DEGREE THREE COHOMOLOGICAL INVARIANTS OF SEMISIMPLE GROUPS

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ABSTRACT. We study the degree 3 cohomological invariants with coefficients in $\mathbb{Q}/\mathbb{Z}(2)$ of a semisimple group over an arbitrary field. A list of all invariants of adjoint groups of inner type is given.

1. INTRODUCTION

1a. **Cohomological invariants.** Let G be a linear algebraic group over a field F (of arbitrary characteristic). The notion of an *invariant* of G was defined in [8] as follows. Consider functor

$$H^1(-,G) : \textsf{\textit{Fields}}_F \longrightarrow \textsf{\textit{Sets}},$$

where $\textsf{\textit{Fields}}_F$ is the category of field extensions of F, taking a field K to the set $H^1(K,G)$ of isomorphism classes of G-torsors over $\operatorname{Spec} K$. Let

$$H : \textsf{\textit{Fields}}_F \longrightarrow \textsf{\textit{Abelian Groups}}$$

be another functor. An *H-invariant* of G is then a morphism of functors

$$I : H^1(-,G) \longrightarrow H.$$

We denote the group of H-invariants of G by $\operatorname{Inv}(G,H)$.

An invariant $I \in \operatorname{Inv}(G,H)$ is called *normalized* if $I(X) = 0$ for the trivial G-torsor X. The normalized invariants form a subgroup $\operatorname{Inv}(G,H)_{\mathrm{norm}}$ of $\operatorname{Inv}(G,H)$ and there is a natural isomorphism

$$\operatorname{Inv}(G,H) \simeq H(F) \oplus \operatorname{Inv}(G,H)_{\mathrm{norm}}.$$

Of particular interest to us is the functor H which takes a field K/F to the Galois cohomology group $H^n\big(K,\mathbb{Q}/\mathbb{Z}(j)\big)$, where the coefficients $\mathbb{Q}/\mathbb{Z}(j)$, $j \geq 0$, are defined as the direct sum of the colimit over n of the Galois modules $\mu_m^{\otimes j}$, where μ_m is the Galois module of m^{th} roots of unity, and a p-component in the case $p = \operatorname{char}(F) > 0$ defined via logarithmic de Rham-Witt differentials (see [13, I.5.7], [14]).

We write $\operatorname{Inv}^n\big(G,\mathbb{Q}/\mathbb{Z}(j)\big)$ for the group of *cohomological invariants of G of degree n with coefficients in $\mathbb{Q}/\mathbb{Z}(j)$.*

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If G is connected, then $\operatorname{Inv}^1\big(G,\mathbb{Q}/\mathbb{Z}(j)\big)_{\mathrm{norm}} = 0$ (see [15, Proposition 31.15]). The degree 2 cohomological invariants with coefficients in $\mathbb{Q}/\mathbb{Z}(1)$ (equivalently, the invariants with values in the Brauer group Br) of a smooth connected group were determined in [1]:

$$\operatorname{Inv}^2(G,\mathrm{Br})_{\mathrm{norm}} = \operatorname{Inv}^2\big(G,\mathbb{Q}/\mathbb{Z}(1)\big)_{\mathrm{norm}} \simeq \operatorname{Pic}(G).$$

In particular, for a semisimple group G we have

$$\operatorname{Inv}^2\big(G,\mathbb{Q}/\mathbb{Z}(1)\big)_{\mathrm{norm}} \simeq \widehat{C}(F),$$

where $\widehat{C}(F)$ is the group of characters defined over F of the kernel C of the universal cover $\widetilde{G} \to G$ by [21, Prop. 6.10].

The group of degree 3 invariants $\operatorname{Inv}^3\big(G,\mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}}$ was determined by Rost in the case when G is simply connected quasi-simple. This group is finite cyclic with a canonical generator called the *Rost invariant* (see [8, Part II]).

In the present paper, based on the results in [18], we extend Rost's result to all semisimple groups.

Theorem. Let G be a semisimple group over a field F. Then there is an exact sequence

$$0 \longrightarrow \mathrm{CH}^2(BG)_{\mathrm{tors}} \longrightarrow H^1\big(F,\widehat{C}(1)\big) \xrightarrow{\sigma}$$
$$\operatorname{Inv}^3\big(G,\mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}} \longrightarrow Q(G)/\operatorname{Dec}(G) \longrightarrow H^2\big(F,\widehat{C}(1)\big).$$

Here BG is the classifying space of G and $Q(G)/\operatorname{Dec}(G)$ is the group defined in Section 3c in terms of the combinatorial data associated with G (the root system, weight and root lattices).

If G is simply connected, the character group $\widehat{C}$ is trivial and we obtain Rost's theorem mentioned above.

The main result has clearer form for adjoint groups G of inner type. In this case every character of C is defined over F, i.e., $\widehat{C} = \widehat{C}(F)$ We show that the group $\operatorname{Inv}^3\big(G,\mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{dec}} := \operatorname{Im}(\sigma)$ of *decomposable* invariants (given by a cup-product with the degree 2 invariants), is canonically isomorphic to $\widehat{C}\otimes F^\times$. The factor group $\operatorname{Inv}^3\big(G,\mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{ind}}$ of $\operatorname{Inv}^3\big(G,\mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}}$ by the decomposable invariants is nontrivial if and only if G has a simple component of type C_n or D_n (when n is divisible by 4), E_6 or E_7. If G is simple, the group of indecomposable invariants is cyclic with a canonical generator restricting to a multiple of the Rost invariant.

We will use the following notation in the paper.

F is the base field,

F_{sep} a separable closure of F,

$\Gamma_F = \operatorname{Gal}(F_{\mathrm{sep}}/F)$.

For a complex A of étale sheaves on a variety X, we write $H^*(X,A)$ for the étale (hyper-)cohomology group of X with values in A.

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2. Preliminaries

2a. **Cohomology of BG.** Let G be a connected algebraic group over a field F and let V be a generically free representation of G such that there is an open G-invariant subscheme $U \subset V$ and a G-torsor $U \to U/G$ such that $U(F) \neq \emptyset$ (see [26, Remark 1.4]).

Let H be a (contravariant) functor from the category of smooth varieties over F to the category of abelian groups. Very often the value $H(U/G)$ is independent (up to canonical isomorphism) of the choice of the representation V provided the codimension of $V \setminus U$ in V is sufficiently large. This is the case, for example, if $H = \mathrm{CH}^i$, the Chow group functor of cycles of codimension i (see [26] or [5]). We write $H(BG)$ for $H(U/G)$ and view U/G as an "approximation" for the "classifying space" BG of G.

We have the two maps $p_i^* : H(U/G) \to H\big((U \times U)/G\big)$, $i = 1, 2$, induced by the projections $p_i : (U \times U)/G \to U/G$. An element $h \in H(U/G)$ is called *balanced* if $p_1^*(h) = p_2^*(h)$. We write $H(U/G)_{\mathrm{bal}}$ for the subgroup of all balanced elements in $H(U/G)$.

Write $\mathcal{H}^n\big(\mathbb{Q}/\mathbb{Z}(j)\big)$ for the Zariski sheaf on a smooth scheme X associated to the presheaf $S \mapsto H^n\big(S, \mathbb{Q}/\mathbb{Z}(j)\big)$.

Let $u \in H^0_{\mathrm{Zar}}\big(U/G, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big)_{\mathrm{bal}}$. Define an invariant $I_u \in \mathrm{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j))$ as follows (see [1]). Let X be a G-torsor over a field extension K/F. Choose a point $x \in (U/G)(K)$ such that X is isomorphic to the pull-back via x of the versal G-torsor $U \to U/G$ and set $I_u(X) = x^*(u)$, where

$$x^* : H^0_{\mathrm{Zar}}\big(U/G, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big) \longrightarrow H^0_{\mathrm{Zar}}\big(\mathrm{Spec}\, K, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big) = H^n\big(K, \mathbb{Q}/\mathbb{Z}(j)\big)$$

is the pull-back homomorphism given by $x : \mathrm{Spec}(K) \to U/G$. The fact that the element u is balanced ensures that $x^*(u)$ does not depend on the choice of the point x (see [1, Lemma 3.2]).

Write $\overline{H}^0_{\mathrm{Zar}}\big(U/G, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big)$ for the factor group of $H^0_{\mathrm{Zar}}\big(U/G, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big)$ by the natural image of $H^n\big(F, \mathbb{Q}/\mathbb{Z}(j)\big)$.

Proposition 2.1. ([1, Corollary 3.4]) The assignment $u \mapsto I_u$ yields an isomorphism

$$\overline{H}^0_{\mathrm{Zar}}\big(U/G, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big)_{\mathrm{bal}} \xrightarrow{\sim} \mathrm{Inv}^n\big(G, \mathbb{Q}/\mathbb{Z}(j)\big)_{\mathrm{norm}}.$$

2b. **The map α_G.** Let G be a semisimple group over F and let C be the kernel of the universal cover $\widetilde{G} \to G$. For a character $\chi \in \widehat{C}(F)$ over F consider the push-out diagram

$$\begin{tikzcd}
1 \arrow[r] & C \arrow[r] \arrow[d, "\chi"'] & \widetilde{G} \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathbb{G}_m \arrow[r] & G' \arrow[r] & G \arrow[r] & 1.
\end{tikzcd}$$

We define a map

$$\alpha_G : H^1(F, G) \longrightarrow \mathrm{Hom}\big(\widehat{C}(F), \mathrm{Br}(F)\big)$$

by $\alpha_G(\xi)(\chi) = \delta(\xi)$, where $\delta : H^1(F, G) \to H^2(F, \mathbb{G}_m) = \mathrm{Br}(F)$ is the connecting map for the bottom row of the diagram.

Example 2.2. Let $G = \mathbf{PGL}_n$. Then $\widehat{C} = \mathbb{Z}/n\mathbb{Z}$ and the map α_G takes the class $[A] \in H^1(F, \mathbf{PGL}_n)$ of a central simple algebra A of degree n to the homomorphism $i + n\mathbb{Z} \mapsto i[A] \in \mathrm{Br}(F)$.

Let C' be the center of G. Recall that there is the *Tits homomorphism* (see [15, Theorem 27.7])

$$\beta_G : \widehat{C}'(F) \longrightarrow \mathrm{Br}(F).$$

A central simple algebra over F representing the class β_G for some $\chi \in \widehat{C}'(F)$ is called a *Tits algebra* of G over F.

In the following proposition we relate the maps α_G and $\beta_{\widetilde{G}}$.

Proposition 2.3. *Let G be a semisimple group, X a G-torsor over F and $\chi \in \widehat{C}'(F)$, where C' is the center of the universal cover $\widetilde{G}$ of G. Let ${}^XG := \mathbf{Aut}_G(X)$ be the twist of G by X and ${}^X\widetilde{G}$ the universal cover of ${}^XG$. Then*

$$\alpha_G(X)(\chi|_C) = \beta_{{}^X\widetilde{G}}(\chi) - \beta_{\widetilde{G}}(\chi),$$

where $C \subset C'$ is the kernel of $\widetilde{G} \to G$.

Proof. By [15, §31], there exist a unique (up to isomorphism) G-torsor Y such that the twist ${}^YG = \mathbf{Aut}_G(Y)$ is quasi-split and $\alpha_G(Y)(\chi|_C) = -\beta_{\widetilde{G}}(\chi)$. If ${}^XY$ is the twist of Y by X, then $\mathbf{Aut}_{{}^XG}({}^XY) \simeq \mathbf{Aut}_G(Y)$ is quasi-split. Hence $\alpha_{{}^XG}({}^XY)(\chi|_C) = -\beta_{{}^X\widetilde{G}}(\chi)$. It follows from [15, Proposition 28.12] that $\alpha_{{}^XG}({}^XY) + \alpha_G(X) = \alpha_G(Y)$. □

2c. **Admissible maps.** Let G be a split simply connected group over F and Π a set of simple roots of G.

Proposition 2.4. (cf. [9, Proposition 5.5]) Let G be a split simply connected group over F, C the center of G. Let Π' be a subset of Π and let G' be the subgroup of G generated by the root subgroups of all roots in Π'. Then G' is a simply connected group and $C \subset G'$ if and only if every fundamental weight w_α for $\alpha \in \Pi \setminus \Pi'$ is contained in the root lattice Λ_r of G.

Proof. The group G' is simply connected by [22, 5.4b]. The images of the co-roots $\alpha^* : \mathbb{G}_m \to T$ for $\alpha \in \Pi'$ generate the maximal torus $T' = G' \cap T$ of G'. Therefore, the character group Ω of the torus T/T' coincides with

$$\{\lambda \in \widehat{T} \quad \text{such that} \quad \langle \lambda, \alpha^* \rangle = 0 \quad \text{for all} \quad \alpha \in \Pi'\}$$

and hence Ω is generated by the fundamental weights w_β for all $\beta \in \Pi \setminus \Pi'$. We have $\widehat{T}' = \Lambda_w/\Omega$ and $\widehat{C} = \Lambda_w/\Lambda_r$. Therefore, $C \subset G' \cap T = T'$ if and only if $\Omega \subset \Lambda_r$. □

A homomorphism $a : \widehat{C}(F) \to \mathrm{Br}(F)$ is called *admissible* if $\operatorname{ind} a(\chi)$ divides $\operatorname{ord}(\chi)$ for every $\chi \in \widehat{C}(F)$.

Example 2.5. Suppose G is the product of split adjoint groups of type A. By Example 2.2, every admissible map belongs to the image of α_G.

Proposition 2.6. *Let G be a split adjoint group over F. Then every admissible map in $\mathrm{Hom}\big(\widehat{C}(F), \mathrm{Br}(F)\big)$ belongs to the image of α_G.*

Proof. Let Π' be the subset of Π of all roots α such that $w_\alpha \in \Lambda_r$ and let G' be the subgroup of $\widetilde{G}$ generated by the root subgroups for all roots in Π'. Then by Proposition 2.4, G' is a simply connected group such that $C \subset G'$. Let C' be the center of G' and set $C'' := C'/C$. By Lemma 2.7 below, the top row in the commutative diagram

$$\begin{tikzcd}
H^1(F, G'/C) \arrow[r] \arrow[d, "\alpha_{G'/C}"'] & H^1(F, G'/C') \arrow[r] \arrow[d, "\alpha_{G'/C'}"'] & \mathrm{Hom}\big(\widehat{C}''(F), \mathrm{Br}(F)\big) \arrow[d, equal] \\
\mathrm{Hom}\big(\widehat{C}(F), \mathrm{Br}(F)\big) \arrow[r, hook] & \mathrm{Hom}\big(\widehat{C}'(F), \mathrm{Br}(F)\big) \arrow[r] & \mathrm{Hom}\big(\widehat{C}''(F), \mathrm{Br}(F)\big)
\end{tikzcd}$$

is exact.

Let $a \in \mathrm{Hom}\big(\widehat{C}(F), \mathrm{Br}(F)\big)$ be an admissible map. Then the image a' of a in $\mathrm{Hom}\big(\widehat{C}'(F), \mathrm{Br}(F)\big)$ is also admissible. Inspection shows that every component of the Dynkin diagram of G' is of type A. (A root α belongs to Π' if and only if the i^{th} row of the inverse C^{-1} of the Cartan matrix is integer, see Section 4b.) By Example 2.5, a' belongs to the image of $\alpha_{G'/C'}$. A diagram chase shows that a belongs to the image of $\alpha_{G'/C}$. The map $\alpha_{G'/C}$ is the composition of $H^1(F, G'/C) \to H^1(F, G)$ and α_G, hence a belongs to the image of α_G. □

Lemma 2.7. *Let $G_1 \to G_2$ be a central isogeny of split semisimple groups with the kernel C_1. Then the sequence*

$$H^1(F, G_1) \longrightarrow H^1(F, G_2) \longrightarrow \mathrm{Hom}\big(\widehat{C}_1(F), \mathrm{Br}(F)\big)$$

with the second map the composition of α_{G_2} and the restriction map on C_1, is exact.

Proof. The group C_1 is diagonalizable as G_1 is split. Let T be a split torus containing C_1 as a subgroup. The push-out diagram

$$\begin{tikzcd}
1 \arrow[r] & C_1 \arrow[r] \arrow[d, "\chi"'] & G_1 \arrow[r] \arrow[d] & G_2 \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & T \arrow[r] & G_3 \arrow[r] & G_2 \arrow[r] & 1
\end{tikzcd}$$

yields a commutative diagram

$$\begin{tikzcd}
H^1(F, G_1) \arrow[r] \arrow[d] & H^1(F, G_2) \arrow[r] \arrow[d, equal] & \mathrm{Hom}\big(\widehat{C}_1(F), \mathrm{Br}(F)\big) \arrow[d, "\chi^*"'] \\
H^1(F, G_3) \arrow[r] & H^1(F, G_2) \arrow[r] & \mathrm{Hom}\big(\widehat{T}(F), \mathrm{Br}(F)\big).
\end{tikzcd}$$

The bottom row is exact as $\mathrm{Hom}\big(\widehat{T}(F), \mathrm{Br}(F)\big) = H^2(F,T)$. The left vertical arrow is surjective since $H^1(F, \mathrm{Coker}(\chi)) = 1$ by Hilbert's Theorem 90. The result follows by diagram chase. $\square$

2d. **The morphism** β_f**.** Let G be a semisimple group, C the kernel of the universal cover $\widetilde{G} \to G$ and $f : X \to \operatorname{Spec} F$ a G-torsor. Write $\mathbb{Z}_f(1)$ for the cone of the natural morphism $\mathbb{Z}_F(1) \to Rf_*\mathbb{Z}_X(1)$ of complexes of étale sheaves over $\operatorname{Spec} F$, where $\mathbb{Z}(1) = \mathbb{G}_m[-1]$. The composition (see [18, §4])

$$\beta_f : \widehat{C} \simeq \tau_{\leq 2}\mathbb{Z}_f(1)[2] \longrightarrow \mathbb{Z}_f(1)[2] \longrightarrow \mathbb{Z}_F(1)[3]$$

yields a homomorphism

$$\beta_f^* : \widehat{C}(F) \longrightarrow H^3\big(F, \mathbb{Z}_F(1)\big) = \mathrm{Br}(F).$$

In the following proposition we relate the maps β_f^* and α_G.

Proposition 2.8. *For a G-torsor $f : X \to \operatorname{Spec} F$, we have $\beta_f^* = \alpha_G(X)$.*

Proof. By [18, Example 6.12], the map β_f^* coincides with the connecting homomorphism for the exact sequence

$$1 \longrightarrow F_{\mathrm{sep}}^\times \longrightarrow F_{\mathrm{sep}}(X)^\times \longrightarrow \mathrm{Div}(X_{\mathrm{sep}}) \longrightarrow \widehat{C}_{\mathrm{sep}} \longrightarrow 0, \tag{2.1}$$

where Div is the divisor group (recall that $\widehat{C}_{\mathrm{sep}} = \mathrm{Pic}(X_{\mathrm{sep}})$).

Consider first the case $G = \mathbf{PGL}_n$ and $X = \mathrm{Isom}(B, M_n)$ is the variety of isomorphisms between a central simple algebra B of degree n and the matrix algebra M_n over F. We have $C = \boldsymbol{\mu}_n$ and $\widehat{C} = \mathbb{Z}/n\mathbb{Z}$. The exact sequence (2.1) for the Severi-Brauer variety S of B in place of X gives the connecting homomorphism $\mathbb{Z} \to \mathrm{Br}(F)$ that takes 1 to the class $[B]$ by [12, Theorem 5.4.10]. A natural map between the two exact sequences induced by the natural morphism $X \to S$ and Example 2.2 yield

$$\beta_f^*(\bar{1}) = [B] = \alpha_{\mathbf{PGL}_n}(X)(\bar{1}). \tag{2.2}$$

Suppose now that $G = \mathbf{PGL}_1(A)$ for a central simple algebra A of degree n. Consider the $\mathbf{PGL}_n$-torsor $Y = \mathrm{Isom}(A, M_n)$. Then G is the twist of $\mathbf{PGL}_n$ by Y. The G-torsor $Z = \mathrm{Isom}(B, A)$ is the twist of X by Y. It follows from [15, Proposition 28.12] that

$$\alpha_G(Z)(\bar{1}) = \alpha_{\mathbf{PGL}_n}(X)(\bar{1}) - \alpha_{\mathbf{PGL}_n}(Y)(\bar{1}) = [B] - [A]. \tag{2.3}$$

The group homomorphism $\mathbf{PGL}_1(B) \times \mathbf{PGL}_1(A^{op}) \to \mathbf{PGL}_1(B \otimes A^{op})$ takes the torsor $Z \times \mathrm{Isom}(A^{op}, A^{op})$ to $V := \mathrm{Isom}(B \otimes A^{op}, A \otimes A^{op})$. Let g and h be the structure morphisms for Z and V, respectively. It follows from (2.2) applied to β_h^* and (2.3) that

$$\beta_g^*(\bar{1}) = \beta_h^*(\bar{1}) = [B] - [A] = \alpha_G(Z)(\bar{1}). \tag{2.4}$$

Now consider the general case. By [25, Théorème 3.3], for every $\chi \in \widehat{C}(F)$, there is a central simple algebra A (of degree n) over F and a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & C & \longrightarrow & \widetilde{G} & \longrightarrow & G & \longrightarrow & 1 \\ & & \Big\downarrow{\scriptstyle\chi} & & \Big\downarrow & & \Big\downarrow & & \\ 1 & \longrightarrow & \boldsymbol{\mu}_n & \longrightarrow & \mathbf{SL}_1(A) & \longrightarrow & \mathbf{PGL}_1(A) & \longrightarrow & 1. \end{array}$$

A G-torsor $f : X \to \operatorname{Spec} F$ yields a $\mathbf{PGL}_1(A)$-torsor, say $k : W \to \operatorname{Spec} F$. We have by (2.4),

$$\beta_f^*(\chi) = \beta_k^*(\bar{1}) = \alpha_{\mathbf{PGL}_1(A)}(W)(\bar{1}) = \alpha_G(X)(\chi). \qquad \square$$

3. The group $\operatorname{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big)$

In this section we determine the group $\operatorname{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big)$ of degree 3 cohomological invariants of a semisimple group G.

Recall first a construction of degree two cohomological invariants of G with coefficients in $\mathbb{Q}/\mathbb{Z}(1)$, or, equivalently, the invariants with values in the Brauer group. Every character $\chi \in \widehat{C}(F)$ yields an invariant I_χ of G of degree 2 with coefficients in $\mathbb{Q}/\mathbb{Z}(1)$ defined by

$$I_\chi(X) = \alpha_G(X)(\chi_K) \in \operatorname{Br}(K).$$

By [1, Theorem 2.4], the assignment $\chi \mapsto I_\chi$ yields an isomorphism

$$\widehat{C}(F) \xrightarrow{\sim} \operatorname{Inv}^2\big(G, \mathbb{Q}/\mathbb{Z}(1)\big)_{\text{norm}}.$$

3a. **Representation ring.** (See [25].) Write $R(G)$ for the representation ring of G, i.e., $R(G)$ is the Grothendieck group of the category of finite dimensional representations of G. As an abelian group $R(G)$ is free with basis the isomorphism classes of irreducible representations.

Consider the weight lattice Λ of G (the character group of a maximal split torus over F_{sep}) as a Γ_F-lattice with respect to the $*$-action (see [24]). Let Γ' be the (finite) factor group of Γ_F acting faithfully on Λ. Write Δ for the semidirect product of the Weyl group W of G and Γ' with respect to the natural action of Γ' on W. The group Δ acts naturally on Λ.

Assigning to a representation of G the formal sum of its weights, we get an injective homomorphism

$$\operatorname{ch} : R(G) \longrightarrow \mathbb{Z}[\Lambda]^\Delta.$$

For any $\lambda \in \Lambda$ write A_λ for the corresponding Tits algebra (over the field of definition of λ) and $\Delta(\lambda)$ for the sum $\sum e^{\lambda'}$ in $\mathbb{Z}[\Lambda]^\Delta$, where λ' runs over the Δ-orbit of λ (we employ the exponential notation for $\mathbb{Z}[\Lambda]$). By [8, Part II, Theorem 10.11], the image of $R(G)$ in $\mathbb{Z}[\Lambda]^\Delta$ is generated by $\operatorname{ind}(A_\lambda) \cdot \Delta(\lambda)$ over all $\lambda \in \Lambda$.

In particular, if G is quasi-split, all Tits algebras are trivial and hence ch is an isomorphism.

Example 3.1. Consider the variety $\mathcal{X}$ of maximal tori in G and the closed subscheme $\mathcal{T} \subset G \times \mathcal{X}$ of all pairs (g, T) with $g \in T$. The generic fiber of the projection $\mathcal{T} \to \mathcal{X}$ is a maximal torus in $G_{F(\mathcal{X})}$, it is called the *generic maximal torus* T_{gen} of G. By [27, Theorem 1], if G is split, the decomposition group of T_{gen} coincides with the Weyl group W. It follows that if G is quasi-split, then Δ is the decomposition group of T_{gen}. Moreover, ch is an isomorphism, hence the restriction homomorphism $R(G) \to R(T_{\mathrm{gen}}) = \mathbb{Z}[\Lambda]^{\Delta}$ is an isomorphism for a quasi-split G.

3b. **Root systems and invariant quadratic forms.** Let $\{\alpha_1, \alpha_2 \dots \alpha_n\}$ be a set of simple roots of an irreducible root system in a vector space V, $\{w_1, w_2, \dots, w_n\}$ the corresponding fundamental weights generating the weight lattice Λ_w and W the Weyl group.

Consider the n-columns $\alpha := (\alpha_1, \alpha_2, \dots, \alpha_n)^t$ and $w := (w_1, w_2, \dots, w_n)^t$. Then $\alpha = Cw$, where $C = (c_{ij})$ is the Cartan matrix (see [2, Chapitre VI]). There is a (unique) W-invariant bilinear form on the dual space V^* such that the length of a short co-root is equal to 1. Let $D := \operatorname{diag}(d_1, d_2, \dots, d_n)$ be the diagonal matrix with d_i the length of the i^{th} co-root. Then DC is a symmetric even integer matrix (i.e., the diagonal terms are even).

Note that if A is a symmetric $n \times n$ matrix over $\mathbb{Q}$, then $\frac{1}{2} w^t A w$ is contained in $\mathsf{Sym}^2(\Lambda_w)$ if and only if the matrix A is even integer.

Consider the integer quadratic form

$$q := \frac{1}{2} w^t DCw \in \mathsf{Sym}^2(\Lambda_w)$$

on Λ_r^*, where Λ_r is the root lattice. Recall that the Weyl group W acts naturally on Λ_w.

Lemma 3.2. *The quadratic form q is W-invariant.*

Proof. Let s_i be the reflection with respect to α_i. It suffices to prove that $s_i(q) = q$. We have $s_i(w) = w - \alpha_i e_i$. Hence

$$\begin{aligned} s_i(q) &= \frac{1}{2}(w - \alpha_i e_i)^t DC(w - \alpha_i e_i) \\ &= q - \alpha_i e_i^t D(Cw - \frac{1}{2}\alpha_i C e_i) \\ &= q - \alpha_i d_i(e_i^t \alpha - \frac{1}{2}\alpha_i e_i^t C e_i) \\ &= q - \alpha_i d_i(\alpha_i - \frac{1}{2}\alpha_i c_{ii}) = q \end{aligned}$$

as $c_{ii} = 2$. ∎

If α_i^* is a short co-root, then $q(\alpha_i^*) = d_i = 1$ since $\langle w_j, \alpha_i^* \rangle = \delta_{ji}$. It follows that q is a (canonical) generator of the cyclic group $\mathsf{Sym}^2(\Lambda_w)^W$.

Example 3.3. For the root system of type A_{n-1}, $n \geq 2$, we have $\Lambda_w = \mathbb{Z}^n / \mathbb{Z}e$, where $e = e_1 + e_2 + \cdots + e_n$. The root lattice Λ_r is generated by the simple roots $\bar{e}_1 - \bar{e}_2, \bar{e}_2 - \bar{e}_3, \dots, \bar{e}_{n-1} - \bar{e}_n$. The Weyl group W is the symmetric group S_n acting naturally on Λ_w. The generator of $\mathsf{Sym}^2(\Lambda_w)^W$ is the form

$$q = -\sum_{i<j} \bar{x}_i \bar{x}_j = \frac{1}{2}\sum_{i=1}^{n} \bar{x}_i^2.$$

The group $\mathsf{Sym}^2(\Lambda_r)^W = \mathsf{Sym}^2(\Lambda_r) \cap \mathsf{Sym}^2(\Lambda_w)^W$ is also cyclic with the canonical generator a positive multiple of q.

Proposition 3.4. *Let m be the smallest positive integer such that the matrix mDC^{-1} is even integer. Then mq is a generator of $\mathsf{Sym}^2(\Lambda_r)^W$.*

Proof. Rewrite q in the form $q = \frac{1}{2}(C^{-1}\alpha)^t DC(C^{-1}\alpha) = \frac{1}{2}\alpha^t DC^{-1}\alpha$. The multiple mq is contained in $\mathsf{Sym}^2(\Lambda_r)$ if and only if the matrix mDC^{-1} is even integer. $\square$

3c. **The groups** Dec(G) $\subset Q(G)$**.** Let A be a lattice. Consider the *abstract total Chern class* homomorphism

$$c_\bullet : \mathbb{Z}[A] \longrightarrow \mathsf{Sym}^\bullet(A)[[t]]^\times$$

defined by $c_\bullet(e^a) = 1 + at$. We define the *abstract Chern class maps*

$$c_i : \mathbb{Z}[A] \longrightarrow \mathsf{Sym}^i(A), \quad i \geq 0,$$

by $c_\bullet(x) = \sum_{i\geq 0} c_i(x)t^i$. Clearly, $c_0(x) = 1$,

$$c_1\Big(\sum_i e^{a_i}\Big) = \sum_i a_i, \quad c_2\Big(\sum_i e^{a_i}\Big) = \sum_{i<j} a_i a_j,$$

c_1 is a homomorphism and

$$c_2(x+y) = c_2(x) + c_1(x)c_1(y) + c_2(y)$$

for all $x, y \in \mathbb{Z}[A]$.

If a group W acts on A, then all the c_i are W-equivariant.

Suppose that $A^W = 0$. Then c_1 is zero on $\mathbb{Z}[A]^W$ and c_2 yields a group homomorphism

$$c_2 : \mathbb{Z}[A]^W \to \mathsf{Sym}^2(A)^W. \tag{3.1}$$

We write Dec(A) for the image of this homomorphism. The group Dec(A) is generated by the *decomposable* elements $\sum_{i<j} a_i a_j$, where $\{a_1, a_2, \dots .a_n\}$ is a W-invariant subset of A. We also have

$$c_2(xy) = \operatorname{rank}(x)c_2(y) + \operatorname{rank}(y)c_2(x) \tag{3.2}$$

for all $x, y \in \mathbb{Z}[A]^W$, where rank $: \mathbb{Z}[A] \to \mathbb{Z}$ is the map $e^a \mapsto 1$. If $S \subset A$ is a finite W-invariant subset, then since $\sum_{x\in S} x \in A^W = 0$, we have

$$c_2\Big(\sum_{a\in S} e^a\Big) = -\frac{1}{2}\sum_{a\in S} a^2. \tag{3.3}$$

Let G be a semisimple group over F. Recall that the weight lattice Λ is a Δ-module (see Section (3a)). Note that $\Lambda^W = 0$, so we have the homomorphism of Γ_F-modules (3.1) with $A = \Lambda$.

Set

$$Q(G) := \mathsf{Sym}^2(\Lambda)^\Delta = \big(\mathsf{Sym}^2(\Lambda)^W\big)^{\Gamma_F}.$$

and write $\operatorname{Dec}(G)$ for the image of the composition

$$\tau : R(G) \xrightarrow{\mathrm{ch}} \mathbb{Z}[\Lambda]^\Delta \xrightarrow{c_2} \mathsf{Sym}^2(\Lambda)^\Delta = Q(G). \tag{3.4}$$

Example 3.5. The map $\tau : R(\mathbf{SL}_n) \to Q(\mathbf{SL}_n)$ takes the class of the tautological representation to the quadratic form $\sum_{i<j} \bar{x}_i\bar{x}_j$ which is the negative of the canonical generator of $Q(\mathbf{SL}_n)$ (see Example 3.3).

It follows from Example 3.5 that if G is a quasi-simple group, then for a representation ρ of G, we have $\tau(\rho) = -N(\rho)q$, where $N(\rho)$ is the Dynkin index of ρ (see [7]). Hence the image of $\operatorname{Dec}(G)$ under τ is equal to $n_G\mathbb{Z}q$, where n_G is the gcd of the Dynkin indexes of all the representations of G. The numbers n_G for split adjoint groups G of types B_n, C_n and E_7 were computed in [7] (see also Section 4b).

A *loop* in G is a group homomorphism $\mathbb{G}_m \to G_{\mathrm{sep}}$ over F_{sep} (see [15, §31]). By [8, Part II, §7]), the group $Q(G)$ has an intrinsic description as the group of all Γ_F-invariant quadratic integral-valued functions on the set of all loops in G. It follows that a homomorphism $G \to G'$ of semisimple groups yields a group homomorphism $Q(G') \to Q(G)$. The functoriality of the Chern class shows that this homomorphism takes $\operatorname{Dec}(G')$ into $\operatorname{Dec}(G)$.

3d. **The key diagram.** Let V be a generically free representation of G such that there is an open G-invariant subscheme $U \subset V$ and a G-torsor $U \to U/G$ such that $U(F) \neq \emptyset$ (see Section 2a). We assume in addition that $V \setminus U$ is of codimension at least 3.

By [14, Th. 1.1], there is an exact sequence

$$0 \longrightarrow \mathrm{CH}^2(U^n/G) \longrightarrow \overline{H}^4\big(U^n/G, \mathbb{Z}(2)\big) \longrightarrow \overline{H}^0_{\mathrm{Zar}}\big(U^n/G, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))\big) \longrightarrow 0$$

for every n. We can view this as an exact sequence of cosimplicial groups. The group $\mathrm{CH}^2(U^n/G)$ is independent of n, so it represents a constant cosimplicial groups $\mathrm{CH}^2(BG)$. Therefore, we have an exact sequence

$$0 \longrightarrow \mathrm{CH}^2(BG) \longrightarrow \overline{H}^4\big(U/G, \mathbb{Z}(2)\big)_{\mathrm{bal}} \longrightarrow \overline{H}^0_{\mathrm{Zar}}\big(U/G, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))\big)_{\mathrm{bal}} \longrightarrow 0.$$

The right group in the sequence is canonically isomorphic to $\operatorname{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}}$ by Proposition 2.1, and hence is independent of V. Therefore, the middle term is also independent of V and we write $\overline{H}^4\big(BG, \mathbb{Z}(2)\big)$ for $\overline{H}^4\big(U/G, \mathbb{Z}(2)\big)_{\mathrm{bal}}$. Therefore, we have the exact row in the following diagram with the exact column given by [18, Theorem 5.3]:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & H^1\big(F, \widehat{C}(1)\big) \arrow[d] \arrow[dr, "\sigma"] & & \\
0 \arrow[r] & \mathrm{CH}^2(BG) \arrow[r] \arrow[dr, "\gamma"'] & \overline{H}^4\big(BG, \mathbb{Z}(2)\big) \arrow[r] \arrow[d] & \mathrm{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}} \arrow[r] & 0 \\
 & & Q(G) \arrow[d, "\theta_G^*"] & & \\
 & & H^2\big(F, \widehat{C}(1)\big), & &
\end{tikzcd}$$

where $\widehat{C}(1)$ is the derived tensor product $\widehat{C} \overset{L}{\otimes} \mathbb{Z}_Y(1)$ in the derived category of étale sheaves on F. Explicitly (see [18, Section 4c]),

$$\widehat{C}(1) = \mathrm{Tor}_1^{\mathbb{Z}}(\widehat{C}_{\mathrm{sep}}, F_{\mathrm{sep}}^{\times}) \oplus (\widehat{C}_{\mathrm{sep}} \otimes F_{\mathrm{sep}}^{\times})[-1].$$

Example 3.6. The group $\mathbf{SL}_n$ is special simply connected, hence $\widehat{C} = 0$ and $\mathrm{Inv}^3\big(\mathbf{SL}_n, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}} = 0$. It follows that we have isomorphisms of infinite cyclic groups

$$\gamma : \mathrm{CH}^2(B\,\mathbf{SL}_n) \xrightarrow{\sim} \overline{H}^4(B\,\mathbf{SL}_n, \mathbb{Z}(2)) \xrightarrow{\sim} Q(\mathbf{SL}_n).$$

The group $\mathrm{CH}^2(B\,\mathbf{SL}_n)$ is generated by c_2 of the tautological representation by [20, §2].

3e. **The map σ.** The map σ is defined as follows (see [18, §5]). Let $f : X \to \operatorname{Spec} K$ be a G-torsor over a field extension K/F, so we have a morphism $\beta_f : \widehat{C} \to \mathbb{Z}_K(1)[3]$ as in Section 2d, and therefore, the composition

$$\widehat{C}(1) = \widehat{C} \overset{L}{\otimes} \mathbb{Z}_F(1) \xrightarrow{\beta_f \overset{L}{\otimes} \mathrm{Id}} \big(\mathbb{Z}_K(1) \overset{L}{\otimes} \mathbb{Z}_F(1)\big)[3] \longrightarrow \mathbb{Z}_K(2)[3],$$

which induces a homomorphism $H^1\big(F, \widehat{C}(1)\big) \to H^4\big(K, \mathbb{Z}(2)\big) = H^3\big(K, \mathbb{Q}/\mathbb{Z}(2)\big)$. Then the value of the invariant $\sigma(\alpha)$ for an element $\alpha \in H^1(F, \widehat{C}(1))$ is equal to the image of α under this homomorphism.

Let $\chi \in \widehat{C}(F)$ and $a \in F^{\times}$. By [18, Remark 5.2], we have $\chi \cup (a) \in H^1(F, \widehat{C}(1))$ and therefore, $\sigma(\chi \cup (a))$ is the invariant taking a G-torsor X over K to $\alpha_G(X)(\chi_K) \cup (a) \in H^3\big(K, \mathbb{Q}/\mathbb{Z}(2)\big)$. Here the cup-product is taken with respect to the pairing

$$\mathrm{Br}(K) \otimes K^{\times} = H^2\big(K, \mathbb{Q}/\mathbb{Z}(1)\big) \otimes H^1\big(K, \mathbb{Z}(1)\big) \longrightarrow H^3\big(K, \mathbb{Q}/\mathbb{Z}(2)\big).$$

3f. **The map γ.** We will determine the map γ in the key diagram.

Lemma 3.7. *The maps γ and $\overline{H}^4\big(BG, \mathbb{Z}(2)\big) \to Q(G)$ are functorial in G.*

Proof. In [18] the map γ is given by the composition

$$\begin{aligned}\mathrm{CH}^2(BG) \longrightarrow H^4(BG, \mathbb{Z}(2)) \xrightarrow{\sim} H^3\big(BG, \mathbb{Z}_f(2)\big) \xrightarrow{\sim} \\ H^3\big(BG, \tau_{\leq 3}\mathbb{Z}_f(2)\big) \longrightarrow H^1_{\mathrm{Zar}}(BG, K_2)^{\Gamma_F} \to D(G),\end{aligned}$$

where $\mathbb{Z}_f(2)$ is the cone of $\mathbb{Z}_{BG}(2) \to Rf_*\mathbb{Z}_{EG}(2)$ for the versal G-torsor $f : EG \to BG$ and the group $D(G)$ containing $Q(G)$ is defined in [18]. The first four homomorphisms are functorial in G, and the last one is functorial as was shown in [8, Page 116] in the case G is simply connected. The proof also goes through for an arbitrary semisimple G. □

Lemma 3.8. *The composition of the second Chern class map*

$$R(G) \longrightarrow K_0(BG) \xrightarrow{c_2} \mathrm{CH}^2(BG)$$

with the diagonal morphism γ in the diagram coincides with the map τ in (3.4) up to sign. The image of γ coincides with $\mathrm{Dec}(G)$.

Proof. As $Q(G)$ injects when the base field gets extended, for the proof of the first statement we may assume that F is separably closed. Let $\rho : G \to \mathbf{SL}_n$ be a representation. Write $x_1, x_2, \dots, x_n$ for the characters of ρ in the weight lattice Λ. Consider the diagram

$$\begin{tikzcd}
R(\mathbf{SL}_n) \arrow[rr, "\tau"] \arrow[dr, "c_2"'] \arrow[dd] & & Q(\mathbf{SL}_n) \arrow[dd] \\
& \mathrm{CH}^2(B\,\mathbf{SL}_n) \arrow[ur, "\gamma"'] \arrow[dd] & \\
R(G) \arrow[rr, "\tau"] \arrow[dr, "c_2"'] & & Q(G) \\
& \mathrm{CH}^2(BG) \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

with the vertical homomorphisms induced by ρ. The vertical faces of the diagram are commutative by Lemma 3.7 and the functoriality of c_2 and the character map ch. By Example 3.5, the top map τ takes the class of the tautological representation ι of $\mathbf{SL}_n$ to the a generator of $Q(\mathbf{SL}_n)$. By Example 3.6, γ in the top of the diagram is an isomorphism taking the canonical generator of $\mathrm{CH}^2(B\,\mathbf{SL}_n)$ to a generator of $Q(\mathbf{SL}_n)$. It follows that $\tau(\iota)$ and $\gamma(c_2(\iota))$ in the top face of the diagram are equal up to sign. The class of ρ in $R(G)$ is the image of τ under the left vertical homomorphism. It follows that $\tau(\rho)$ and $\gamma(c_2(\rho))$ in the bottom face of the diagram are also equal to sign.

The second statement follows from the first and the surjectivity of the second Chern class map $R(G) \to \mathrm{CH}^2(BG)$ (see [6, Appendix C] and [26, Corollary 3.2]). □

3g. **Main theorem.** The following theorem describes the group of degree 3 cohomological invariants with coefficients in $\mathbb{Q}/\mathbb{Z}(2)$ of an arbitrary semisimple group.

Theorem 3.9. *Let G be a semisimple group over a field F. Then there is an exact sequence*

$$0 \longrightarrow \mathrm{CH}^2(BG)_{\mathrm{tors}} \longrightarrow H^1\big(F, \widehat{C}(1)\big) \xrightarrow{\sigma}$$
$$\mathrm{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}} \longrightarrow Q(G)/\operatorname{Dec}(G) \xrightarrow{\theta_G^*} H^2\big(F, \widehat{C}(1)\big).$$

Proof. Follows from the key diagram above and Lemma 3.8 as $Q(G)$ is torsion free and $H^1\big(F, \widehat{C}(1)\big)$ is torsion. □

Remark 3.10. The map θ_G^* is trivial if G is split or adjoint of inner type (see [18, Proposition 4.1 and Remark 5.5]).

The exact sequence in Theorem 3.9 is functorial in G. More precisely, let $G \to G'$ be a homomorphism of semisimple groups extending to a homomorphism $C \to C'$ of the kernels of the universal covers. By Lemma 3.7, the diagram

$$\begin{tikzcd}
H^1\big(F, \widehat{C}'(1)\big) \arrow[r, "\sigma'"] \arrow[d] & \mathrm{Inv}^3\big(G', \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}} \arrow[r] \arrow[d] & Q(G')/\operatorname{Dec}(G') \arrow[d] \\
H^1\big(F, \widehat{C}(1)\big) \arrow[r, "\sigma"] & \mathrm{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}} \arrow[r] & Q(G)/\operatorname{Dec}(G)
\end{tikzcd}$$

is commutative.

Write $\mathrm{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{dec}}$ for the image of σ. We call these invariants *decomposable*. Thus, we have an exact sequence

$$0 \longrightarrow \mathrm{CH}^2(BG)_{\mathrm{tors}} \longrightarrow H^1\big(F, \widehat{C}(1)\big) \xrightarrow{\sigma} \mathrm{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{dec}} \longrightarrow 0.$$

We don't know if the group $\mathrm{CH}^2(BG)_{\mathrm{tors}}$ is trivial, but it is always finite.

Proposition 3.11. *The group $\mathrm{CH}^2(BG)$ is finitely generated. In particular, $\mathrm{CH}^2(BG)_{\mathrm{tors}}$ is finite.*

Proof. By [25, Théorème 3.3] and Section 3a, we have

$$\mathbb{Z}[\Lambda_r]^{\Delta} \subset R(G) \subset \mathbb{Z}[\Lambda_w].$$

The Noetherian ring $\mathbb{Z}[\Lambda_r]$ is finite over $\mathbb{Z}[\Lambda_r]^{\Delta}$, hence $\mathbb{Z}[\Lambda_r]^{\Delta}$ is Noetherian. The $\mathbb{Z}[\Lambda_r]^{\Delta}$-algebra $\mathbb{Z}[\Lambda_w]$ is finite, hence so is $R(G)$. It follows that the ring $R(G)$ is Noetherian. Let I be the kernel of the rank map $R(G) \to \mathbb{Z}$. Since I is finitely generated, the factor group $R(G)/I^2$ is finitely generated. By (3.2), the second Chern class factors through a surjective homomorphism $R(G)/I^2 \to \mathrm{CH}^2(BG)$, whence the result. □

We will show in Section 4a that the group $\mathrm{CH}^2(BG)_{\mathrm{tors}}$ is trivial if G is adjoint of inner type.

The factor group

$$\operatorname{Inv}^3\big(G,\mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{ind}} := \operatorname{Inv}^3\big(G,\mathbb{Q}/\mathbb{Z}(2)\big)/\operatorname{Inv}^3\big(G,\mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{dec}}$$

is called the group of *indecomposable* invariants. Thus, we have an exact sequence

$$0 \longrightarrow \operatorname{Inv}^3\big(G,\mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{ind}} \longrightarrow Q(G)/\operatorname{Dec}(G) \xrightarrow{\theta_G^*} H^2\big(F,\widehat{C}(1)\big).$$

If G is simply connected quasi-simple, all decomposable invariants are trivial, and the group $\operatorname{Inv}^3\big(G,\mathbb{Q}/\mathbb{Z}(2)\big) = \operatorname{Inv}^3\big(G,\mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{ind}} \simeq Q(G)/\operatorname{Dec}(G)$ is cyclic generated by the *Rost invariant* R_G. The order of the *Rost number* n_G of R_G is determined in [8, Part II].

4. Groups of inner type

Let G be a semisimple group over F. A group G' is called an *inner form of* G if there is a G-torsor X over F such that G' is the twist of G by X, or equivalently, $G' \simeq \mathbf{Aut}_G(X)$. The choice of the torsor X yields a canonical bijection $\varphi : H^1(K,G') \xrightarrow{\sim} H^1(K,G)$ for every field extension K/F (see [15, Proposition 8.8]). Therefore, we have an isomorphism $\operatorname{Inv}^n\big(G,\mathbb{Q}/\mathbb{Z}(j)\big) \xrightarrow{\sim} \operatorname{Inv}^n\big(G',\mathbb{Q}/\mathbb{Z}(j)\big)$. Note that this isomorphism does not preserve normalized invariants as φ does not preserve trivial torsors. Precisely, φ takes the class of a trivial torsor to the class of X. We modify the isomorphism to get an isomorphism

$$\operatorname{Inv}^n\big(G,\mathbb{Q}/\mathbb{Z}(j)\big)_{\mathrm{norm}} \xrightarrow{\sim} \operatorname{Inv}^n\big(G',\mathbb{Q}/\mathbb{Z}(j)\big)_{\mathrm{norm}}, \tag{4.1}$$

taking an invariant I of G to an invariant I' of G' satisfying

$$I'(X') = I(\varphi(X')) - I(X).$$

4a. **Decomposable invariants.** Let G be a semisimple group of inner type. Then $\widehat{C}$ is a diagonalizable finite group.

Lemma 4.1. *There is a natural isomorphism* $H^1\big(F,\widehat{C}(1)\big) \simeq \widehat{C}\otimes F^\times$.

Proof. Write $\widehat{C} \simeq R/S$, where R and S are lattices. In the exact sequence

$$H^1\big(F,S(1)\big) \longrightarrow H^1\big(F,R(1)\big) \longrightarrow H^1\big(F,\widehat{C}(1)\big) \longrightarrow H^2\big(F,S(1)\big)$$

the first two terms are $S\otimes F^\times$ and $R\otimes F^\times$, respectively, and the last term is equal to $S\otimes H^2\big(F,\mathbb{Z}(1)\big) = 0$ by Hilbert's Theorem 90. The result follows. ☐

Recall that under the isomorphism in Lemma 4.1, the map σ in Theorem 3.9 is defined as follows. For every $\chi \in \widehat{C}$ and $a \in F^\times$, the invariant $\sigma(\chi\cup(a))$ takes a G-torsor X over a field extension K/F to $\alpha_G(X)(\chi_K)\cup(a) \in H^3(K,\mathbb{Q}/\mathbb{Z}(2))$ (see Section 3e).

Theorem 4.2. *Let G be a semisimple adjoint group of inner type over a field F. Then the homomorphism*

$$\sigma : \widehat{C} \otimes F^\times \longrightarrow \operatorname{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{dec}}$$

is an isomorphism. Equivalently, the group $\mathrm{CH}^2(BG)$ is torsion-free.

Proof. As G is an inner form of a split group, by (4.1), we may assume that G is split. The group $\widehat{C}$ is a direct sum of cyclic subgroups generated by $\chi_1, \dots, \chi_m$, respectively. Let $a_1, \dots, a_m \in F^\times$ be such that the element $u := \sum \chi_i \otimes a_i$ belongs to the kernel of σ. It suffices to show that $a_i \in (F^\times)^{s_i}$, where $s_i := \operatorname{ord}(\chi_i)$ for all i.

Fix an integer i. For a field extension K/F and any $\rho \in H^1(K, \mathbb{Q}/\mathbb{Z})$ of order s_i, consider the admissible map $f : \widehat{C} \to \operatorname{Br}(K(t))$ for the field $K(t)$ of rational functions over K, defined by

$$f(\chi_j) = \begin{cases} \rho \cup (t), & \text{in } \operatorname{Br}(K(t)) \text{ if } j = i; \\ 0, & \text{otherwise.} \end{cases}$$

By Proposition 2.6, there is a G-torsor X over $K(t)$ satisfying $\alpha_G(X)(\chi_j) = f(\chi_j)$ for all j. As $u \in \operatorname{Ker}(\sigma)$, we have

$$0 = \sigma(u)(X) = \sum_j \alpha_G(X)(\chi_j) \cup (a_j) = \rho \cup (t) \cup (a_i)$$

in $H^3(K(t), \mathbb{Q}/\mathbb{Z}(2))$. Taking residue at t (see [8, Part II, Appendix A]),

$$H^3_{nr}\big(K(t), \mathbb{Q}/\mathbb{Z}(2)\big) \longrightarrow H^2\big(K, \mathbb{Q}/\mathbb{Z}(1)\big) = \operatorname{Br}(K),$$

we get $\rho \cup (a_i) = 0$ in $\operatorname{Br}(K)$. By Lemma 4.3 below, we have $a \in (F^\times)^{s_i}$. □

Lemma 4.3. *Let $a \in F^\times$ and $s > 0$ be such that for every field extension K/F and every $\rho \in H^1(K, \mathbb{Q}/\mathbb{Z})$ of order s one has $\rho \cup (a) = 0$ in $H^2\big(K, \mathbb{Q}/\mathbb{Z}(1)\big) = \operatorname{Br}(K)$. Then $a \in (F^\times)^s$.*

Proof. Let $H = \mathbb{Z}/s\mathbb{Z}$. Choose an H-torsor $X \to Y$ with smooth Y, $\operatorname{Pic}(X) = 0$ and $F[X]^\times = F^\times$. (For example, take an approximation of $EH \to BH$.) By [3] or [17], there is an exact sequence

$$\operatorname{Pic}(X)^H \longrightarrow H^2(H, F[X]^\times) \longrightarrow \operatorname{Br}(Y),$$

which yields an injective map $F^\times/F^{\times s} \to \operatorname{Br}(F(Y))$ as $H^2(H, F[X]^\times) = H^2(H, F^\times) = F^\times/F^{\times s}$ and $\operatorname{Br}(Y)$ injects into $\operatorname{Br}(F(Y))$ by [19, Corollary 2.6]. This map takes a to $\rho \cup (a)$, where $\rho \in H^1(F(Y), \mathbb{Q}/\mathbb{Z})$ corresponds to the cyclic extension $F(X)/F(Y)$. As $\rho \cup (a) = 0$ by assumption, we have $a \in (F^\times)^s$. □

4b. **Indecomposable invariants.** In this section we compute the groups of indecomposable invariants of adjoint groups of inner type.

$$\boxed{\text{Type } A_{n-1}}$$

In the split case we have $G = \mathbf{PGL}_n$, the projective general linear group, $n \geq 2$, $\Lambda_w = \mathbb{Z}^n/\mathbb{Z}e$, where $e = e_1 + e_2 + \cdots + e_n$. The root lattice is generated by the simple roots $\bar{e}_1 - \bar{e}_2, \bar{e}_2 - \bar{e}_3, \ldots, \bar{e}_{n-1} - \bar{e}_n$, $\widehat{C} = \Lambda_w/\Lambda_r \simeq \mathbb{Z}/n\mathbb{Z}$. The generator of $\mathsf{Sym}^2(\Lambda_w)^W$ is the form

$$q = -\sum_{i<j} \bar{x}_i\bar{x}_j = \frac{1}{2}\sum \bar{x}_i^2.$$

The matrix D (see Section 3b) is the identity matrix I_n. The inverses of Cartan matrices here and below are taken from [4, Appendix F]:

$$C^{-1} = \frac{1}{n}\begin{pmatrix} n-1 & n-2 & n-3 & \vdots & 2 & 1 \\ n-2 & 2(n-2) & 2(n-3) & \vdots & 4 & 2 \\ n-3 & 2(n-3) & 3(n-3) & \vdots & 6 & 3 \\ \cdots & \cdots & \cdots & & \cdots & \cdots \\ 2 & 4 & 6 & \vdots & 2(n-2) & n-2 \\ 1 & 2 & 3 & \vdots & n-2 & n-1 \end{pmatrix}$$

By Proposition 3.4,

$$Q(G) = \mathsf{Sym}^2(\Lambda_r)^W = \begin{cases} 2n\mathbb{Z}q, & \text{if } n \text{ is even;} \\ n\mathbb{Z}q, & \text{if } n \text{ is odd.} \end{cases}$$

If $a := \sum_{i,j=1}^n e^{\bar{x}_i - \bar{x}_j} \in \mathbb{Z}[\Lambda_r]^W$, we have by (3.3),

$$c_2(a) = \frac{1}{2}\sum(\bar{x}_i - \bar{x}_j)^2 = n\sum \bar{x}_i^2 = 2nq \in \operatorname{Dec}(G).$$

It follows that $\operatorname{Dec}(G) = Q(G)$ if n is even.

Suppose that n is odd. If $b = \sum_{i=1}^n e^{n\bar{x}_i} \in \mathbb{Z}[\Lambda_r]^W$, we have by (3.3),

$$c_2(b) = \frac{1}{2}\sum(n\bar{x}_i)^2 = n^2 q \in \operatorname{Dec}(G).$$

As n is odd, $\gcd(2n, n^2) = n$, hence $nq \in \operatorname{Dec}(G)$ and again $\operatorname{Dec}(G) = Q(G)$.

Thus, $\operatorname{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} = Q(G)/\operatorname{Dec}(G) = 0$.

A G-torsor is given by a central simple algebra A of degree n (here and below see [15]). The twist of G by A is the group $\mathbf{PGL}_1(A)$. The Tits classes of algebras for this group are the multiples of $[A]$ in $\operatorname{Br}(F)$. In view of Proposition 2.3 and 4.1, we have

Theorem 4.4. *Let $G = \mathbf{PGL}_1(A)$ for a central simple algebra A over F. Then*

$$\operatorname{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}} \simeq F^\times/F^{\times n}.$$

An element $x \in F^\times$ corresponds to the invariant taking a central simple algebra A' of degree n to the cup-product $([A'] - [A]) \cup (x)$.

Type B_n

In the split case we have $G = \mathbf{O}^+_{2n+1}$, the special orthogonal group, $n \geq 2$, $\Lambda_w = \mathbb{Z}^n + \mathbb{Z}e$, where $e = \frac{1}{2}(e_1 + e_2 + \cdots + e_n)$, $\Lambda_r = \mathbb{Z}^n$ and $\widehat{C} \simeq \mathbb{Z}/2\mathbb{Z}$. The generator of $\mathit{Sym}^2(\Lambda_w)^W$ is the form $q = \frac{1}{2}\sum_i x_i^2$ and $D = \operatorname{diag}(1, 1, \dots 1, 2)$,

$$C^{-1} = \begin{pmatrix} 1 & 1 & 1 & \vdots & 1 & 1 & 1 \\ 1 & 2 & 2 & \vdots & 2 & 2 & 2 \\ 1 & 2 & 3 & \vdots & 3 & 3 & 3 \\ \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\ 1 & 2 & 3 & \vdots & n-2 & n-2 & n-2 \\ 1 & 2 & 3 & \vdots & n-2 & n-1 & n-1 \\ 1/2 & 1 & 3/2 & \vdots & (n-2)/2 & (n-1)/2 & n/2 \end{pmatrix}$$

By Proposition 3.4, $Q(G) = \mathit{Sym}^2(\Lambda_r)^W = 2\mathbb{Z}q$.
If $a := \sum_{i=1}^n (e^{x_i} + e^{-x_i}) \in \mathbb{Z}[\Lambda_r]^W$, we have

$$c_2(a) = \frac{1}{2}\sum(x_i^2 + (-x_i)^2) = 2q \in \operatorname{Dec}(G).$$

It follows that $\operatorname{Dec}(G) = Q(G)$, so $\operatorname{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\text{ind}} = Q(G)/\operatorname{Dec}(G) = 0$.

A G-torsor is given by the similarity class of a nondegenerate quadratic form p of dimension $2n+1$. The twist of G by p is the special orthogonal group $\mathbf{O}^+(p)$ of the form p. The only nontrivial Tits class of algebras for this group is the class of the even Clifford algebra $C_0(p)$ of p. In view of Proposition 2.3 and 4.1, we have

Theorem 4.5. *Let $G = \mathbf{O}^+(p)$ for a nondegenerate quadratic form p of dimension $2n+1$. Then*

$$\operatorname{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big)_{\text{norm}} \simeq F^\times/F^{\times 2}.$$

An element $x \in F^\times$ corresponds to the invariant taking the similarity class of a nondegenerate quadratic form p' of dimension $2n+1$ to the cup-product $([C_0(p')] - [C_0(p)]) \cup (x)$.

Type C_n

In the split case we have $G = \mathbf{PGSp}_{2n}$, the projective symplectic group, $n \geq 3$, $\Lambda_w = \mathbb{Z}^n$, Λ_r consists of all $\sum a_i e_i$ with $\sum a_i$ even, $\widehat{C} \simeq \mathbb{Z}/2\mathbb{Z}$. The generator of $\mathit{Sym}^2(\Lambda_w)^W$ is $q = \sum_i x_i^2$. $D = \mathrm{diag}(2, 2, \dots 2, 1)$ and

$$C^{-1} = \begin{pmatrix} 1 & 1 & 1 & \vdots & 1 & 1 & 1/2 \\ 1 & 2 & 2 & \vdots & 2 & 2 & 1 \\ 1 & 2 & 3 & \vdots & 3 & 3 & 3/2 \\ \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\ 1 & 2 & 3 & \vdots & n-2 & n-2 & (n-2)/2 \\ 1 & 2 & 3 & \vdots & n-2 & n-1 & (n-1)/2 \\ 1 & 2 & 3 & \vdots & n-2 & n-1 & n/2 \end{pmatrix}$$

By Proposition 3.4,

$$Q(G) = \mathit{Sym}^2(\Lambda_r)^W = \begin{cases} \mathbb{Z}q, & \text{if } n \equiv 0 \text{ modulo } 4; \\ 2\mathbb{Z}q, & \text{if } n \equiv 2 \text{ modulo } 4; \\ 4\mathbb{Z}q, & \text{if } n \text{ is odd.} \end{cases}$$

If $a := \sum_i (e^{2x_i} + e^{-2x_i}) \in \mathbb{Z}[\Lambda_r]^W$, we have

$$c_2(a) = \sum (2x_i)^2 = 4q \in \mathrm{Dec}(G).$$

It follows that $\mathrm{Dec}(G) = Q(G)$ if n is odd.

Suppose that n is even. If $b := \sum_{i \neq j} (e^{x_i + x_j} + e^{x_i - x_j}) \in \mathbb{Z}[\Lambda_r]^W$, we have

$$c_2(b) = \frac{1}{2} \sum_{i \neq j} [(x_i - x_j)^2 + (x_i + x_j)^2] = 2(n-1)q \in \mathrm{Dec}(G).$$

As n is even, $\gcd(4, 2(n-1)) = 2$, we have $2q \in \mathrm{Dec}(G)$. On the other hand, by [8, Part II, Lemma 14.2], $\mathrm{Dec}(G) \subset 2q\mathbb{Z}$, therefore, $\mathrm{Dec}(G) = 2q\mathbb{Z}$.

It follows that

$$\mathrm{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{ind}} = Q(G)/\,\mathrm{Dec}(G) = \begin{cases} (\mathbb{Z}/2\mathbb{Z})q, & \text{if } n \equiv 0 \text{ modulo } 4; \\ 0, & \text{otherwise.} \end{cases}$$

A G-torsor is given by a pair (A, σ), where A is a central simple algebra of degree $2n$ and σ is a symplectic involution on A. The twist of G by (A, σ) is the projective symplectic group $\mathbf{PGSp}(A, \sigma)$. The only nontrivial Tits class of algebras for this group is the class of the algebra A. In view of Proposition 2.3 and 4.1, we have

Theorem 4.6. *Let $G = \mathbf{PGSp}(A, \sigma)$ for a a central simple algebra of degree $2n$ with symplectic involution σ. Then*

$$\mathrm{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{dec}} \simeq F^\times / F^{\times 2}.$$

An element $x \in F^\times$ corresponds to the invariant taking a pair (A', σ') to the cup-product $([A'] - [A]) \cup (x)$.

If n is not divisible by 4, we have $\mathrm{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}} = \mathrm{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{dec}}$. If n is divisible by 4, the group $\mathrm{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{ind}}$ is cyclic of order 2.

In the case n is divisible by 4 and $\mathrm{char}(F) \neq 2$ an invariant I of order 2 generating $\mathrm{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{ind}}$ was constructed in [11, §4]. Thus, in this case we have

$$\mathrm{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}} = \mathrm{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{dec}} \oplus (\mathbb{Z}/2\mathbb{Z})I \simeq F^\times/F^{\times 2} \oplus (\mathbb{Z}/2\mathbb{Z}).$$

$\boxed{\text{Type } D_n}$

In the split case we have $G = \mathbf{PGO}^+_{2n}$, the projective orthogonal group, $n \geq 4$, $\Lambda_w = \mathbb{Z}^n + \mathbb{Z}e$, where $e = \frac{1}{2}(e_1 + e_2 + \cdots + e_n)$, Λ_r consists of all $\sum a_i e_i$ with $\sum a_i$ even, $\widetilde{C}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if n is even and to $\mathbb{Z}/4\mathbb{Z}$ if n is odd. The generator of $\mathsf{Sym}^2(\Lambda_w)^W$ is the form $q = \frac{1}{2}\sum_i x_i^2$ and $D = I_n$,

$$C^{-1} = \begin{pmatrix} 1 & 1 & 1 & \vdots & 1 & 1/2 & 1/2 \\ 1 & 2 & 2 & \vdots & 2 & 1 & 1 \\ 1 & 2 & 3 & \vdots & 3 & 3/2 & 3/2 \\ \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\ 1 & 2 & 3 & \vdots & n-2 & (n-2)/2 & (n-2)/2 \\ 1/2 & 1 & 3/2 & \vdots & (n-2)/2 & n/4 & (n-2)/4 \\ 1/2 & 1 & 3/2 & \vdots & (n-2)/2 & (n-2)/4 & n/4 \end{pmatrix}$$

By Proposition 3.4,

$$Q(G) = \mathsf{Sym}^2(\Lambda_r)^W = \begin{cases} 2\mathbb{Z}q, & \text{if } n \equiv 0 \text{ modulo } 4; \\ 4\mathbb{Z}q, & \text{if } n \equiv 2 \text{ modulo } 4; \\ 8\mathbb{Z}q, & \text{if } n \text{ is odd.} \end{cases}$$

If $a := \sum_i (e^{2x_i} + e^{-2x_i}) \in \mathbb{Z}[\Lambda_r]^W$, we have

$$c_2(a) = \sum (2x_i)^2 = 8q \in \mathrm{Dec}(G).$$

It follows that $\mathrm{Dec}(G) = Q(G)$ if n is odd.

Suppose that n is even. If $b := \sum_{i \neq j} (e^{x_i + x_j} + e^{x_i - x_j}) \in \mathbb{Z}[\Lambda_r]^W$, we have

$$c_2(b) = \frac{1}{2}\sum_{i \neq j} \big[(x_i - x_j)^2 + (x_i + x_j)^2\big] = 4(n-1)q \in \mathrm{Dec}(G).$$

As n is even, $\gcd(8, 4(n-1)) = 4$, we have $4q \in \mathrm{Dec}(G)$. On the other hand, by [8, Part II, Lemma 15.2], $\mathrm{Dec}(G) \subset 4\mathbb{Z}q$, therefore, $\mathrm{Dec}(G) = 4\mathbb{Z}q$.

It follows that

$$\mathrm{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{ind}} = Q(G)/\operatorname{Dec}(G) = \begin{cases} (2\mathbb{Z}/4\mathbb{Z})q, & \text{if } n \equiv 0 \text{ modulo } 4; \\ 0, & \text{otherwise.} \end{cases}$$

A G-torsor is given by a quadruple (A, σ, f, e), where A is a central simple algebra of degree $2n$, (σ, f) is a quadratic pair on A of trivial discriminant and e an idempotent in the center of the Clifford algebra $C(A, \sigma, f)$. The twist of G by (A, σ, f, e) is the projective orthogonal group $\mathbf{PGO}^+(A, \sigma, f)$. The nontrivial Tits classes of algebras for this group are the class of the algebra A and the classes of the two components $C^{\pm}(A,\sigma,f)$ of the Clifford algebra. In view of Proposition 2.3 and 4.1, we have

Theorem 4.7. *Let $G=\mathbf{PGO}^+(A,\sigma,f)$ for a a central simple algebra of degree $2n$ with quadratic pair (σ,f) of trivial discriminant. Then*

$$\mathrm{Inv}^3\big(G,\mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{dec}}\simeq\begin{cases}(F^\times/F^{\times 2})\oplus(F^\times/F^{\times 2}), & \text{if } n \text{ is even;}\\ F^\times/F^{\times 4}, & \text{if } n \text{ is odd.}\end{cases}$$

If n is even and $x^+,x^-\in F^\times$, then the corresponding invariant takes a quadruple (A',σ',f',e') to

$$\big([C^+(A',\sigma',f')]-[C^+(A,\sigma,f)]\big)\cup(x^+)+\big([C^-(A',\sigma',f')]-[C^-(A,\sigma,f)]\big)\cup(x^-).$$

If n is even and $x\in F^\times$, then the corresponding invariant takes a quadruple (A',σ',f',e') to $\big([C^+(A',\sigma',f')]-[C^+(A,\sigma,f)]\big)\cup(x)$.

If n is not divisible by 4, we have $\mathrm{Inv}^3\big(G,\mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}}=\mathrm{Inv}^3\big(G,\mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{dec}}$. If n is divisible by 4, the group $\mathrm{Inv}^3\big(G,\mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{ind}}$ is cyclic of order 2.

In the case n is divisible by 4 and $\mathrm{char}(F)\neq 2$ we sketch below a construction of a nontrivial indecomposable invariant I of order 2 for a split adjoint group $G=\mathbf{PGO}^+_{2n}$. A G-torsor X over F is given by a triple (A,σ,e), where A is a central simple algebra over F with an orthogonal involution σ of trivial discriminant and e is a nontrivial idempotent of the center of the Clifford algebra of (A,σ) (see [15, §29F]). We need to determine the value of $I(X)$ in $H^3\big(F,\mathbb{Q}/\mathbb{Z}(2)\big)$.

We have $G=\mathbf{Aut}(A,\sigma,e)=\mathbf{PGO}^+(A,\sigma)$. The exact sequence

$$1\longrightarrow\boldsymbol{\mu}_2\longrightarrow\mathbf{O}^+(A,\sigma)\longrightarrow\mathbf{PGO}^+(A,\sigma)\longrightarrow 1,$$

where $\mathbf{O}^+(A,\sigma)$ is the special orthogonal group, yields an exact sequence

$$H^1\big(F,\mathbf{O}^+(A,\sigma)\big)\xrightarrow{\varphi}H^1\big(F,\mathbf{PGO}^+(A,\sigma)\big)\xrightarrow{\delta}\mathrm{Br}(F).$$

The reduction method used in [11] for the construction of an indecomposable degree 3 invariant for a symplectic involution works as well in the orthogonal case. It reduces the general situation to the case $\mathrm{ind}(A)\leq 4$. In this case the algebra A is isomorphic to $M_2(B)$ for a central simple algebra B as $2n$ is divisible by 8 and hence it admits a hyperbolic involution σ'. By [15, Proposition 8.31], one of the two components of the Clifford algebra $C(A,\sigma')$ is split. Let e' be the corresponding idempotent in the center of $C(A,\sigma')$. (If both components split, then A is split by [15, Theorem 9.12], and we let e' be any of the two idempotents.)

The element $\delta(A,\sigma',e')$ is trivial, hence $(A,\sigma',e')=\varphi(v)$ for some $v\in H^1\big(F,\mathbf{O}^+(A,\sigma)\big)$. The set $H^1\big(F,\mathbf{O}^+(A,\sigma)\big)$ is described in the [15, §29.27] as the set of equivalence classes of pairs $(a,x)\in A\times F$ such that a is σ-symmetric invertible element and $x^2=\mathrm{Nrd}(a)$. Thus, $v=(a,x)$ for such a pair (a,x) and we set $I(X)=[A]\cup(x)$.

$\boxed{\text{Type } E_6}$

We have $\widehat{C} \simeq \mathbb{Z}/3\mathbb{Z}$ and $D = I_6$,

$$C^{-1} = \frac{1}{3}\begin{pmatrix} 4 & 5 & 6 & 4 & 2 & 3 \\ 5 & 10 & 12 & 8 & 4 & 6 \\ 6 & 12 & 18 & 12 & 6 & 9 \\ 4 & 8 & 12 & 10 & 5 & 6 \\ 2 & 4 & 6 & 5 & 4 & 3 \\ 3 & 6 & 9 & 6 & 3 & 6 \end{pmatrix}$$

By Proposition 3.4, $Q(G) = \mathsf{Sym}^2(\Lambda_r)^W = 3\mathbb{Z}q$.

Write $\delta_i \in \mathbb{Z}[\Lambda_w]^W$ for the sum of elements in the W-orbit of e^{w_i}. We have $c_2(\delta_1) = 6q$, $c_2(\delta_2) = 24q$, $c_2(\delta_3) = 150q$ by [16, §2] and $\operatorname{rank}(\delta_1) = [W(E_6) : W(D_5)] = 27$, $\operatorname{rank}(\delta_3) = [W(E_6) : W(A_1 + A_4)] = 216$. Note that δ_2 and $\delta_1 w_3$ belong to $\mathbb{Z}[\Lambda_r]^W$. By (3.2),

$$c_2(\delta_1\delta_3) = \operatorname{rank}(\delta_1)c_2(\delta_3) + \operatorname{rank}(\delta_3)c_2(\delta_1) = 27 \cdot 150q + 216 \cdot 6q = 5346q.$$

As $\gcd(24, 5346) = 6$, we have $6q \in \operatorname{Dec}(G)$. On the other hand, $c_2(\delta_i) \in 6\mathbb{Z}q$ for all i by [16, §2], hence $\operatorname{Dec}(G) = 6\mathbb{Z}q$. Thus,

$$\operatorname{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big)_{\text{ind}} = Q(G)/\operatorname{Dec}(G) = (3\mathbb{Z}/6\mathbb{Z})q.$$

Note that the exponents of the groups $\operatorname{Inv}^3(G)_{\text{dec}}$ and $\operatorname{Inv}^3(G)_{\text{ind}}$ are relatively prime.

Theorem 4.8. *Let G be an adjoint group of type E_6 of inner type. Then*

$$\operatorname{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big)_{\text{norm}} \simeq (F^\times/F^{\times 3}) \oplus (\mathbb{Z}/2\mathbb{Z}).$$

It follows from the computation that the pull-back of the generator of $\operatorname{Inv}^3(G)_{\text{ind}}$ to $\operatorname{Inv}^3(\widetilde{G})_{\text{norm}}$ is 3 times the Rost invariant $R_{\widetilde{G}}$. This was observed in [10, Proposition 7.2] in the case $\operatorname{char}(F) \neq 2$.

$\boxed{\text{Type } E_7}$

We have $\widehat{C} \simeq \mathbb{Z}/2\mathbb{Z}$ and $D = I_7$,

$$C^{-1} = \frac{1}{2}\begin{pmatrix} 4 & 6 & 8 & 6 & 4 & 2 & 4 \\ 6 & 12 & 16 & 12 & 8 & 4 & 8 \\ 8 & 16 & 24 & 18 & 12 & 6 & 12 \\ 6 & 12 & 18 & 15 & 10 & 5 & 9 \\ 4 & 8 & 12 & 10 & 8 & 4 & 6 \\ 2 & 4 & 6 & 5 & 4 & 3 & 3 \\ 4 & 8 & 12 & 9 & 6 & 3 & 7 \end{pmatrix}$$

By Proposition 3.4, $Q(G) = \mathsf{Sym}^2(\Lambda_r)^W = 4\mathbb{Z}q$.

We have $c_2(\delta_1) = 36q$ and $c_2(\delta_7) = 12q$ by [16, §2] and $\operatorname{rank}(\delta_7) = [W(E_7) : W(E_6)] = 56$. Note that δ_1 and δ_7^2 belong to $\mathbb{Z}[\Lambda_r]^W$.

By (3.2),

$$c_2(\delta_7^2) = 2\operatorname{rank}(\delta_7)c_2(\delta_7) = 2\cdot 56\cdot 12q = 1344.$$

As $\gcd(36, 1344) = 12$, we have $12q \in \operatorname{Dec}(G)$. On the other hand, $c_2(\delta_i) \in 12\mathbb{Z}q$ for all i by [16, §2], hence $\operatorname{Dec}(G) = 12\mathbb{Z}q$. Thus,

$$\operatorname{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} = Q(G)/\operatorname{Dec}(G) = (4\mathbb{Z}/12\mathbb{Z})q.$$

Theorem 4.9. *Let G be an adjoint group of type E_7 of inner type. Then*

$$\operatorname{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}} \simeq (F^\times/F^{\times 2}) \oplus (\mathbb{Z}/3\mathbb{Z}).$$

It follows from the computation that the pull-back of the generator of $\operatorname{Inv}^3(G)_{\mathrm{ind}}$ to $\operatorname{Inv}^3(\widetilde{G})_{\mathrm{norm}}$ is 4 times the Rost invariant $R_{\tilde{G}}$. This was observed in [10, Proposition 7.2] in the case $\operatorname{char}(F) \neq 3$.

Every inner semisimple group of the types G_2, F_4 and E_8 is simply connected. Then the group $\operatorname{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ is of order 2, 6 and 60, respectively (see [8, Part II]).

Recall that the groups $\operatorname{Inv}^3(G)_{\mathrm{ind}}$ are all the same for all twisted forms of G. This is not the case for $\operatorname{Inv}^3(\widetilde{G})_{\mathrm{ind}} = \operatorname{Inv}^3(\widetilde{G})$. Write $\widetilde{G}_{\mathrm{gen}}$ for a "generic" twisted form of $\widetilde{G}$ (see [10, §6]). For such groups the Rost number $n_{\tilde{G}_{\mathrm{gen}}}$ is the largest possible. Their values can be found in [8, Part II].

Theorem 4.10. *Let G be an adjoint semisimple group of inner type, $\widetilde{G} \to G$ a universal cover. Then the map*

$$\operatorname{Inv}^3(G)_{\mathrm{ind}} \simeq \operatorname{Inv}^3(G_{\mathrm{gen}})_{\mathrm{ind}} \longrightarrow \operatorname{Inv}^3(\widetilde{G}_{\mathrm{gen}})_{\mathrm{ind}} = \operatorname{Inv}^3(\widetilde{G}_{\mathrm{gen}}) = (\mathbb{Z}/n_{\tilde{G}_{\mathrm{gen}}}\mathbb{Z})R_{\tilde{G}_{\mathrm{gen}}}$$

is injective. In the case G is simple, the group $\operatorname{Inv}^3(G)_{\mathrm{ind}}$ is nonzero only in the following cases:

C_n, n is divisible by 4: $\operatorname{Inv}^3(G)_{\mathrm{ind}} = (\mathbb{Z}/2\mathbb{Z})R_{\tilde{G}}$,

D_n, n is divisible by 4: $\operatorname{Inv}^3(G)_{\mathrm{ind}} = (2\mathbb{Z}/4\mathbb{Z})R_{\tilde{G}}$,

E_6: $\operatorname{Inv}^3(G)_{\mathrm{ind}} = (3\mathbb{Z}/6\mathbb{Z})R_{\tilde{G}}$,

E_7: $\operatorname{Inv}^3(G)_{\mathrm{ind}} = (4\mathbb{Z}/12\mathbb{Z})R_{\tilde{G}}$.

5. Restriction to the generic maximal torus

Let G be a semisimple group over F and T_{gen} the generic maximal torus of G defined over $F(\mathcal{X})$, where $\mathcal{X}$ is the variety of maximal tori in G (see Example 3.1). We can restrict invariants of G to invariant of T_{gen} via the composition

$$\operatorname{Inv}^n\big(G, \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow \operatorname{Inv}^n\big(G_{F(\mathcal{X})}, \mathbb{Q}/\mathbb{Z}(j)\big) \xrightarrow{\mathrm{Res}} \operatorname{Inv}^n\big(T_{\mathrm{gen}}, \mathbb{Q}/\mathbb{Z}(j)\big).$$

The degree 3 invariants of algebraic tori have been studied in [1].

Suppose that G is quasi-split. Then the character group of T_{gen} is isomorphic to the weight lattice Λ with the Δ-action (see Example 3.1). The exact sequence $0 \to \Lambda \to \Lambda_w \to \widehat{C} \to 0$, Example 3.1, Theorem 3.9 and [1, Theorem 4.3] yield a diagram

$$\begin{tikzcd} H^1\big(F, \widehat{C}(1)\big) \arrow[r] \arrow[d] & \operatorname{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}} \arrow[r] \arrow[d] & \mathbb{Z}[\Lambda]^{\Delta}/\operatorname{Dec}(\Lambda) \arrow[d, equal] \\ H^2\big(F(\mathcal{X}), \widehat{T}_{\mathrm{gen}}(1)\big) \arrow[r] & \operatorname{Inv}^3\big(T_{\mathrm{gen}}, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}} \arrow[r] & \mathbb{Z}[\Lambda]^{\Delta}/\operatorname{Dec}(\Lambda). \end{tikzcd}$$

Theorem 5.1. *Let G be a quasi-split group over a perfect field F, T_{gen} the generic maximal torus. Then the homomorphism*

$$\operatorname{Inv}^n\big(G, \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow \operatorname{Inv}^n\big(T_{\mathrm{gen}}, \mathbb{Q}/\mathbb{Z}(j)\big)$$

is injective, i.e., every invariant of G is determined by its restriction on the generic maximal torus.

Proof. Consider the morphism $\mathcal{T} \to \mathcal{X}$ as in Example 3.1. Let V be a generically free representation of G such that there is an open G-invariant subscheme $U \subset V$ and a G-torsor $U \to U/G$. The group scheme $\mathcal{T}$ over $\mathcal{X}$ acts naturally on $U \times \mathcal{X}$. Consider the factor scheme $(U \times \mathcal{X})/\mathcal{T}$. In fact, we can view this as a variety as follows. Let T_0 be a quasi-split maximal torus in G. The Weyl group W of T_0 acts on $(U/T_0) \times (G/T_0)$ by $w(T_0u, gT_0) = (T_0wu, gw^{-1}T_0)$. Then $(U \times \mathcal{X})/\mathcal{T}$ can be viewed as a factor variety $\big((U/T_0) \times (G/T_0)\big)/W$. Note that the function field of $(U \times \mathcal{X})/\mathcal{T}$ is isomorphic to the function field of $U_{F(\mathcal{X})}/T_{\mathrm{gen}}$ over $F(\mathcal{X})$.

We claim that the natural morphism

$$f : (U \times \mathcal{X})/\mathcal{T} \longrightarrow U/G$$

is surjective on K-points for any field extension K/F. A K-point of U/G is a G-orbit $O \subset U$ defined over K. As F is perfect, by [23, Theorem 11.1], there is a maximal torus $T \subset G$ and a T-orbit $O' \subset O$ defined over K. Then the pair (O', T) determines a point of $\big((U \times \mathcal{X})/\mathcal{T}\big)(K)$ over O. The claim is proved.

It follows from the claim that the generic fiber of f has a rational point (over $F(U/G)$). Therefore, the natural homomorphism

$$H^n\big(F(U/G), \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H^n\big(F(\mathcal{X})(U_{F(\mathcal{X})}/T_{\mathrm{gen}}), \mathbb{Q}/\mathbb{Z}(j)\big) \tag{5.1}$$

is injective.

Let $I \in \operatorname{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j))$ be an invariant with trivial restriction on T_{gen}. Let p_{gen} be the generic fiber of $p : U \to U/G$ and let q_{gen} be the generic fiber of $q : U_{F(\mathcal{X})} \to U_{F(\mathcal{X})}/T_{\mathrm{gen}}$. Then the pull-back of p_{gen} with respect to the field extension $F(\mathcal{X})(U_{F(\mathcal{X})}/T_{\mathrm{gen}})/F(U/G)$ is isomorphic to the pull-back of q_{gen} under the change of group homomorphism $T_{\mathrm{gen}} \to G$. It follows that

$$0 = \operatorname{Res}(I)(q_{\mathrm{gen}}) = I(p_{\mathrm{gen}})_{F(\mathcal{X})(U_{F(\mathcal{X})}/T_{\mathrm{gen}})}.$$

As (5.1) is injective, we have $I(p_{\mathrm{gen}}) = 0$ in $H^n\big(F(U/G), \mathbb{Q}/\mathbb{Z}(j)\big)$ and hence $I = 0$ by [8, Part II, Theorem 3.3] or [1, Theorem 2.2]. $\square$

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