Rationally isomorphic Azumaya algebras with involution over semilocal Bézout domains

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Abstract

Let R be a commutative domain in which 2 is invertible, and F its fraction field. Let $(\mathcal{A},\sigma)$ and $(\mathcal{A}',\sigma')$ be R−algebras with involution. They are said to be rationally isomorphic if $(\mathcal{A}\otimes_R F, \sigma\otimes_R \mathrm{id}_F) \cong_F (\mathcal{A}'\otimes_R F, \sigma'\otimes_R \mathrm{id}_F)$. The main result of this paper states that for R a semilocal Bézout domain, rationally isomorphic R−algebras with involution are isomorphic.

Let R be a commutative domain in which 2 is invertible, and F its fraction field. Let $(\mathcal{A},\sigma)$ and $(\mathcal{A}',\sigma')$ be R−algebras with involution. They are said to be *rationally isomorphic* if $(\mathcal{A}\otimes_R F, \sigma\otimes_R \mathrm{id}_F) \cong_F (\mathcal{A}'\otimes_R F, \sigma'\otimes_R \mathrm{id}_F)$. Under which conditions on R can we conclude that $(\mathcal{A},\sigma)$ and $(\mathcal{A}',\sigma')$ are already isomorphic as R−algebras with involution? If any pair of R−algebras with involution that is rationally isomorphic, is already isomorphic over R, we say that R *has the RIII property*.
Consider the linear algebraic group $G = \mathrm{Aut}(\mathcal{A},\sigma)$. The question whether $(\mathcal{A},\sigma) \cong_R (\mathcal{A}',\sigma')$ translates to the question whether principal homogeneous spaces over G that are trivial over F, are already trivial over R. This problem has been studied for R a discrete valuation ring, and more generally for regular local rings. For these rings it goes back to a conjecture of Grothendieck and Serre, stating that principal G−homogeneous spaces over a Noetherian regular integral k−scheme X, k a field, that are rationally trivial are locally trivial. In [N], Y. Nisnevich proved the conjecture for schemes of dimension one and regular Henselian local schemes of any dimension. Nisnevich's work includes, in the case R is a discrete valuation ring, a postive answer to the question considered in this paper. Nisnevich's proof is based on the relation between étale cohomology and adélic invariants of G. Other ingredients are the fact that the statement holds for complete discrete valuation rings (an unpublished theorem of Tits), and weak approximation (cf. [H]). In [Pa], I. Panin proved a purity theorem on multipliers for $\mathrm{Aut}(\mathcal{A},\sigma)$, which together with the result for discrete valuations, gives a positive answer to the conjecture for automorphism groups of algebras with involution.

In the rest of the introduction, we will always assume that 2 is invertible in the rings we consider. In [B] the first author considers Azumaya algebras with involution under specialisation and the question whether rationally isomorphic algebras with involution have isomorphic reductions. The question whether for algebras with involution over valuation rings, rational isomorphism implies isomorphism, is natural in this context. In [B, (8.7)] it is shown that Henselian valuation

rings have the RIII property. In this paper we will show that semilocal Bézout domains have the RIII property. We will first treat the case of finite Krull dimension and then show that the general case can be reduced to this case. Our result includes in particular that valuation rings have the RIII property. Although this also covers the case of discrete valuation rings, and more generally semilocal principal ideal domains, we will treat this class of rings (the Noetherian case) separately, since the proof then simplifies in different places. Furthermore, the role played by a Cassels–Pfister type result for algebras with involution becomes transparent.
Using the relation between $\varepsilon-$hermitian spaces and adjoint involutions we obtain, as a corollary to our main result, that "rational similarity implies similarity" for $\varepsilon-$hermitian spaces over an Azumaya algebra with involution without zero divisors over a semilocal Bézout domain. In particular this implies that rationally similar symmetric or skew–symmetric bilinear spaces over semilocal Bézout domains, are similar.

The structure of the paper is as follows. In the first section we state the problem formally and recall from [B] some reduction that can be made in order to show the RIII property holds. We also show that proving that the RIII property holds for two involutions on a fixed algebra, is equivalent to a statement on multipliers. We then prove certain characterisations of the multipliers of an algebra with involution over a semilocal Bézout domain of finite Krull dimension. This characterisation is then used to prove the RIII property for such rings. In section 3 the characterisation is first obtained in the Noetherian case (discrete valuation rings and semilocal principal ideal domains), using that valuation rings are elementary divisor domains, and a Cassels–Pfister type result for algebras with involution over semilocal principal ideal domains, which is proven in section 2. In section 4 we treat the general case of semilocal Bézout domains, first considering the case of finite Krull dimension. The characterisation of the multipliers is then obtained using some result for Henselian valuation rings (treated in [B]), and a norm argument based on Paulo Ribenboim's approximation theorem for valuations.

We fix some notation for the rest of the article. F will denote a field of characteristic different from 2. R will be a commutative semilocal domain in which 2 is invertible, with fraction field F, and S will be either equal to R or to a commutative separable $R-$algebra that is free of dimension 2 as an $R-$module. In the latter case, we call S a *free separable quadratic $R-$algebra*. If S is a domain, we denote its fraction field by K.

Let T be a commutative ring and M a finitely generated, free $T-$module. We will use the term *dimension* for the rank of M over T, and denote this by $\dim_T(M)$.

1 Hermitian spaces, algebras with involution and multipliers

In this section we define $R-$algebras with involution and elaborate on the properties of rationally isomorphic $R-$algebras with involution, also by interpreting them as adjoint involutions of hermitian spaces. Therefore, we zoom in on some properties of hermitian spaces first.

Let C be a (not necessarily commutative) ring with unit. We assume $2 \in C^{\times}$. Let θ be an *involution* on C, i.e. an anti–automorphism of C of order at most 2. Let $\varepsilon = \pm 1$. An *$\varepsilon-$hermitian*

module over (C,θ) is a pair (V,h) where V is a finitely generated, projective right C–module, and $h : V \times V \to C$ a bi–additive map such that for all $x, y \in V$ and all $\alpha, \beta \in C$, the following hold: $h(x\alpha, y\beta) = \theta(\alpha)h(x,y)\beta$ and $h(y,x) = \varepsilon\theta(h(x,y))$. The form h is called *hermitian* if $\varepsilon = 1$ and *skew–hermitian* if $\varepsilon = -1$. If $\theta = \mathrm{id}_C$, h is called *a bilinear form*.
Let $V^* = \mathrm{Hom}_C(V,C)$. This is a left C–module. Define the right C–module ${}^\theta V^*$ by ${}^\theta V^* = \{{}^\theta\varphi \mid \varphi \in V^*\}$ with the operations ${}^\theta\varphi + {}^\theta\psi = {}^\theta(\varphi+\psi)$, $({}^\theta\varphi)\alpha = {}^\theta(\theta(\alpha)\varphi)$ for all $\varphi, \psi \in V^*$ and all $\alpha \in C$. Then h is called *non–singular* if the adjoint transformation

$$\widehat{h} : V \to {}^\theta V^* : x \mapsto {}^\theta\varphi, \quad \text{where } \varphi(y) = h(x,y) \text{ for all } y \in V,$$

is an isomorphism of right C–modules. We call (V,h) *an ε–hermitian space* if h is non–singular.

Suppose V is free over C with basis $\mathfrak{B} = (e_1, \ldots, e_n)$. Then h defines a matrix $C_h = (h(e_i, e_j)_{i,j}) \in M_n(\Delta)$. Define the dual basis $\mathfrak{B}^\# = (e_1^\#, \ldots, e_n^\#)$ by the property $e_i^\#(e_j) = \delta_{ij}$. Then $({}^\theta e_1^\#, \ldots, {}^\theta e_n^\#)$ is a C–basis for ${}^\theta V^*$. The matrix of $\hat{h}$ with respect to the bases $\mathfrak{B}, \mathfrak{B}^\#$ is given by εC_h. Hence, $\hat{h}$ is an isomorphism if and only C_h is invertible. If h is non–singular, we may consider the elements ${}^\theta e_1^\#, \ldots, {}^\theta e_n^\#$ as elements of V.

Let U be a C–submodule of V. The *orthogonal complement of U*, which is equal to $\{x \in V \mid h(x,y) = 0 \text{ for all } y \in U\}$, will be denoted by $U^\perp$. The subspace U is called *totally isotropic* if $U \subset U^\perp$. An ε–hermitian module (V,h) is called *isotropic* if it contains a nonzero totally isotropic subspace U, and *anisotropic* otherwise. Equivalently, (V,h) is isotropic if there exists an element $0 \neq x \in V$ such that $h(x,x) = 0$. (V,h) is called *hyperbolic* if it contains a direct summand U such that $U^\perp = U$.

Two ε–hermitian modules (V,h), (V',h') over (C,θ) are called *isometric*, denoted by $(V,h) \simeq (V',h')$ or $h \simeq h'$, if there is a C–linear bijection $\varphi : V \to V'$ such that $h(x,y) = h'(\varphi(x), \varphi(y))$ for all $x, y \in V$.

We recall some facts on R–algebras with involution from [B].

1.1 Proposition.

(a) *Suppose $S \neq R$. Then S has an R–basis $(1,z)$ with $z^2 = az + b$, for certain $a, b \in R$. Let $f(x) = x^2 - ax - b \in R[x]$. Then the discriminant of $f(x)$ is a unit in R. Furthermore, S is a domain if and only if $f(x)$ is irreducible in $R[x]$.*

(b) *Suppose R is integrally closed in F. Then S is the integral closure of R in $S \otimes_R F$. Furthermore S is a domain if and only if $S \otimes_R F$ is a field, and the latter is then the fraction field of S. If S is not a domain, then $S \cong R \times R$.*

Proof. See [B, (2.7)]. □

1.2 Proposition. *There is a unique involution ι on S such that $R = \{x \in S \mid \iota(x) = x\}$. If $S \neq R$ then this involution is given by $\iota : S \to S : c + dz \mapsto c + d(a - z)$.*

Proof. See [B, (2.9)]. □

In the special case where $S \cong R \times R$, ι is given by the switch map.

Let $\mathcal{A}$ be an Azumaya algebra over S and σ an $R-$linear involution on $\mathcal{A}$ such that, if $Z(\mathcal{A}) \neq R$, then σ restricts to the involution ι on S. If $S = R$, σ is called *an involution of the first kind*, otherwise, it is called *an involution of the second kind*. We call the pair $(\mathcal{A}, \sigma)$ an *$R-$algebra with involution*. If S is not a domain then we call $(\mathcal{A}, \sigma)$ *degenerate*.

If R is a field, then σ is called *isotropic* if there exists a nonzero $x \in \mathcal{A}$ such that $\sigma(x)x = 0$, and *anisotropic* otherwise. σ is called *hyperbolic* if there is an idempotent $x \in \mathcal{A}$ such that $\sigma(x) = 1 - x$.

Let R' be a commutative semilocal domain that is also an $R-$algebra. We write $(\mathcal{A}, \sigma)_{R'} = (\mathcal{A}_{R'}, \sigma_{R'}) = (\mathcal{A} \otimes_R R', \sigma \otimes_R \mathrm{id}_{R'})$. The center of $\mathcal{A}_{R'}$ is equal to $S \otimes_R R'$.

1.3 Proposition. *Let $(\mathcal{A}, \sigma)$ be an $R-$algebra with involution and let R' a commutative semilocal domain that is an $R-$algebra. Then $(\mathcal{A}, \sigma)_{R'}$ is an $R'-$algebra with involution.*

Proof. See [B, (2.10)]. □

Let $(\mathcal{A}, \sigma)$ and $(\mathcal{A}', \sigma')$ be $R-$algebras with involution. We say $(\mathcal{A}, \sigma)$ *and* $(\mathcal{A}', \sigma')$ *are isomorphic over R* if there exists an isomorphism $\varphi : \mathcal{A} \to \mathcal{A}'$ of $R-$algebras such that $\varphi \circ \sigma = \sigma' \circ \varphi$. We denote this by $(\mathcal{A}, \sigma) \cong_R (\mathcal{A}', \sigma')$.

Let $\mathcal{B}$ be an Azumaya algebra over R. The map $\mathrm{sw} : \mathcal{B} \times \mathcal{B}^{\mathrm{op}} \to \mathcal{B} \times \mathcal{B}^{\mathrm{op}} : (a, b) \mapsto (b, a)$ defines an involution of the second kind on $\mathcal{B} \times \mathcal{B}^{\mathrm{op}}$, called *the switch involution*.

1.4 Proposition. *Let $\mathcal{A}$ be an Azumaya algebra wit center $R \times R$. Then there exist Azumaya algebras $\mathcal{A}_1, \mathcal{A}_2$ over R such that $\mathcal{A} \cong \mathcal{A}_1 \times \mathcal{A}_2$. Furthermore, if σ is an involution of the second kind on $\mathcal{A}$, then $\mathcal{A}_2 \cong \mathcal{A}_1^{\mathrm{op}}$ and $(\mathcal{A}, \sigma) \cong_R (\mathcal{A}_1 \times \mathcal{A}_1^{\mathrm{op}}, \mathrm{sw})$.*

Proof. See [B, (2.11)]. □

1.5 Corollary. *Suppose R is a field and let $(\mathcal{A}, \sigma)$ be a degenerate $R-$algebra with involution. Then σ is hyperbolic.*

Proof. By Proposition 1.1 (b), $Z(\mathcal{A}) \cong R \times R$ and Proposition 1.4 yields that there exists an idempotent $x \in \mathcal{A}$ such that $\sigma(x) = 1 - x$. □

1.6 Proposition. *Let $\mathcal{A}$ be an Azumaya algebra over S. Then $\mathcal{A}$ is free as an $R-$module.*

Proof. See [B, (2.12)]. □

In this paper we will work with algebras with involution over a semilocal Bézout domain. These are semilocal domains in which the finitely generated ideals are principal (the defining property of a Bézout domain). Semilocal Bézout domains are exactly those semilocal domains whose localisations at their finitely many maximal ideals are valuation rings, cf. [B, (2.2)].

The following characterisation of semilocal Bézout domains of finite Krull dimension is easily obtained.

1.7 Lemma. *R is a semilocal Bézout domain of finite Krull dimension if and only if R is the intersection of finitely many valuation rings of F of finite rank (= Krull dimension).*

If R is a semilocal Bezout domain then R−algebras with involution have additional properties.

1.8 Proposition. *Suppose R is semilocal Bézout domain and assume S is a domain. Then S is a semilocal Bézout domain. Let Δ be an Azumaya algebra over S without zero divisors. Every finitely generated, torsion–free left or right Δ−module is free.*

Proof. See [B, (2.4), (2.8) and (2.16)]. □

1.9 Proposition. *Suppose R is a semilocal Bézout domain.*

(a) *Let $(\mathcal{C}, \theta)$ be an R−algebra with involution with center a domain. Let (V, h) be an ε−hermitian space over $(\mathcal{C}, \theta)$. There exists a unique involution σ on $\mathrm{End}_{\mathcal{C}}(V)$ such that $\sigma(a) = \theta(a)$ for all $a \in Z(\mathcal{C})$, and for all $x, y \in V$ and all $f \in \mathrm{End}_{\mathcal{C}}(V)$ we have that*

$$h(x, f(y)) = h(\sigma(f)(x), y).$$

We denote this involution by ad_h. Then $(\mathrm{End}_{\mathcal{C}}(V), \mathrm{ad}_h)$ is an R−algebra with involution, called the adjoint algebra with involution of h, *and denoted by $\mathrm{Ad}(h)$. If θ is of the first kind (resp. of the second kind), then ad_h is of the first kind (resp. of the second kind).*

(b) *Let $(\mathcal{A}, \sigma)$ be an R−algebra with involution with center a domain. Then there exists an R−algebra with involution (Δ, θ) without zero divisors, with $Z(\Delta) = Z(\mathcal{A})$ and θ of the same kind as σ, and an ε−hermitian space (V, h) over (Δ, θ), such that $(\mathcal{A}, \sigma) \cong_R \mathrm{Ad}(h)$.*

Proof. See [B, (4.6)] for (a) and [B, (4.7)] for (b). □

1.10 Notation. Let $(\mathcal{C}, \theta)$ be an R−algebra with involution and (V, h) an ε−hermitian module over $(\mathcal{C}, \theta)$. We denote by $(V, h)_F = (V_F, h_F)$ the ε−hermitian module over $(\mathcal{C}, \theta)_F$ obtained by extending scalars from R to F.

1.11 Proposition. *Let R be a commutative semilocal Bézout domain with fraction field F. Let (Δ, θ) be an R−algebra with involution without zero divisors. The following hold.*

(a) *Let (V, h) be an ε−hermitian space over (Δ, θ). If $(V, h)_F$ is isotropic (metabolic), then (V, h) is already isotropic (metabolic).*

(b) *Suppose $2 \in R^\times$. Then hyperbolic ε−hermitian spaces of the same dimension over (Δ, θ) are isometric.*

(c) *Suppose $2 \in R^\times$. Let $(V, h), (V', h')$ be ε−hermitian spaces over (Δ, θ). If $(V, h)_F \simeq_F (V', h')_F$ then $(V, h) \simeq (V', h')$.*

Proof. See [B, (4.5)]. The proof of (c) uses a general Witt cancellation result of Keller (see [K, (VI.5.7.2)] for more details). □

Let $(\mathcal{A},\sigma)$ and $(\mathcal{A}',\sigma')$ be two $R-$algebras with involution. We say they have the "rational isomorphism implies isomorphism" property, denoted by RIII, if the following holds:

$$\text{if } (\mathcal{A},\sigma)_F \cong_F (\mathcal{A}',\sigma')_F \text{ then } (\mathcal{A},\sigma) \cong_R (\mathcal{A}',\sigma').$$

We say that R *has the RIII property* if any pair $((\mathcal{A},\sigma),(\mathcal{A}',\sigma'))$ of $R-$algebras with involution has the RIII property. In [B, (8.7)], the author showed that if R is a Henselian valuation ring, then it has the RIII property. In this paper we will extend the results of [B] and show that if R is a semilocal Bézout domain, then it has the RIII property. A crucial ingredient of the proof will be the following result for Henselian valuation rings from [B].

1.12 Proposition. *Suppose R is a Henselian valuation ring of F. Denote its residue field by κ. Let $(\mathcal{A},\sigma)$ be an $R-$algebra with involution. Then $(\mathcal{A},\sigma)_\kappa$ is hyperbolic if and only if $(\mathcal{A},\sigma)_F$ is hyperbolic.*

Proof. See [B, (7.7)]. □

As in [B], we will use this result in the following way. Let $\mathcal{O}$ be a valuation ring of F. Let F^s be a separable closure of F and let $\mathcal{O}^s$ be an extension of $\mathcal{O}$ to F^s. Let $G = \{\rho \in \mathrm{Gal}(F^s/F) \mid \rho(\mathcal{O}^s) = \mathcal{O}^s\}$. Then $((F^s)^G, \mathcal{O}^s \cap (F^s)^G)$ is Henselian by [EP, (3.2.15)]. We denote it by $(F^h,\mathcal{O}^h)$, it is called *a Henselisation* of $(F,\mathcal{O})$. By [EP, (5.2.5)], $(F,\mathcal{O}) \subset (F^h,\mathcal{O}^h)$ is *an immediate extension*, i.e. $(F,\mathcal{O})$ and $(F^h,\mathcal{O}^h)$ have isomorphic value groups and residue fields.

1.13 Corollary. *Suppose R is a valuation ring of F. Denote its residue field by κ. Let (F^h,R^h) be a Henselisation of (F,R). Let $(\mathcal{A},\sigma)$ be an $R-$algebra with involution. Then $(\mathcal{A},\sigma)_\kappa$ is hyperbolic if and only if $(\mathcal{A},\sigma)_{F^h}$ is hyperbolic.*

Proof. Since R^h still has residue field κ, the statement follows immediately from Proposition 1.12. □

We recall several reductions of the RIII problem explained in [B].

1.14 Proposition. *Suppose R is a semilocal Bézout domain. If pairs of $R-$algebras with involution of the form $((\mathcal{A},\sigma),(\mathcal{A},\sigma'))$, where $(\mathcal{A},\sigma)_F \cong_F (\mathcal{A}',\sigma')_F$ via a $Z(\mathcal{A}_F)-$isomorphism, have the RIII property, then R has the RIII property.*

Proof. This is shown in [B, (8.3)] in the case R is a valuation ring, but it goes through for semilocal Bézout domains. □

1.15 Proposition. *Suppose R is a semilocal Bézout domain. Let $((\mathcal{A},\sigma),(\mathcal{A}',\sigma'))$ be a pair of $R-$algebras with involution. Assume that $(\mathcal{A},\sigma)_F \cong_F (\mathcal{A}',\sigma')_F$ and $Z(\mathcal{A}) \cong R\times R$. Then $(\mathcal{A},\sigma) \cong (\mathcal{A}',\sigma')$.*

Proof. By Proposition 1.14, in order to show the claim we may assume that $\mathcal{A}' = \mathcal{A}$. Since all involutions of the second kind on $\mathcal{A}$ are isomorphic over R by Proposition 1.4, the statement follows. □

1.16 Proposition. *Suppose R is a semilocal Bézout domain. Let (Δ, θ) be an R−algebra with involution without zero divisors, and let (V, h) be an ε−hermitian space over (Δ, θ). Let $(\mathcal{A}, \sigma) = \mathrm{Ad}(h)$. Let furthermore $s \in \mathcal{A}^\times$ be such that $\sigma(s) = s$ and let $\sigma' = \mathrm{Int}(s) \circ \sigma$. Define $h' : V \times V \to \Delta$ by $h'(x, y) = h(s^{-1}(x), y)$ for all $x, y, \in V$. Then (V, h') is an ε−hermitian space over (Δ, θ) such that $(\mathcal{A}, \sigma') = \mathrm{Ad}(h')$, and the following are equivalent:*

(i) $(\mathcal{A}, \sigma) \cong_R (\mathcal{A}, \sigma')$ *through a $Z(\mathcal{A})$−automorphism.*

(ii) *There exists elements $e \in R^\times$ and $g \in \mathcal{A}^\times$ such that $es = \sigma(g)g$.*

(iii) *There exists $u \in R^\times$ such that $h' \simeq_R uh$.*

Furthermore, if $u \in R^\times$, then $h' \simeq_R uh$ if and only if there exists an element $g \in \mathcal{A}^\times$ such that $us = \sigma(g)g$.

Proof. See [B, (4.9)]. □

The following proposition is shown in [B, (8.5)] for valuation rings, but the proof goes through for semilocal Bézout domains.

1.17 Proposition. *Suppose R is a semilocal Bézout domain and assume S is a domain. Let $\mathcal{A}$ be an Azumaya algebra over S, and σ, σ' involutions of the first or second kind on $\mathcal{A}$. Suppose $(\mathcal{A}, \sigma)_F \cong_F (\mathcal{A}, \sigma')_F$. Then there exists $s \in \mathcal{A}^\times$ such that $\sigma(s) = s$ and $\sigma' = \mathrm{Int}(s) \circ \sigma$.*

Proposition 1.16 yields the equivalence between similarity of certain hermitian spaces and isomorphism of their adjoint algebras with involution. This result holds in fact without constraints on the hermitian spaces, as is shown below.

1.18 Proposition. *Suppose R is a semilocal Bézout domain and let (Δ, θ) be an R−algebra with involution without zero divisors. Let (V, h) and (V', h') be two ε−hermitian spaces over (Δ, θ). Then there exists $u \in R^\times$ such that $h' \simeq_R uh$ if and only if $\mathrm{Ad}(h)$ and $\mathrm{Ad}(h')$ are isomorphic through a $Z(\Delta)$−isomorphism.*

Proof. Suppose first there exists $u \in R^\times$ and a Δ−linear bijection $\varphi : V \to V'$ such that $uh(x, y) = h'(\varphi(x), \varphi(y))$, for all $x, y \in V$. Then one easily checks that $\mathrm{End}_\Delta(V) \to \mathrm{End}_\Delta(V'); f \mapsto \varphi \circ f \circ \varphi^{-1}$ defines a $Z(\Delta)$−isomorphism of algebras with involution: $\mathrm{Ad}(h) \to \mathrm{Ad}(h')$.
Suppose conversely that there exists an isomorphism $\beta : \mathrm{Ad}(h) \to \mathrm{Ad}(h')$ that is the identity on $Z(\Delta)$. Then $\mathrm{End}_\Delta(V) \cong \mathrm{End}_\Delta(V')$ and hence V and V' have the same dimension over Δ. Hence, there is a Δ−linear bijection $\psi : V \to V'$. Define an ε−hermitian form $\tilde{h} : V' \times V' \to \Delta$ by $\tilde{h}(\psi(x), \psi(y)) = h(x, y)$, for all $x, y \in V$. Then $\tilde{h} \simeq_R h$ and therefore, $\mathrm{Ad}(\tilde{h}) \cong_R \mathrm{Ad}(h) \cong_R \mathrm{Ad}(h')$ via $Z(\Delta)$−isomorphisms. Since $\mathrm{ad}_{h'}$ and $\mathrm{ad}_{\tilde{h}}$ are involutions on $\mathrm{End}_\Delta(V')$ and $\mathrm{ad}_{\tilde{h}_F}$ and $\mathrm{ad}_{h'_F}$ are isomorphic, it follows from Proposition 1.17 that there exists $s \in \mathrm{End}_\Delta(V')^\times$ such that $\mathrm{ad}_{\tilde{h}} = \mathrm{Int}(s) \circ \mathrm{ad}_{h'}$ and $\mathrm{ad}_{h'}(s) = s$. Define an ε−hermitian form $h'' : V' \times V' \to \Delta$ by

$$h''(x', y') = h'(s^{-1}(x'), y'),$$

for all $x', y' \in V'$. One easily checks that $\mathrm{ad}_{h''} = \mathrm{ad}_{\tilde{h}}$. Furthermore, we have that $v = \widehat{h''}^{-1} \circ \widehat{\tilde{h}} \in \mathrm{End}_\Delta(V')^\times$ and, by definition of v, $h''(v(x'), y') = \tilde{h}(x', y')$ for all $x', y' \in V'$. It follows that $\mathrm{ad}_{h''} = \mathrm{ad}_{\tilde{h}} = \mathrm{Int}(v^{-1}) \circ \mathrm{ad}_{h''}$. Hence, $v \in Z(\Delta)$, and we get $\tilde{h} = \theta(v)h''$. Since $\tilde{h}$ and h'' are both ε–hermitian, it follows that $\theta(v) = v$ and hence $v \in R^\times$. It follows that

$$h'' \simeq_R v^{-1}\tilde{h} \simeq_R v^{-1}h.$$

It follows that $\mathrm{Ad}(h'')$ and $\mathrm{Ad}(h)$, and hence $\mathrm{Ad}(h'')$ and $\mathrm{Ad}(h')$ are isomorphic through a $Z(\Delta)$–isomorphism. By Proposition 1.16, this means there exists $e \in R^\times, g \in \mathrm{End}_\Delta(V')^\times$ such that

$$es = \mathrm{ad}_{h'}(g)g.$$

For all $x', y' \in V'$, we get that

$$h''(x', y') = h'(s^{-1}(x'), y') = eh'(g^{-1}\,\mathrm{ad}_{h'}(g^{-1})(x'), y') = eh'(\mathrm{ad}_{h'}(g^{-1})(x'), \mathrm{ad}_{h'}(g^{-1})(y')).$$

This means that $h'' \simeq_R eh'$. So, putting everything together, we obtain $h' \simeq_R e^{-1}h'' \simeq_R e^{-1}v^{-1}h$, and $e^{-1}v^{-1} \in R^\times$. This yields the statement. □

The next result will be crucial in order to show that semilocal Bézout domains have the RIII property.

1.19 Proposition. *Suppose R is a valuation ring of F, with residue field κ. Let $(\mathcal{A}, \sigma)$ be an R–algebra with involution. Let $e \in F^\times$, $s \in \mathcal{A}^\times$ and $g \in \mathcal{A}_F^\times$ be such that $es = \sigma_F(g)g$. Let (F^h, R^h) be a Henselisation of (F, R). If $e \notin F^{\times 2}R^\times$ then $(\mathcal{A}, \sigma)_{F^h}$ and $(\mathcal{A}, \sigma)_\kappa$ are hyperbolic.*

Proof. If $Z(\mathcal{A})$ is not a domain then $(\mathcal{A}, \sigma)$ is hyperbolic over F^h and κ by Proposition 1.5. See [B, (8.6)] for the case where $Z(\mathcal{A})$ is a domain and combine it with Corollary 1.13. □

Let (B, τ) be an F–algebra with involution. An element $f \in B$ is called *a similitude* if $\tau(f)f \in F^\times$. The set of similitudes for (B, τ) forms a group, denoted by $\mathrm{Sim}(B, \tau)$. If $f \in \mathrm{Sim}(B, \tau)$ then $\mu(f) = \sigma(f)f \in F^\times$ is called *a multiplier* of (B, τ). The multipliers of (B, τ) form a subgroup of $F^\times$, denoted by $G(B, \tau)$.

1.20 Proposition. *Let (C, θ) be an F–algebra with involution of any kind. Let $(V, \tilde{h})$ be an ε–hermitian space over (C, θ) and let $(B, \tau) \cong (\mathrm{End}_C(V), \mathrm{ad}_{\tilde{h}})$. Then*

$$G(B, \tau) = \{\alpha \in F^\times \mid \tilde{h} \simeq \alpha\tilde{h}\}.$$

Proof. We have that $\tilde{h} \simeq \alpha\tilde{h}$ if and only if there exist a C–linear bijection $\varphi : V \to V$ such that

$$\tilde{h}(x, y) = \alpha\tilde{h}(\varphi(x), \varphi(y)) = \alpha\tilde{h}((\tau(\varphi)\varphi)(x), y), \quad \text{for all } x, y \in V.$$

The non–singularity of $\tilde{h}$ implies that this equality holds if and only if $\tau(\varphi)\varphi = \alpha^{-1}$, and hence $\alpha^{-1} \in G(B, \tau)$. Since $G(B, \tau)$ is a group, it follows that $\alpha \in G(B, \tau)$. □

1.21 Proposition. *Let (B, τ) be an F–algebra with involution of any kind.*

(a) If (B,τ) is hyperbolic then $G(B,\tau) = F^\times$.

(b) Let L/F be a finite extension. Then

$$N_{L/F}(G((B,\tau)_L)) \subset G(B,\tau).$$

Proof. (a) follows using the characterisation of $G(B,\tau)$ in terms of ε–hermitian forms from Proposition 1.20. If $Z(B)$ is a domain, then (b) holds by [KMRT, (12.21)]. If $Z(B)$ is not a domain, then (b) holds since $G(B,\tau) = F^\times$ by (a). □

Using the language of multipliers, property (ii) in Proposition 1.16 can be replaced by another (seemingly weaker) property,

1.22 Corollary. *In the situation of Proposition 1.16, (i), (ii) and (iii) are also equivalent to*

(ii') There exist elements $e \in G((\mathcal{A},\sigma)_F)R^\times$ and $g \in \mathcal{A}_F^\times$ such that $es = \sigma(g)g$.

Furthermore, if $e \in F^\times$ is such that there exists an element $g \in \mathcal{A}$ such that $es = \sigma(g)g$ and $(\mathcal{A},\sigma) \cong_R (\mathcal{A},\sigma')$, then $e \in G((\mathcal{A},\sigma)_F)R^\times$.

Proof. For the first statement, it suffices to prove that (ii') is equivalent to (iii). It follows directly from Proposition 1.16 that (iii) implies (ii'), (since (ii) is stronger). So assume (ii') holds. By Proposition 1.16, $es = \sigma_F(g)g$ yields that $h'_F \cong_F eh_F$. Write $e = au$ with $a \in G((\mathcal{A},\sigma)_F)$ and $u \in R^\times$. By Proposition 1.20, we have that $h_F \simeq_F ah_F$ and hence, it follows that $h'_F \simeq_F uh_F$. By Proposition 1.11 (b), we get that $h' \simeq_R uh$.

Let $e \in F^\times$ be such that there exists $g \in \mathcal{A}$ such that $es = \sigma(g)g$ and assume that $(\mathcal{A},\sigma) \cong_R (\mathcal{A},\sigma')$. Then $g \in \mathcal{A}_F^\times$ and Proposition 1.16 yields that $h'_F \simeq_F eh_F$. Since $(\mathcal{A},\sigma) \cong_R (\mathcal{A},\sigma')$ by hypothesis, invoking Proposition 1.16 once more yields that there exists $u \in R^\times$ such that $h'_F \simeq_F uh_F$. It follows that $euh_F \simeq_F h_F$. By Proposition 1.20, the latter means exactly that $eu \in G((\mathcal{A},\sigma)_F)$. □

2 Cassels–Pfister type theorems for involutions and hermitian forms

In this section, we assume R is a semilocal principal ideal domain. We furthermore fix an R–algebra with involution without zero divisors (Δ,θ), and an ε–hermitian space (V,h) over (Δ,θ). Furthermore, we let $(\mathcal{A},\sigma) = \mathrm{Ad}(h)$ and $D = \Delta_F$. Note that $Z(\mathcal{A}) = Z(\Delta)$ is a domain.

A *(Δ–)lattice in* $(V,h)_F$ is a finitely generated, right Δ–submodule of V_F containing a D–basis of V_F. Lattices in $(V,h)_F$ are torsion–free Δ–modules, and hence free Δ–modules by Proposition 1.8.

Let $\mathcal{L}$ be a lattice in $(V,h)_F$. The *dual of* $\mathcal{L}$ is defined as

$$\mathcal{L}^\# = \{v \in V_F \mid h_F(v,\mathcal{L}) \subset \Delta\}.$$

$\mathcal{L}$ is called *integral (with respect to h_F)* if $\mathcal{L} \subset \mathcal{L}^\#$ and *self–dual* or *unimodular* if $\mathcal{L} = \mathcal{L}^\#$.

2.1 Proposition. *Let $\mathcal{L}$ be an integral lattice in $(V,h)_F$ and denote the restriction of h_F to $\mathcal{L}$ by $h_{\mathcal{L}}$. The following are equivalent:*

(i) $(\mathcal{L}, h_{\mathcal{L}})$ *is an ε−hermitian space over (Δ, θ).*

(ii) $\mathcal{L}^{\#} = \mathcal{L}$.

Proof. Let $\mathfrak{B} = (f_1, \dots, f_n)$ be a Δ−basis for $\mathcal{L}$. Consider the dual basis ${}^{\theta}\mathfrak{B}^{\#} = ({}^{\theta}f_1^{\#}, \dots, {}^{\theta}f_n^{\#})$ as elements of V_F. Suppose (i) holds. Then the elements of ${}^{\theta}\mathfrak{B}^{\#}$ belong to $\mathcal{L}$. Suppose $h_F(v, \mathcal{L}) \subset \Delta$ for some $v \in V_F$. We can write $v = \sum_{i=1}^{n} {}^{\theta}e_i^{\#} x_i$, with the $x_i \in D$. Then $h_F(v, e_j) = \sum_{i=1}^{n} \theta_F(x_i)\delta_{ij} = \theta_F(x_j) \in \Delta$ and hence $v \in \mathcal{L}$. So, $\mathcal{L}$ is unimodular.

Suppose $\mathcal{L} = \mathcal{L}^{\#}$. Since $h_F({}^{\theta}f_i^{\#}, f_j) = \delta_{ij} \in \Delta$ and $\mathcal{L}$ is unimodular, we have that ${}^{\theta}\mathfrak{B}^{\#}$ belongs to $\mathcal{L}$. The matrix of $\hat{h}_{\mathcal{L}}$ is the matrix of base change from $\mathfrak{B}$ to ${}^{\theta}\mathfrak{B}^{\#}$, and hence invertible over Δ. This yields (i). □

2.2 Corollary. *V is a unimodular lattice in $(V,h)_F$.*

Proof. It is clear that V is an integral lattice in $(V,h)_F$. Since (V,h) is a ε-hermitian space, the statement follows from Proposition 2.1. □

We refer to [T] for proofs of the following known facts on lattices.

2.3 Proposition.

(a) Let $\mathcal{L}$ be a lattice in $(V,h)_F$. Then $\mathcal{L}^{\#}$ is a lattice in $(V,h)_F$ and $\mathcal{L}^{\#\#} = \mathcal{L}$.

(b) Let $\mathcal{L}_1, \mathcal{L}_2$ be lattices in $(V,h)_F$. Then

$$(\mathcal{L}_1 + \mathcal{L}_2)^{\#} = \mathcal{L}_1^{\#} \cap \mathcal{L}_2^{\#} \quad \text{and} \quad (\mathcal{L}_1 \cap \mathcal{L}_2)^{\#} = \mathcal{L}_1^{\#} + \mathcal{L}_2^{\#}.$$

Note that this implies that the intersection of two lattices in $(V,h)_F$ is again a lattice.

(c) Let $\mathcal{L}_1, \mathcal{L}_2$ be lattices in $(V,h)_F$. If $\mathcal{L}_1 \subset \mathcal{L}_2$, then $\mathcal{L}_2^{\#} \subset \mathcal{L}_1^{\#}$.

2.4 Lemma. *The unimodular lattices in $(V,h)_F$ are exactly the maximal integral lattices in $(V,h)_F$.*

2.5 Proposition. *Let $\mathcal{L}$ be a unimodular lattice in $(V,h)_F$. Then there is an isometry u of $(V,h)_F$ such that $u(V) = \mathcal{L}$.*

Proof. Let $h_{\mathcal{L}}$ denote the restriction of h_F to $\mathcal{L}$. The hermitian spaces $(\mathcal{L}, h_{\mathcal{L}})$ and (V,h) become isometric over F since they become two representations of the same form. Proposition 1.11 (c) yields that $(\mathcal{L}, h_{\mathcal{L}})$ and (V,h) are isometric over R. This means that there is a bijective Δ−linear map $u : V \to \mathcal{L}$ such that $h_{\mathcal{L}}(u(x), u(y)) = h(x,y)$. Extending scalars to F, u defines an isometry of $(V,h)_F$ with $u(V) = \mathcal{L}$. □

We can now prove a Cassels–Pfister type theorem for algebras with involution over semilocal principal ideal domains.

2.6 Theorem (Involution CP for semilocal principal ideal domains). *Let $f \in \mathcal{A}_F$ be such that $\sigma_F(f)f \in \mathcal{A}$. Then there exists an element $u \in \mathcal{A}_F^\times$ such that $\sigma_F(u)u = 1$ and $uf \in \mathcal{A}$.*

Proof. We follow the proof of the main theorem of [T].

We have that $(\mathcal{A}, \sigma)_F \cong_F (\mathrm{End}_D(V_F), \mathrm{ad}_{h_F})$. Let $f \in \mathrm{End}_D(V_F)$ be such that $\sigma_F(f)f \in \mathrm{End}_\Delta(V)$. Then we can write $f = d^{-1}\tilde{f}$ for some $\tilde{f} \in \mathrm{End}_\Delta(V)$ and $d \in R$. For all $m, m' \in V$, we have that

$$h_F(f(m), f(m')) = h_F(\sigma_F(f)f(m), m') \in \Delta,$$

since $\sigma_F(f)f(m) \in V$ and $V = V^\#$.

Note that $f(V)$ is not necessarily a lattice in $(V, h)_F$. However, $f(V) + dV$ is a lattice and it is also integral since $df \in \mathrm{End}_\Delta(V)$. Since R is a principal ideal domain, every integral lattice in $(V, h)_F$ is contained in a unimodular lattice. This can be seen as follows. Let $\mathcal{L}_1 \subset \mathcal{L}_2 \subset \ldots$ be a chain of integral lattices, we have that $\mathcal{L}_i \subset \mathcal{L}_1^\#$, for all i. Since $\mathcal{L}_1^\#$ is a lattice over Δ, Δ is finitely generated over R, and R is a Noetherian ring, it follows that $\mathcal{L}_1^\#$ is a Noetherian $R-$module. Since all $\mathcal{L}_i$ are $R-$submodules of $\mathcal{L}_1^\#$, it follows that the chain $\mathcal{L}_1 \subset \mathcal{L}_2 \subset \ldots$ must stop. Hence, any integral lattice in $(V, h)_F$ is contained in a maximal integral lattice.

Let $\mathcal{L}$ be a maximal integral lattice in $(V, h)_F$ containing $f(V) + dV$. Then $\mathcal{L}$ is unimodular by Lemma 2.4. Proposition 2.5 implies that there is an isometry u of $(V, h)_F$ such that $u(V) = \mathcal{L}$. It follows that $f(V) \subset \mathcal{L} = u(V)$, so $u^{-1}f(V) \subset V$ and hence $u^{-1}f \in \mathrm{End}_\Delta V$. Since u is an isometry of $(V, h)_F$, we have that

$$h_F(\sigma_F(u^{-1})u^{-1}(x), y) = h_F(u^{-1}(x), u^{-1}(y)) = h_F(x, y),$$

for all $x, y \in V_F$. It follows that $\sigma_F(u^{-1})u^{-1} = 1$. This proves the statement. □

2.7 Corollary (Hermitian CP for semilocal principal ideal domains). *If there exists an element $x \in V_F$ such that $h_F(x, x) \in \Delta$, then there exists an element $x' \in V$ such that $h(x', x') = h_F(x, x)$.*

Proof. Let $(e_1, \ldots, e_n)$ be a $\Delta-$basis for V. Let $\delta = h_F(x, x)$, with $x \in V_F$. Consider the element $f \in \mathrm{End}_D(V_F)$ defined by $f(e_1) = x$ and $f(e_i) = 0$, for $i = 2, \ldots, n$. Since $h_F((\mathrm{ad}_h(f)f)(e_j), e_i) = h_F(f(e_j), f(e_i)) = 0$ for $i \neq 1$ or $j \neq 1$, and $h_F((\mathrm{ad}_h(f)f)(e_1), e_1) = h_F(f(e_1), f(e_1)) = h_F(x, x) \in \Delta$, it follows that $h_F((\mathrm{ad}_h(f)f)(y), y) \in \Delta$ for all $y \in V$. Therefore, $(\mathrm{ad}_h(f)f)(y) \in V^\# = V$, and hence $\mathrm{ad}_h(f)f \in \mathrm{End}_\Delta(V)$.

By Theorem 2.6 there exists an element $u \in \mathrm{End}_D(V_F)^\times$ with $\mathrm{ad}_h(u)u = 1$ such that $uf \in \mathrm{End}_\Delta(V)$. Since $\mathrm{ad}_h(u)u = 1$, we have $h_F(u(y), u(y)) = h_F((\mathrm{ad}_h(u)u)(y), y) = h_F(y, y)$ for all $y \in V_F$ and therefore in particular

$$h_F(u(x), u(x)) = h_F(x, x) = \delta.$$

So, we have found an element $z = u(x) = u(f(e_1)) \in V$ representing δ. □

2.8 Remarks.

(a) The conclusion of Theorem 2.6 also holds if R is any valuation ring of F and Δ is a valuation ring of D. The proof of Theorem 2.6 goes through completely provided that every integral lattice is contained in a unimodular (= maximal integral) lattice. If R is not discrete then it is not Noetherian and the proof of the latter fact is fairly technical. It uses the elementary divisor property for finitely generated modules over Δ.

(b) Theorem 2.6 is used in the proof of Proposition 3.5, which will be the crucial result in order to show that semilocal principal ideal domains have the RIII property. The proofs given of the analogous results for algebras with involution over semilocal Bézout domains of finite Krull dimension (see section 4), rely heavily on Proposition 1.19. The proof of this proposition, cf. [B, (8.6)], uses the Cassels–Pfister property as formulated in Corollary 2.7, but for hermitian spaces over Azumaya algebras that are in addition (non–commutative) valuation rings. A direct proof of this CP version is given in [B, (4.16)].

3 Semilocal principal ideal domains

In this section we assume R is a semilocal principal ideal domain. Let $(\mathcal{A},\sigma)$ be an R–algebra with involution. The main result in this section, Proposition 3.5, gives a characterisation of the multipliers of $(\mathcal{A},\sigma)_F$ up to units in R. These multipliers are described by local conditions, i.e. in terms of the discrete valuation rings of F lying over R. As consequence of this characterisation, it follows that R has the RIII property. In particular we obtain a different proof of Nisnevich' theorem, stating that discrete valuation rings in which 2 is invertible have the RIII property. The results in this section will also be covered by the more general results in the next section, but since the proofs simplify when working with discrete valuation rings, we treat this case separately.

3.1 Lemma. *Let (B,τ) be an F–algebra with involution. The following are equivalent.*

(i) (B,τ) is hyperbolic.

(ii) There is an element $b \in B$, such that $\tau(b)b = 0$ and $\dim_F bB \geqslant \frac{1}{2}\dim_F B$.

Proof. See [T, (2.1)]. □

3.2 Proposition. *Let $\mathcal{O}$ a discrete valuation ring of F. Denote the residue field of $\mathcal{O}$ by κ. Let $(\mathcal{A},\sigma)$ be an $\mathcal{O}$–algebra with involution. Suppose there exist elements $e \in \mathcal{O}$, $s \in \mathcal{A}^\times$ and $g \in \mathcal{A}_F^\times$ such that $es = \sigma(g)g$. If $e \notin F^{\times 2}\,\mathcal{O}^\times$, then $(\mathcal{A},\sigma)_\kappa$ is hyperbolic.*

Proof. Note that, if $Z(\mathcal{A}) \cong \mathcal{O}\times\mathcal{O}$ then $(\mathcal{A},\sigma)_\kappa$ is degenerate and hence automatically hyperbolic by Proposition 1.5.

Let v be a discrete valuation on F with valuation ring $\mathcal{O}$. Without loss of generality, we may assume that $v(e) = 1$. For let π be a uniformiser for v, then multiplying both sides of $\sigma(g)g = es$ with an appropriate even power of π and using that σ is the identity on $\mathcal{O}$, we obtain $\pi us = \sigma_F(g')g'$, with $u \in \mathcal{O}^\times, g' \in \mathcal{A}_F^\times$. By Theorem 2.6, there exists $\tilde{g} \in \mathcal{A}$ such that $\pi us = \sigma(\tilde{g})\tilde{g}$. By

abuse of notation, we denote $\tilde{g}$ again by g in the rest of the proof.
Let $n \in \mathbb{N}$ be such that $\dim_{\mathcal{O}} \mathcal{A} = \dim_F \mathcal{A}_F = n^2$ if σ is of the first kind, and such that $\dim_{\mathcal{O}} \mathcal{A} = \dim_F \mathcal{A}_F = 2n^2$ if σ is of the second kind. Let $L/Z(\mathcal{A}_F)$ be a splitting field of $\mathcal{A}_F$ and let V be a discrete valuation ring of L lying over $\mathcal{O}$. Let w be a discrete valuation with valuation ring V and let Π be a uniformiser for w. Denote the residue field of V by $\tilde{\kappa}$. We denote the maps $\mathcal{O} \to \kappa$ and the induced map $\mathcal{A} \to \mathcal{A}_\kappa$ by $\overline{}$. The commutativity of the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\sigma"] \arrow[d, "-"'] & \mathcal{A} \arrow[d, "-"] \\ \mathcal{A}_\kappa \arrow[r, "\sigma_\kappa"'] & \mathcal{A}_\kappa \end{tikzcd}$$

implies that $\overline{\sigma(g)} = \sigma_\kappa(\overline{g})$ and since $v(e) = 1$, it follows that $0 = \sigma_\kappa(\overline{g})\overline{g}$. So, in order to show that $(\mathcal{A}, \sigma)_\kappa$ is hyperbolic, by Lemma 3.1, it suffices to show that

$$\dim_\kappa \overline{g}\mathcal{A}_\kappa \geqslant \frac{1}{2} \dim_\kappa \mathcal{A}_\kappa = \frac{n^2}{2}.$$

The commutative diagram

$$\begin{tikzcd} \mathcal{O} \arrow[r] \arrow[d, "-"'] & V \arrow[d, "-"] \\ \kappa \arrow[r] & \tilde{\kappa} \end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\varphi"] \arrow[d, "-"'] & \mathcal{A} \otimes_{\mathcal{O}} V \arrow[d, "-"] \\ \mathcal{A}_\kappa \arrow[r, "\psi"] & \mathcal{A}_\kappa \otimes_\kappa \tilde{\kappa}. \end{tikzcd}$$

We have that $\mathcal{A}_L \cong M_n(L)$ if σ is of the first kind, and by [KMRT, (2.15)], $\mathcal{A}_L \cong M_n(L) \times M_n(L) \cong M_n(L) \times M_n(L)^{\mathrm{op}}$ if σ is of the second kind. It follows from [B, (2.20), (2.22)] that $\mathcal{A} \otimes_{\mathcal{O}} V \cong M_n(V)$, and hence also $\mathcal{A} \otimes_V \tilde{\kappa} \cong M_n(\tilde{\kappa})$, if σ is of the first kind, and from [B, (2.23)] that $\mathcal{A} \otimes_{\mathcal{O}} V \cong M_n(V) \times M_n(V) \cong M_n(V) \times M_n(V)^{\mathrm{op}}$, and hence also $\mathcal{A} \otimes_V \tilde{\kappa} \cong M_n(\tilde{\kappa}) \times M_n(\tilde{\kappa}) \cong M_n(\tilde{\kappa}) \times M_n(\tilde{\kappa})^{\mathrm{op}}$, if σ is of the second kind. Note that the involution σ_V on $\mathcal{A} \otimes_{\mathcal{O}} V$ then corresponds to the switch involution on $M_n(V) \times M_n(V)^{\mathrm{op}}$.

Since

$$\dim_{\tilde{\kappa}} \psi(\overline{g})(\mathcal{A} \otimes_\kappa \tilde{\kappa}) = \dim_{\tilde{\kappa}}(\overline{g} \otimes 1)(\mathcal{A}_\kappa \otimes_\kappa \tilde{\kappa}) = \dim_{\tilde{\kappa}}(\overline{g}\mathcal{A}_\kappa \otimes_\kappa \tilde{\kappa}) = \dim_\kappa \overline{g}\mathcal{A}_\kappa$$

it suffices to show that

$$\dim_{\tilde{\kappa}} \psi(\overline{g})(\mathcal{A} \otimes_\kappa \tilde{\kappa}) = \dim_{\tilde{\kappa}} \overline{\varphi(g)}(\mathcal{A} \otimes_\kappa \tilde{\kappa}) \geqslant \frac{n^2}{2}.$$

Suppose σ is of the first kind. It is well–known that $\dim_{\tilde{\kappa}} \overline{\varphi(g)}\mathcal{A}_{\tilde{\kappa}} = \operatorname{rank}(\overline{\varphi(g)}) \cdot n$. Since V is a valuation ring, it is an elementary divisor domain by [Ka, p. 480]. Hence, there are matrices $P, Q \in M_n(V)^\times$ such that

$$\varphi(g) = P \operatorname{diag}(d_1, \ldots, d_n) Q,$$

with $d_1,\dots,d_n \in V$ (see [Ka, p. 465 (1)]). Let $C = \mathrm{diag}(d_1,\dots,d_n)$. It follows that $\overline{\varphi(g)} = \overline{PCQ}$. Since $\overline{P},\overline{Q} \in M_n(\tilde\kappa)^\times$, we have that $\mathrm{rank}(\overline{\varphi(g)}) = \mathrm{rank}(\overline{C})$. The latter is obtained by subtracting the number of $\overline{d_i} = 0$ from n. Let us denote this number by ℓ. Then ℓ is equal to the number of d_i that are divisible by Π. This number is at most $w(d_1) + \ldots + w(d_n) = w(\det(\varphi(g)))$, since $\det(P),\det(Q) \in V^\times$. Taking determinants of the relation $e\varphi(s) = \varphi(\sigma(g))\varphi(g) = \sigma_V(\varphi(g))\varphi(g)$ yields

$$e^n \det(\varphi(s)) = \det(\varphi(g))^2,$$

by [KMRT, (2.2)]. Since $s \in \mathcal{A}^\times$, it follows that $\det(\varphi(s)) \in V^\times$ and hence

$$w(\det(\varphi(g))) = \frac{n}{2}.$$

So, we get that

$$\dim_{\tilde\kappa} \overline{\varphi(g)}(\mathcal{A} \otimes_\kappa \tilde\kappa) = n \cdot \mathrm{rank}(\overline{\varphi(g)}) = n(n-\ell) \geqslant n[n - w(\det(\varphi(g)))] = n^2/2.$$

Suppose σ is of the second kind. Let $g' \in M_n(V)$ and $g'' \in M_n(V)^{\mathrm{op}}$ be such that $\varphi(g) = (g', g'') \in \mathcal{A} \otimes_{\mathcal{O}} V \cong M_n(V) \times M_n(V)^{\mathrm{op}}$. It follows that

$$\dim_{\tilde\kappa} \overline{\varphi(g)}(\mathcal{A} \otimes_\kappa \tilde\kappa) = \dim_{\tilde\kappa} \overline{g'} M_n(\tilde\kappa) + \dim_{\tilde\kappa} \overline{g''} M_n(\tilde\kappa)^{\mathrm{op}}.$$

Invoking the elementary divisor property of V, we find matrices $P', Q', P'', Q'' \in M_n(V)^\times$ and elements $d_1',\dots,d_n', d_1'',\dots,d_n'' \in V$ such that

$$g' = P' \,\mathrm{diag}(d_1',\dots,d_n') Q' \qquad \text{and} \qquad g'' = P'' \,\mathrm{diag}(d_1'',\dots,d_n'') Q''.$$

We have that $\dim_{\tilde\kappa} \overline{g'} M_n(\tilde\kappa) = \mathrm{rank}(\overline{g'}) \cdot n = n(n-\ell')$, where ℓ' is the number of indices in $i \in \{1,\dots,n\}$ such that $\overline{d_i'} = 0$, and $\dim_{\tilde\kappa} \overline{g''} M_n(\tilde\kappa) = \mathrm{rank}(\overline{g''}) \cdot n = n(n-\ell'')$, where ℓ'' is the number of indices in $i \in \{1,\dots,n\}$ such that $\overline{d_i''} = 0$. We have that ℓ' is equal to the number of d_i' divisible by Π and ℓ'' is equal to the number of d_i'' divisible by Π. As in the reasoning in the first kind case, we get that

$$\dim_{\tilde\kappa} \overline{g'} M_n(\tilde\kappa) \geqslant n(n - w(\det(g'))) \qquad \text{and} \qquad \dim_{\tilde\kappa} \overline{g''} M_n(\tilde\kappa)^{\mathrm{op}} \geqslant n(n - w(\det(g''))).$$

So, it follows that

$$\dim_{\tilde\kappa} \overline{\varphi(g)}(\mathcal{A} \otimes_\kappa \tilde\kappa) \geqslant n(n - w(\det(g'))) + n(n - w(\det(g''))) = 2n^2 - n(w(\det(g'g''))). \qquad (\star)$$

Let $s', s'' \in M_n(V)$ be such that $\varphi(s) = (s', s'')$. Since $s \in \mathcal{A}^\times$, it follows that $s', s'' \in M_n(V)^\times$. Using the fact that σ_V acts as the switch involution on $M_n(V) \times M_n(V)^{\mathrm{op}}$, we get $(es', es'') = (g'', g')(g', g'') = (g''g', g' * g'') = (g''g', g''g')$ in $M_n(V) \times M_n(V)^{\mathrm{op}}$. It follows that $es' = g''g'$ in $M_n(V)$, and hence, taking determinants, $e^n \det(s') = \det(g''g')$. Applying w and using that $\det(s') \in V^\times$, it follows that $w(\det(g''g')) = n$ (since $w(e) = v(e) = 1$). Plugging this in into $(\star)$ yields

$$\dim_{\tilde\kappa} \overline{\varphi(g)}(\mathcal{A} \otimes_\kappa \tilde\kappa) \geqslant 2n^2 - n(w(\det(g'g''))) = n^2 \geqslant n^2/2.$$

□

3.3 Corollary. *Let $\mathcal{O}$ be a discrete valuation ring of F. Denote its residue field by κ. Let $(\mathcal{A},\sigma)$ be an $\mathcal{O}$−algebra with involution. Let $e \in G((\mathcal{A},\sigma)_F)O^\times$. If $e \notin F^{\times 2}\mathcal{O}^\times$, then $(\mathcal{A},\sigma)_\kappa$ is hyperbolic.*

Proof. By assumption, there exist $u \in \mathcal{O}^\times$ and $g \in \mathcal{A}_F^\times$ such that $eu = \sigma_F(g)g$. Proposition 3.2 yields the statement. □

The converse of the corollary, if $(\mathcal{A},\sigma)_\kappa$ is hyperbolic then every element of $F^\times$ is a multiplier times a unit in $\mathcal{O}$, also holds. This will be shown in proposition 3.5.

3.4 Proposition. *Let (B,τ) an F−algebra with involution. Let E/F be an algebraic field extension such that τ_E is hyperbolic. Then there is a finite separable subextension L/F over which τ becomes hyperbolic.*

Proof. Let $(e_1,\dots,e_n)$ be an F−basis for B. Then it is a E−basis for B_E. Since τ_E is hyperbolic, there is an idempotent $x \in B_E$ such that $\tau_E(x) = 1-x$. Write $x = \sum_{i=1}^n e_i x_i$, with the x_i in E. Then τ already becomes hyperbolic over $F(x_1,\dots,x_n)$. Since all the x_i are algebraic over F, this is a finite extension of F. Since $\mathrm{char}(F) \neq 2$, we get that τ already becomes hyperbolic over the separable closure of F in $F(x_1,\dots,x_n)$ (see [KMRT, (9.16)]). □

By assumption R is a semilocal principal ideal domain. The localisation of R at a maximal ideal is a valuation ring that is moreover a principal ideal domain, and therefore a discrete valuation ring.

3.5 Proposition. *Let $(\mathcal{A},\sigma)$ be an R−algebra with involution. Let $e \in F^\times$. The following conditions are equivalent:*

(i) $e \in G((\mathcal{A},\sigma)_F)R^\times$.

(ii) For each discrete valuation ring $\mathcal{O}$ of F lying over R such that $e \notin F^{\times 2}\mathcal{O}^\times$, we have that $(\mathcal{A},\sigma)$ becomes hyperbolic over the residue field of $\mathcal{O}$.

Proof. That (i) implies (ii) follows Corollary 3.3.

Assume that (ii) holds. Let $\pi_1,\dots,\pi_r$ be generators for the (finitely many) maximal ideals of R, and let $\mathcal{O}_1,\dots,\mathcal{O}_r$ be the localisations of R at $(\pi_1),\dots,(\pi_r)$ respectively. These are discrete valuation rings and $R = \mathcal{O}_1 \cap \ldots \cap \mathcal{O}_r$. Since R is a unique factorisation domain, there exist $u \in R^\times$ and $\alpha_1,\dots,\alpha_r \in \mathbb{Z}$ such that $e = u\pi_1^{\alpha_1}\cdots\pi_r^{\alpha_r}$. Since $F^{\times 2} \subset G((\mathcal{A},\sigma)_F)$ and $G((\mathcal{A},\sigma)_F)$ is a group, in order to prove (i) holds, it suffices to show that for each $i \in \{1,\dots,r\}$ such that $e \notin F^{\times 2}\mathcal{O}_i^\times$, i.e. α_i is odd, there exists a generator $\tilde{\pi}_i$ for the prime ideal (π_i) with $\tilde{\pi}_i \in G((\mathcal{A},\sigma)_F)$.
Let $\mathcal{O} \in \{\mathcal{O}_1,\dots,\mathcal{O}_r\}$ be arbitrary such that $e \notin F^{\times 2}\mathcal{O}^\times$. Let $\pi \in \{\pi_1,\dots,\pi_r\}$ be the prime element corresponding to $\mathcal{O}$ and denote the residue field of $\mathcal{O}$ by κ. Let $(F^h,\mathcal{O}^h)$ be a Henselisation of $(F,\mathcal{O})$. Recall that this is an immediate extension and therefore $\kappa^h \cong \kappa$. Therefore, for any subfield $F \subset L \subset F^h$, the residue field of $\mathcal{O}^h|_L$ is also isomorphic to κ. Since $(\mathcal{A},\sigma)_\kappa$ is hyperbolic by hypothesis, it follows from Corollary 1.13 that $(\mathcal{A},\sigma)_{F^h}$ is hyperbolic as well. Proposition 3.4 yields that there is a finite separable subextension $F \subset L \subset F^h$ over which σ becomes hyperbolic. Let $V = \mathcal{O}^h|_L$. Let R' be the integral closure of R in L. Since R is a Dedekind domain and

L/F is a finite separable extension, R' is also a Dedekind domain by [FT, (II.5)]. Furthermore, R' is the intersection of the valuation rings of L lying over $\mathcal{O}_1, \ldots, \mathcal{O}_r$. There are only finitely many such valuation rings and therefore R' is a semilocal Dedekind domain, and hence a principal ideal domain by [L2, (I.15)]. Let (Π) be the principal prime ideal of R' corresponding to the valuation ring V. Taking norms of ideals, we have that $N((\Pi)) = (N(\Pi)) = (\pi)^{f_V}$, with $f_V = [\kappa_V : \kappa]$ the relative residue degree, cf. [L2, (I.22)]. Since, as we saw above, $\kappa_V = \kappa$, it follows that $f_V = 1$, and hence, $\tilde{\pi} = N(\Pi)$ is a generator for the prime ideal (π). Since $(\mathcal{A}, \sigma)_L$ is hyperbolic, $G(\mathcal{A}, \sigma)_L = L^\times$ by Proposition 1.21 (a). Invoking part (b) of the same Proposition then implies that $\tilde{\pi} \in G((\mathcal{A}, \sigma)_F)$. This proves the statement. □

3.6 Remark. If the center of $\mathcal{A}$ is not a domain then the properties (i) and (ii) in Proposition 3.5 both hold for trivial reasons. For if $(\mathcal{A}, \sigma)$ is degenerate then $(\mathcal{A}, \sigma)_F$ is also degenerate and hence hyperbolic. Proposition 1.21 (a) yields that $G((\mathcal{A}, \sigma)_F) = F^\times$. Furthermore, $(\mathcal{A}, \sigma)$ remains degenerate over the valuation rings lying over R and hence, $(\mathcal{A}, \sigma)$ is automatically hyperbolic over the residue fields of these valuation rings.

3.7 Remark. In the next section it is shown that Proposition 3.5 holds more generally for semilocal Bezout domains of finite Krull dimension. Although the proof presented here for semilocal principal ideal domains has the same structure as the one that will be given in the next section, it is somewhat simpler. First of all for discrete valuations we need not invoke the Henselisation to prove that (i) implies (ii). For the converse we do use the Henselisation (Corollary 1.13) but the norm argument in the above proof is simpler than the norm argument we will present in the next section, since we can use the properties of norms of ideals in prinicipal ideal domains. This makes the norm argument much more direct.

Using Proposition 3.5 together with the reductions made in section 1, we can now show semilocal principal ideal domains have the RIII property.

3.8 Theorem. *Suppose R is a semilocal principal ideal domain. Then R has the RIII property.*

Proof. Let $((\mathcal{A}, \sigma), (\mathcal{A}', \sigma'))$ be a pair of R−algebras with involution such that $(\mathcal{A}, \sigma)_F \cong_F (\mathcal{A}', \sigma')'_F$. By Proposition 1.14, we may assume that $\mathcal{A}' = \mathcal{A}$ and that $(\mathcal{A}, \sigma)_F \cong_F (\mathcal{A}, \sigma')'_F$ are isomorphic through a $Z(\mathcal{A}_F)$−isomorphism. Furthermore, by Proposition 1.15, we may assume that $Z(\mathcal{A})$ is a domain. By Proposition 1.16, there exist nonzero elements $e \in F^\times, s \in \mathcal{A}^\times, g \in \mathcal{A}_F^\times$ such that $es = \sigma_F(g)g$. Combining Propositions 3.2, 3.5 and Corollary 1.22 yields that $(\mathcal{A}, \sigma) \cong_R (\mathcal{A}, \sigma')$. □

4 Semilocal Bézout domains

In this section we assume R is a semilocal Bézout domain. Let $(\mathcal{A}, \sigma)$ be an R−algebra with involution. We will show that the characterisation of the elements in $G((\mathcal{A}, \sigma)_F)R^\times$, as obtained Proposition 3.5 for semilocal principal ideal domains, holds more generally for semilocal Bézout domains of finite Krull dimension. As a consequence we obtain our main result saying that semilocal Bézout domains of finite Krull dimension have the RIII property. Using this we can then prove that R also has the RIII property if it does not necessarily have finite Krull dimension.

In order to obtain a local description of the elements in $G((\mathcal{A},\sigma)_F)R^\times$ if R has finite Krull dimension, the hyperbolicity result in the Henselian case (see Proposition 1.19) will play an important role. The norm argument on ideals from the previous section will be replaced by a norm argument on elements, based on the following strong approximation theorem of Paulo Ribenboim.

4.1 Theorem. *Let E be a field and $v_1,\dots,v_m$ pairwise incomparable valuations on E with respective valuation rings $\mathcal{O}_1,\dots,\mathcal{O}_m$. For $i=1,\dots,m$, let Γ_i be the value group of $\mathcal{O}_i$. Let V_{ij} be the smallest overring of $\mathcal{O}_i$ and $\mathcal{O}_j$ in E and let Δ_{ij} be the convex subgroup of Γ_i such that Γ_i/Δ_{ij} is the value group of V_{ij}. Then $\Gamma_i/\Delta_{ij}\cong\Gamma_j/\Delta_{ji}$. Let θ_{ij} be the quotient map $\Gamma_i\to\Gamma_i/\Delta_{ij}$. Let $(\gamma_1,\dots,\gamma_m)\in\Gamma_1\times\ldots\times\Gamma_m$ be such that $\theta_{ij}(\gamma_i)=\theta_{ji}(\gamma_j)$ under the identification $\Gamma_i/\Delta_{ij}=\Gamma_j/\Delta_{ji}$. Then there exists an element $x\in E$ such that $v_i(x)=\gamma_i$ for $i=1,\dots,m$.*

Proof. See [R, Théorème 5']. □

If $\mathcal{O}_1,\dots,\mathcal{O}_m$ are pairwise independent valuation rings, then $\Delta_{ij}=\Gamma_i$ and one gets the well–known classical approximation theorem.

4.2 Corollary. *Let E be a field and $v_1,\dots,v_m$ pairwise independent valuations with respective value groups $\Gamma_1,\dots,\Gamma_m$. Let $(\gamma_1,\dots,\gamma_m)\in\Gamma_1\times\ldots\times\Gamma_m$. Then there exists an element $x\in E$ such that $v_i(x)=\gamma_i$ for $i=1,\dots,m$.*

Proof. See [EP, (2.4.1)], where a stronger form is given. □

4.3 Proposition. *Different valuation rings of rank 1 of the same field are independent.*

Proof. See [EP, (2.3.2)]. □

4.4 Lemma. *Let E/F be a finite field extension. Let $\mathcal{O}'$ be a valuation ring of E lying over $\mathcal{O}$. Let β be a prime ideal of $\mathcal{O}'$ different from the maximal ideal of $\mathcal{O}'$. Then $\beta\cap\mathcal{O}$ is a prime ideal of $\mathcal{O}$ that is not maximal.*

Proof. Since β is a prime ideal of $\mathcal{O}'$, it is clear that $\beta\cap\mathcal{O}$ is a prime ideal of $\mathcal{O}$. Let T be the integral closure of $\mathcal{O}$ in E. Then T is the intersection of the finitely many valuation rings of E lying over $\mathcal{O}$. Let $\mathcal{M}'$ be the maximal ideal of $\mathcal{O}'$. Then $\mathfrak{m}'=\mathcal{M}'\cap T$ is a maximal ideal of T and $\mathcal{O}'=T_{\mathfrak{m}'}$ by [EP, (3.2.6), (3.2.7)]. By [L1, (IX.1.11)], $\beta\cap T$ is a maximal ideal if and only if $\beta\cap\mathcal{O}$ is a maximal ideal. Suppose $\beta\cap T$ is a maximal ideal. Then it is necessarily equal to $\mathfrak{m}'$, since $\mathcal{O}'$ can only contain one maximal ideal of T by [EP, (3.2.7)]. Then β contains $\mathfrak{m}'\mathcal{O}_1=\mathfrak{m}'T_{\mathfrak{m}'}$, which is equal to $\mathcal{M}'$, and hence, $\beta=\mathcal{M}'$, a contradiction. So, it follows that $\beta\cap\mathcal{O}$ is not maximal. □

4.5 Lemma. *Let $\mathcal{O}$ be a valuation ring of F and let $(\mathcal{A},\sigma)$ be an $\mathcal{O}$–algebra with involution. Let $(F^h,\mathcal{O}^h)$ be a Henselisation of $(F,\mathcal{O})$. If $(\mathcal{A}_{F^h},\sigma_{F^h})$ is hyperbolic then there exists a finite subextension $F\subset M\subset F^h$ with the following properties:*

(a) *σ_M is hyperbolic;*

(b) *let v be a valuation on F with valuation ring $\mathcal{O}$ and let $v_1 = v^h|_M, v_2, \ldots, v_m$ be the different valuations on M extending v. Then for $i = 2, \ldots, m$, there exist $n_i \in \mathbb{N}$ such that for all $x \in M$*

$$v(N_{M/F}(x)) = v_1(x) + \sum_{i=2}^{m} n_i v_i(x).$$

Proof. Since $(\mathcal{A}_{F^h}, \sigma_{F^h})$ is hyperbolic, Proposition 3.4 implies the existence of a finite subextension $F \subset L \subset F^h$ such that $(\mathcal{A}_L, \sigma_L)$ is hyperbolic.

Let N/F be the Galois closure of L in F^s. Let v^s be a valuation on F^s extending v. Let $v^h = v^s|_{F^h}$ and $w = v^s|_N$. Since N/F is a Galois extension, all valuation rings of N lying over $\mathcal{O}$ are conjugate to $\mathcal{O}_w$ by [EP, (3.2.15)]. Let $H = \{\tau \in \mathrm{Gal}(N/F) \mid \tau(\mathcal{O}_w) = \mathcal{O}_w\}$ and let M be the fixed field of H. Then $M = N \cap F^h$ (for an explicit argument see the proof of [EP, (5.2.5)]). So, we have that $L \subset M$ and therefore, σ_M is hyperbolic.
Let $\{\rho_1 = \mathrm{id}_N, \rho_2, \ldots, \rho_t\}$ be a set of representatives for the right cosets of $\mathrm{Gal}(N/F)/H$. By Galois theory, the restrictions of the ρ_i to M are exactly the different F–embeddings of M in F^s. We have that $w \circ (\rho_i)|_M$ is a valuation on M with valuation ring $\rho_i^{-1}(\mathcal{O}_w) \cap M$. It is possible that $\rho_i^{-1}(\mathcal{O}_w) \cap M = \rho_j^{-1}(\mathcal{O}_w) \cap M$ for $i \neq j$, but by the proof of [EP, (3.3.1)], it follows that $\mathcal{O}_w \cap M \neq \rho_i^{-1}(\mathcal{O}_w) \cap M$ if $i \neq 1$. This means that $w|_M \neq w \circ (\rho_i)|_M$ if $i \neq 1$. We have that $v_1 = v^h|_M = w|_M$ and $\{v_2, \ldots, v_m\} = \{w \circ (\rho_2)|_M, \ldots, w \circ (\rho_t)|_M\}$, and hence, $v_i \neq v_1$ if $i \neq 1$.

Let $x \in M$. Then $N_{M/F}(x) = x\rho_2(x)\cdots\rho_t(x)$ by definition.

$$v(N_{M/F}(x)) = w(N_{M/F}(x)) = w(x\rho_2(x)\cdots\rho_t(x)) = w(x) + w(\rho_2(x)) + \ldots + w(\rho_t(x)).$$

By the reasoning above, it follows that there exist $n_1, \ldots, n_m \in \mathbb{N}$ such that

$$N_{M/F}(x) = v_1(x) + \sum_{i=2}^{m} n_i v_i(x).$$

□

Let V_{ij} be as in Theorem 4.1 for some pair (i, j), $i \neq j$. Then $v_{ij} : E \to \Gamma_i/\Delta_{ij}; x \mapsto v_i(x) \bmod \Delta_{ij}$ defines a valuation on v_{ij} on E with valuation ring V_{ij}. Let $\gamma_i \in \Gamma_i$ as in Theorem 4.1 and let $b \in E$ be such that $v_i(b) = \gamma_i$. Then $\theta_{ij}(\gamma_i) = v_{ij}(b)$.

4.6 Lemma. *Let $\mathcal{O}_1, \ldots, \mathcal{O}_r$ be different valuation rings of F and let $R = \mathcal{O}_1 \cap \ldots \cap \mathcal{O}_r$. Suppose $x \in F$ is such that $x \in F^{\times 2}\mathcal{O}_i^\times$ for $i = 1, \ldots, r$. Then $x \in F^{\times 2}R^\times$.*

Proof. Let $w_1, \ldots, w_r$ be valuations on F with respective valuation rings $\mathcal{O}_1, \ldots, \mathcal{O}_r$. Denote their respective value groups by $\Gamma_1, \ldots, \Gamma_r$. The hypothesis implies that there exists a tuple $(\gamma_1, \ldots, \gamma_r) \in \Gamma_1 \times \ldots \times \Gamma_r$ such that $w_1(x) = 2\gamma_1, \ldots, w_r(x) = 2\gamma_r$. In the notations of Theorem 4.1, we have that $\theta_{ij}(2\gamma_i) = v_{ij}(x) = v_{ji}(x) = \theta_{ji}(2\gamma_j)$. Since the $\theta_{ij} : \Gamma_i \to \Gamma_i/\Delta_{ij}$ are group homomorphisms and since Γ_i/Δ_{ij} is an ordered abelian group and therefore torsion–free, it follows that $\theta_{ij}(\gamma_i) = \theta_{ji}(\gamma_j)$. Therefore, we can apply Theorem 4.1 to find an element $a \in F$ such that $w_1(a) = \gamma_1, \ldots, w_r(a) = \gamma_r$. Then $w_i(a^2x^{-1}) = 0$ for $i = 1, \ldots, r$, which means that $a^2x^{-1} \in R^\times$. This proves the claim. □

4.7 Proposition. *Suppose R has Krull dimension 1. Let $(\mathcal{A},\sigma)$ be an R−algebra with involution. Let $e \in F^\times$. If for each maximal ideal $\mathfrak{m}$ of R such that $e \notin F^{\times 2}R^\times_{\mathfrak{m}}$, we have that $(\mathcal{A},\sigma)$ becomes hyperbolic over $R/\mathfrak{m}$, then $e \in G((\mathcal{A},\sigma)_F)R^\times$.*

Proof. Let $\mathfrak{m}_1,\dots,\mathfrak{m}_r$ be the maximal ideals of R. For $i = 1,\dots,r$, we denote $R_{\mathfrak{m}_i}$ by $\mathcal{O}_i$. Note that the residue field of $\mathcal{O}_i$ is isomorphic to $R/\mathfrak{m}_i$. Let $w_1,\dots,w_r$ be valuations on F with respective valuation rings $\mathcal{O}_1,\dots,\mathcal{O}_r$. Let $V_e = \{i \in \{1,\dots,r\} \mid e \notin F^{\times 2}\,\mathcal{O}_i^\times\}$.

For each $j \in V_e$, we have that σ becomes hyperbolic over the residue field of $\mathcal{O}_j$ by assumption, and hence, σ becomes hyperbolic over a Henselisation $(F_j^h,\mathcal{O}_j^h)$ of $(F,\mathcal{O}_j)$ by Corollary 1.13. Then there exists a finite subextension $F \subset M_j \subset F_j^h$ with the properties as in Proposition 4.5. Let $\{v_{j1},\dots,v_{jn_j}\}$ be the set of valuations on M_j lying over some w_i, starting with the ones lying over w_j, that is, let $\ell_j \in \{1,\dots,n_j\}$ be such that $v_{j1},\dots,v_{j\ell_j}$ are the valuations lying over w_j, and such that $v_{j1} = w_j^h|_{M_j}$. Since the w_i have rank 1, they are independent, and hence the v_{jk} are pairwise independent by Lemma 4.3, since they all have rank 1 by [EP, (3.2.5)]. By Corollary 4.2, there exists an element $x_j \in M_j$ such that $v_{j1}(x_j) = -w_j(e)$ and $v_{jk}(x_j) = 0$ for $k = 2,\dots,n_j$. Then by Proposition 4.5, we have that $w_j(N_{M_j/F}(x_j)) = v_{j1}(x_j) = -w_j(e)$ and $w_i(N_{M_j/F}(x_j)) = 0$ for $i \neq j$. Let $y = \prod_{j\in V_e} N_{M_j/F}(x_j)$. We have that $w_j(y) = -w_j(e)$ for $j \in V_e$ and $w_j(y) = 0$ for $j \notin V_e$. It follows that $ye \in F^{\times 2}\,\mathcal{O}_j^\times$ for $j = 1,\dots,r$. Then Lemma 4.6 yields that $ye \in F^{\times 2}R^\times$. Since σ becomes hyperbolic over M_j, invoking Proposition 1.21 (a) and (b) yields that $N_{M_j/F}(x_j) \in G((\mathcal{A},\sigma)_F)$. Since $G((\mathcal{A},\sigma)_F)$ is a group and $F^{\times 2} \subset G((\mathcal{A},\sigma)_F)$, it follows that $e \in G((\mathcal{A},\sigma)_F)R^\times$. □

4.8 Theorem. *Suppose R has finite Krull dimension. Let $(\mathcal{A},\sigma)$ be an R−algebra with involution. Let $e \in F^\times$. The following are equivalent.*

(i) $e \in G((\mathcal{A},\sigma)_F)R^\times$.

(ii) For each valuation ring $\mathcal{O}$ of F containing R, such that $e \notin F^{\times 2}\,\mathcal{O}^\times$, we have that $(\mathcal{A},\sigma)$ becomes hyperbolic over the residue field of $\mathcal{O}$.

Proof. That (i) implies (ii) follows from Proposition 1.19.

Let $\mathfrak{m}_1,\dots,\mathfrak{m}_r$ be the maximal ideals of R. For $i = 1,\dots,r$, we denote $R_{\mathfrak{m}_i}$ by $\mathcal{O}_i$. Let $w_1,\dots,w_r$ be valuations on F with respective valuation rings $\mathcal{O}_1,\dots,\mathcal{O}_r$. Denote their respective value groups by $\Gamma_1,\dots,\Gamma_r$. We will show that (ii) implies (i) by induction on the sum of the ranks of the $\mathcal{O}_i$. We will denote this sum by m.

We have that $m \geqslant r$, since $\mathcal{O}_1,\dots,\mathcal{O}_r$ all have rank at least 1. If $r = m$, then they all have rank 1, and hence R has Krull dimension 1. Proposition 4.7 then yields the statement.

Suppose now $r > m$. Then at least one of the $\mathcal{O}_i$ has rank bigger than 1. Without loss of generality, we may assume that the rank of $\mathcal{O}_1$ is bigger than 1. Let V_1 be the smallest overring of $\mathcal{O}_1$ in F strictly bigger than $\mathcal{O}_1$ (which exists since $\mathcal{O}_1$ has finite rank). Then $\mathrm{rank}(V_1) < \mathrm{rank}(O_1)$. Let $R' = V_1 \cap (\cap_{i=2}^m \mathcal{O}_i)$. Since $\mathrm{rank}(V_1) + \sum_{i=2}^r \mathrm{rank}(\mathcal{O}_i) < m$ and $R \subset V_1$, by induction we have that $e \in G((\mathcal{A},\sigma)_F)u$, for some $u \in R'^\times$. Put $v = u^{-1}$, then $ev \in G((\mathcal{A},\sigma)_F)$. Let $g \in \mathcal{A}_F^\times$ be such that $ev = \sigma_F(g)g$. We have that $w_i(v) = 0$ for $i = 2,\dots,r$. If $w_1(v) \in 2\Gamma_1$ we are done, since then, by Lemma 4.6, $v \in R^\times F^{\times 2}$ and $e \in G((\mathcal{A},\sigma)_F)R^\times$ follows. So suppose $w_1(v) \notin 2\Gamma_1$. Let $(F_1^h, \mathcal{O}_1^h)$ be a Henselisation of $(F, \mathcal{O}_1)$. If $w_1(e) \in 2\Gamma_1$ then $w_1(\sigma_F(g)g) \notin 2\Gamma_1$. Then σ becomes hyperbolic over F_1^h by Proposition 1.19. If $w_1(e) \notin 2\Gamma_1$ then σ is hyperbolic over the residue field of $\mathcal{O}_1$ by assumption, and hence σ is hyperbolic over F_h^1 by Corollary 1.13. In any case, there exists a finite field extension $F \subset M_1 \subset F_1^h$ with the properties as in Proposition 4.5. Let $\{v_{11}, \dots, v_{1n}\}$ be the set of valuations on M_1 lying over some w_i, starting with the ones lying over w_1, that is, let $\ell \in \{1, \dots, n\}$ be such that $v_{11}, \dots, v_{1\ell}$ are the valuations lying over w_1, and such that $v_{11} = w_1^h|_{M_1}$. For $k = 2, \dots, n$, let V_{1k} be the smallest overring of $\mathcal{O}_{v_{11}}$ and $\mathcal{O}_{v_{1k}}$ and let Δ_{1k} be the corresponding convex subgroup of $\Gamma_1 \cong \Gamma_{v_{11}}$ such that the value group of V_{1k} is Γ_1/Δ_{1k}. We have that $V_{12}, \dots, V_{1n}$ are linearly ordered. Without loss of generality, we may assume that $V_{12} = \cap_{k=2}^n V_{1k}$. It follows that $\Delta_{12} = \cap_{k=2}^n \Delta_{1k}$. Let β be the maximal ideal of V_{12}. Then $V_{12} = (\mathcal{O}_{v_{11}})_\beta$ and hence β is a prime ideal of $\mathcal{O}_{v_{11}}$ that is not maximal. Let $\tilde{V} = V_{12} \cap F$. Then $\tilde{V}$ is a valuation ring of F containing $\mathcal{O}_1$. We have that $\beta \cap \tilde{V}$ is the maximal ideal of $\tilde{V}$. Furthermore, it is clear that $\beta \cap \tilde{V} = \beta \cap \mathcal{O}_1$. Hence, $\tilde{V} = (\mathcal{O}_1)_{\beta \cap \mathcal{O}_1}$ and since $\beta \cap \mathcal{O}_1$ is a prime ideal of $\mathcal{O}_1$ that is not maximal by Lemma 4.4, we have that $\mathcal{O}_1 \subsetneq \tilde{V}$. Let $\tilde{\Delta}$ be the convex subgroup of Γ_1 corresponding to $\tilde{V}$, that is, such that the value group of $\tilde{V}$ is isomorphic to $\Gamma_1/\tilde{\Delta}$. Then by [EP, (2.3.1)], we have that $\Delta_{12} \subset \tilde{\Delta}$. By [EP, (3.2.5)], $\tilde{V}$ and V_{12} have the same rank. This implies that $\tilde{\Delta} = \Delta_{12}$.

Since V_1 is the smallest overring of $\mathcal{O}_1$, we have that $V_1 \subset \tilde{V}$. Let $\tilde{\Delta}$ be the convex subgroup of Γ_1 corresponding to V_1, then this means that $\tilde{\Delta} \subset \Delta_{12}$. Since $v \in V_1^\times$, we get that $w_1(v) \in \tilde{\Delta} \subset \Delta_{12}$. Then we can apply Theorem 4.1 to find an element $x_1 \in M_1$ with $v_{1k}(x_1) = -w_1(v)$ for $k = 1$ and $v_{1k}(x_1) = 0$ otherwise. Proposition 4.5 then implies that $w_i(N_{M_1/F}(x_1)) = -w_1(v)$ for $i = 1$ and $w_i(N_{M_1/F}(x_1)) = 0$ otherwise. It follows that $vN_{M_1/F}(x_1) \in R^\times$. Furthermore, since σ is hyperbolic over M_1, Proposition 1.21 (a) yields that $x_1 \in G(\mathcal{A}_{M_1}, \sigma_{M_1})$ and hence $N_{M_1/F}(x_1) \in G((\mathcal{A},\sigma)_F)$ by Proposition 1.21 (b). Then $evN_{M_1/F}(x_1) \in G((\mathcal{A},\sigma)_F)$, since $ev \in G((\mathcal{A},\sigma)_F)$ and $G((\mathcal{A},\sigma)_F)$ is a group. It follows that $e \in G((\mathcal{A},\sigma)_F)R^\times$, as desired. □

4.9 Remarks.

(a) As in the Noetherian case, see Remark 3.6, the properties (i) and (ii) of Theorem 4.8 both hold for trivial reasons if $Z(\mathcal{A})$ is not a domain. The reasoning is the same.

(b) Note that, whereas it suffices to impose the hyperbolicity condition on the localisations of R at its maximal ideals in Proposition 4.7, we need to impose the condition on all valuation rings containing R in Theorem 4.8, for the induction proof to work.

The above theorem implies that if R has finite Krull dimension, algebras with involution that are rationally isomorphic, through a central isomorphism, are isomorphic.

4.10 Proposition. *Suppose R has finite Krull dimension. Let $((\mathcal{A},\sigma),(\mathcal{A},\sigma'))$ be a pair of R-algebras with involution and assume $Z(\mathcal{A})$ is a domain. If $(\mathcal{A},\sigma)_F \cong_F (\mathcal{A},\sigma')_F$ through a $Z(\mathcal{A}_F)$-isomorphism, then $(\mathcal{A},\sigma) \cong_R (\mathcal{A},\sigma')$ through a $Z(\mathcal{A})$-isomorphism.*

Proof. By Proposition 1.17, there exists an element $s \in \mathcal{A}^\times$ such that $\sigma(s) = s$ and $\sigma' = \mathrm{Int}(s) \circ \sigma$. By Proposition 1.16, there exist elements $e \in F^\times$ and $g \in \mathcal{A}_F^\times$ such that $es = \sigma_F(g)g$. Combining Proposition 1.19, Theorem 4.8 and Corollary 1.22 yields that $(\mathcal{A},\sigma) \cong_R (\mathcal{A},\sigma')$, through a $Z(\mathcal{A})$-isomorphism. □

Note that, in view of Propositions 1.14 and 1.15, the above already implies that if R has finite Krull dimension, then R has the RIII property. We can show more generally that R also has the RIII property if it does not necessarily have finite Krull dimension, by reducing to the case of finite Krull dimension.

4.11 Proposition. *Let F_0 be the prime field of F. Let $(\mathcal{A},\sigma)$ be an R–algebra with involution. Then there exists a finitely generated field extension $\tilde{F}/F_0$ and an $(R\cap\tilde{F})$–algebra with involution $(\tilde{\mathcal{A}},\tilde{\sigma})$ such that $R\cap\tilde{F}$ is a semilocal Bézout domain of finite Krull dimension and $(\mathcal{A},\sigma)\cong_R(\tilde{\mathcal{A}},\tilde{\sigma})_R$.*

Proof. Since R is a semilocal Bézout domain, $\mathcal{A}$ is free as an R–module by Proposition 1.6. Let $(e_1,\dots,e_n)$ be an R–basis for $\mathcal{A}$. Then it is an F–basis for $\mathcal{A}_F$. In fact, by [K, (I.1.3.5)], R is a direct summand of $\mathcal{A}$, and hence, we may assume that $e_1=1$. For $i,j,k=1,\dots,n$, let $\varepsilon_{ijk}\in R$ be such that $e_ie_j=\sum_{k=1}^n e_k\varepsilon_{ijk}$. By [K, (III.5.1.2)], since $\mathcal{A}$ is separable over R, there exists an idempotent $x\in\mathcal{A}\otimes_R\mathcal{A}^{\mathrm{op}}$ such that under the map $m:\mathcal{A}\otimes_R\mathcal{A}^{\mathrm{op}}\to\mathcal{A}; a\otimes b\mapsto ab$, we have that $m(x)=1$ and $(a\otimes 1)x=(1\otimes a)x$, for all $a\in\mathcal{A}$. We write $x=\sum_{i=1}^{\ell}x_i\otimes y_i$ and $x_i=\sum_{k=1}^n e_kx_{ki}$ and $y=\sum_{k=1}^n e_ky_{ki}$. Furthermore, for $i,j=1,\dots,n$, there exist elements $\alpha_{ij}\in R$ such that $\sigma(e_i)=\sum_{j=1}^n e_j\alpha_{ij}$. Suppose that $Z(\mathcal{A})$ is a free separable quadratic R–algebra. By Proposition 1.1, $Z(\mathcal{A})=R[z]$, for some $z\in Z(\mathcal{A})$ satisfying $z^2=az+b$, with $a,b\in R$. If the polynomial $f(x)=x^2-ax-b\in R[x]$ is reducible, let y be a root of $f(x)$ in R. Let $\tilde{F}$ be the field obtained from F_0 by adjoining the elements $\alpha_{ij},\varepsilon_{ijk},x_{ki},y_{ki}$, and if $Z(\mathcal{A})\neq R$ by also adjoining a,b, and furthermore if $f(x)$ is reducible, we also adjoin y. Then $\tilde{F}$ is finitely generated over F_0. Let $\tilde{R}=R\cap\tilde{F}$ and $R_0=R\cap F_0$. Since R is a semilocal Bézout domain, it is the intersection of finitely many valuation rings of F. Then $\tilde{R}$ is the intersection of finitely many valuation rings of $\tilde{F}$ and is therefore a semilocal Bézout domain with fraction field $\tilde{F}$. Similarly, R_0 is a semilocal Bézout domain with fraction field F_0.

We define an algebra $\tilde{\mathcal{A}}$ over $\tilde{R}$ as follows. We let $\tilde{\mathcal{A}}$ be the free $\tilde{R}$–module with basis $(e_1,\dots,e_n)$. Then $\tilde{\mathcal{A}}$ is multiplicatively closed in $\mathcal{A}$, since all $\varepsilon_{ijk}\in\tilde{R}$ by construction. Furthermore, σ restricts to an involution on $\tilde{\mathcal{A}}$, which we denote by $\tilde{\sigma}$. By construction, we have that $x\in\tilde{\mathcal{A}}\otimes_{\tilde{R}}\tilde{\mathcal{A}}^{\mathrm{op}}$, and hence $\tilde{\mathcal{A}}$ is separable over $\tilde{R}$ by [K, (III.5.1.2)]. Furthermore, we have that $Z(\tilde{\mathcal{A}})=Z(\mathcal{A})\cap\tilde{\mathcal{A}}$. Suppose first $Z(\mathcal{A})=R$ and let $c\in R\cap\tilde{A}$. Then there exist $c_1,\dots,c_n\in\tilde{R}$ such that $c=\sum_{i=1}^n e_ic_i$. Considering c as element of $\mathcal{A}$ and using that $e_1=1$, it follows that $c_1=c$ and it follows that $c\in\tilde{R}$. Hence, $Z(\tilde{\mathcal{A}})=\tilde{R}$.
Suppose $Z(\mathcal{A})=R[z]$. Since $z\in\tilde{\mathcal{A}}$ by construction of $\tilde{F}$, it is clear that $\tilde{R}[z]\subset Z(\tilde{\mathcal{A}})$. Suppose $c+dz\in\tilde{\mathcal{A}}$, with $c,d\in R$. By construction of $\tilde{F}$, there exist $z_1,\dots,z_n\in\tilde{R}$ such that $z=\sum_{i=1}^n e_iz_i$. Since $z\notin R$, at least one of $z_2,\dots,z_n$ is nonzero, say z_2. Then $c+dz=(c+dz_1)+\sum_{i=2}^n e_i(dz_i)$, and it follows that $c+dz_1,dz_2,\dots,dz_n\in\tilde{R}$. Since $z_2\in\tilde{R}$, it follows that $d\in\tilde{F}$, and then also $c\in\tilde{F}$. Hence, $c+dz\in\tilde{F}(z)\cap\tilde{\mathcal{A}}$. Since $\tilde{\mathcal{A}}$ is finite–dimensional over $\tilde{R}$, it is integral over $\tilde{R}$. So, $\tilde{F}(z)\cap\tilde{\mathcal{A}}$ is contained in the integral closure of $\tilde{R}$ in $\tilde{F}(z)$. Suppose $\tilde{R}[z]$ is a domain. Since 2 and the discriminant of $f(x)$ are units in $\tilde{R}$ by construction of $\tilde{F}$ and $\tilde{R}$ is integrally closed in $\tilde{F}$, a standard argument shows that $\tilde{R}[z]$ is the integral closure of $\tilde{R}$ in $\tilde{F}(z)$.
Suppose $\tilde{R}[z]$ is not a domain. By construction of $\tilde{F}$, $f(x)$ is then reducible in $\tilde{R}[x]$ (and separable), which implies that $\tilde{F}(z)\cong\tilde{F}\times\tilde{F}$ and $\tilde{R}[z]\cong\tilde{R}\times\tilde{R}$, and $\tilde{R}\times\tilde{R}$ is the integral closure of $\tilde{R}$ in $\tilde{F}\times\tilde{F}$. So, we get $Z(\tilde{A})=\tilde{R}[z]$.

The above shows that $(\tilde{\mathcal{A}},\tilde{\sigma})$ is a $\tilde{R}$–algebra with involution and it is clear that $(\mathcal{A},\sigma) \cong_R (\tilde{\mathcal{A}},\tilde{\sigma})_R$.

It remains to show that $\tilde{R}$ has finite Krull dimension. By Proposition 1.7, it suffices to show that $\tilde{R}$ is an intersection of valuation rings of $\tilde{F}$ of finite rank. If F_0 is a finite field, then F_0 only has one valuation ring, namely itself, and hence $R_0 = F_0$. If $F_0 = \mathbb{Q}$ then [EP, 2.1.4] yields that any non–trivial valuation ring of $\mathbb{Q}$ is discrete. Since R_0 is the intersection of finitely many valuation rings of F_0, it therefore has finite Krull dimension . We have that $\tilde{R}$ is the intersection of valuation rings of $\tilde{F}$ lying over valuation rings of F_0. Let $\tilde{\mathcal{O}}$ be a valuation ring of $\tilde{F}$ and $\mathcal{O}' = \tilde{\mathcal{O}} \cap F_0$. Then, by [EP, 3.4.4],

$$\mathrm{rank}(\tilde{\mathcal{O}}) \leqslant \mathrm{tr.deg}(\tilde{F}/F_0) + \mathrm{rank}(\mathcal{O}').$$

Since the right hand side is finite, it follows that $\tilde{\mathcal{O}}$ has finite rank. Hence, $\tilde{R}$ has finite Krull dimension. □

4.12 Proposition. *Let $((\mathcal{A},\sigma),(\mathcal{A},\sigma'))$ be a pair of R–algebras with involution and assume $Z(\mathcal{A})$ is a domain. If $(\mathcal{A},\sigma)_F \cong_F (\mathcal{A},\sigma')_F$ through a $Z(\mathcal{A}_F)$–isomorphism, then $(\mathcal{A},\sigma) \cong_R (\mathcal{A},\sigma')$ through a $Z(\mathcal{A})$-isomorphism.*

Proof. By Proposition 1.17, there exists an element $s \in \mathcal{A}^\times$ such that $\sigma(s) = s$ and $\sigma' = \mathrm{Int}(s) \circ \sigma$. By Proposition 1.16, the isomorphism $(\mathcal{A},\sigma)_F \cong_F (\mathcal{A},\sigma')_F$ yields the existence of elements $e \in F^\times$ and $g \in \mathcal{A}_F^\times$ such that $es = \sigma_F(g)g$. Let F_0 be the prime field of F and let $\tilde{F},\tilde{\mathcal{A}},\tilde{\sigma}$ be as in Proposition 4.11. Let $(e_1,\dots,e_n)$ be an R–basis for $\mathcal{A}$. There exist $s_1,\dots,s_n \in R$ and $g_1,\dots,g_n \in F$ such that $s = \sum_{i=1}^n e_i s_i$ and $g = \sum_{i=1}^n e_i g_i$. We adjoin $s_1,\dots,s_n,g_1,\dots,g_n$ to $\tilde{F}$. This field we obtain is still a finitely generated field extension of F_0. By abuse of notation, we denote it again by $\tilde{F}$. Let $\tilde{R} = R \cap \tilde{F}$. Then $\tilde{R}$ is a semilocal Bézout domain with fraction field $\tilde{F}$.

By construction of $\tilde{F}$, the equation $es = \sigma_F(g)g$ is already defined in $\tilde{\mathcal{A}}_{\tilde{F}}$. Furthermore, σ' restricts to an involution on $\tilde{\mathcal{A}}$, which we denote by $\tilde{\sigma'}$. Then, by Proposition 1.16, we have that $(\tilde{\mathcal{A}},\tilde{\sigma})_{\tilde{F}} \cong_{\tilde{F}} (\tilde{\mathcal{A}},\tilde{\sigma'})_{\tilde{F}}$, through a $Z(\tilde{\mathcal{A}}_F)$–isomorphism. Since $\tilde{R}$ has finite Krull dimension by Proposition 4.11, Proposition 4.10 yields that $(\tilde{\mathcal{A}},\tilde{\sigma}) \cong_{\tilde{R}} (\tilde{\mathcal{A}},\tilde{\sigma'})$, through a $Z(\tilde{\mathcal{A}})$–isomorphism, and scalar extension to R then yields that $(\mathcal{A},\sigma)$ and $(\mathcal{A},\sigma')$ are $Z(\mathcal{A})$–isomorphic.

4.13 Theorem. *R has the RIII property.*

Proof. Let $((\mathcal{A},\sigma),(\mathcal{A}',\sigma'))$ be a pair of R–algebras with involution. Suppose $(\mathcal{A},\sigma)_F \cong_F (\mathcal{A}',\sigma')_F$. By Proposition 1.14 we may assume that $\mathcal{A}' = \mathcal{A}$ and that $(\mathcal{A},\sigma)_F \cong_F (\mathcal{A}',\sigma')_F$ via a $Z(\mathcal{A}_F)$–isomorphism. Furthermore, we may assume $Z(\mathcal{A})$ is a domain by Proposition 1.15. The theorem then follows from Proposition 4.12 □

4.14 Corollary. *Let (Δ,θ) be an R–algebra with involution without zero divisors. Let (V,h) and (V',h') be ε–hermitian spaces over (Δ,θ). Suppose there exists $e \in F^\times$ such that $h'_F \simeq_F eh_F$. Then there exists $u \in R^\times$ such that $h' \simeq_R uh$.*

Proof. Since by assumption, h'_F and h_F are similar, it follows from Propostion 1.18, applied in the case $R = F$, that $\mathrm{Ad}(h_F) \cong_F \mathrm{Ad}(h'_F)$ through a $Z(\Delta_F)$–isomorphism. It follows from Proposition 4.12 that $\mathrm{Ad}(h) \cong_R \mathrm{Ad}(h')$, through a $Z(\Delta)$–isomorphism. Applying the other implication given by Proposition 1.18, we obtain that there exists $u \in R^\times$ such that $h' \simeq_R uh$. □

4.15 Remark. This corollary states that "rational similarity implies similarity" for ε–hermitian spaces over (Δ, θ). As a special case of the above corollary, namely taking $\Delta = R$ and $\theta = \mathrm{id}$, we obtain that rationally similar non–singular symmetric or skew–symmetric bilinear spaces are similar.

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