let (Δ, θ) be an $\mathcal{O}$–algebra with involution without zero divisors. Let $\varepsilon = \pm 1$. Let (V, h) be an ε–hermitian space over (Δ, θ). If $(V, h)_F$ is isotropic (resp. metabolic), then (V, h) is already isotropic (resp. metabolic).*

Proof. Suppose that $(V, h)_F$ is isotropic. Let $0 \neq x \in V_F$ be such that $h_F(x, x) = 0$. There exists a nonzero $r \in \mathcal{O}$ such that $rx \in V$. Then $rx \neq 0$ and $h(rx, rx) = 0$. Hence, (V, h) is isotropic. Suppose that $(V, h)_F$ is metabolic, but (V, h) non–metabolic. By Proposition 3.4, we can decompose $(V, h) \simeq (V_1, h_1) \perp (V_2, h_2)$, with (V_1, h_1) anisotropic and (V_2, h_2) metabolic. Then (V_1, h_1) remains anisotropic over F by the first part of the proof. But this means that $(V, h)_F$ is not metabolic, a contradiction. This proves the statement. □

In the situation of Proposition 3.4, suppose that $\mathcal{O} = F$ and $\mathrm{char}(F) \neq 2$. Then Δ is a division algebra, and the decomposition of (V, h) into an anisotropic and a hyperbolic ε–hermitian space over (Δ, θ) is unique up to isometry, by the Witt cancellation result of [15, (I.6.3.4)]. The *Witt index of* (V, h) is then defined as half the Δ–dimension of the hyperbolic part of (V, h).

3.2. Adjoint involutions

In this section, we zoom in on the "adjoint algebra with involution" of a (skew–)hermitian space over an Azumaya algebra with involution over a valuation ring. We characterise isomorphic involutions on a fixed Azumaya algebra in terms of (skew–)hermitian spaces they are adjoint to. These results will be used in section 7, in order to prove that isomorphism of Azumaya algebras with involution over a Henselian valuation ring in which 2 is invertible, can be detected rationally.

3.7 Proposition. *Let $(\mathcal{C}, \theta)$ be an $\mathcal{O}$–algebra with involution with center a domain. Let $\varepsilon = \pm 1$ and let (V, h) be an ε–hermitian space over $(\mathcal{C}, \theta)$. There exists a unique involution σ on $\mathrm{End}_{\mathcal{C}}(V)$ such that $\sigma(a) = \theta(a)$ for all $a \in Z(\mathcal{C})$, and for all $x, y \in V$ and all $f \in \mathrm{End}_{\mathcal{C}}(V)$ we have that*

$$h(x, f(y)) = h(\sigma(f)(x), y).$$

We denote this involution by ad_h. Then $(\mathrm{End}_{\mathcal{C}}(V), \mathrm{ad}_h)$ is an $\mathcal{O}$–algebra with involution with center $Z(\mathcal{C})$, called the adjoint algebra with involution of h, *and denoted by $\mathrm{Ad}(h)$. If θ is of the first (resp. second) kind, then ad_h is of the first (resp. second) kind.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of $Z(\mathcal{C})$. Then $\mathrm{End}_{\mathcal{C}}(V)/\mathfrak{m}\,\mathrm{End}_{\mathcal{C}}(V) \cong \mathrm{End}_{\mathcal{C}/\mathfrak{m}\mathcal{C}}(V/\mathfrak{m}V)$ by [15, (III.5.1.8)]. Since V is a finitely generated, projective $\mathcal{C}$–module and $\mathcal{C}$ is finite–dimensional over $Z(\mathcal{C})$ by Proposition 2.6 (a), V is finitely generated and faithful as a module over $Z(\mathcal{C})$, and hence, $V/\mathfrak{m}V$ is nonzero. This implies that $\mathrm{End}_{\mathcal{C}/\mathfrak{m}\mathcal{C}}(V/\mathfrak{m}V)$ is a central simple algebra over $Z(\mathcal{C})/\mathfrak{m}$. Hence, $\mathrm{End}_{\mathcal{C}}(V)$ is an Azumaya algebra over $Z(\mathcal{C})$. One easily checks that σ is an involution on $\mathrm{End}_{\mathcal{C}}(V)$ of the same kind as θ, and the uniqueness of ad_h follows from the fact that h is non–singular. □

The converse of Proposition 3.7 also holds.

3.8 Proposition. *Let $(\mathcal{A}, \sigma)$ be an $\mathcal{O}$–algebra with involution with center a domain. Then the following hold:*

- *(a) Every Azumaya algebra over $Z(\mathcal{A})$ Brauer equivalent to $\mathcal{A}$ carries an involution of the same kind as σ.*
- *(b) There exists an Azumaya algebra Δ over $Z(\mathcal{A})$ without zero divisors such that for every involution θ on Δ of the same kind as σ, there exists an ε–hermitian space (V, h) over (Δ, θ), with $\varepsilon \in \{1, -1\}$, such that $(\mathcal{A}, \sigma) \cong_{Z(\mathcal{A})} \mathrm{Ad}(h)$.*

Proof. We denote $Z(\mathcal{A})$ by S and its fraction field by K. Let $\mathcal{C}$ be an Azumaya algebra over S Brauer equivalent to $\mathcal{A}$. If σ is of the first kind, then $\mathcal{A}_F$, and hence also $\mathcal{C}_F$, is of exponent 2 in $\mathrm{Br}(K)$ by [16, (3.1) (1)]. By Proposition 2.11, $\mathcal{C}$ is of exponent 2 in $\mathrm{Br}(S)$. Suppose that σ is of the second kind. Then the corestriction of $\mathcal{A}_F$ is split by [16, (3.1) (2)] and hence, so is the corestriction of $\mathcal{C}_F$. Invoking Proposition 2.11 once more, we obtain that the corestriction of $\mathcal{C}$ is also split. Since S is a semilocal domain by Proposition 2.5, [24, (4.4)] yields that there exists an involution θ on $\mathcal{C}$ of the same kind as σ. This proves (a).

Proposition 2.10 yields that there exists an Azumaya algebra Δ over S without zero divisors, and a finite–dimensional right Δ–module V such that $\mathcal{A} \cong \mathrm{End}_{\Delta}(V) \cong \mathrm{M}_n(\Delta)$ as S–algebras, with $n = \dim_\Delta(V)$. In the rest of the proof, we identify $\mathcal{A}$ with $\mathrm{M}_n(\Delta)$. By (a), there exists an involution θ on Δ of the same kind as σ. We define an involution $\star$ on $\mathcal{A}$ by $(d_{ij})^\star_{ij} = (\theta(d_{ij}))^t_{ij}$, where t denotes the transpose involution. By Proposition 2.7, there exists $s \in \mathcal{A}^\times$ such that $s^\star = \varepsilon s$, with $\varepsilon = \pm 1$, and $\sigma = \mathrm{Int}(s) \circ \star$. Let $(e_1, \ldots, e_n)$ be a Δ–basis for V. Let $h_\star : V \times V \to \Delta$ be the hermitian form over (Δ, θ) defined by the $(n \times n)$ identity matrix with respect to the basis $(e_1, \ldots, e_n)$. We have that $\star = \mathrm{ad}_{h_\star}$ and it is clear that $h_\star$ is non–singular. We define $h : V \times V \to \Delta$ by $h(x,y) = h_\star(s^{-1}(x), y)$. For all $x, y \in V$, we have that

$$\begin{aligned} h(y,x) &= h_\star(s^{-1}(y), x) = \theta(h_\star(x, s^{-1}(y))) = \theta(h_\star((s^{-1})^\star(x), y)) \\ &= \varepsilon\theta(h_\star(s^{-1}(x), y) = \varepsilon\theta(h(x,y)), \end{aligned}$$

and it is clear that $h(x\alpha, y\beta) = \theta(\alpha)h(x,y)\beta$ for all $\alpha, \beta \in \Delta$. Furthermore, h is non–singular since $s \in \mathcal{A}^\times$. So, (V, h) is an ε–hermitian space and

$$h(\sigma(f)(x), y) = h_\star(s^{-1}\sigma(f)(x), y) = h_\star(f^\star(s^{-1}(x)), y) = h_\star(s^{-1}(x), f(y)) = h(x, f(y)).$$

□

3.9 Proposition. *Let $(\mathcal{A}, \sigma)$ be an $\mathcal{O}$–algebra with involution with center a domain. Let furthermore $s \in \mathcal{A}^\times$ be such that $\sigma(s) = s$ and let $\sigma' = \mathrm{Int}(s) \circ \sigma$. Then the following are equivalent:*

(i) $(\mathcal{A}, \sigma) \cong_{Z(\mathcal{A})} (\mathcal{A}, \sigma')$.

(ii) There exist elements $u \in \mathcal{O}^\times$ and $g \in \mathcal{A}^\times$ such that $us = \sigma(g)g$.

Proof. Suppose that (i) holds. Let $\varphi : \mathcal{A} \to \mathcal{A}$ be a $Z(\mathcal{A})$–automorphism such that $\sigma' \circ \varphi = \varphi \circ \sigma$. By Proposition 2.6 (b), there exists an element $g \in \mathcal{A}^\times$ such that $\varphi = \mathrm{Int}(\sigma(g))$. We get that

$$\mathrm{Int}(\sigma(g)) \circ \sigma = \mathrm{Int}(s) \circ \sigma \circ \mathrm{Int}(\sigma(g)) = \mathrm{Int}(sg^{-1}) \circ \sigma.$$

This implies that $\mathrm{Int}(\sigma(g)) = \mathrm{Int}(sg^{-1})$ and hence, there exists $u \in Z(\mathcal{A})^\times$ such that $\sigma(g) = usg^{-1}$. In other words $us = \sigma(g)g$. It follows that $\sigma(u)s = \sigma(s)\sigma(u) = \sigma(us) = us$. Since $s \in \mathcal{A}^\times$, it follows that $u \in \mathcal{O}$. Since $u \in Z(\mathcal{A})^\times$, we have that u is in fact an element of $\mathcal{O}^\times$. This proves (ii). For the converse, we can just go backwards through the proof of (i) $\Rightarrow$ (ii). □

3.10 Proposition. *Let (Δ, θ) be an $\mathcal{O}$–algebra with involution without zero divisors. Let $\varepsilon = \pm 1$ and let (V, h) be an ε–hermitian space over (Δ, θ). Let $(\mathcal{A}, \sigma) = \mathrm{Ad}(h)$. Let furthermore $s \in \mathcal{A}^\times$ be such that $\sigma(s) = s$ and let $\sigma' = \mathrm{Int}(s) \circ \sigma$. Define $h' : V \times V \to \Delta$ by $h'(x,y) = h(s^{-1}(x), y)$ for all $x, y, \in V$. Then (V, h') is an ε–hermitian space over (Δ, θ) such that $(\mathcal{A}, \sigma') = \mathrm{Ad}(h')$. Let $u \in \mathcal{O}^\times$. Then the following are equivalent:*

(i) $(V, h') \simeq (V, uh)$.

(ii) There exists $g \in \mathcal{A}^\times$ such that $us = \sigma(g)g$.

Furthermore, if $u \in \mathcal{O}^\times$ is such that (i) and (ii) hold, then $(\mathcal{A}, \sigma) \cong_{Z(\mathcal{A})} (\mathcal{A}, \sigma')$.

Proof. Suppose that $(V, h') \simeq (V, uh)$. Then there exists a Δ–linear bijection $\varphi : V \to V$ such that $h'(x,y) = uh(\varphi(x), \varphi(y))$. Then $\varphi \in \mathrm{End}_\Delta(V) = \mathcal{A}$ and it follows that

$$h(s^{-1}(x), y) = h'(x,y) = h(u\sigma(\varphi)\varphi(x), y)$$

for all $x, y \in V$. The non–singularity of h yields that $s^{-1} = u\sigma(\varphi)\varphi$, i.e. $us = \varphi^{-1}\sigma(\varphi^{-1})$. Conversely, suppose that there exists $g \in \mathcal{A}^\times$ such that $us = \sigma(g)g$. Then

$$h'(x,y) = h(s^{-1}(x), y) = uh(g^{-1}\sigma(g^{-1})(x), y) = uh(\sigma(g^{-1})(x), \sigma(g^{-1})(y)).$$

This yields that $(V, h') \simeq (V, uh)$. The last statement of the proposition follows immediately from Proposition 3.9. □

3.11 Proposition. *Let $\mathcal{A}$ be an Azumaya algebra with center $\mathcal{O}$ or a separable quadratic $\mathcal{O}$–algebra that is a domain. Let σ and σ' be two $\mathcal{O}$–linear involutions of the first or second kind on $\mathcal{A}$. Suppose that $(\mathcal{A}, \sigma)_F \cong_F (\mathcal{A}, \sigma')_F$. Then there exists $s \in \mathcal{A}^\times$ such that $\sigma(s) = s$ and $\sigma' = \mathrm{Int}(s) \circ \sigma$.*

Proof. By Proposition 2.7, there exists an element $s \in \mathcal{A}^\times$ such that $\sigma(s) = \pm s$ and $\sigma' = \mathrm{Int}(s) \circ \sigma$. Since σ_F and σ'_F are isomorphic, they must be of the same kind and type. If $\mathrm{char}(F) = 2$, then we automatically get that $\sigma(s) = s$. If $\mathrm{char}(F) \neq 2$, then [16, (2.7) (3)] yields that we necessarily have $\sigma(s) = \sigma_F(s) = s$. □

4. Valuations and value functions

In the sequel, we will work with non–commutative valuation rings. We recall below the definition and some basic facts, and in the commutative case, the equivalent concept of a place from one field to another. We further include some basics of the theory of value functions on vector spaces and algebras over a valued field, which was recently developed in [23, 26]. We will use value functions in order to connect, for an algebra with involution over a valuation ring, the induced structures over the fraction field and residue field of the valuation ring.

Throughout this section, we fix a field F. Let D be an F–division algebra. A *valuation ring* of D is a subring Λ of D such that for all $x \in D$, we have that $x \in \Lambda$ or $x^{-1} \in \Lambda$, and furthermore, Λ is invariant under conjugation with elements of D. (In the literature, Λ is sometimes called an invariant valuation ring of D.) Let Γ be a totally ordered abelian group and let ∞ be a symbol of a set strictly containing Γ, and satisfying $\gamma < \infty$ and $\infty = \infty + \infty = \infty + \gamma = \gamma + \infty$ for all $\gamma \in \Gamma$. A map $w : D \to \Gamma \cup \{\infty\}$ is called *a valuation on D* if $w^{-1}(\{\infty\}) = \{0\}$, $w(a+b) \geqslant \min(w(a), w(b))$ and $w(ab) = w(a) + w(b)$, for all $a, b \in D$. The ring $\mathcal{O}_D = \{a \in D \mid w(a) \geqslant 0\}$ is a valuation ring of D. The valuation ring $\mathcal{O}_D$ has a unique maximal left (and right) ideal M_D, which is equal to $\{a \in D \mid w(a) > 0\}$. One calls $\Gamma_D = w(D^\times)$ *the value group of w*, and one can show that $\Gamma_D \cong D^\times / \mathcal{O}_D{}^\times$ (see e.g. [28, p. 388]). Given a valuation ring Λ of D, there exists a valuation on D with valuation ring precisely Λ (see e.g. [28, p. 388]). So, there is a one-to-one correspondence between valuations on D and valuation rings of D.

Let L be a field and let $L^\infty = L \cup \{\infty\}$, with the field operations of L extended to L^∞ by $\infty + x = x + \infty = \infty$ for any $x \in L$, $x \cdot \infty = \infty \cdot x = \infty$ for any $0 \neq x \in L^\infty$, whereas $\infty + \infty, 0 \cdot \infty$ and $\infty \cdot 0$ are not defined. A *place* from F to L is a map $\lambda : F \to L^\infty$ such that $\lambda(1) = 1$,

$\lambda(xy) = \lambda(x)\lambda(y)$ and $\lambda(x+y) = \lambda(x) + \lambda(y)$ for all $x, y \in F$, whenever the right hand sides are defined.
Given a place $\lambda : F \to L^\infty$, the set $\mathcal{O}_\lambda = \{x \in F \mid \lambda(x) \neq \infty\}$ is a valuation ring of F with maximal ideal $\mathfrak{m}_\lambda = \{x \in F \mid \lambda(x) = 0\}$. The place λ identifies the residue field $\mathcal{O}_\lambda/\mathfrak{m}_\lambda$ with a subfield of L. Conversely, let $\mathcal{O}$ be a valuation ring of F with residue field κ. Setting $\lambda(a) = \overline{a}$ for all $a \in \mathcal{O}$ and $\lambda(a) = \infty$ for all $a \in F \smallsetminus O$ defines a place $\lambda : F \to \kappa^\infty$.

For the rest of this section, we fix a valuation v on F. We denote its valuation ring by $\mathcal{O}$ and its maximal ideal by $\mathfrak{m}$.

For every $\gamma \in v(\Gamma)$, let $F_v^{\geqslant\gamma} = \{a \in F \mid v(a) \geqslant \gamma\}$ and $F_v^{>\gamma} = \{a \in F \mid v(a) > \gamma\}$. We then set

$$\mathrm{gr}_v(F) = \bigoplus_{\gamma \in v(\Gamma)} F_v^{\geqslant\gamma}/F_v^{>\gamma}.$$

Let V be a finite–dimensional F–vector space. A map $\alpha : V \to \Gamma \cup \{\infty\}$ is called a *v–value function on V* if $\alpha^{-1}(\{\infty\}) = \{0\}$, $\alpha(xa) = \alpha(x) + v(a)$ and $\alpha(x+y) \geqslant \min(\alpha(x), \alpha(y))$ for all $x, y \in V$ and all $a \in F$.

For every $\gamma \in v(\Gamma)$, let $V_\alpha^{\geqslant\gamma} = \{x \in V \mid \alpha(x) \geqslant \gamma\}$ and $V_\alpha^{>\gamma} = \{x \in V \mid \alpha(x) > \gamma\}$. Then

$$\mathrm{gr}_\alpha(V) = \bigoplus_{\gamma \in v(\Gamma)} V_\alpha^{\geqslant\gamma}/V_\alpha^{>\gamma}.$$

is a graded $\mathrm{gr}_v(F)$–module.

Let B be a finite–dimensional F–algebra. A v–value function α on B is called *surmultiplicative* if $\alpha(1) = 0$ and $\alpha(ab) \geqslant \alpha(a) + \alpha(b)$ for all $a, b \in B$. In this case, $\mathrm{gr}_\alpha(B)$ has the structure of a graded $\mathrm{gr}_v(F)$–algebra.

4.1 Example. Let V be an finite–dimensional F–vector space and $\mathfrak{B} = (b_1, \ldots, b_n)$ an F–basis for V. It is an easy verification that the map

$$v_{\mathfrak{B}} : \sum_{i=1}^{n} b_i x_i \mapsto \min_{1 \leqslant i \leqslant n}(v(x_i)), \qquad \text{for } x_1, \ldots, x_n \in F,$$

is a v–value function on V.

We call a v–value function on a finite–dimensional F–vector space V *a v–norm* if there exists an F–basis $\mathfrak{B}$ for V such that $\alpha = v_{\mathfrak{B}}$; the basis $\mathfrak{B}$ is then called *a splitting basis for α.*

4.2 Remark. Since we work with value functions having the same value group as the underlying valuation, our definition of v–norm is the same as the one given in [23].

4.3 Proposition. *Let $\mathcal{V}$ be a finite–dimensional $\mathcal{O}$–module and let $V = \mathcal{V} \otimes_{\mathcal{O}} F$. Then there is a unique v–norm α on V such that $V_\alpha^{\geqslant 0} = \mathcal{V}$.*

Proof. Let $\mathfrak{B}$ be an $\mathcal{O}$–basis of $\mathcal{V}$. Then it is an F–basis of V, and it is clear that $V_{v_{\mathfrak{B}}}^{\geqslant 0} = \mathcal{V}$. Let α be any v–norm on V such that $V_\alpha^{\geqslant 0} = \mathcal{V}$. Let $\mathfrak{B}'$ be a splitting basis for α. Then $\mathfrak{B}'$ is an $\mathcal{O}$–basis of $\mathcal{V}$. Using the matrix of base change from $\mathfrak{B}$ to $\mathfrak{B}'$ (whose entries lie in $\mathcal{O}$), one easily obtains that $v_{\mathfrak{B}}(x) \geqslant \alpha(x)$ for all $x \in V$. Interchanging the roles of $\mathfrak{B}$ and $\mathfrak{B}'$ yields the other inequality. □

4.4 Proposition. *Let V be a finite–dimensional F−vector space and $\mathfrak{B} = (b_1, \dots, b_n)$ an F−basis for V. Let $x \in V$ be such that there exists an index $i \in \{1, \dots, n\}$ such that $x - b_i \in V^{>0}_{v_\mathfrak{B}}$. Then the family obtained from $\mathfrak{B}$ by replacing b_i by x is also a splitting basis for $v_\mathfrak{B}$.*

Proof. Since $\mathfrak{B}$ is a splitting basis for $v_\mathfrak{B}$ and $v_\mathfrak{B}(x - b_i) > 0$, we can write $x - b_i = \sum_{j=1}^n b_j x_j$, with $x_1, \dots, x_n \in \mathfrak{m}$. It follows that $v_\mathfrak{B}(x) = 0 = v_\mathfrak{B}(b_i(1 + x_i))$. Hence, by [23, Corollary 2.3 (iii)], the family obtained from $\mathfrak{B}$ by replacing b_i by x is also a splitting basis for $v_\mathfrak{B}$. □

4.5 Proposition. *Let T be the intersection of finitely many valuation rings of F. Let $\mathcal{V}$ be a finite–dimensional T−module and let $V = \mathcal{V} \otimes_T F$. Let furthermore W be a nonzero F−subspace of V. Then $W \cap \mathcal{V}$ is free as a T−module and*

$$\dim_F(W) = \dim_T(W \cap \mathcal{V}).$$

Proof. By assumption, there exist valuation rings $\mathcal{O}_1, \dots, \mathcal{O}_\ell$ of F such that $T = \mathcal{O}_1 \cap \dots \cap \mathcal{O}_\ell$, and we may assume that they are pairwise incomparable. Let $v_1, \dots, v_\ell$ be corresponding valuations on F. For $i = 1, \dots, \ell$, let $\mathfrak{M}_i$ be the unique maximal ideal of $\mathcal{O}_i$ and $\mathcal{M}_i = \mathfrak{M}_i \cap T$. By [9, (3.2.6), (3.2.7)] we have that $\mathcal{O}_i = T_{\mathcal{M}_i}$ for $i = 1, \dots, \ell$, and $\mathcal{M}_1, \dots, \mathcal{M}_\ell$ are the different maximal ideals of T. Furthermore, $T/\mathcal{M}_i$ is naturally isomorphic to $T_{\mathcal{M}_i}/\mathcal{M}_i T_{\mathcal{M}_i} = \mathcal{O}_i/\mathfrak{M}_i$ via $a \mod \mathcal{M}_i \mapsto \frac{a}{1} \mod \mathcal{M}_i T_{\mathcal{M}_i}$, for $i = 1, \dots, \ell$.

Let $\mathfrak{B} = (e_1, \dots, e_n)$ be a T−basis for $\mathcal{V}$. Let $i \in \{1, \dots, \ell\}$. Then $\mathfrak{B}$ is an $\mathcal{O}_i$−basis for $\mathcal{V}_i = \mathcal{V}\mathcal{O}_i \subset V$. We consider the v_i−norm $\alpha_i = (v_i)_\mathfrak{B}$ on $V = \mathcal{V}_i F$. We have that $V^{\geqslant 0}_{\alpha_i} = \mathcal{V}_i$. We set $\mathcal{W} = W \cap \mathcal{V}$. By [23, Proposition 2.5], $\alpha_i|_W$ is a v_i−norm. Let $(d^i_1, \dots, d^i_r)$ be a splitting basis for $\alpha_i|_W$. We prove that there is a common splitting basis for $\alpha_1|_W, \dots, \alpha_\ell|_W$.

Since $\mathcal{M}_1, \dots, \mathcal{M}_\ell$ are pairwise different maximal ideals of T, they are pairwise coprime. By the Chinese Remainder Theorem, the natural isomorphisms $T/\mathcal{M}_i \to \mathcal{O}_i/\mathfrak{M}_i$ for $i = 1, \dots, \ell$, and [17, (XVI.2.7)], the T−homomorphism

$$\varphi : \mathcal{W} \to \mathcal{W}\mathcal{O}_1/\mathcal{W}\mathfrak{M}_1 \times \dots \times \mathcal{W}\mathcal{O}_\ell/\mathcal{W}\mathfrak{M}_\ell$$

is surjective. We show that $\mathcal{W}\mathcal{O}_i = W^{\geqslant 0}_{\alpha_i}$ for $i = 1, \dots, \ell$. It is clear that $\mathcal{W}\mathcal{O}_i \subset W^{\geqslant 0}_{\alpha_i}$. Conversely, let $x \in W^{\geqslant 0}_{\alpha_i}$. Then there exists $t \in T \setminus \mathcal{M}_i$ such that $xt \in \mathcal{W}$. Since $1/t \in T_{\mathcal{M}_i} = \mathcal{O}_i$, it follows that $x \in \mathcal{W}\mathcal{O}_i$. Multiplying $\mathcal{W}\mathcal{O}_i = W^{\geqslant 0}_{\alpha_i}$ by $\mathfrak{M}_i$, we obtain that $\mathcal{W}\mathfrak{M}_i = W^{>0}_{\alpha_i}$. By the surjectivity of φ, there exist $f_1, \dots, f_r \in \mathcal{W}$ such that

$$\varphi(f_j) = (\overline{d^1_j}, \dots, \overline{d^\ell_j}) \in \prod_{i=1}^{\ell} W^{\geqslant 0}_{\alpha_i}/W^{>0}_{\alpha_i}.$$

By Proposition 4.4, $(f_1, \dots, f_r)$ is a splitting basis for $\alpha_1|_W, \dots, \alpha_\ell|_W$. It follows that

$$\mathcal{W} \subset W^{\geqslant 0}_{\alpha_1} \cap \dots \cap W^{\geqslant 0}_{\alpha_\ell} = \left\{ \sum_{j=1}^{r} f_j x_j \mid x_1, \dots, x_\ell \in \mathcal{O}_1 \cap \dots \cap \mathcal{O}_\ell \right\} \subset \mathcal{W},$$

since $T = \mathcal{O}_1 \cap \dots \cap \mathcal{O}_\ell$. Hence, $\mathcal{W}$ is free over T and $\dim_T(\mathcal{W}) = \dim_F(W)$. □

4.6 Remark. In the sequel, we wil only apply Proposition 4.5 in the case where T is the intersection of at most two valuation rings. Note furthermore that the statement of Proposition 4.5 can also be proved by using that an intersection of finitely many valuation rings of a field is a so–called elementary divisor domain.

In the previous results we only used value functions on vector spaces. In the last result of this section, we prove the existence of a special value function on an F–algebra with involution, obtained by scalar extension from an $\mathcal{O}$–algebra with involution. This result will be used in section 6 when we consider Azumaya algebras with involution under specialisation with respect to a Henselian valuation ring.

Let B be a finite–dimensional F–algebra and α a surmultiplicative v–norm on B. Then α is said to be a *v–gauge* if $\mathrm{gr}_\alpha(B)$ is *graded semisimple*, i.e. $\mathrm{gr}_\alpha(B)$ does not contain any nonzero nilpotent homogeneous two–sided ideals. Let (B,τ) be an F–algebra with involution. A v–gauge α on B is called *τ–invariant* if $\alpha(\tau(x)) = \alpha(x)$ for all $x \in B$.

4.7 Proposition. *Let $(\mathcal{A},\sigma)$ be an $\mathcal{O}$–algebra with involution. Then there exists a unique σ_F–invariant v–gauge α on $\mathcal{A}_F$ such that $(\mathcal{A}_F)_\alpha^{\geqslant 0} = \mathcal{A}$ and $(\mathcal{A}_F)_\alpha^{>0} = \mathfrak{m}\mathcal{A}$.*

Proof. By Proposition 2.6 (a), $\mathcal{A}$ is free over $\mathcal{O}$. Let $\mathfrak{B} = (e_1,\dots,e_n)$ be an $\mathcal{O}$–basis for $\mathcal{A}$. Then $\mathfrak{B}$ is an F–basis for $\mathcal{A}_F$. It is clear that $(\mathcal{A}_F)_{v_\mathfrak{B}}^{\geqslant 0} = \mathcal{A}$ and $(\mathcal{A}_F)_{v_\mathfrak{B}}^{>0} = \mathfrak{m}\mathcal{A}$. We show that $v_\mathfrak{B}$ is a v–gauge and that it is σ_F–invariant. The fact that $v_\mathfrak{B}$ is the unique gauge with this property follows from Proposition 4.3. Since $1 \in \mathcal{A}$, we have that $v_\mathfrak{B}(1) \geqslant 0$ and hence $v_\mathfrak{B}(1) = 0$ since $\mathcal{A} \neq \mathfrak{m}\mathcal{A}$. In order to show that $v_\mathfrak{B}$ is surmultiplicative, by [26, Lemma 1.2], it suffices to show that $v_\mathfrak{B}(e_ie_j) \geqslant v_\mathfrak{B}(e_i) + v_\mathfrak{B}(e_j) = 0$ for all $i,j \in \{1,\dots,n\}$. Since $\mathcal{A} = (\mathcal{A}_F)_{v_\mathfrak{B}}^{\geqslant 0}$ is multiplicatively closed, this is clearly satisfied.

We next verify that $v_\mathfrak{B}$ is σ_F–invariant. Let $i \in \{1,\dots,n\}$. There exist $d_{i1},\dots,d_{in} \in \mathcal{O}$ such that $\sigma(e_i) = \sum_{k=1}^n e_kd_{ik}$. Let $(x_1,\dots,x_n) \in F^n$ be arbitrary. Then for $k = 1,\dots,n$, we have that $v(\sum_{i=1}^n x_id_{ik}) \geqslant \min_{1\leqslant i\leqslant n}(v(x_i))$. Furthermore, $\sigma_F(\sum_{i=1}^n e_ix_i) = \sum_{i=1}^n \sigma(e_i)x_i = \sum_{k=1}^n e_k(\sum_{i=1}^n x_id_{ik})$, and hence

$$v_\mathfrak{B}\left(\sigma_F\left(\sum_{i=1}^n e_ix_i\right)\right) = \min_{1\leqslant k\leqslant n}\left(v\left(\sum_{i=1}^n x_id_{ik}\right)\right) \geqslant \min_{1\leqslant i\leqslant n}(v(x_i)) = v_\mathfrak{B}\left(\sum_{i=1}^n e_ix_i\right).$$

This yields that $v_\mathfrak{B}(x) = v_\mathfrak{B}(\sigma_F^2(x)) \geqslant v_\mathfrak{B}(\sigma_F(x)) \geqslant v_\mathfrak{B}(x)$, for all $x \in \mathcal{A}_F$. This proves the σ_F–invariance of $v_\mathfrak{B}$.

In order to have that $v_\mathfrak{B}$ is a v–gauge, all that remains to be shown is that the graded algebra $\mathrm{gr}_{v_\mathfrak{B}}(\mathcal{A}_F)$ is semisimple. Suppose for the sake of contradiction that $\mathrm{gr}_{v_\mathfrak{B}}(\mathcal{A}_F)$ contains a nonzero homogeneous two–sided nilpotent ideal I. Let $I_0 = I \cap \mathrm{gr}_{v_\mathfrak{B}}(\mathcal{A}_F)_0 = I \cap \mathcal{A}/\mathfrak{m}\mathcal{A}$. For a nonzero $x \in B$, we write $\tilde{x} = x + (\mathcal{A}_F)_{v_\mathfrak{B}}^{>v_\mathfrak{B}(x)}$. Let a be a nonzero element of $\mathcal{A}_F$ such that $\tilde{a} \in I$. Since $v_\mathfrak{B}$ has the same value group as v, there exists $u \in F$ such that $v_\mathfrak{B}(a) = -v(u)$. Then $v_\mathfrak{B}(au) = v_\mathfrak{B}(a) + v(u) = 0$ and since I is an ideal of $\mathrm{gr}_{v_\mathfrak{B}}(\mathcal{A}_F)$, we have that $\tilde{a}\tilde{u} \in I$, and furthermore, $\tilde{a}\tilde{u} = \widetilde{au}$. Since $\widetilde{au} \neq 0$, this implies that $I_0 \neq 0$. However, this is not possible since $\mathcal{A}/\mathfrak{m}\mathcal{A}$ is semisimple. Hence, $v_\mathfrak{B}$ is a v–gauge. □

5. Specialisation of involutions and the index

In this section, we prove the first main specialisation result of this article. We fix fields F and L, a place $\lambda : F \to L^\infty$, and we denote the valuation ring of F associated to λ by $\mathcal{O}$. We don't make any assumptions on the characteristic of F and L.

Let $(\mathcal{A},\sigma)$ be an $\mathcal{O}$–algebra with involution. By Proposition 2.1, $(\mathcal{A},\sigma)_F$ is an F–algebra with involution and $(\mathcal{A},\sigma)_L$ an L–algebra with involution. We show in Theorem 5.7 that isotropic right ideals of $(\mathcal{A},\sigma)_F$ specialise under λ to isotropic right ideals of $(\mathcal{A},\sigma)_L$, in a way that preserves the dimension.

In order to treat the isotropy behaviour of first and second kind involutions in a uniform way, we introduce the notion of "balanced ideals" and extend some notions and results from [16] to balanced ideals.

Throughout this section, we fix an F–algebra with involution (B,τ). We first extend the notions of degree and Schur index of a central simple algebra to B. As in [16], we let $\deg(B)$ be the square root of $\dim_F(B)/\dim_F(Z(B))$. If B is simple, we let $\operatorname{ind}(B)$ be the usual Schur index of B as a central simple $Z(B)$–algebra. If B is not simple, then by Proposition 2.2, there exists a central simple F–algebra E such that $B \cong E \times E^{\mathrm{op}}$. In that case we set $\operatorname{ind}(B) = \operatorname{ind}(E)$.

Suppose that there exists a central simple F–algebra E such that $B \cong E \times E^{\mathrm{op}}$. Let I be a right ideal of B. Then I corresponds to a right ideal $I_1 \times I_2^{\mathrm{op}}$ of $E \times E^{\mathrm{op}}$, with I_1 a right ideal of E and I_2 a left ideal of E.

We call a right ideal I of B *balanced* if it is free as a $Z(B)$–module. If $Z(B)$ is a field, then all right ideals of B are balanced. If $Z(B) \cong F \times F$, then the following lemma describes what the balanced ideals of B look like.

5.1 Lemma. *Let T be a domain and let M and N be finite–dimensional T–modules. Then $M \times N$ is free as a $(T \times T)$–module if and only if $\dim_T(M) = \dim_T(N)$, and in that case $\dim_{T\times T}(M \times N) = \dim_T(M) = \dim_T(N)$.*

Proof. This follows from the fact that $(T \times T)^n \cong T^n \times T^n$ as $(T \times T)$–modules for all $n \in \mathbb{N}$, and that $T^n \times T^m$ is not a free $(T \times T)$–module if $n \neq m$. □

Let I be a balanced right ideal of B. Then $\dim_{Z(B)}(I)$ is divisible by $\deg(B)\operatorname{ind}(B)$, by Proposition 5.1 and [16, pp. 5–6]. We call

$$\operatorname{rdim}(I) = \frac{\dim_{Z(B)}(I)}{\deg(B)}$$

the *reduced dimension of I*. It extends the notion of reduced dimension for right ideals of central simple algebras from [16] to cover the semisimple case as well. If $B \cong E \times E^{\mathrm{op}}$ for a central simple F–algebra E, and I is a balanced right ideal of B that corresponds to the ideal $I_1 \times I_2^{\mathrm{op}}$ of $E \times E^{\mathrm{op}}$, then $\operatorname{rdim}(I) = \frac{\dim_F(I_1)}{\deg(E)} = \operatorname{rdim}(I_1)$ by Lemma 5.1.

Replacing 'right' by 'left' in the above, we get analogous results for left ideals of B. Applying the general results for modules from section 4 to right ideals, we obtain the following theorem.

5.2 Theorem. *Let $\mathcal{A}$ be an Azumaya algebra with center either $\mathcal{O}$ or a separable quadratic $\mathcal{O}$–algebra. If $Z(\mathcal{A})$ is not a domain, assume that there exists an Azumaya algebra $\mathcal{B}$ over $\mathcal{O}$ such that $\mathcal{A} \cong \mathcal{B} \times \mathcal{B}^{\mathrm{op}}$. Then $\deg(\mathcal{A}_F) = \deg(\mathcal{A}_L)$. Let furthermore I be a balanced right ideal of $\mathcal{A}_F$. Then $I \cap \mathcal{A}$ is free as a $Z(\mathcal{A})$–module and $(I \cap \mathcal{A}) \otimes_{\mathcal{O}} L$ is a balanced right ideal of $\mathcal{A}_L$ of the same reduced dimension as I.*

Proof. Since $\mathcal{A}$ is free as an $\mathcal{O}$-module by Proposition 2.6 (a), we have that $\dim_F(\mathcal{A}_F) = \dim_{\mathcal{O}}(\mathcal{A}) = \dim_L(\mathcal{A}_L)$. This clearly implies that $\deg(\mathcal{A}_F) = \deg(\mathcal{A}_L)$.

We now prove the statement on the ideals. If $I = 0$, then there is nothing to prove. So, in the rest of the proof, we may assume that $I \neq 0$. It is clear that $(I \cap \mathcal{A}) \otimes_{\mathcal{O}} L$ is a right ideal of $\mathcal{A}_L$. Suppose first that $Z(\mathcal{A})$ is a domain. Then $Z(\mathcal{A})$ is a valuation ring or the intersection of two valuation rings of $Z(\mathcal{A}_F)$, by Proposition 2.5. Furthermore, $\mathcal{A}$ is free as a $Z(\mathcal{A})$-module by Proposition 2.6 (a). We have that $\mathcal{A} \otimes_{Z(\mathcal{A})} Z(\mathcal{A}_F) \cong \mathcal{A} \otimes_{\mathcal{O}} F$ as $Z(\mathcal{A}_F)$-modules. Since I is a $Z(\mathcal{A}_F)$-subspace of $\mathcal{A}_F$, we can apply Proposition 4.5 to obtain that $I \cap \mathcal{A}$ is free as a $Z(\mathcal{A})$-module and

$$\dim_{Z(\mathcal{A})}(I \cap \mathcal{A}) = \dim_{Z(\mathcal{A}_F)}(I).$$

Assume that $Z(\mathcal{A})$ is not a domain. By assumption, there exists an Azumaya algebra $\mathcal{B}$ over $\mathcal{O}$ such that $\mathcal{A} \cong \mathcal{B} \times \mathcal{B}^{\mathrm{op}}$. Then $\mathcal{B}$ is free as an $\mathcal{O}$-module by Proposition 2.6 (a). We have that $\mathcal{A}_F \cong (\mathcal{B} \otimes_{\mathcal{O}} F) \times (\mathcal{B} \otimes_{\mathcal{O}} F)^{\mathrm{op}}$. Under this isomorphism, we identify I with a right ideal $I_1 \times I_2^{\mathrm{op}}$ of $(\mathcal{B} \otimes_{\mathcal{O}} F) \times (\mathcal{B} \otimes_{\mathcal{O}} F)^{\mathrm{op}}$, where I_1 is a right ideal of $\mathcal{B} \otimes_{\mathcal{O}} F$ and I_2 a left ideal of $\mathcal{B} \otimes_{\mathcal{O}} F$. Then $I \cap \mathcal{A} = (I_1 \cap \mathcal{B}) \times (I_2 \cap \mathcal{B})^{\mathrm{op}}$. Since I is balanced, Lemma 5.1 yields that $\dim_F(I_1) = \dim_F(I_2)$. By Proposition 4.5, we have that $I_1 \cap \mathcal{B}$ and $I_2 \cap \mathcal{B}$ are free as $\mathcal{O}$-modules and

$$\dim_{\mathcal{O}}(I_1 \cap \mathcal{B}) = \dim_F(I_1) = \dim_F(I_2) = \dim_{\mathcal{O}}(I_2 \cap \mathcal{B}).$$

Applying Lemma 5.1 to $T = \mathcal{O}$ yields that $I \cap \mathcal{A}$ is free as a $Z(\mathcal{A})$-module and $\dim_{Z(\mathcal{A})}(I \cap \mathcal{A}) = \dim_{Z(\mathcal{A}_F)}(I)$.

We have the following isomorphisms of $Z(\mathcal{A}_L)$-modules:

$$(I \cap \mathcal{A}) \otimes_{Z(\mathcal{A})} Z(\mathcal{A}_L) \cong ((I \cap \mathcal{A}) \otimes_{Z(\mathcal{A})} Z(\mathcal{A})) \otimes_{\mathcal{O}} L \cong (I \cap \mathcal{A}) \otimes_{\mathcal{O}} L.$$

Since $I \cap \mathcal{A}$ is free as a $Z(\mathcal{A})$-module, it follows that $(I \cap \mathcal{A}) \otimes_{\mathcal{O}} L$ is free as a $Z(\mathcal{A}_L)$-module. In other words, $(I \cap \mathcal{A}) \otimes_{\mathcal{O}} L$ is a balanced right ideal of $\mathcal{A}_L$.

It remains to prove the claim about the reduced dimensions. It follows from the above that

$$\dim_{Z(\mathcal{A}_F)}(I) = \dim_{Z(\mathcal{A})}(I \cap \mathcal{A}) = \dim_{Z(\mathcal{A}_L)}[(I \cap \mathcal{A}) \otimes_{\mathcal{O}} L]$$

Since $\deg(\mathcal{A}_F) = \deg(\mathcal{A}_L)$, this yields that $\mathrm{rdim}(I) = \mathrm{rdim}((I \cap \mathcal{A}) \otimes_{\mathcal{O}} L)$. □

For a right ideal I of B, the left ideal $I^0 = \{x \in B \mid xI = 0\}$ is *the annihilator of* I. Similarly, for a left ideal J of B, the right ideal $J^0 = \{x \in B \mid Jx = 0\}$ is *the annihilator of* J.

5.3 Proposition. *Let I be a balanced right (resp. left) ideal of B. Then I^0 is a balanced left (resp. right) ideal of B and* $\mathrm{rdim}(I) + \mathrm{rdim}(I^0) = \deg(B)$.

Proof. If $Z(B)$ is a field, this is the statement of [16, (1.14)]. Assume that $Z(B) \cong F \times F$ and $B \cong E \times E^{\mathrm{op}}$, for some central simple F-algebra E. Let I be a right ideal of B. The proof for left ideals is analogous. We identify I with a right ideal $I_1 \times I_2^{\mathrm{op}}$ of $E \times E^{\mathrm{op}}$, where I_1 (resp. I_2) is a right (resp. left) ideal of E. It is easily seen that $I^0 \cong I_1^0 \times (I_2^0)^{\mathrm{op}}$, and using this together with [16, (1.14)] yields the statement. □

A right (resp. left) ideal I of B is called *isotropic* with respect to τ if $I \subset \tau(I)^0$. The algebra with involution (B,τ), or τ itself, is called *isotropic* if B contains a nonzero isotropic right ideal, and *anisotropic* otherwise. Note that τ is isotropic if and only if there is a nonzero element $x \in B$ such that $\tau(x)x = 0$. The algebra with involution (B,τ), or τ itself, is called *hyperbolic* if there exists an idempotent $x \in B$ such that $\tau(x) = 1 - x$, and is called *metabolic* if (B,τ) contains an isotropic balanced right ideal I of reduced dimension $\deg(B)/2$. By Proposition 5.3, the latter is equivalent to $I = \tau(I)^0$.

5.4 Proposition. *Let I be an isotropic balanced right ideal of (B,τ). Then* $\mathrm{rdim}(I) \leqslant \deg(B)/2$.

Proof. If $Z(B)$ is a field, this follows from [16, (6.2)]. Let us consider the case $(B,\tau) \cong (E \times E^{\mathrm{op}}, \mathrm{sw}_E)$, with E a central simple F−algebra. We identify I with a right ideal $I_1 \times I_2^{\mathrm{op}}$ of $E \times E^{\mathrm{op}}$, where I_1 (resp. I_2) is a right (resp. left) ideal of E. Since I is balanced, we have that $\mathrm{rdim}(I) = \mathrm{rdim}(I_1) = \mathrm{rdim}(I_2)$. Since I is isotropic, it follows that $I_2 \subset I_1^0$. By Proposition 5.3,

$$2\,\mathrm{rdim}(I) = \mathrm{rdim}(I_1) + \mathrm{rdim}(I_2) \leqslant \mathrm{rdim}(I_1) + \mathrm{rdim}(I_1^0) = \deg(B),$$

whence the statement. □

5.5 Proposition. *Suppose that (B,τ) is degenerate. Then τ is hyperbolic, and for any $\ell \in \mathbb{N}$ such that $0 \leqslant \ell \leqslant \frac{\deg(B)}{2\,\mathrm{ind}(B)}$, there exists an isotropic balanced right ideal of (B,τ) of reduced dimension $\ell \cdot \mathrm{ind}(B)$.*

Proof. We have that $(B,\tau) \cong (E \times E^{\mathrm{op}}, \mathrm{sw}_E)$, for some central simple F−algebra E. The element $(1,0) \in E \times E^{\mathrm{op}}$ is idempotent and $\mathrm{sw}_E(1,0) = (0,1) = (1,1) - (1,0)$. Hence, τ is hyperbolic. Let $\ell \in \mathbb{N}$ be such that $0 \leqslant \ell \leqslant \frac{\deg(B)}{2\,\mathrm{ind}(B)}$. By the characterisation of left and right ideals of E in [16, (1.12)] there exists a right ideal I_1 of E with $\mathrm{rdim}(I_1) = \ell \cdot \mathrm{ind}(B) = \ell \cdot \mathrm{ind}(E) \leqslant \deg(B)/2$ and a left ideal I_2 of E inside I_1^0 of the same reduced dimension as I_1. Then $I_1 \times I_2^{\mathrm{op}}$ is an isotropic balanced right ideal of $E \times E^{\mathrm{op}}$ of reduced dimension $\ell \cdot \mathrm{ind}(B)$. □

5.6 Proposition. *Let (D,θ) be an F−algebra with involution and assume that D is a division algebra. Let $\varepsilon = \pm 1$ and let (V,h) be an ε−hermitian space over (D,θ). Let I be a right ideal of* $\mathrm{End}_D(V)$. *Then* $I = \mathrm{Hom}_D(V,W)$, *for some right D−subspace W of V, and the following hold:*

(a) *I is isotropic for* ad_h *if and only if W is totally isotropic for h.*

(b) ad_h *is isotropic (resp. metabolic) if and only if h is isotropic (resp. metabolic).*

Proof. See [16, (1.13)] for the fact that every right ideal I of $\mathrm{End}_D(V)$ is of the form $\mathrm{Hom}_D(V,W)$, for some right D−subspace W of V. For (a) see [2, (1.6)], and for the statement on metabolicity in (b), see [8, (4.8)]. □

We can now prove the first specialisation theorem for isotropic right ideals. Almost all the work was done in Theorem 5.2, so the proof is now very short.

5.7 Theorem. *Let $(\mathcal{A},\sigma)$ be an $\mathcal{O}$−algebra with involution. Let I be an isotropic balanced right ideal of $(\mathcal{A},\sigma)_F$. Then $I \cap \mathcal{A}$ is free as a $Z(\mathcal{A})$−module and $(I \cap \mathcal{A}) \otimes_{\mathcal{O}} L$ is an isotropic balanced right ideal of $(\mathcal{A},\sigma)_L$ of the same reduced dimension as I.*

Proof. It is clear that if I is an isotropic right ideal of $(\mathcal{A},\sigma)_F$, then $(I\cap\mathcal{A})\otimes_{\mathcal{O}} L$ is an isotropic right ideal of $(\mathcal{A},\sigma)_L$. The rest of the statement follows from Theorem 5.2. $\square$

The following corollary is now immediate.

5.8 Corollary. *Let $(\mathcal{A},\sigma)$ be an $\mathcal{O}$−algebra with involution. If σ_F is isotropic (resp. metabolic), then σ_L is isotropic (resp. metabolic) as well.*

We define the *index of* (B,τ) to be

$$\mathrm{ind}(B,\tau) = \{\mathrm{rdim}(I) \mid I \text{ an isotropic balanced right ideal of } (B,\tau)\}.$$

This definition coincides with the one given in [16, p. 73]. This is clear if $Z(B)$ is a field and the degenerate case follows from Proposition 5.5. Using the definition of the index together with [16, (6.7)] yields, if $\mathrm{char}(F)\neq 2$, that τ is metabolic if and only if it is hyperbolic.

5.9 Proposition. *Let (D,θ) be an F−algebra with involution and assume that D is a division algebra. Let $\varepsilon=\pm 1$ and let (V,h) be an ε−hermitian space over (D,θ). Then*

$$\mathrm{ind}(\mathrm{Ad}(h)) = \{\mathrm{ind}(D)\cdot d \mid 0\leqslant d\leqslant i_w(h)\}.$$

Proof. Using the fact that for every D−subspace W of V, we have that $\mathrm{rdim}(\mathrm{Hom}_D(V,W)) = \dim_D(W)\deg(D)$, the statement follows immediately from Proposition 5.6. $\square$

We recast the result of Theorem 5.7 in terms of the index.

5.10 Corollary. *Let $(\mathcal{A},\sigma)$ be an $\mathcal{O}$−algebra with involution. Then*

$$\mathrm{ind}((\mathcal{A},\sigma)_F)\subset\mathrm{ind}((\mathcal{A},\sigma)_L).$$

Proof. Let $0\neq i\in\mathrm{ind}((\mathcal{A},\sigma)_F)$ and let I be an isotropic balanced right ideal of $(\mathcal{A},\sigma)_F$ of reduced dimension i. By Theorem 5.7, $(I\cap\mathcal{A})\otimes_{\mathcal{O}} L$ is an isotropic balanded right ideal of $(\mathcal{A},\sigma)_L$ and $\mathrm{rdim}((I\cap\mathcal{A})\otimes_{\mathcal{O}} L)=i$. It follows that $i\in\mathrm{ind}((\mathcal{A},\sigma)_L)$. $\square$

5.11 Remark. If the characteristic of the fields involved in the place is different from 2, and we are in the "geometric setting", then there is another proof of Corollary 5.10. Let k be a field with $\mathrm{char}(k)\neq 2$, and (C,ρ) a k−algebra with involution with $\deg(C)\geqslant 3$. Let k_1/k and k_2/k be field extensions and $\lambda: k_1\to k_2^\infty$ a k−place. Let $\mathcal{O}_\lambda$ be the valuation ring of k_1 corresponding to λ. Then $(C,\rho)_{\mathcal{O}_\lambda}$ is an $\mathcal{O}_\lambda$−algebra with involution. The inclusion $\mathrm{ind}((C,\rho)_{k_1})\subset\mathrm{ind}((C,\rho)_{k_2})$ can then be shown using certain k−varieties associated to (C,ρ), studied in [19, 20]. The rational points of these varieties over a field extension M/k are isotropic balanced right ideals of $(C,\rho)_M$ of a certain reduced dimension. The inclusion $\mathrm{ind}((C,\rho)_{k_1})\subset\mathrm{ind}((C,\rho)_{k_2})$ then translates into a statement of "rational points that carry over under places".

6. Henselian valuation rings

Throughout this section, we fix a field F and a valuation ring $\mathcal{O}$ of F. We further assume that $\mathcal{O}$ is *Henselian*, i.e. $\mathcal{O}$ extends uniquely to a valuation ring in any separable closure of F. We denote the maximal ideal of $\mathcal{O}$ by $\mathfrak{m}$, its residue field by κ, and we let v be a valuation on F with valuation ring $\mathcal{O}$.

If $2 \in \mathcal{O}^\times$, then a symmetric bilinear space over $\mathcal{O}$ that is isotropic (resp. hyperbolic) over the residue field of $\mathcal{O}$, is already isotropic (resp. hyperbolic) over F. This is shown in the proof of [25, (6.2.4)], where the statement assumes that $\mathcal{O}$ is discrete, but the proof does not. We prove an analogue of this result for $\mathcal{O}$−algebras with involution. This complements the statement on the index in Corollary 5.10, and will be an important ingredient of the proof of the isomorphism result for Azumaya algebras over Henselian valuation rings in Theorem 7.3.

6.1 Proposition. *Let $(\mathcal{A}, \sigma)$ be an $\mathcal{O}$−algebra with involution with center a domain S. We denote the fraction field of S by K. The following hold:*

(a) *S is the unique extension of $\mathcal{O}$ in K, its value group is equal to the one of $\mathcal{O}$ and its residue field is a separable quadratic extension of κ.*

(b) *Suppose that $\mathcal{A}$ does not have zero divisors. Then $\mathcal{A}$ is a valuation ring of $\mathcal{A}_F$. Furthermore, the value groups of $\mathcal{O}$ and $\mathcal{A}$ are equal.*

Proof. Since $\mathcal{O}$ is Henselian, there is a unique valuation ring of K extending $\mathcal{O}$. Proposition 2.5 then yields (a). Suppose that $\mathcal{A}$ does not have zero divisors. Then $\mathcal{A}_F$ does not have zero divisors either and hence, it is a division algebra with center K. By (a), S is a valuation ring. Note that S is also Henselian. By [28, Corollary 2.2], S extends to a valuation ring V of $\mathcal{A}_F$. It follows from [12, (2.5)] that $V = \mathcal{A}$ (and in this case, the proof of [12, (2.5)] in fact simplifies). The equality of the value groups follows from (a) together with [28, Theorem 3.2]. □

6.2 Corollary. *Let $(\mathcal{A}, \sigma)$ be an $\mathcal{O}$−algebra with involution with center a domain S. There exists an Azumaya algebra Δ without zero divisors over S that is a valuation ring of Δ_F, an $\mathcal{O}$−linear involution θ on Δ of the same kind as σ, and an ε−hermitian space (V, h) over (Δ, θ), with $\varepsilon \in \{1, -1\}$, such $(\mathcal{A}, \sigma) \cong_S \mathrm{Ad}(h)$. Furthermore, the value groups of Δ and $\mathcal{O}$ are equal.*

Proof. This follows immediately from Proposition 3.8 together with Proposition 6.1. □

6.3 Proposition. *Let $\mathcal{A}$ be an Azumaya algebra with center $\mathcal{O}$ or a separable quadratic $\mathcal{O}$−algebra that is a domain. Then* $\mathrm{ind}(\mathcal{A}_F) = \mathrm{ind}(\mathcal{A}_\kappa)$.

Proof. The result follows from the fact that $\mathcal{A}$ is Brauer equivalent to a valuation ring Δ of a division algebra Brauer equivalent to $\mathcal{A}_F$, by Corollary 6.2, and that $\Delta_\kappa \cong \Delta/\mathfrak{m}\Delta$ is a division algebra, since $\mathfrak{m}\Delta$ is the unique maximal left and right ideal of Δ. □

Using the previous proposition, we can already treat the degenerate case.

6.4 Corollary. *Let $(\mathcal{A}, \sigma)$ be a degenerate $\mathcal{O}$−algebra with involution. Then* $\mathrm{ind}((\mathcal{A}, \sigma)_F) = \mathrm{ind}((\mathcal{A}, \sigma)_\kappa)$.

Proof. Since $Z(\mathcal{A})$ is not a domain, we have that $Z(\mathcal{A}) \cong \mathcal{O} \times \mathcal{O}$ by Proposition 2.3. By Proposition 2.2, there exists an Azumaya algebra $\mathcal{B}$ over $\mathcal{O}$ such that $(\mathcal{A}, \sigma) \cong_{\mathcal{O}} (\mathcal{B} \times \mathcal{B}^{\mathrm{op}}, \mathrm{sw}_{\mathcal{B}})$. By Proposition 6.3, we have that $\mathrm{ind}(\mathcal{A}_F) = \mathrm{ind}(\mathcal{B}_F) = \mathrm{ind}(\mathcal{B}_\kappa) = \mathrm{ind}(\mathcal{A}_\kappa)$. Since $\deg(\mathcal{A}_F) = \deg(\mathcal{A}_\kappa)$ by Theorem 5.2, the equality $\mathrm{ind}((\mathcal{A}, \sigma)_F) = \mathrm{ind}((\mathcal{A}, \sigma)_\kappa)$ now follows from Proposition 5.5. □

6.5 Proposition. *Assume that $2 \in \mathcal{O}^\times$. Let $(\mathcal{A}, \sigma)$ be an $\mathcal{O}$−algebra with involution with center a domain. Then σ_F is isotropic if and only σ_κ is isotropic.*

Proof. We denote $Z(\mathcal{A})$ by S and its fraction field by K. Let Δ be as in Corollary 6.2. Let w be a valuation on D with valuation ring Δ and let v_S be the restriction of w to K. Then S is the valuation ring of v_S. By Proposition 6.1, v, v_S and w have the same value group.
The statement of the proposition follows from [27, Corollary 2.3], provided that $\mathcal{A}_F$ is tame over F in the sense of [27, p. 121], and that there exists a σ_F−invariant v−gauge α on $\mathcal{A}_F$ such that $(\mathcal{A}_F)_\alpha^{\geqslant 0} = \mathcal{A}$ and $(\mathcal{A}_F)_\alpha^{>0} = \mathfrak{m}\mathcal{A}$. The existence of the gauge follows from Proposition 4.7. The tameness condition for $\mathcal{A}_F$ means that K/F is tame and $\mathcal{A}_F$ splits over the maximal tamely ramified extension of K. The fact that K/F is tame follows from the equality of the value groups of v and v_S, which even implies that (K, v_S) is an unramified extension of (F, v). Since the value groups of v and w are equal, any maximal subfield of D yields an unramified, and therefore also tamely ramified, extension of (K, v_S) splitting $\mathcal{A}_F$. It follows that $\mathcal{A}_F$ is tame over F. □

6.6 Corollary. *Assume that $2 \in \mathcal{O}^\times$. Let (Δ, θ) be an $\mathcal{O}$−algebra with involution without zero divisors. Let $\varepsilon = \pm 1$ and let (V, h) be an ε−hermitian space over (Δ, θ). Then $i_w(h_F) = i_w(h_\kappa)$. In particular, h_F is isotropic (resp. hyperbolic) if and only if h_κ is isotropic (resp. hyperbolic).*

Proof. Let $(\mathcal{A}, \sigma) = \mathrm{Ad}(h)$. By Proposition 6.5, we have that σ_F is isotropic if and only if σ_κ is isotropic. Proposition 5.6 yields that h_F is isotropic if and only if h_κ is isotropic. Suppose that h_F is not hyperbolic. Then, by Proposition 3.4, we can write $(V, h) \simeq (V_1, h_1) \perp (V_2, h_2)$, with (V_1, h_1) (resp. (V_2, h_2)), an anisotropic (resp. hyperbolic) ε−hermitian space over (Δ, θ). Since h_1 is anisotropic, $(h_1)_F$ is also anisotropic, by Proposition 3.6. By the first part of the proof, we obtain that $(h_1)_\kappa$ is anisotropic. It follows that $i_w(h_F) = i_w(h_\kappa)$, as desired. □

6.7 Theorem. *Assume that $2 \in \mathcal{O}^\times$. Let $(\mathcal{A}, \sigma)$ be an $\mathcal{O}$−algebra with involution. Then*

$$\mathrm{ind}((\mathcal{A}, \sigma)_F) = \mathrm{ind}((\mathcal{A}, \sigma)_\kappa).$$

In particular, σ_F is isotropic (resp. hyperbolic) if and only if σ_κ is isotropic (resp. hyperbolic).

Proof. If $Z(\mathcal{A})$ is not a domain, this is the statement of Corollary 6.4. So, assume that $Z(\mathcal{A})$ is a domain. Let (Δ, θ) and (V, h) be as in Corollary 6.2. In order to prove the statement, by Proposition 5.9, it suffices to show that $\mathrm{ind}(\mathcal{A}_F) = \mathrm{ind}(\mathcal{A}_\kappa)$ and $i_w(h_F) = i_w(h_\kappa)$. The claim about the Schur indices follows from Proposition 6.3 and the claim about the Witt indices from Corollary 6.6. □

We can use Theorem 6.7 to prove a hyperbolicity result for ε−hermitian spaces over $\mathcal{O}$−algebras with involution without zero divisors, which will in turn be used in the proof of the isomorphism result in Theorem 7.3.

6.8 Proposition. *Assume that $2 \in \mathcal{O}^\times$. Let (Δ, θ) be an $\mathcal{O}$−algebra with involution without zero divisors. Let $\varepsilon = \pm 1$ and let (V, h) be an ε−hermitian space over (Δ, θ). Excluding the case $\theta = \mathrm{id}_\Delta$ and $\varepsilon = -1$, there exists $x \in V$ such that $h(x, x) \in \Delta^\times$.*

Proof. Let $\mathfrak{B} = (e_1, \ldots, e_n)$ be a Δ−basis for V. If one of $h(e_1, e_1), \ldots, h(e_n, e_n) \in \Delta^\times$, then we are done. So, suppose that $h(e_1, e_1), \ldots, h(e_n, e_n) \in \Delta \setminus \Delta^\times$. Note that $\Delta \setminus \Delta^\times = \mathfrak{m}\Delta$, since Δ is a valuation ring by Proposition 6.1 (b). By the same result, the value groups of Δ and $\mathcal{O}$ are equal, and since $\theta \neq \mathrm{id}_\Delta$ if $\varepsilon = -1$, this implies that there exists $d \in \Delta^\times$ such that $\theta(d) = \varepsilon d$. Let C be the matrix of h with respect to $\mathfrak{B}$. Since h is non–singular, there exist $\lambda_1, \ldots, \lambda_n \in \Delta$ such that the first entry of $(d\lambda_1, \ldots, d\lambda_n)C$ is d. It follows that $d = \sum_{i=1}^n d\lambda_i h(e_i, e_1) = h(\sum_{i=1}^n e_i \theta(d\lambda_i), e_1)$. Let $x = \sum_{i=1}^n e_i \theta(d\lambda_i)$. If $h(x, x) \in \Delta^\times$, then we are done. Otherwise, $h(e_1 + x, e_1 + x) = h(e_1, e_1) + h(x, x) + 2d \in \Delta^\times$, and we are also done. □

6.9 Corollary. *Assume that $2 \in \mathcal{O}^\times$. Let (Δ, θ) be an $\mathcal{O}$−algebra with involution without zero divisors. Let $\varepsilon = \pm 1$ and let (V, h) and (V', h') be two ε−hermitian spaces over (Δ, θ). Suppose that there exists a scalar $e \in \mathcal{O}$ such that $(V, eh)_F \simeq (V', h')_F$. If $e \notin F^{\times 2}\mathcal{O}^\times$, then h and h' are hyperbolic.*

Proof. If $\theta = \mathrm{id}_\Delta$ and $\varepsilon = -1$, then h and h' are hyperbolic by Proposition 3.4, since a skew–hermitian space over $(\Delta, \mathrm{id}_\Delta)$ is necessarily isotropic. So, for the rest of the proof, we assume that $\varepsilon = 1$ if $\theta = \mathrm{id}_\Delta$. Let $\mathfrak{B}' = (e_1', \ldots, e_n')$ be a Δ−basis for V'. Then $\mathfrak{B}'$ is a Δ_F−basis for V_F'. By assumption, there exists a bijective Δ_F−linear map $\varphi : V_F \to V_F'$ such that for all $x \in V_F$, we have that $eh_F(x, x) = h_F'(\varphi(x), \varphi(x))$. By Proposition 6.8, there exists $x \in V$ such that $h(x, x) \in \Delta^\times$. We write $\varphi(x) = \sum_{i=1}^n e_i' y_i$, with $y_1, \ldots, y_n \in D$. Since Δ is a valuation ring with the same value group as $\mathcal{O}$, by Proposition 6.1 (b), there exist $a_1, \ldots, a_n \in F$ such that $a_i y_i \in \Delta^\times$. We may assume that $v(a_1) = \max_{1 \leqslant i \leqslant n}(v(a_i))$. Then $y = \varphi(x)a_1 \in V'$. Since $(\bar{e}_1', \ldots, \bar{e}_n')$ is a Δ_κ−basis for $V_\kappa' \cong V'/\mathfrak{m}V'$, and $a_1 y_1 \notin \mathfrak{m}\Delta$, we have that $\bar{y} \in V'/\mathfrak{m}V'$ is nonzero. Since $e \notin F^{\times 2}\mathcal{O}^\times$ and $h(x, x) \in \Delta^\times$, it follows that $ea_1^2 \in \mathfrak{m}\Delta$, which implies that h_κ' is isotropic.

Since $2 \in \mathcal{O}^\times$, it follows from Corollary 6.6 that h_F' is isotropic as well. Suppose that h_F' is non–hyperbolic. Then h_F is also non–hyperbolic, and by Proposition 3.4, we can decompose $(V, h) \simeq (V_1, h_1) \perp (V_2, h_2)$ and $(V', h') \simeq (V_1', h_1') \perp (V_2', h_2')$, with (V_1, h_1) and (V_1', h_1') anisotropic ε−hermitian spaces over (Δ, θ), and (V_2, h_2) and (V_2', h_2') hyperbolic ε−hermitian spaces over (Δ, θ). It follows that

$$(V_1', h_1')_F \perp (V_2', h_2')_F \simeq (V_1, eh_1)_F \perp (V_2, eh_2)_F.$$

We have that $(h_1')_F$ and $e(h_1)_F$ anisotropic by Proposition 3.6, and $(h_2')_F$ and $e(h_2)_F$ hyperbolic. The Witt cancellation property for ε−hermitian spaces over division rings (see [15, (I.6.3.4)]) yields that $(V_1', h_1')_F \simeq (V_1, eh_1)_F$. However, the reasoning above now yields that $(h_1')_F$ is isotropic, a contradiction. Hence, h_F' is hyperbolic, and then clearly h_F is hyperbolic as well. Proposition 3.6 yields that h' and h are already hyperbolic. □

6.10 Remark. J.–P. Tignol has suggested a different proof of Theorem 6.9, using results on hermitian forms on graded algebras induced by value functions on algebras. This proof is written out in [3, (4.26)–(4.28)].

We can use Proposition 6.8 to prove that, for ε−hermitian spaces over an $\mathcal{O}$−algebra with involution without zero divisors, isometry can be detected rationally.

6.11 Proposition. *Assume that* $2 \in \mathcal{O}^\times$. *Let* (Δ, θ) *be an* $\mathcal{O}$*–algebra with involution without zero divisors. Let* $\varepsilon = \pm 1$ *and let* (V, h) *and* (V', h') *be two* ε*–hermitian spaces over* (Δ, θ). *If* $(V', h')_F \simeq (V, h)_F$, *then* $(V', h') \simeq (V, h)$.

Proof. If $\theta = \mathrm{id}_\Delta$ and $\varepsilon = -1$, then (V, h) and (V', h') are both hyperbolic by Proposition 3.4. Since $(V', h')_F \simeq (V, h)_F$ implies that $\dim_\Delta(V) = \dim_\Delta(V')$, it follows that $(V', h') \simeq (V, h)$. So, for the rest of the proof, we assume that $\theta \neq \mathrm{id}_\Delta$ if $\varepsilon = -1$. It follows from Propositions 6.8 and 3.2 that there exist $\alpha_1, \ldots, \alpha_n, \alpha'_1, \ldots, \alpha'_n \in \Delta^\times$ such that $h \simeq \langle \alpha_1, \ldots, \alpha_n \rangle_\theta$ and $h' \simeq \langle \alpha'_1, \ldots, \alpha'_n \rangle_\theta$. By assumption, we have that $\langle \alpha_1, \ldots, \alpha_n \rangle_{\theta_F} \simeq \langle \alpha'_1, \ldots, \alpha'_n \rangle_{\theta_F}$. By mimicking a proof of M. Kneser of a representation result for quadratic forms (see [5, (4.5)]), one can show that there exist $\beta'_2, \ldots, \beta'_n \in \Delta^\times$ such that $h' \simeq \langle \alpha_1, \beta'_2, \ldots, \beta'_n \rangle_\theta$. The proof of this representation result is written out in [3, (2.50)]. Since Δ_F is a division algebra, the Witt cancellation property of [15, (I.6.3.4)] yields that $\langle \alpha_2, \ldots, \alpha_n \rangle_{\theta_F} \simeq \langle \beta'_2, \ldots, \beta'_n \rangle_{\theta_F}$. The statement now follows by induction on $\dim_\Delta(V)$. □

6.12 Remark. If we drop the Henselian assumption on $\mathcal{O}$, then one can still show that (skew–) hermitian spaces over an $\mathcal{O}$–algebra with involution without zero divisors, that become isometric over F, are already isometric over $\mathcal{O}$. This follows from a general Witt cancellation result by B. Keller (see [15, (VI.5.7.2)]), the proof of which is a lot more involved than the one above.

7. Good reduction

We fix fields F and L and a place $\lambda : F \to L^\infty$. We denote the valuation ring of F associated to λ by $\mathcal{O}$ and its residue field by κ.

As mentioned in the introduction, if a symmetric bilinear space over F has good reduction with respect to λ, then its adjoint algebra with involution is obtained by scalar extension from an $\mathcal{O}$–algebra with involution. Therefore, it is natural to make the following definition. Let (B, τ) be an F–algebra with involution. Then we say that (B, τ) *has good reduction with respect to* λ if there exists an $\mathcal{O}$–algebra with involution $(\mathcal{A}, \sigma)$ such that $(B, \tau) \cong (\mathcal{A}, \sigma)_F$. We call $(\mathcal{A}, \sigma)$ *a* λ*–unimodular representation of* (B, τ).
We can consider $(\mathcal{A}, \sigma)_L$ as a residue algebra with involution for (B, τ). The question then naturally arises whether this residue algebra with involution is determined up to L–isomorphism. We show that this is the case if $\mathrm{char}(L) \neq 2$, i.e. $2 \in \mathcal{O}^\times$. To this end, it suffices to show in the case where $\mathcal{O}$ is Henselian and $2 \in \mathcal{O}^\times$, and given two $\mathcal{O}$–algebras with involution $(\mathcal{A}, \sigma)$ and $(\mathcal{A}', \sigma')$, that $(\mathcal{A}, \sigma)_F \cong_F (\mathcal{A}', \sigma')_F$ implies $(\mathcal{A}, \sigma)_\kappa \cong_\kappa (\mathcal{A}', \sigma')_\kappa$. In fact, we will prove something stronger in that case, namely that $(\mathcal{A}, \sigma)_F \cong_F (\mathcal{A}', \sigma')_F$ implies $(\mathcal{A}, \sigma) \cong (\mathcal{A}', \sigma')$. Using the results for Henselian valuation rings from section 6, we will extend the aforementioned isomorphism result to the case of general valuation rings in which 2 is invertible in the forthcoming article [4], together with J. Van Geel. This type of isomorphism problems has been studied in the literature for other kinds of rings as well (see Remark 7.4).

We start with a reduction of the isomorphism problem to the case of one algebra with two involutions.

7.1 Proposition. *Suppose that for all* $\mathcal{O}$*–algebras with involution* $(\mathcal{A}, \sigma)$ *the following holds: if* σ' *is an* $\mathcal{O}$*–linear involution on* $\mathcal{A}$ *such that there is a* $Z(\mathcal{A}_F)$*–isomorphism* $(\mathcal{A}, \sigma)_F \to (\mathcal{A}, \sigma')_F$, *then* $(\mathcal{A},\sigma) \cong_{\mathcal{O}} (\mathcal{A},\sigma')$*. Then for all pairs* $((\mathcal{A},\sigma),(\mathcal{A}',\sigma'))$ *of* $\mathcal{O}$*−algebras with involution, we have that* $(\mathcal{A},\sigma)_F \cong_F (\mathcal{A}',\sigma')_F$ *implies* $(\mathcal{A},\sigma) \cong_{\mathcal{O}} (\mathcal{A}',\sigma')$*.*

Proof. Let $\varphi : (\mathcal{A},\sigma)_F \to (\mathcal{A}',\sigma')_F$ be an isomorphism of F−algebras with involution. Then φ restricts to an F−isomorphism $Z(\mathcal{A}_F) \to Z(\mathcal{A}'_F)$. Since $Z(\mathcal{A})$ is the integral closure of R in $Z(\mathcal{A}_F)$, and $Z(\mathcal{A}')$ is the integral closure of R in $Z(\mathcal{A}'_F)$, by Proposition 2.3, and since φ is an R−isomorphism, it follows that $Z(\varphi(\mathcal{A})) = \varphi(Z(\mathcal{A})) = Z(\mathcal{A}')$. If we consider $\mathcal{A}'$ as an Azumaya algebra over $Z(\mathcal{A})$ via φ, then $\varphi : \mathcal{A}_F \to \mathcal{A}'_F$ is a $Z(\mathcal{A}_F)$−isomorphism. By Proposition 2.11 and Corollary 2.12, it follows that there exists an isomorphism of $Z(\mathcal{A})$−algebras $\psi : \mathcal{A} \to \mathcal{A}'$. Let $\tilde{\sigma} = \psi^{-1} \circ \sigma' \circ \psi$. Then ψ is an isomorphism of R−algebras with involution from $(\mathcal{A},\tilde{\sigma})$ to $(\mathcal{A}',\sigma')$. We have that $\varphi^{-1} \circ \psi_F : (\mathcal{A},\tilde{\sigma})_F \to (\mathcal{A},\sigma)_F$ is an isomorphism of F−algebras with involution that is $Z(\mathcal{A}_F)$−linear. The hypothesis now yields that $(\mathcal{A},\tilde{\sigma}) \cong_{\mathcal{O}} (\mathcal{A},\sigma)$, and hence, $(\mathcal{A},\sigma) \cong_{\mathcal{O}} (\mathcal{A}',\sigma')$. □

Using the reduction in Proposition 7.1, it is easily seen that degenerate rationally isomorphic $\mathcal{O}$−algebras with involution are isomorphic.

7.2 Proposition. *Let* $(\mathcal{A},\sigma)$ *and* $(\mathcal{A}',\sigma')$ *be degenerate* $\mathcal{O}$*−algebras with involution. Suppose that* $(\mathcal{A},\sigma)_F \cong_F (\mathcal{A}',\sigma')_F$*. Then* $(\mathcal{A},\sigma) \cong (\mathcal{A}',\sigma')$*.*

Proof. By Proposition 7.1, in order to show the claim we may assume that $\mathcal{A}' = \mathcal{A}$. Since $Z(\mathcal{A}) \cong \mathcal{O} \times \mathcal{O}$ by assumption, all involutions of the second kind on $\mathcal{A}$ are isomorphic over $\mathcal{O}$ by Proposition 2.2. This yields the statement. □

We now prove the main theorem of this section.

7.3 Theorem. *Assume that* $\mathcal{O}$ *is Henselian and that* $2 \in \mathcal{O}^\times$*. Let* $(\mathcal{A},\sigma)$ *and* $(\mathcal{A}',\sigma')$ *be* $\mathcal{O}$*−algebras with involution. If* $(\mathcal{A},\sigma)_F \cong_F (\mathcal{A}',\sigma')_F$*, then* $(\mathcal{A},\sigma) \cong_{\mathcal{O}} (\mathcal{A}',\sigma')$*.*

Proof. By Proposition 7.1, in order to show that $(\mathcal{A},\sigma) \cong_{\mathcal{O}} (\mathcal{A}',\sigma')$, we may assume that $\mathcal{A}' = \mathcal{A}$ and that $(\mathcal{A},\sigma)_F \cong_{Z(\mathcal{A}_F)} (\mathcal{A},\sigma')_F$. If $Z(\mathcal{A})$ is not a domain, then we are done by Proposition 7.2. So, suppose that $Z(\mathcal{A})$ is a domain. By Proposition 3.11, there exists $s \in \mathcal{A}^\times$ such that $\sigma(s) = s$ and $\sigma' = \mathrm{Int}(s) \circ \sigma$. By Proposition 3.9, there exist elements $e \in F^\times$ and $g \in \mathcal{A}_F^\times$ such that $es = \sigma_F(g)g$. Let (Δ,θ) and (V,h) be as in Corollary 6.2 such that $(\mathcal{A},\sigma) \cong_{Z(\mathcal{A})} \mathrm{Ad}(h)$. Identifying $(\mathcal{A},\sigma)$ and $\mathrm{Ad}(h)$ through this isomorphism, we consider s as element of $\mathrm{End}_\Delta(V)^\times$ and g as element of $\mathrm{End}_{\Delta_F}(V_F)^\times$. Then $(\mathcal{A},\sigma') = \mathrm{Ad}(h')$, where $h' : V \times V \to \Delta$ is defined by $h'(x,y) = h(s^{-1}(x),y)$ for all $x,y \in V$. By Proposition 3.10, we have that $(V,h')_F \simeq (V,eh)_F$. Suppose that $e \notin F^{\times 2}\mathcal{O}^\times$. Then Corollary 6.9 yields that h and h' are hyperbolic. It follows that $(V,h)_F \simeq (V,h')_F$. Suppose that $e \in F^{\times 2}\mathcal{O}^\times$. Since $(V,h')_F \simeq (V,eh)_F$ and $(V,h)_F \simeq (V,a^2h)_F$, for all $a \in F^\times$, we get that $(V,h')_F \simeq (V,uh)_F$ for some $u \in \mathcal{O}^\times$. So, in both cases, there exists an element $v \in \mathcal{O}^\times$ such that $(V,h')_F \simeq (V,vh)_F$. By Proposition 6.11, it follows that $(V,h') \simeq (V,vh)$. Proposition 3.10 now yields that $(\mathcal{A},\sigma) \cong_{\mathcal{O}} (\mathcal{A},\sigma')$. □

7.4 Remark. Theorem 7.3 holds in particular for complete discrete valuation rings in which 2 is invertible. This result is not new. It is an (unpublished) result of J. Tits, see [21]. There, it is used by Y. Nisnevich to prove a result on algebraic groups which implies, in the case where $\mathcal{O}$ is a discrete valuation ring with $2 \in \mathcal{O}^\times$, that isomorphism of $\mathcal{O}$−algebras with involution can be detected over F. The latter result is in turn used by I. Panin to prove that isomorphism of algebras with involution over a regular local ring containing a field of characteristic different from 2, can be detected rationally (see [22]).

Let F^s be a separable closure of F and let $\mathcal{O}^s$ be an extension of $\mathcal{O}$ to F^s. Let $G = \{\rho \in \mathrm{Gal}(F^s/F) \mid \rho(\mathcal{O}^s) = \mathcal{O}^s\}$. Then $((F^s)^G, \mathcal{O}^s \cap (F^s)^G)$ is Henselian by [9, (3.2.15)]. We denote it by $(F^h, \mathcal{O}^h)$; it is called *a Henselisation* of $(F, \mathcal{O})$. By [9, (5.2.5)], $(F, \mathcal{O}) \subset (F^h, \mathcal{O}^h)$ is *an immediate extension*, i.e. $(F, \mathcal{O})$ and $(F^h, \mathcal{O}^h)$ have isomorphic value groups and residue fields.

7.5 Corollary. *Assume that* $2 \in \mathcal{O}^\times$. *Let* $(\mathcal{A}, \sigma)$ *and* $(\mathcal{A}', \sigma')$ *be* $\mathcal{O}$*−algebras with involution. If* $(\mathcal{A}, \sigma)_F \cong_F (\mathcal{A}', \sigma')_F$, *then* $(\mathcal{A}, \sigma)_\kappa \cong_\kappa (\mathcal{A}', \sigma')_\kappa$.

Proof. Let $(F^h, \mathcal{O}^h)$ be a Henselisation of $(F, \mathcal{O})$. Since $(\mathcal{A}, \sigma)_F \cong_F (\mathcal{A}', \sigma')_F$, it follows that $(\mathcal{A}, \sigma)_{F^h} \cong_{F^h} (\mathcal{A}', \sigma')_{F^h}$ as well. Theorem 7.3 then yields that $(\mathcal{A}, \sigma)_{\mathcal{O}^h} \cong_{\mathcal{O}^h} (\mathcal{A}', \sigma')_{\mathcal{O}^h}$. Since $(F, \mathcal{O}) \subset (F^h, \mathcal{O}^h)$ is an immediate extension, the residue field of $\mathcal{O}^h$ is isomorphic to κ and hence, by scalar extension to κ, we get that $(\mathcal{A}, \sigma)_\kappa \cong_\kappa (\mathcal{A}', \sigma')_\kappa$. □

7.6 Corollary. *Assume that* $2 \in \mathcal{O}^\times$. *Let* $(\mathcal{A}, \sigma)$ *and* $(\mathcal{A}', \sigma')$ *be* $\mathcal{O}$*−algebras with involution. If* $(\mathcal{A}, \sigma)_F \cong_F (\mathcal{A}', \sigma')_F$, *then* $(\mathcal{A}, \sigma)_L \cong_L (\mathcal{A}', \sigma')_L$.

Proof. We have that L contains up to isomorphism the residue field of $\mathcal{O}$. The statement now follows immediately from Corollary 7.5. □

Let (B, τ) be an F−algebra with involution with good reduction with respect to λ. Let $(\mathcal{A}, \sigma)$ be a λ−unimodular representation of (B, τ). If $2 \in \mathcal{O}^\times$, we set $\lambda_*(B, \tau) = (\mathcal{A}, \sigma)_L$. By Corollary 7.6, $\lambda_*(B, \tau)$ is well defined up to L−isomorphism, and we call it *the residue algebra with involution of* (B, τ).

We can now formulate Corollary 5.10 as follows.

7.7 Corollary. *Assume that* $2 \in \mathcal{O}^\times$. *Let* (B, τ) *be an* F*−algebra with involution with good reduction with respect to* λ. *Then* $\mathrm{ind}(B, \tau) \subset \mathrm{ind}(\lambda_*(B, \tau))$.

7.8 Remark. The good reduction definition for algebras with involution does not completely generalise the one for symmetric bilinear spaces, but it does so up to similarity. Namely, let (V, b) be a symmetric bilinear space over F. Then $\mathrm{Ad}(b)$ has good reduction with respect to λ if and only if (V, b) has up to similarity good reduction with respect to λ. The sufficient condition is clear. Suppose conversely that there exists an $\mathcal{O}$−algebra with involution $(\mathcal{A}, \sigma)$ such that $(\mathcal{A}, \sigma)_F \cong_F \mathrm{Ad}(b)$. Then $\mathcal{A}$ is split by Proposition 2.11, and by Proposition 3.8, there exists a symmetric bilinear space $(\mathcal{V}, \varphi)$ over $\mathcal{O}$ such that $(\mathcal{A}, \sigma) \cong_{\mathcal{O}} \mathrm{Ad}(\varphi)$. It follows from [16, p. 1] that there exists $u \in F^\times$ such that $(V, ub) \simeq (\mathcal{V}, \varphi)_F$.

Using Remark 7.8 together with Proposition 5.6, it is now easily seen that Theorem 1.1 also follows from Corollary 7.7.

Acknowledgements.
I would like to thank my advisors Karim Johannes Becher and Jan Van Geel for their support and ideas. I am very grateful to Jean–Pierre Tignol for interesting discussions, for his suggestion to consider the Henselian case, and for proofreading an earlier version of this article. Further thanks go to Andrew Dolphin, David Leep and Manuel Ojanguren, who has been a great help in understanding Azumaya algebras with involution over valuation rings.

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