ISOTROPY OF PRODUCTS OF QUADRATIC FORMS OVER FIELD EXTENSIONS

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Abstract. The isotropy of products of Pfister forms is studied. In particular, an improved lower bound on the value of their first Witt index is obtained. This result and certain of its corollaries are applied to the study of the weak isotropy index (or equivalently, the sublevel) of arbitrary quadratic forms. The relationship between this invariant and the level of the form is investigated. The problem of determining the set of values of the weak isotropy index of a form as it ranges over field extensions is addressed, with both admissible and inadmissible integers being determined. An analogous investigation with respect to the level of a form is also undertaken. An examination of the weak isotropy index and the level of round and Pfister forms concludes this article.

1. Introduction

We study the isotropy of products of quadratic forms over field extensions in distinct, but interrelated, contexts, treating of the isotropy of products of Pfister forms over their function fields and the behaviour of the weak isotropy index and level of forms over field extensions.

Given the central role of Pfister forms within the theory of quadratic forms, the isotropy behaviour of their products with other forms has been a topic of long-standing interest. In particular, given quadratic forms π and q over a field F such that π is similar to an anisotropic Pfister form, in the case where the product of π and q is isotropic over F, it has long been known that the Witt index of this product is a multiple of the dimension of π. Thus, if the product of π and q is anisotropic over F, it follows that the Witt index of this product over the generic extension that makes it isotropic (its first Witt index) is at least the dimension of π.

The main result in the opening section of this article, Theorem 2.6, improves upon this lower bound, establishing that the first Witt index of the product of π and q is at least the first Witt index of q times the dimension of π. Whereas this bound is not always attained (as per Example 2.8), we can establish attainment for certain forms over ordered fields (Proposition 2.9) and for forms with maximal splitting (Proposition 2.10). As corollaries, we can show that the maximal splitting property is preserved under multiplication by Pfister forms (Corollary 2.11) and certain Pfister neighbours (Corollary 2.12).

We next consider the isotropy of the product of a form q with forms of type $\langle 1, \ldots, 1\rangle$, that is, the isotropy of the orthogonal sum of a number of copies of q. Given a form q over F, the *weak isotropy index of* q, denoted $wi(q)$, as introduced by Becher in [B], is the least number n such that the orthogonal sum of n copies of q is isotropic over F. If no such n exists, then $wi(q)$ is infinite. In the third, fourth and sixth sections of this article, we study the behaviour of this invariant. We formulate these results in terms of the *q-sublevel of* F, denoted $\underline{s}_q(F)$, defined by Berhuy, Grenier-Boley and Mahmoudi in [BG-BM], bearing in mind the relation $\underline{s}_q(F) = wi(q) - 1$. Having established relationships between $\underline{s}_q(F)$ and a number of other invariants in

Section 3, we devote the fourth section to studying the possible values of this invariant as it ranges over field extensions. In particular, for q a form over some ground field F, we seek to determine the entries of the set $\{\underline{s}_q(K) \mid K/F \text{ field extension}\}$. Drawing upon known results regarding isotropy over function fields of forms, we establish criteria for the containment of integers within this set, and determine entries without placing restrictions on q (see Proposition 4.1). Restricting to forms q of a specific type, we identify further entries (see Proposition 4.7 and Theorem 4.8) and, indeed, in certain cases obtain a complete determination of the set. In particular, in Corollary 4.9 we establish the existence of forms q over ordered fields F that $\{0, \ldots, \underline{s}_q(F)\} = \{\underline{s}_q(K) \mid K/F \text{ field extension}\}$. In the complementary direction, for all forms q satisfying a certain condition, which cannot be relaxed in general, we apply our aforementioned lower bound on the first Witt index of multiples of Pfister forms to show that all the integers contained in certain intervals do not belong to the set $\{\underline{s}_q(K) \mid K/F \text{ field extension}\}$ (see Theorem 4.12). Moreover, these intervals of inadmissible integers cannot, in general, be extended in either direction.

Letting R be a non-trivial ring, the *level of* R, denoted $s(R)$, is the least positive integer n such that -1 is a sum of n squares in R if such an n exists, and is infinite otherwise. Interest in this invariant first arose on account of the Artin-Schreier theorem, which states that a field F has an ordering if and only if $s(F) = \infty$, and its behaviour with respect to various classes of rings continues to be a topic of study. In [BG-BM], Berhuy, Grenier-Boley and Mahmoudi introduced the concept of the level of a field with respect to a form q, or the q-level, as a generalisation of the level, and undertook a wide-ranging investigation of this invariant. For q a form over F, the *q-level of* F, denoted $s_q(F)$, is the least $n \in \mathbb{N}$ such that the orthogonal sum of n copies of q represents -1 over F, and is infinite if no such n exists. Continuing on from the investigations undertaken in [BG-BM], we devote Section 5 to considerations of the behaviour of this invariant with respect to field extensions, seeking a determination of the set $\{s_q(K) \mid K/F \text{ field extension}\}$. In this regard, we establish analogues of our results with respect to the q-sublevel. In particular, for all $n \in \mathbb{N}$, we establish the existence of an n-dimensional form q over an ordered field that can attain any prescribed positive integer as its q-level over a suitable extension (see Theorem 5.3).

In the final section of this article, we establish bounds on the q-sublevel and q-level of Pfister neighbours. Moreover, we compare and contrast the behaviour of the q-sublevel and q-level with respect to round and Pfister forms, classes of forms for which these invariants are known to coincide. For q a Pfister form, we conclude with a treatment of the q-sublevel with respect to function fields of quadratic forms, which generalises and strengthens certain results established in [BG-BM] with respect to quadratic field extensions.

Henceforth, we will let F denote a field of characteristic different from two (indeed, if $\mathrm{char}(F) = 2$ then every anisotropic quadratic form q over F satisfies $\underline{s}_q(F) = 1$). The term "form" will refer to a regular quadratic form. Every form over F can be diagonalised. Given $a_1, \ldots, a_n \in F^\times$ for $n \in \mathbb{N}$, one denotes by $\langle a_1, \ldots, a_n\rangle$ the n-dimensional quadratic form $a_1X_1^2 + \ldots + a_nX_n^2$. If p and q are forms over F, we denote by $p \perp q$ their orthogonal sum and by $p \otimes q$ their tensor product. For $n \in \mathbb{N}$, we will denote the orthogonal sum of n copies of q by $n \times q$. We use aq to denote $\langle a\rangle \otimes q$ for $a \in F^\times$. We write $p \simeq q$ to indicate that p and q are isometric, and say that p and q are *similar* (over F) if $p \simeq aq$ for some $a \in F^\times$. For q a form over F and K/F a field extension, we will employ the notation q_K when viewing q as a form over K via the canonical embedding. A form p is a *subform of* q if $q \simeq p \perp r$ for some form r, in which case we will write $p \subset q$. A form q *represents* $a \in F$ if there exists a vector v such that $q(v) = a$. We denote by $D_F(q)$ the set of values in $F^\times$ represented by q. A form over F is *isotropic* if it represents zero non-trivially, and *anisotropic* otherwise. A form q over F is *universal* if $D_F(q) = F^\times$. In particular, isotropic forms are universal [L, Theorem I.3.4]. Every form q has a decomposition $q \simeq q_{\mathrm{an}} \perp i_W(q) \times \langle 1, -1\rangle$ where the anisotropic form q_{an} and the integer $i_W(q)$ are uniquely determined. A form q is *hyperbolic* if q_{an} is trivial, whereby $i_W(q) = \frac{1}{2}\dim q$. Two anisotropic forms p and q over F are *isotropy equivalent* if for every field extension K/F we have that p_K is isotropic if and only if q_K is isotropic. The following basic fact (see [L, Exercise I.16]) will be employed frequently.

Lemma 1.1. *If $\tau \subset \varphi$ with $\dim \tau \geqslant \dim \varphi - i_W(\varphi) + 1$, then τ is isotropic.*

An *ordering of F* is a set $P \subset F^\times$ such that $P \cup -P = F^\times$ and $x + y, xy \in P$ for all $x, y \in P$. We will let X_F denote the space of orderings of F. If X_F is non-empty, we say that F is a *formally real field*. For $a \in F^\times$ a sum of squares in $F^\times$, denoted $a \in \sum {F^\times}^2$, the *length of a*, $\ell_F(a)$, is the least number of squares in $F^\times$ that sum to a (we set $\ell_F(a) = \infty$ if $a \notin \sum {F^\times}^2$). The *Pythagoras number of F* is $p(F) = \sup\{\ell_F(a) \mid a \in \sum {F^\times}^2\}$. Given a form q over F and an ordering $P \in X_F$, the *signature of q at P*, denoted $\mathrm{sgn}_P(q)$, is the number of coefficients in a diagonalisation of q that are in P minus the number that are not in P. A form q over F is *indefinite at $P \in X_F$* if $|\mathrm{sgn}_P(q)| < \dim q$. For F a field without orderings, the *u-invariant of F* is $u(F) = \sup\{\dim q \mid q \text{ is an anisotropic form over } F\}$. For $n, m \in \mathbb{N}$, we will often invoke the identity $\left\lceil \frac{n}{m} \right\rceil = \left\lfloor \frac{n-1}{m} \right\rfloor + 1$.

For $n \in \mathbb{N}$, an *n-fold Pfister form* over F is a form isometric to $\langle 1, a_1\rangle \otimes \ldots \otimes \langle 1, a_n\rangle$ for some $a_1, \ldots, a_n \in F^\times$ (the form $\langle 1\rangle$ is the 0-fold Pfister form). Isotropic Pfister forms are hyperbolic [L, Theorem X.1.7]. A form τ over F is a *neighbour* of a Pfister form π if $\tau \subset a\pi$ for some $a \in F^\times$ and $\dim \tau > \frac{1}{2}\dim \pi$. An anisotropic form q is isotropy equivalent to a Pfister form π if and only if q is a neighbour of π [H, Proposition 2]. A form q over F is a *group form* if $D_F(q)$ is a subgroup of $F^\times$. A form q over F is *round* if $D_F(q) = \{a \in F^\times \mid aq \simeq q\}$, the group of similarity factors of q. Pfister forms are round (see [L, Theorem X.1.8]). Indeed, Witt's Round Form Theorem [L, Theorem X.1.14] states that the product of a Pfister form and a round form is round.

For a form q over F with $\dim q = n \geqslant 2$ and $q \not\simeq \langle 1, -1\rangle$, the *function field $F(q)$ of q* is the quotient field of the integral domain $F[X_1, \ldots, X_n]/(q(X_1, \ldots, X_n))$ (this is the function field of the affine quadric $q(X) = 0$ over F). To avoid case distinctions, we set $F(q) = F$ if $\dim q \leqslant 1$ or $q \simeq \langle 1, -1\rangle$. The integer $i_W(q_{F(q)})$ (which is positive for all forms q of dimension greater than one) is called the *first Witt index of q*, and is denoted by $i_1(q)$. For all forms p over F and all extensions K/F such that q_K is isotropic, we have that $i_W(p_{F(q)}) \leqslant i_W(p_K)$ (see [Kn, Proposition 3.1 and Theorem 3.3]). In particular, we have that $i_1(q) \leqslant i_W(q_K)$ for all extensions K/F such that q_K is isotropic. An anisotropic form q is said to have *maximal splitting* if $\dim q - i_1(q)$ is a power of two. As per [L, Theorem X.4.1], $F(q)$ is a purely-transcendental extension of F if and only if q is isotropic over F. On account of this fact, one can see that two anisotropic forms p and q over F are *isotropy equivalent* if and only if $p_{F(q)}$ and $q_{F(p)}$ are isotropic. The behaviour of orderings with respect to function field extensions is governed by the following result due to Elman, Lam and Wadsworth [ELW, Theorem 3.5] and, independently, Knebusch [GS, Lemma 10].

Theorem 1.2. *Let q be a form over a formally real field F such that $\dim q \geqslant 2$. Then $P \in X_F$ extends to $F(q)$ if and only if q is indefinite at P.*

[H, Theorem 1] and [KM, Theorem 4.1] represent important isotropy criteria for function fields of quadratic forms. We will regularly invoke these results throughout this article, and therefore recall them below.

Theorem 1.3. (*Hoffmann*) *Let p and q be forms over F such that p is anisotropic. If $\dim p \leqslant 2^n < \dim q$ for some integer $n \geqslant 0$, then $p_{F(q)}$ is anisotropic.*

Theorem 1.4. (*Karpenko, Merkurjev*) *Let p and q be anisotropic forms over F such that $p_{F(q)}$ is isotropic. Then*

(*i*) $\dim p - i_1(p) \geqslant \dim q - i_1(q)$;
(*ii*) $\dim p - i_1(p) = \dim q - i_1(q)$ *if and only if $q_{F(p)}$ is isotropic.*

2. The isotropy of multiples of Pfister forms

Since the isotropy of scalar multiples of Pfister forms is well understood (indeed, an anisotropic form q of dimension at least two is a scalar multiple of a Pfister form if and only if q is hyperbolic over $F(q)$, see [EKM, Corollary 23.4]), we will restrict our attention to products of Pfister forms with forms of dimension at least two. In his thesis [R, Théorème 6.4.2], Roussey established the following.

Theorem 2.1. (*Roussey*) *Let p and q be two forms over F of dimension at least two and let π be similar to a Pfister form over F. If p is isotropic over $F(q)$, then $\pi \otimes p$ is isotropic over $F(\pi \otimes q)$.*

In a similar vein to the above, we note that the corresponding result with respect to hyperbolicity also holds, having been established by Fitzgerald [F, Theorem 3.2].

The following example, communicated to me by Karim Becher, demonstrates that the converse of the above theorem does not hold in general.

Example 2.2. Let $q \simeq \langle 1,1,1,7\rangle$ and $\pi \simeq \langle 1,1,1,1\rangle$ over $F = \mathbb{Q}$. Since $\det q \notin \mathbb{Q}^2$, the form q is not similar to a 2-fold Pfister form. Thus, $i_1(q) = 1$ by [EKM, Corollary 23.4]. Hence, Theorem 1.4 (*i*) implies that $\langle 1,1,1\rangle$ is anisotropic over $\mathbb{Q}(q)$. As $7 \in D_{\mathbb{Q}}(\pi)$, we have that $7\langle 1,1,1,1\rangle \simeq \langle 1,1,1,1\rangle$, and thus that $\pi \otimes q \simeq 16 \times \langle 1\rangle$. Since $\pi \otimes \langle 1,1,1\rangle$ is a Pfister neighbour of $16 \times \langle 1\rangle$, we have that $\pi \otimes \langle 1,1,1\rangle$ is isotropic over $\mathbb{Q}(\pi \otimes q)$.

While the converse of Theorem 2.1 does not generally hold, we can establish it in a restricted setting.

Proposition 2.3. *Let p and q be forms of dimension at least two such that $\pi \otimes p$ and $\pi \otimes q$ are anisotropic over F, where π is similar to a Pfister form. Suppose that q has maximal splitting and that q is isotropic over $F(p)$. If $\pi \otimes p$ is isotropic over $F(\pi \otimes q)$, then p is isotropic over $F(q)$.*

Proof. Since q has maximal splitting, it follows that $\dim q - i_1(q) = 2^n$ for some integer $n \geqslant 0$. As $\pi \otimes p$ is isotropic over $F(\pi \otimes q)$, Theorem 1.3 enables us to conclude that $\dim p > 2^n$. Since q is isotropic over $F(p)$, Theorem 1.4 (*i*) implies that $\dim p - i_1(p) = 2^n$, whereby Theorem 1.4 (*ii*) implies that p is isotropic over $F(q)$. □

The following classical result regarding the isotropy of multiples of Pfister forms is well known. This result, as formulated below, is a consequence of a theorem of Elman and Lam (see [EL, Theorem 1.4]). Wadsworth and Shapiro [WS, Theorem 2] established that it holds, more generally, for multiples of round forms.

Theorem 2.4. *Let π be an anisotropic Pfister form over F and let q be another form over F. If $\pi \otimes q$ is isotropic, then there exist forms q_1 and q_2 over F such that $\pi \otimes q_1$ is anisotropic, q_2 is hyperbolic, and $\pi \otimes q \simeq \pi \otimes q_1 \perp \pi \otimes q_2$. In particular, $i_W(\pi \otimes q) = (\dim \pi) i_W(q_2)$.*

With respect to the above theorem, we clearly have that $i_W(q_2) \geqslant i_W(q)$. These quantities do not appear to satisfy any stronger relation however (indeed, the form q may be anisotropic). The following is an immediate corollary of Theorem 2.4.

Corollary 2.5. *Let q a form of dimension at least two and π similar to a Pfister form be such that $\pi \otimes q$ is anisotropic over F. Then $i_1(\pi \otimes q) \geqslant \dim \pi$.*

Invoking Theorem 2.1, we obtain the following refinement of the above bound.

Theorem 2.6. *Let q a form of dimension at least two and π similar to a Pfister form be such that $\pi \otimes q$ is anisotropic over F. Then $i_1(\pi \otimes q) \geqslant (\dim \pi) i_1(q)$.*

Proof. If $i_1(q) = 1$, then the statement is precisely Corollary 2.5. Hence, we may assume that $i_1(q) > 1$. Let $q' \subset q$ over F of dimension $\dim q - i_1(q) + 1$. Lemma 1.1 implies that q' is isotropic over $F(q)$. Hence, $\pi \otimes q'$ is isotropic over $F(\pi \otimes q)$ by Theorem 2.1. As $\pi \otimes q' \subset \pi \otimes q$, we have that $\pi \otimes q'$ is anisotropic over F by assumption and, furthermore, that $\pi \otimes q$ is isotropic over $F(\pi \otimes q')$, whereby $\pi \otimes q'$ and $\pi \otimes q$ are isotropy equivalent. Invoking Theorem 1.4 (i), we have that $\dim(\pi \otimes q') - i_1(\pi \otimes q') = \dim(\pi \otimes q) - i_1(\pi \otimes q)$, whereby $i_1(\pi \otimes q) = i_1(\pi \otimes q') + \dim \pi(\dim q - \dim q') = i_1(\pi \otimes q') + \dim \pi(i_1(q) - 1)$. Since $i_1(\pi \otimes q') \geqslant \dim \pi$ by Corollary 2.5, we have that $i_1(\pi \otimes q) \geqslant (\dim \pi) i_1(q)$. □

Corollary 2.7. *Let q a form of dimension at least two and π similar to a Pfister form be such that $\pi \otimes q$ is anisotropic over F. If $p \subset \pi \otimes q$ over F of codimension less than $(\dim \pi) i_1(q)$, then p is isotropic over $F(\pi \otimes q)$.*

Proof. Theorem 2.6 implies that $p \subset \pi \otimes q$ of codimension less than $i_1(\pi \otimes q)$, whereby Lemma 1.1 implies that p is isotropic over $F(\pi \otimes q)$. □

With respect to Theorem 2.6, the following example shows that $i_1(\pi \otimes q)$ can exceed $(\dim \pi) i_1(q)$.

Example 2.8. As in Example 2.2, let $q \simeq \langle 1, 1, 1, 7 \rangle$ and $\pi \simeq \langle 1, 1, 1, 1 \rangle$ over $F = \mathbb{Q}$. As before, $i_1(q) = 1$ and $\pi \otimes q$ is isometric to the Pfister form $16 \times \langle 1 \rangle$. Hence, we have that $i_1(\pi \otimes q) = 8 > (\dim \pi) i_1(q) = 4$.

For certain forms q over F, we can determine the value of $i_1(\pi \otimes q)$. Over formally real fields F, we have the following result.

Proposition 2.9. *Let q a form of dimension at least two and π similar to a Pfister form be such that $\pi \otimes q$ is anisotropic over a formally real field F. Let $P \in X_F$ be an ordering such that π is definite at P and q is indefinite at P, whereby $|\mathrm{sgn}_P(q)| \leqslant \dim q - 2i_1(q)$. If $|\mathrm{sgn}_P(q)| = \dim q - 2i_1(q)$, then $i_1(\pi \otimes q) = (\dim \pi) i_1(q)$.*

Proof. Since π is (positive) definite at P and $|\mathrm{sgn}_P(q)| = \dim q - 2i_1(q)$, it follows that $|\mathrm{sgn}_P(\pi \otimes q)| = \dim \pi(\dim q - 2i_1(q))$. Hence, Theorem 1.2 implies that P extends to $K = F(\pi \otimes q)$. Over K, $(\pi \otimes q)_K \simeq ((\pi \otimes q)_K)_{\mathrm{an}} \perp i_1(\pi \otimes q) \times \langle 1, -1 \rangle_K$, whereby a comparison of signatures with respect to P yields that $i_1(\pi \otimes q) \leqslant (\dim \pi) i_1(q)$. Invoking Theorem 2.6, we also have that $i_1(\pi \otimes q) \geqslant (\dim \pi) i_1(q)$, whereby the result follows. □

In the case where the form q has maximal splitting, the value of $i_1(\pi \otimes q)$ again coincides with our lower bound.

Proposition 2.10. *Let q a form of dimension at least two and π similar to a Pfister form be such that $\pi \otimes q$ is anisotropic over F. If q has maximal splitting, then $i_1(\pi \otimes q) = (\dim \pi) i_1(q)$.*

Proof. Let $\dim q = 2^n + k$ for some integers n and k such that $0 < k \leqslant 2^n$. Hence, $\dim(\pi \otimes q) = 2^n \dim \pi + k \dim \pi$, where $0 < k \dim \pi \leqslant 2^n \dim \pi$. As $i_1(q) = k$, Theorem 2.6 implies that $i_1(\pi \otimes q) \geqslant k \dim \pi$. Let $\vartheta \subset \pi \otimes q$ over F such that $\dim \vartheta = 2^n \dim \pi$. If $i_1(\pi \otimes q) > k \dim \pi$, then Lemma 1.1 implies that ϑ is isotropic over $F(\pi \otimes q)$, contradicting Theorem 1.3. Thus, $i_1(\pi \otimes q) = (\dim \pi) i_1(q)$. □

Corollary 2.11. *Let q a form of dimension at least two and π similar to a Pfister form be such that $\pi \otimes q$ is anisotropic over F. If q has maximal splitting, then $\pi \otimes q$ has maximal splitting.*

Proof. Proposition 2.10 implies that $\dim(\pi \otimes q) - i_1(\pi \otimes q) = \dim \pi(\dim q - i_1(q))$. Since $\dim q - i_1(q) = 2^k$ for some integer $k \geqslant 0$, it follows that $\dim(\pi \otimes q) - i_1(\pi \otimes q) = 2^{n+k}$ for some $n \in \mathbb{N}$. Hence, $\pi \otimes q$ has maximal splitting. □

The preceding results can be extended to multiples of Pfister neighbours of sufficiently small codimension.

Corollary 2.12. *Let q a form of dimension at least two and π similar to a Pfister form be such that $\pi \otimes q$ is anisotropic over F. Let τ be a neighbour of π such that $\dim \tau > \dim \pi - \frac{(\dim \pi) i_1(q)}{\dim q}$. If q has maximal splitting, then $i_1(\tau \otimes q) = (\dim \pi) i_1(q) - (\dim q)(\dim \pi - \dim \tau)$, whereby $\tau \otimes q$ has maximal splitting.*

Proof. As per Corollary 2.7, since $\dim(\tau \otimes q) > \dim(\pi \otimes q) - (\dim \pi) i_1(q)$, it follows that $\tau \otimes q$ is isotropic over $F(\pi \otimes q)$. Thus, $\tau \otimes q$ is isotropy equivalent to $\pi \otimes q$, whereby Theorem 1.4 (*ii*) implies that $i_1(\tau \otimes q) = i_1(\pi \otimes q) - (\dim q)(\dim \pi - \dim \tau)$. Invoking Corollary 2.11, it follows that $i_1(\tau \otimes q) = (\dim \pi) i_1(q) - (\dim q)(\dim \pi - \dim \tau)$, whereby $\dim(\tau \otimes q) - i_1(\tau \otimes q) = \dim \pi(\dim q - i_1(q)) = 2^m$ for some $m \in \mathbb{N}$, whereby $\tau \otimes q$ has maximal splitting. □

The following example shows that the dimension condition in the preceding result cannot be relaxed in general. This example furthermore demonstrates that the anisotropic product of two Pfister neighbours, both necessarily having maximal splitting, need not have maximal splitting.

Example 2.13. Let F be a field such that $\langle 1, 1, 1, d, d, d \rangle$ is anisotropic over F for some $d \in F^\times$. Let $K = F((x))((y))$ be the iterated Laurent series field in two variables over F. Consider the Pfister neighbours $\tau_1 \simeq \langle 1, 1, 1 \rangle$ and $\tau_2 \simeq \langle d \rangle \perp \langle 1, -x, -y, xy \rangle$ over K. Applying Springer's Theorem for complete discretely valuated fields [L, Theorem VI.1.4], firstly with respect to the y-adic valuation and subsequently with respect to the x-adic valuation, one sees that the form $\tau_1 \otimes \tau_2$ is anisotropic over K. Suppose $\tau_1 \otimes \tau_2$ has maximal splitting. Since $\dim(\tau_1 \otimes \tau_2) = 15$, [H, Corollary 3] implies that $\tau_1 \otimes \tau_2$ is a neighbour of some 4-fold Pfister form π over K. Comparing determinants, we have that $\tau_1 \otimes \tau_2 \perp \langle d \rangle \simeq a\pi$ for some $a \in K$. As $a\pi \in I^3 K$, it has trivial Clifford invariant (see [L, Corollary V.3.4]), whereby the Clifford invariant of $\tau_1 \otimes \tau_2 \perp \langle d \rangle$ must also be trivial. Hence, applying [L, V.(3.13)], we obtain that $C_0(\tau_1 \otimes \tau_2)$, the even Clifford algebra of $\tau_1 \otimes \tau_2$, belongs to the trivial class in the Brauer group over K. However, applying [L, V.(3.13)] to a decomposition of $\tau_1 \otimes \tau_2$, we see that $C_0(\tau_1 \otimes \tau_2)$ is Brauer equivalent to $(-1, -1)_K \otimes_K (x, y)_K$, a product of two quaternion algebras. As above, iterated applications of Springer's Theorem [L, Theorem VI.1.4] with respect to the y-adic and x-adic valuations enable us to conclude that the form $\langle 1, 1, 1, x, y, -xy \rangle$ is anisotropic over K, whereby [L, Theorem III.4.8] implies that $(-1, -1)_K \otimes_K (x, y)_K$

is a biquaternion division algebra over K, and hence non-trivial in the Brauer group of K. Thus, we may conclude that $\tau_1 \otimes \tau_2$ does not have maximal splitting.

3. Basic properties of the weak isotropy index

By definition, the q-sublevel and q-level of F satisfy

$$\underline{s}_q(F) = \inf\{n \in \mathbb{N} \cup \{0\} \mid (n+1) \times q \text{ is isotropic over } F\}$$

and

$$s_q(F) = \inf\{n \in \mathbb{N} \mid \langle 1 \rangle \perp n \times q \text{ is isotropic over } F\}.$$

An important distinction between these concepts is the fact that the q-sublevel is invariant with respect to scaling, whereas the q-level is generally not. For example, if q is a form over a formally real field F such that $s_q(F) < \infty$ and $\mathrm{sgn}_P(q) = -\dim q$ for some $P \in X_F$, it follows that $s_{-q}(F) = \infty$.

If q is an isotropic form over F, then $\underline{s}_q(F) = 0$ and $s_q(F) = 1$. Thus, we will restrict our attention to forms q that are anisotropic over F. Our opening result records some basic properties of the q-sublevel of a field, by establishing analogues of statements in [BG-BM, Lemma 3.1 and Proposition 3.3] concerning the q-level.

Proposition 3.1. *Let q be an anisotropic form over F.*

- *(i)* $1 \leqslant \underline{s}_q(F) \leqslant s(F)$.
- *(ii)* *If $q' \subset aq$ for some $a \in F^\times$, then* $\underline{s}_q(F) \leqslant \underline{s}_{q'}(F)$.
- *(iii)* *If K/F is a field extension, then* $\underline{s}_q(K) \leqslant \underline{s}_q(F)$.
- *(iv)* *If K/F is a field extension whose degree is odd, then* $\underline{s}_q(K) = \underline{s}_q(F)$.
- *(v)* *If K/F is a purely transcendental field extension, then*
$$\underline{s}_q(K) = \underline{s}_q(F) = \underline{s}_q(F(\!(x)\!)) = \underline{s}_{q\perp\langle x\rangle}(F(\!(x)\!)).$$
- *(vi)* *For every $n \in \mathbb{N}$, we have that* $\underline{s}_{n\times q}(F) = \left\lfloor \frac{\underline{s}_q(F)}{n} \right\rfloor$.
- *(vii)* *If $\underline{s}_q(F) < \infty$, then* $\underline{s}_q(F) \leqslant p(F) - 1$.
- *(viii)* *If F is not formally real, then* $\underline{s}_q(F) \leqslant \left\lceil \frac{u(F)}{\dim q} \right\rceil \leqslant u(F)$.

Proof. (i), (ii) and (iii) easily follow from the definition of the q-sublevel of a field, while (iv) can be proven by invoking Springer's Theorem [L, Theorem VII.2.7]. Statement (v) follows from invoking Springer's Theorem [L, Theorem VI.1.4] for complete discretely valuated fields

To prove (vi), we note that $\left(\left\lceil \frac{\underline{s}_q(F)+1}{n} \right\rceil\right) n \times q$ is isotropic, whereby $\underline{s}_{n\times q}(F) \leqslant \left\lceil \frac{\underline{s}_q(F)+1}{n} \right\rceil - 1$. Since $\left\lceil \frac{\underline{s}_q(F)+1}{n} \right\rceil - 1 = \left\lfloor \frac{\underline{s}_q(F)}{n} + 1 \right\rfloor - 1 = \left\lfloor \frac{\underline{s}_q(F)}{n} \right\rfloor$, we have that $\left(\left\lceil \frac{\underline{s}_q(F)+1}{n} \right\rceil - 1\right) n \times q$ is anisotropic, establishing (vi).

To prove (vii), we may assume that $p(F) < \infty$. Since $(\underline{s}_q(F)+1) \times q$ is isotropic, we have that $p(F) \times q$ is isotropic. Hence, $\underline{s}_q(F) \leqslant p(F) - 1$. Statement $(viii)$ follows from the fact that $\left(\left\lceil \frac{u(F)}{\dim q} \right\rceil + 1\right) \times q$ is isotropic. $\square$

Remark 3.2. We remark that all of the bounds in Proposition 3.1 can be attained. As $\underline{s}_{\langle 1\rangle}(F) = s(F)$, letting $q \simeq \langle 1 \rangle$ over a field F that is not formally real, one realises the upper bound in (i). Invoking (v), one sees that the upper bound in (ii) can be attained in the case where q' is a proper subform of q. The attainability of the upper bound in (iii) can be deduced from (iv) or (v). The upper bound in (vii) can be realised by letting $q \simeq \langle 1 \rangle$ over a field F of finite Pythagoras number satisfying $p(F) = s(F)+1$ (see [P, Ch. 7, Proposition 1.5]). Finally, as per [BG-BM, Remark 3.4], letting $q \simeq \langle 1\rangle$ over a field F such that $s(F) = u(F) = 2^m$ for some integer $m \geqslant 0$, one realises the upper bounds in $(viii)$.

As was observed in [BG-BM, Lemma 3.1 (8)], if q is an anisotropic form over F such that $1 \in D_F(q)$, then $\underline{s}_q(F) \leqslant s_q(F)$, since $\langle 1\rangle \perp s_q(F) \times q \subset (s_q(F)+1) \times q$ in this case. Indeed, more generally, we have the following relation.

Proposition 3.3. *Let q be an anisotropic form over F and $a \in F^\times$. Then*

$$\underline{s}_q(F) = \inf\{s_{aq}(F) \mid a \in D_F(q)\}.$$

Proof. We note that $\underline{s}_q(F) = \inf\{n \in \mathbb{N} \mid (n+1) \times q \text{ is isotropic over } F\}$ in this situation, as q is anisotropic over F by assumption.

As $(n+1) \times q$ is isotropic over F if and only if there exists $a \in F^\times$ such that $a \in D_F(q)$ and $-a \in D_F(n \times q)$, we can conclude that

$$\underline{s}_q(F) = \inf\{n \in \mathbb{N} \mid -a \in D_F(n \times q) \text{ for some } a \in D_F(q)\}.$$

Hence, we have that $\underline{s}_q(F) = \inf\{n \in \mathbb{N} \mid -1 \in D_F(n \times aq) \text{ for some } a \in D_F(q)\}$. Thus, we can conclude that $\underline{s}_q(F) = \inf\{s_{aq}(F) \mid a \in D_F(q)\}$. □

The *q-length of $a \in F^\times$* is $\ell_q(a) := \inf\{n \in \mathbb{N} \mid n \times q \perp \langle -a\rangle \text{ is isotropic over } F\}$. The *Pythagoras q-number of F* is $p_q(F) := \sup\{\ell_q(a) \mid a \in F^\times \text{ with } \ell_q(a) < \infty\}$. For q an anisotropic form over F such that $\underline{s}_q(F) < \infty$, the form $(\underline{s}_q(F)+1) \times q$ is isotropic (and hence universal) over F, whereby we have that $s_q(F) \leqslant \underline{s}_q(F)+1$ (as per [BG-BM, Lemma 3.1 (7)]). Indeed, we have the following result.

Proposition 3.4. *For q an anisotropic form over F, the following are equivalent:*

(i) $\underline{s}_q(F) < \infty$.
(ii) $s_q(F) < \infty$ *and* $s_{-q}(F) < \infty$.
(iii) $p_q(F) < \infty$ *and* $p_q(F) \times q$ *is universal.*

Proof. Assuming (i), we have that $(\underline{s}_q(F)+1) \times q$ is isotropic, and thus universal, whereby $p_q(F) \times q$ is universal and $p_q(F) \leqslant \underline{s}_q(F)+1$, establishing (iii).

Assuming (iii), we have that $\{-1, 1\} \subset D_F(p_q(F) \times q)$, whereby $s_q(F) \leqslant p_q(F)$ and $s_{-q}(F) \leqslant p_q(F)$, establishing (ii).

Assuming (ii), we have that $(s_q(F) + s_{-q}(F)) \times q$ is isotropic, whereby $\underline{s}_q(F) \leqslant s_q(F) + s_{-q}(F) - 1$, establishing (i). □

With respect to the above result, we note the existence of forms q over fields F such that $s_q(F) < \infty$ and $\underline{s}_q(F) = \infty$ (see Remark 5.4).

Proposition 3.5. *Let q be an anisotropic form over F such that $\underline{s}_q(F) < \infty$. Then $p_q(F) - 1 \leqslant \underline{s}_q(F) \leqslant p_q(F)$.*

Proof. As per the above proof, $p_q(F) - 1 \leqslant \underline{s}_q(F)$. Moreover, since $p_q(F) \times q$ is universal, we have that $(p_q(F)+1) \times q$ is isotropic, whereby $\underline{s}_q(F) \leqslant p_q(F)$. □

Remark 3.6. Letting $q \simeq \langle 1\rangle$ over a field F such that $\underline{s}_q(F) < \infty$, Proposition 3.5 states that $p(F) - 1 \leqslant s(F) \leqslant p(F)$. As per [P, Ch. 7, Proposition 1.5], there exist fields K and L over F satisfying $s(K) = p(K) - 1 < \infty$ and $s(L) = p(L) < \infty$.

4. Values of the weak isotropy index

In this section, we study the behaviour of the q-sublevel (or equivalently, the weak isotropy index) with respect to field extensions. In particular, for q an anisotropic form over F, we will study the set $\{\underline{s}_q(K) \mid K/F \text{ field extension}\}$. Clearly, $\underline{s}_q(F)$ always belongs to this set, with the remaining entries being less than $\underline{s}_q(F)$.

We begin by seeking to show that certain prescribed integers belong to the above set. As motivated earlier, function fields of associated quadratic forms are the natural field extensions to consider in this regard. Indeed, an integer $m \leqslant \underline{s}_q(F)$ is an element of the set $\{\underline{s}_q(K) \mid K/F \text{ field extension}\}$ if and only if the form $m \times q$ is anisotropic over $F((m+1)\times q)$. Invoking Theorem 1.3, if m is such that $m \dim q \leqslant 2^n < (m+1)\dim q$ for some integer $n \geqslant 0$, then $m \in \{\underline{s}_q(K) \mid K/F \text{ field extension}\}$. Our opening result determines those integers to which this observation applies.

Proposition 4.1. *Let q be an anisotropic form over F. An integer $m \leqslant \underline{s}_q(F)$ is such that $m \dim q \leqslant 2^n < (m+1)\dim q$ if and only if $m = \left\lfloor \frac{2^n}{\dim q} \right\rfloor$ for some integer $n \geqslant 0$. In particular, an anisotropic form $\left\lfloor \frac{2^n}{\dim q} \right\rfloor \times q$ over F remains anisotropic over $F\left(\left(\left\lfloor \frac{2^n}{\dim q} \right\rfloor + 1\right) \times q\right)$, whereby $\left\lfloor \frac{2^n}{\dim q} \right\rfloor \in \{\underline{s}_q(K) \mid K/F \text{ field extension}\}$.*

Proof. Let $m \leqslant \underline{s}_q(F)$ be such that $m \dim q \leqslant 2^n < (m+1)\dim q$ for some integer $n \geqslant 0$. Since $m \dim q \leqslant 2^n$, it follows that $m \leqslant \frac{2^n}{\dim q}$, and thus $m \leqslant \left\lfloor \frac{2^n}{\dim q} \right\rfloor$ as m is an integer. Moreover, as $2^n < (m+1)\dim q$, we have that $m > \frac{2^n}{\dim q} - 1$. Hence, we have that $m \geqslant \left\lfloor \frac{2^n}{\dim q} \right\rfloor$, and thus we can conclude that $m = \left\lfloor \frac{2^n}{\dim q} \right\rfloor$.

Conversely, letting $m = \left\lfloor \frac{2^n}{\dim q} \right\rfloor$, we clearly have that $\dim(m \times q) \leqslant 2^n$. Moreover, as $\left\lfloor \frac{2^n}{\dim q} \right\rfloor = \left\lceil \frac{2^n+1}{\dim q} \right\rceil - 1$, letting $m = \left\lfloor \frac{2^n}{\dim q} \right\rfloor$ gives us that $\dim((m+1)\times q) > 2^n$. The last statement now follows from applying Theorem 1.3. □

Corollary 4.2. *Let q be an anisotropic form over F of dimension at least two. Then $\{0,1\} \subseteq \{\underline{s}_q(K) \mid K/F \text{ field extension}\}$.*

Proof. Since $q_{F(q)}$ is isotropic, one has that $0 \in \{\underline{s}_q(K) \mid K/F \text{ field extension}\}$. For $\dim q = 2^{n-1} + k$ for some integers n and k such that $0 < k \leqslant 2^{n-1}$, Proposition 4.1 implies that $\left\lfloor \frac{2^n}{\dim q} \right\rfloor = 1 \in \{s_q(K) \mid K/F \text{ field extension}\}$. □

Corollary 4.3. *Let q be an anisotropic form over F of dimension 2^n for some integer $n \geqslant 0$. Then $2^k \in \{\underline{s}_q(K) \mid K/F \text{ field extension}\}$ for all integers $k \geqslant 0$ such that $2^k \leqslant \underline{s}_q(F)$.*

Proof. This follows immediately from Proposition 4.1. □

With respect to certain forms q and integers m, Proposition 4.1 enables us to determine whether or not m is in $\{\underline{s}_q(K) \mid K/F \text{ field extension}\}$.

Proposition 4.4. *Let q be a form over F such that $(m+1)\times q$ has maximal splitting for some integer $m < \underline{s}_q(F)$. Then $m \in \{\underline{s}_q(K) \mid K/F \text{ field extension}\}$ if and only if $m = \left\lfloor \frac{2^n}{\dim q} \right\rfloor$ for some integer $n \geqslant 0$. In particular, if $m \times q$ is anisotropic over $F((m+1)\times q)$, then $m = \left\lfloor \frac{2^n}{\dim q} \right\rfloor$ for some integer $n \geqslant 0$.*

Proof. For $m = \left\lfloor \frac{2^n}{\dim q} \right\rfloor$ for some integer $n \geqslant 0$, Proposition 4.1 implies that we have $m \in \{\underline{s}_q(K) \mid K/F \text{ field extension}\}$.

Conversely, let K/F be such that $\underline{s}_q(K) = m$. As $m \times q$ is anisotropic over K, Lemma 1.1 implies that $\dim(m \times q) \leqslant \dim((m+1) \times q) - i_W((m+1) \times q_K)$. We recall that $i_W((m+1) \times q_K) \geqslant i_1((m+1) \times q)$, as the form $(m+1) \times q$ is isotropic over K. Thus, we have that $\dim(m \times q) \leqslant \dim((m+1) \times q) - i_1((m+1) \times q)$. Since $(m+1) \times q$ has maximal splitting, $\dim((m+1) \times q) - i_1((m+1) \times q) = 2^n$ for some integer $n \geqslant 0$. Hence, it follows that $\dim(m \times q) \leqslant 2^n < \dim((m+1) \times q)$. Invoking Proposition 4.1, we have that $m = \left\lfloor \frac{2^n}{\dim q} \right\rfloor$. □

Remark 4.5. If q is similar to an n-fold Pfister form over F for $n \geqslant 0$, then for all integers $m \geqslant 0$ we have that $(m+1) \times q$ is a neighbour of a Pfister form similar to $2^k \times q$ for some integer $k \geqslant 0$. Thus, the form $(m+1) \times q$ has maximal splitting for all integers $m \geqslant 0$ in this case, whereby Proposition 4.4 can be applied to give a complete determination of $\{\underline{s}_q(K) \mid K/F \text{ field extension}\}$ (see Remark 6.2).

Theorem 1.4 provides another criterion for the admissibility of a prescribed integer in the set $\{\underline{s}_q(K) \mid K/F \text{ field extension}\}$.

Proposition 4.6. *For q an anisotropic form over F, let $n \geqslant 0$ be an integer such that $n < \underline{s}_q(F)$. Then $n \times q$ is anisotropic over $F((n+1) \times q)$ if and only if $i_1((n+1) \times q) - i_1(n \times q) \neq \dim q$.*

Proof. Since $(n+1) \times q$ is isotropic over $F(n \times q)$, this follows immediately from Theorem 1.4 (*ii*). □

Proposition 4.7. *Let q be an anisotropic form over F. For $n \geqslant 0$ an integer such that $2^n \leqslant \underline{s}_q(F)$, let $m = \left\lfloor 2^n - \frac{i_1(2^n \times q)}{\dim q} \right\rfloor$. In particular, if q has maximal splitting, then $m = \left\lfloor 2^n - \frac{2^n i_1(q)}{\dim q} \right\rfloor$. Then $m \times q$ stays anisotropic over $F((m+1) \times q)$, whereby $m \in \{\underline{s}_q(K) \mid K/F \text{ field extension}\}$.*

Proof. Given an integer $n \geqslant 0$ such that $2^n \leqslant \underline{s}_q(F)$, let $m = \left\lfloor 2^n - \frac{i_1(2^n \times q)}{\dim q} \right\rfloor$, whereby Proposition 2.10 implies that $m = \left\lfloor 2^n - \frac{2^n i_1(q)}{\dim q} \right\rfloor$ if q has maximal splitting. As $m \leqslant 2^n - 1$, we have that $(m+1) \times q \subset 2^n \times q$, which is anisotropic over F. As $m+1 > 2^n - \frac{2^n i_1(q)}{\dim q}$, we have that $\dim((m+1) \times q) > 2^n \dim q - i_1(2^n \times q)$, whereby $(m+1) \times q$ is isotropic over $F(2^n \times q)$ by Lemma 1.1. Thus, Theorem 1.4 (*ii*) implies that $\dim((m+1) \times q) - i_1((m+1) \times q) = 2^n \dim q - i_1(2^n \times q)$. As $\dim(m \times q) \leqslant 2^n \dim q - i_1(2^n \times q)$, Theorem 1.4 (*i*) implies that $m \times q$ is anisotropic over $F((m+1) \times q)$. □

Although there exists an explicit determination of the possible values of the first Witt index of a given form in terms of its dimension (see [K]), pinpointing the exact value taken by this invariant remains problematic. Moreover, a determination of the precise value of $i_1(q)$ for a given form q does not enable one to determine $i_1(m \times q)$, for general integers m. Thus, for our purposes, the criteria provided by Proposition 4.6 and the general statement in Proposition 4.7 are difficult to apply. In the case where the function field of the given form has an ordering however, the signature of the form with respect to this ordering imposes a natural bound on the first Witt index of the form and those of its multiples. Hence, for certain forms q over certain fields F, we can apply Proposition 4.6 to determine further entries of $\{\underline{s}_q(K) \mid K/F \text{ field extension}\}$.

Theorem 4.8. *Let q be an anisotropic form over a formally real field F such that $|\mathrm{sgn}_P(q)| = \dim q - 2$ for some $P \in X_F$. Then, for $m' := \min\{\dim q - 1, \underline{s}_q(F) - 1\}$, we have that $\{0, \ldots, m'\} \cup \{\underline{s}_q(F)\} \subseteq \{\underline{s}_q(K) \mid K/F$ field extension$\}$. In particular, for all $m \geqslant 0$ such that $m \leqslant m'$, we have that $m \times q$ is anisotropic over $F((m+1) \times q)$.*

Proof. As above, $\underline{s}_q(F) \in \{\underline{s}_q(K) \mid K/F$ field extension$\}$.

Let $m \geqslant 0$ be an integer such that $m \leqslant m'$. As $m \leqslant \underline{s}_q(F) - 1$, we have that $(m+1) \times q$ is anisotropic over F. Clearly, $|\mathrm{sgn}_P((m+1) \times q)| = (m+1)\dim q - 2(m+1)$, whereby Theorem 1.2 implies that P extends to $K = F((m+1) \times q)$. Since $((m+1) \times q)_K \simeq (((m+1) \times q)_K)_{\mathrm{an}} \perp i_1((m+1) \times q) \times \langle 1, -1 \rangle_K$, a comparison of signatures with respect to P yields that $i_1((m+1) \times q) \leqslant m+1$. As $m+1 \leqslant \dim q$, we have that $i_1((m+1) \times q) < \dim q + i_1(m \times q)$, whereby Proposition 4.6 implies that $\underline{s}_q(K) = m$. □

If $\underline{s}_q(F) \leqslant \dim q$ for q as above, our determination of $\{\underline{s}_q(K) \mid K/F$ field extension$\}$ is complete.

Corollary 4.9. *Let q be an anisotropic form over a formally real field F such that $|\mathrm{sgn}_P(q)| = \dim q - 2$ for some $P \in X_F$ and $\underline{s}_q(F) \leqslant \dim q$. Then we have that $\{0, \ldots, \underline{s}_q(F)\} = \{\underline{s}_q(K) \mid K/F$ field extension$\}$. In particular, for all $m \geqslant 0$ such that $m \times q$ is anisotropic over F, we have that $m \times q$ is anisotropic over $F((m+1) \times q)$.*

Proof. Since $\underline{s}_q(F) \leqslant \dim q$, the result follows from invoking Theorem 4.8. □

For the forms q treated in Theorem 4.8, the following example demonstrates that, in general, a complete determination of $\{\underline{s}_q(K) \mid K/F$ field extension$\}$ remains outstanding in cases where $\underline{s}_q(F) > \dim q$.

Example 4.10. Let q be a 4-dimensional form over a formally real field F such that $|\mathrm{sgn}_P(q)| = 2$ for some $P \in X_F$ and $\underline{s}_q(F) \geqslant 5$ (for example, for F_0 a formally real field, one can let $F = F_0(X_1, X_2, X_3, X_4)$ and $q \simeq \langle X_1, X_2, X_3, X_4 \rangle$, whereby we have that $\underline{s}_q(F) = \infty$ and $|\mathrm{sgn}_P(q)| = 2$ for $P \in X_F$ such that $\{X_1, -X_2, -X_3, -X_4\} \subset P$). By Theorem 4.8, we have that $\{0,1,2,3\} \cup \{\underline{s}_q(F)\} \subseteq \{\underline{s}_q(K) \mid K/F$ field extension$\}$. For $K = F(5 \times q)$, it follows from Theorem 1.3 that $4 \times q$ is anisotropic over K, whereby $\underline{s}_q(K) = 4$. Hence, we can conclude that $\{0,1,2,3\} \cup \{\underline{s}_q(F)\} \subsetneq \{\underline{s}_q(K) \mid K/F$ field extension$\}$.

For certain forms satisfying $\underline{s}_q(F) > \dim q$ however, Theorem 4.8 does provide a complete determination of $\{\underline{s}_q(K) \mid K/F$ field extension$\}$.

Example 4.11. Let q be a 3-dimensional form over a formally real field F such that q is indefinite at $P \in X_F$ and $\underline{s}_q(F) = 4$. This can be achieved, for example, by letting $F = F_0(X)$ and $q \simeq \langle 1, -a, X \rangle$ for F_0 a formally real field such that $p(F_0) \geqslant 5$ and $a \in {F_0}^{\times}$ such that $\ell_{F_0}(a) = 5$ (whereby $\langle 1 \rangle \perp 4 \times \langle -a \rangle$ is anisotropic over F_0, implying that its associated Pfister form $4 \times \langle 1, -a \rangle$ is anisotropic over F_0, and thus that $4 \times q$ is anisotropic over F by [L, Exercise IX.1]). By Theorem 4.8, $\{0,1,2,4\} \subseteq \{\underline{s}_q(K) \mid K/F$ field extension$\}$. Let K be any extension of F such that $4 \times q$ is isotropic over K. Since $4 \times q \simeq (4 \times \langle 1 \rangle) \otimes q$, we have that $i_W((4 \times q)_K) \geqslant 4$ by Theorem 2.4. Hence, $3 \times q$ is isotropic over K by Lemma 1.1, whereby $\underline{s}_q(K) \leqslant 2$. Thus, $\{\underline{s}_q(K) \mid K/F$ field extension$\} = \{0,1,2,4\}$.

For an arbitrary form q over F, in order to establish that certain integers less than $\underline{s}_q(F)$ do not belong to the set $\{\underline{s}_q(K) \mid K/F$ field extension$\}$, function fields of associated quadratic forms are once again the appropriate extensions to consider. To this end, we can invoke our lower bound on the first Witt index of multiples of Pfister forms, Theorem 2.6, to establish the following result.

Theorem 4.12. *For q an anisotropic form over F, let ℓ and n be positive integers satisfying $2^n\ell \leqslant \underline{s}_q(F)$, where n is the 2-adic order of $2^n\ell$.*

(i) *If $\dim q < i_1(2^n\ell \times q)$, then for $m \in \left(2^n\ell - \frac{i_1(2^n\ell\times q)}{\dim q}, 2^n\ell\right)$ we have that $m \times q$ is isotropic over $F((m+1)\times q)$, whereby $m \notin \{\underline{s}_q(K) \mid K/F$ field extension$\}$.*

(ii) *If $\dim q < 2^n i_1(\ell \times q)$, then for $m \in \left(2^n\ell - \frac{2^n i_1(\ell\times q)}{\dim q}, 2^n\ell\right)$ we have that $m \times q$ is isotropic over $F((m+1)\times q)$, whereby $m \notin \{\underline{s}_q(K) \mid K/F$ field extension$\}$.*

Proof. (i) Since $2^n\ell \leqslant \underline{s}_q(F)$, we have that $2^n\ell \times q$ is anisotropic over F. Moreover, as $m < 2^n\ell$, we have that $(m+1)\times q \subset 2^n\ell \times q$. Let K/F be such that $(m+1)\times q$ is isotropic over K, whereby $i_W((2^n\ell \times q)_K) \geqslant i_1(2^n\ell \times q)$. If $m > 2^n\ell - \frac{i_1(2^n\ell\times q)}{\dim q}$, then $m \times q \subset 2^n\ell \times q$ of codimension less than $i_W((2^n\ell \times q)_K)$. Thus, Lemma 1.1 implies that $m \times q$ is isotropic over K, whereby $\underline{s}_q(K) \leqslant m - 1$.

(ii) This statement represents a special case of (i), since $i_1(2^n\ell \times q) \geqslant 2^n i_1(\ell \times q)$ by Theorem 2.6. □

If $\dim q \geqslant i_1(2^n\ell \times q)$, then both of the intervals in the preceding result are empty. Indeed, subject to the weaker condition that $\dim q \geqslant 2^n i_1(\ell\times q)$ holds, the following example establishes the existence of forms q over F such that every integer less than $\underline{s}_q(F)$ is attainable as $\underline{s}_q(K)$ for some extension K/F.

Example 4.13. For the sake of clarity, we will consider the case where $\ell = 1$. Let q be an anisotropic form over a formally real field F such that $\dim q = \underline{s}_q(F) = 2^n$ for some positive integer n and $|\mathrm{sgn}_P(q)| = \dim q - 2$ for some ordering $P \in X_F$ (an explicit example of such a form q directly follows this argument). Since $|\mathrm{sgn}_P(q)| = \dim q - 2$, it follows that $i_1(q) = 1$ whereby $2^n i_1(q) = 2^n = \dim q$. Hence $\{0, \ldots, \underline{s}_q(F)\} = \{\underline{s}_q(K) \mid K/F$ field extension$\}$ by Corollary 4.9.

We can construct an explicit example of such a form q as follows: Let $K(x)$ be a formally real field such that $s(K(x)) \geqslant 2^{2n}$. Letting $q \simeq (2^n - 1)\times\langle 1\rangle \perp \langle x\rangle$, we have that $sgn_P(q) = 2^n - 2$ where the ordering $P \in X_{K(x)}$ is such that $-x \in P$. Invoking Springer's Theorem [L, Theorem VI.1.4] with respect to the x-adic valuation, the form $(2^n + 1) \times q$ is anisotropic over $K(x)$, as $s(K) \geqslant s(K(x)) \geqslant 2^{2n}$. Letting $F = K(x)((2^n + 1) \times q)$, the ordering P extends to F, by Theorem 1.2, and the form $2^n \times q$ is anisotropic over F, by Theorem 1.3, whereby $\underline{s}_q(F) = 2^n$ as desired.

Remark 4.14. Regarding Theorem 4.12, we can invoke our earlier results to establish that the intervals of integers not belonging to the set $\{\underline{s}_q(K) \mid K/F$ field extension$\}$ cannot, in general, be extended in either direction. Let q be a form over F and let $n \in \mathbb{N}$ be such that $2^n \leqslant \underline{s}_q(F)$. Proposition 4.7 states that $\left\lfloor 2^n - \frac{i_1(2^n\times q)}{\dim q}\right\rfloor \in \{\underline{s}_q(K) \mid K/F$ field extension$\}$ and $\left\lfloor 2^n - \frac{2^n i_1(q)}{\dim q}\right\rfloor \in \{\underline{s}_q(K) \mid K/F$ field extension$\}$ whenever q has maximal splitting. Moreover, as per Corollary 4.3, we have that $2^n \in \{\underline{s}_q(K) \mid K/F$ field extension$\}$ whenever q is a form of two-power dimension.

As a consequence of Theorem 4.12, one can see that Theorem 4.8 does not hold for arbitrary forms.

Example 4.15. Let π be a 3-fold Pfister form over F such that $4 \times \pi$ is anisotropic. Let q be a 6-dimensional neighbour of π, whereby $i_1(q) = 2$ and $\underline{s}_q(F) \geqslant 4$. Hence, we can conclude that $3 \notin \{\underline{s}_q(K) \mid K/F$ field extension$\}$ by Theorem 4.12. Similarly, if we have that $\underline{s}_q(F) \geqslant 32$ for q as above, then $3, 6, 7, 11, \ldots, 15, 22, \ldots, 31 \notin \{\underline{s}_q(K) \mid K/F$ field extension$\}$.

5. Values of the q-level

In analogy with the preceding section, for q an anisotropic form over F, we study the behaviour of the q-level with respect to field extensions, seeking to determine the entries of the set $\{s_q(K) \mid K/F \text{ field extension}\}$. As with the q-sublevel, $s_q(F)$ always belongs to this set, with the remaining entries being less than $s_q(F)$.

As per [BG-BM, Proposition 3.13 (2)], if an integer $m \leqslant s_q(F)$ is such that $1+(m-1)\dim q \leqslant 2^n < 1+m\dim q$ for some integer $n \geqslant 0$, then it follows that $s_q(K) = m$ for $K = F(\langle 1\rangle \perp m \times q)$. As with the q-sublevel, we can establish the values of m to which this criterion applies. In the statement of the following result, we combine this observation with other analogues of our results with respect to the q-sublevel.

Theorem 5.1. *Let $m \in \mathbb{N}$ such that $m < s_q(F)$ for q an anisotropic form over F.*

(i) $1+(m-1)\dim q \leqslant 2^n < 1+m\dim q$ *for some integer $n \geqslant 0$ if and only if $m = \left\lceil \frac{2^n}{\dim q} \right\rceil$. In particular, for $m = \left\lceil \frac{2^n}{\dim q} \right\rceil$, the form $\langle 1\rangle \perp (m-1)\times q$ remains anisotropic over $F(\langle 1\rangle \perp m\times q)$.*

(ii) $\{1,2\} \subseteq \{s_q(K) \mid K/F \text{ field extension}\}$.

(iii) If $\dim q = 2^n$ for some integer $n \geqslant 0$, then $2^k \in \{s_q(K) \mid K/F \text{ field extension}\}$ for all integers $k \geqslant 0$ such that $2^k \leqslant s_q(F)$.

(iv) If $\langle 1\rangle \perp m\times q$ has maximal splitting, then $m \in \{s_q(K) \mid K/F \text{ field extension}\}$ if and only if $m = \left\lceil \frac{2^n}{\dim q} \right\rceil$ for some integer $n \geqslant 0$.

(v) If $\dim q > i_1(\langle 1\rangle \perp m\times q) - i_1(\langle 1\rangle \perp (m-1)\times q)$, then $s_q(F(\langle 1\rangle \perp m\times q)) = m$.

Proof. (i) Suppose that $1+(m-1)\dim q \leqslant 2^n < 1+m\dim q$ for $n \geqslant 0$. Since $1+(m-1)\dim q \leqslant 2^n$, we have that $m \leqslant \frac{2^n+\dim q-1}{\dim q}$. Moreover, as $2^n < 1+m\dim q$, it follows that $m > \frac{2^n-1}{\dim q}$. Thus, since $m \in \mathbb{N}$, it follows that $m \geqslant \left\lfloor \frac{2^n-1}{\dim q} \right\rfloor + 1 = \left\lfloor \frac{2^n+\dim q-1}{\dim q} \right\rfloor$. Hence, we have that $m = \left\lfloor \frac{2^n+\dim q-1}{\dim q} \right\rfloor = \left\lceil \frac{2^n}{\dim q} \right\rceil$. Conversely, letting $m = \left\lceil \frac{2^n}{\dim q} \right\rceil$, we clearly have that $\dim(\langle 1\rangle \perp m\times q) > 2^n$. Moreover, as $\left\lceil \frac{2^n}{\dim q} \right\rceil = \left\lfloor \frac{2^n-1}{\dim q} \right\rfloor + 1$, it follows that $\dim(\langle 1\rangle \perp (m-1)\times q) \leqslant 2^n$.

Statements (ii) and (iii) are immediate corollaries of (i). Arguing in an analogous manner to the proof of Proposition 4.4, one can also establish Statement (iv) as a corollary of (i). Statement (v) follows from invoking Theorem 1.4 (i). □

Remark 5.2. If q is an n-fold Pfister form over F for $n \geqslant 0$, then for all $m \in \mathbb{N}$ we have that $\langle 1\rangle \perp m\times q$ is a neighbour of the Pfister form $2^k \times q$, where $2^{k-1} \leqslant m < 2^k$. Thus, for all $m \in \mathbb{N}$, the form $\langle 1\rangle \perp m\times q$ has maximal splitting, whereby Theorem 5.1 (iv) is applicable for all $m < s_q(F)$ (see Remark 6.2). Unlike the situation with respect to the q-sublevel however, this observation does not apply to all forms that are similar to q.

As per Theorem 4.8, over formally real fields F we can invoke the signatures of q to bound the Witt indices of forms containing q. Applying this methodology to considerations of the q-level, we can establish, for all $n \in \mathbb{N}$, the existence of n-dimensional forms q over F that can attain any prescribed integer less than their level over F as their level over a suitable extension.

Theorem 5.3. *Let q be a form over a formally real field F such that $\mathrm{sgn}_P(q) = -\dim q$ for some $P \in X_F$. Then $\{1,\ldots,s_q(F)\} = \{s_q(K) \mid K/F \text{ field extension}\}$.*

Proof. Clearly $\{s_q(K) \mid K/F \text{ field extension}\} \subseteq \{1, \ldots, s_q(F)\}$, as $s_q(K) \leqslant s_q(F)$.

Let $n \in \mathbb{N}$ be such that $n < s_q(F)$, whereby $\langle 1 \rangle \perp n \times q$ is anisotropic over F. Let $K = F(\langle 1 \rangle \perp n \times q)$. Since $\langle 1 \rangle \perp n \times q$ is indefinite with respect to P, Theorem 1.2 implies that P extends to K. Moreover, as $|\mathrm{sgn}_P(\langle 1 \rangle \perp n \times q)| = n \dim q - 1$, we have that $i_1(\langle 1 \rangle \perp n \times q) = 1$. Thus, Theorem 1.4 (i) implies that $\langle 1 \rangle \perp (n-1) \times q$ is anisotropic over K, whereby $s_q(K) = n$. □

Remark 5.4. One can invoke the above proof to establish that, in general, the q-level does not impose an upper bound on the q-sublevel. Let q be a form over a formally real field F such that $\mathrm{sgn}_P(q) = -\dim q$ for some $P \in X_F$. As per the above proof, for $n \in \mathbb{N}$ such that $n < s_q(F)$, one has that $s_q(K) = n$ for $K = F(\langle 1 \rangle \perp n \times q)$. As $\mathrm{sgn}_P(q) = -\dim q$ and P extends to K, it follows that $\underline{s}_q(K) = \infty$. Furthermore, there exist field extensions L/K such that $s_q(L) = n$ and $\underline{s}_q(L)$ takes arbitrarily larger finite values. In particular, letting $m \in \mathbb{N}$ be such that $1 + (n-1)\dim q \leqslant m \dim q \leqslant 2^r < (m+1)\dim q$ for some $r \in \mathbb{N}$, if one sets $L = K((m+1) \times q)$, then Theorem 1.3 implies that $s_q(L) = n$ and $\underline{s}_q(L) = m$.

Remark 5.5. As per [BG-BM, Corollary 3.14], for all forms q over F of dimension 1 or 2 (respectively 3) such that $s_q(F) = \infty$, all integers of the form 2^k (respectively $\frac{2^{2k}+2}{3}$ and $\frac{2^{2k+1}+1}{3}$) belong to $\{s_q(K) \mid K/F \text{ field extension}\}$. We note that these values can be recovered by invoking Theorem 5.1 (i). Indeed, for all forms q over F of dimension $r \in \mathbb{N}$ such that $s_q(F) = \infty$, Theorem 5.1 (i) implies that all integers $\left\lceil \frac{2^n}{\dim q} \right\rceil$ belong to $\{s_q(K) \mid K/F \text{ field extension}\}$. Conversely, we can establish that the only integers belonging to $\{s_q(K) \mid K/F \text{ field extension}\}$ for all forms q over F of dimension r such that $s_q(F) = \infty$ are those of the form $\left\lceil \frac{2^n}{\dim q} \right\rceil$. This observation follows from invoking Theorem 5.1 (iv) in conjunction with the following example, wherein for all $r, m \in \mathbb{N}$, the existence is established of forms q over F of dimension r such that $s_q(F) = \infty$ and $\langle 1 \rangle \perp m \times q$ has maximal splitting.

Example 5.6. Assuming the existence of forms q over F such that $s_q(F) = \infty$, we can conclude that F is a formally real field. Thus, $q \simeq n \times \langle 1 \rangle$ is an n-dimensional form over F such that $s_q(F) = \infty$ for all $n \in \mathbb{N}$. Moreover, for all $m \in \mathbb{N}$, the form $\langle 1 \rangle \perp m \times q$ is a Pfister neighbour of $2^r \times \langle 1 \rangle$, for $2^{r-1} \leqslant mn < 2^r$, whereby it has maximal splitting.

Whereas $\underline{s}_q(F) \leqslant s_q(F) + s_{-q}(F) - 1$ for q an anisotropic form over F, Remark 5.4 demonstrates that finiteness of $s_q(F)$ does not imply that of $\underline{s}_q(F)$. Thus, in general, we cannot hope to invoke Theorem 2.6 to show that certain integers do not belong to $\{s_q(K) \mid K/F \text{ field extension}\}$ (indeed, as per Theorem 5.3, there exist anisotropic forms q that can take any prescribed integer as their level over a suitable extension). For those forms q such that finiteness of their level implies finiteness of their sublevel, such as forms q with $1 \in D(q)$ for example, one can argue as per the proof of Theorem 4.12 to establish an analogous result.

6. The weak-isotropy index of round forms and Pfister neighbours

In [BG-BM, Proposition 4.1], it was shown that $\underline{s}_q(F) = s_q(F)$ for q an anisotropic group form over F. Thus, in our considerations of Pfister forms q, we will state all results in terms of $s_q(F)$, bearing in mind that the same statements hold for $\underline{s}_q(F)$.

In [BG-BM, Proposition 4.3], it was shown that the $s_q(F)$ is either a power of two or is infinite in the case where q is a round form over F. Furthermore, in the case where q is a Pfister form over F, the following result [BG-BM, Theorem 4.4] was established.

Theorem 6.1. *(Berhuy, Grenier-Boley, Mahmoudi) If q is an anisotropic Pfister form over F, then $\{s_q(K) \mid K/F$ field extension$\} = \{1, \ldots, 2^i, \ldots, s_q(F)\}$.*

Remark 6.2. As before, if q is an anisotropic n-fold Pfister form over F, then the forms $\langle 1\rangle \perp m \times q$ and $(m+1) \times q$ have maximal splitting for all $m \in \mathbb{N}$, whereby Theorem 6.1 can be recovered by invoking Proposition 4.4 or Theorem 5.1 (iv) (indeed, in the case of the sublevel, the statement holds for all forms similar to q).

As per [BG-BM, Remark 4.5], the round form $q \simeq \langle 1,1,1\rangle$ over $\mathbb{R}$ is no longer round over certain extensions $K/\mathbb{R}$, whereby one can show that $3 \in \{s_q(K) \mid K/\mathbb{R}\}$, implying that Theorem 6.1 does not hold for round forms. Similarly, for $q \simeq \langle 1,1,1\rangle$, one can show that $4 \notin \{s_q(K) \mid K/\mathbb{R}\} \cup \{\underline{s}_q(K) \mid K/\mathbb{R}\}$ and that $5 \in \{\underline{s}_q(K) \mid K/\mathbb{R}\}$.

Proposition 6.3. *Let q be an anisotropic Pfister form over F and let τ be a neighbour of q. Then $\underline{s}_q(F) \leqslant \underline{s}_\tau(F) \leqslant \left\lfloor \frac{\underline{s}_q(F)\dim q}{\dim \tau}\right\rfloor$.*

Proof. Proposition 3.1 (ii) implies that $\underline{s}_q(F) \leqslant \underline{s}_\tau(F)$.

In order to prove the remaining inequality, we may assume that $\underline{s}_q(F) < \infty$, whereby Theorem 6.1 implies that $\underline{s}_q(F) = 2^n$ for some integer $n \geqslant 0$. Thus, the Pfister form $2^{n+1} \times q$ is hyperbolic over F. Consider the form $\vartheta \simeq \left(\left\lfloor \frac{\underline{s}_q(F)\dim q}{\dim \tau}\right\rfloor + 1\right) \times \tau$ over F. Since $\left\lfloor \frac{\underline{s}_q(F)\dim q}{\dim \tau}\right\rfloor \leqslant 2^{n+1} - 1$, we have that $\vartheta \subset 2^{n+1} \times q$. Moreover, as $\left\lfloor \frac{\underline{s}_q(F)\dim q}{\dim \tau}\right\rfloor = \left\lceil \frac{\underline{s}_q(F)\dim q+1}{\dim \tau}\right\rceil - 1$, the form ϑ is such that $\dim\vartheta \geqslant \underline{s}_q(F)\dim q + 1 = 2^n \dim q + 1$, whereby Lemma 1.1 implies that ϑ is isotropic, establishing the result. □

Clearly, the above bounds are attained in the case where τ is similar to q.

In [BG-BM, Corollary 4.6], it was established that $s_q(F) \leqslant s_\tau(F) \leqslant 2s_q(F)$ for τ a subform of a Pfister form q such that $\dim \tau > \frac{1}{2}\dim q$. This upper bound can be extended to Pfister neighbours and, as a corollary of Proposition 6.3, refined.

Corollary 6.4. *Let q be an anisotropic Pfister form over F and let τ be a neighbour of q. Then $s_\tau(F) \leqslant \left\lceil \frac{s_q(F)\dim q+1}{\dim \tau}\right\rceil$.*

Proof. As $s_\tau(F) \leqslant \underline{s}_\tau(F)+1$, Proposition 6.3 implies that $s_\tau(F) \leqslant \left\lfloor \frac{s_q(F)\dim q}{\dim \tau}\right\rfloor + 1$. Since $\left\lfloor \frac{s_q(F)\dim q}{\dim \tau}\right\rfloor = \left\lceil \frac{s_q(F)\dim q+1}{\dim \tau}\right\rceil - 1$, the result follows. □

As per [BG-BM, Example 4.7], for $p \neq 2$ a prime number and $F = \mathbb{Q}_p$, the upper bound can be attained by letting q be the unique anisotropic 4-dimensional form over F with pure subform τ, whereby $s_\tau(F) = 2$ and $s_q(F) = 1$.

In [BG-BM, Lemma 4.8], for q a Pfister form over F, the following lower bound on the q-level over a quadratic extension of F was established. Addressing [BG-BM, Remark 4.24], we show that this result holds for round forms. Additionally, we establish a related upper bound.

Proposition 6.5. *Let q be an anisotropic round form over F and let $K = F(\sqrt{d})$ be a quadratic field extension of F. Then we have that $s_q(K) \leqslant \ell_q(-d) \leqslant 2s_q(K)$.*

Proof. As q is a round form over F, we have that $1 \in D_F(q)$. Hence, we have that $-d \in D_F(\ell_q(-d) \times q)$, and thus that $-1 \in D_K(\ell_q(-d) \times q)$. Thus, $s_q(K) \leqslant \ell_q(-d)$.

If $s_q(K) = s_q(F)$, then $(s_q(K)+1) \times q$ is isotropic (and hence universal) over F, as $1 \in D_F(q)$. Thus, $\ell_q(-d) \leqslant s_q(K) + 1 \leqslant 2s_q(K)$.

If $s_q(K) < s_q(F)$, then $\langle 1\rangle \perp s_q(K) \times q$ is anisotropic over F and isotropic over K. Thus, there exists $a \in F^\times$ such that $a\langle 1, -d\rangle \subset \langle 1\rangle \perp s_q(K) \times q$ by [L, Theorem VII.3.1]. Letting $s_q(K) = 2^n + k$ for integers n and k such that $0 \leqslant k < 2^n$, we have that $a\langle 1, -d\rangle \subset 2^{n+1} \times q$ since $1 \in D_F(q)$. By Witt's Round Form Theorem, $2^{n+1} \times q$ is a round form. Hence, since $a \in D_F(2^{n+1} \times q)$, we can conclude that $\langle 1, -d\rangle \subset a(2^{n+1} \times q) \simeq 2^{n+1} \times q$. Thus, $\ell_q(-d) \leqslant 2^{n+1} \leqslant 2s_q(K)$. □

Thus, we have the following analogue of [BG-BM, Proposition 4.10] for round forms.

Proposition 6.6. *Let q be an anisotropic round form over F and let $d \in F$ be an element such that $\ell_q(-d) = n$. If $K = F(\sqrt{d})$, we have that $2^{r-1} \leqslant s_q(K) < 2^{r+1}$, where r is determined by $2^r \leqslant n < 2^{r+1}$.*

Proof. This follows as an immediate corollary of Proposition 6.5. □

Broadening our considerations from quadratic field extensions to arbitrary function fields of quadratic forms, we highlight a sufficient condition for the weak-isotropy index of a Pfister form to remain unchanged upon passing to such extensions.

Proposition 6.7. *Let q be an anisotropic Pfister form over F and φ an anisotropic form over F. If $\dim\varphi > (s_q(F))\dim q$, then $s_q(F(\varphi)) = s_q(F)$.*

Proof. As $\dim\varphi > (s_q(F))\dim q$, we necessarily have that $s_q(F)$ is finite, whereby Theorem 6.1 implies that $s_q(F) = 2^k$ for some integer $k \geqslant 0$. Hence, $s_q(F) \times q$ is an anisotropic Pfister form over F. Suppose that $s_q(F(\varphi)) < s_q(F)$, whereby $s_q(F) \times q$ becomes hyperbolic over $F(\varphi)$. Invoking [L, Theorem X.4.5], we have that $a\varphi \subset s_q(F) \times q$ for every $a \in D_F(\varphi)$, whereby it follows that $\dim\varphi \leqslant (s_q(F))\dim q$. Thus, our statement follows by contraposition. □

Given the preceding result, for q a Pfister form over F, it is justified to restrict our study of the behaviour of the q-level with respect to function field extensions $F(\varphi)$ where φ is an anisotropic form over F such that $\dim\varphi \leqslant (s_q(F))\dim q$. In this setting, our concluding results establish analogues of the preceding results with respect to quadratic field extensions, thereby generalising and strengthening [BG-BM, Lemma 4.8] and [BG-BM, Proposition 4.10].

In analogy with the q-length of $a \in F^\times$, for q and φ forms over F, we define the *q-length of φ* to be $\ell_q(\varphi) = \min\{n \in \mathbb{N} \mid \varphi \subset n \times q\}$ if such numbers n exist, and set it to be infinite otherwise.

Proposition 6.8. *Let q be an anisotropic Pfister form over F and φ an anisotropic form over F such that $\dim\varphi \leqslant (s_q(F))\dim q$. Then $s_q(F(\varphi)) \leqslant \ell_q(a\varphi) \leqslant 2s_q(F(\varphi))$ for every $a \in D_F(\varphi)$.*

Proof. Since $a\varphi \subset \ell_q(a\varphi) \times q$, the form $\ell_q(a\varphi) \times q$ is isotropic over $F(\varphi)$, whereby $s_q(F(\varphi)) \leqslant \ell_q(a\varphi)$. To establish the remaining inequality, we may assume that $s_q(F(\varphi)) < \infty$.

If $s_q(F(\varphi)) = s_q(F)$, then the Pfister form $2s_q(F(\varphi)) \times q$ is hyperbolic over F. Thus, for $a \in D_F(\varphi)$, we have that $a\varphi \subset a\varphi \perp -a\varphi \subset 2s_q(F(\varphi)) \times q$ since $\dim\varphi \leqslant (s_q(F))\dim q$, establishing the result in this case.

If $s_q(F(\varphi)) < s_q(F)$, then we have that $2s_q(F(\varphi)) \leqslant s_q(F)$ by Theorem 6.1. As $2s_q(F(\varphi)) \times q$ is a Pfister form, it follows that it is anisotropic over F, as otherwise Lemma 1.1 would imply that $\langle 1\rangle \perp s_q(F(\varphi)) \times q$ is isotropic over F, a contradiction in this case. Since $\langle 1\rangle \perp s_q(F(\varphi)) \times q$ is isotropic over $F(\varphi)$, it follows that $2s_q(F(\varphi)) \times q$ becomes hyperbolic over $F(\varphi)$. Thus, invoking [L, Theorem X.4.5], we have that $a\varphi \subset 2s_q(F(\varphi)) \times q$ for every $a \in D_F(\varphi)$, establishing the result. □

The following example shows that the above bounds can be attained.

Example 6.9. If $a\varphi \subset q$, then clearly $s_q(F(\varphi)) = \ell_q(a\varphi) = 1$. Next, let F be a field of q-level at least two. If $a\varphi \simeq 2 \times q$, then $\ell_q(a\varphi) = 2$. Moreover, as the Pfister form $2 \times q$ is hyperbolic over $F(\varphi)$ in this case, we have that $\langle 1 \rangle \perp q$ is isotropic over $F(\varphi)$ by Lemma 1.1. Thus, we have that $\ell_q(a\varphi) = 2s_q(F(\varphi))$ for $a\varphi \simeq 2 \times q$.

Proposition 6.10. *Let q be an anisotropic Pfister form over F and φ an anisotropic form over F such that $\dim \varphi \leqslant (s_q(F)) \dim q$. If $\ell_q(a\varphi) = n$ for some $a \in D_F(\varphi)$ where $2^r < n \leqslant 2^{r+1}$, then $s_q(F(\varphi)) = 2^r$.*

Proof. We will first prove that $s_q(F(\varphi)) \leqslant 2^r$. If $s_q(F) \leqslant 2^r$, then this is clear. Hence, we may assume that $s_q(F) \geqslant 2^{r+1}$. In this case, the Pfister form $2^{r+1} \times q$ is anisotropic over F, as otherwise Lemma 1.1 would imply that $\langle 1 \rangle \perp 2^r \times q$ is isotropic over F, a contradiction. Since $a\varphi \subset n \times q$ for some $a \in D_F(\varphi)$, we have that $2^{r+1} \times q$ is hyperbolic over $F(\varphi)$, and thus that $\langle 1 \rangle \perp 2^r \times q$ is isotropic over $F(\varphi)$ by Lemma 1.1. Hence, we have that $s_q(F(\varphi)) \leqslant 2^r$.

Proposition 6.8 implies that $n \leqslant 2s_q(F(\varphi))$. As $s_q(F(\varphi))$ is necessarily a 2-power by Theorem 6.1, we can conclude that $2^{r+1} \leqslant 2s_q(F(\varphi))$. Hence, $s_q(F(\varphi)) = 2^r$. □

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