PERIOD-INDEX AND u-INVARIANT QUESTIONS FOR FUNCTION FIELDS OVER COMPLETE DISCRETELY VALUED FIELDS

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ABSTRACT. Let K be a complete discretely valued field with residue field κ and F the function field of a curve over K. Let p be the characteristic of κ and ℓ a prime not equal to p. If the Brauer ℓ-dimensions of all finite extensions of κ are bounded by d and the Brauer ℓ-dimensions of all extensions of κ of transcendence degree at most 1 are bounded by $d+1$, then it is known that the Brauer ℓ-dimension of F is at most $d+2$ ([S1], [HHK1]). In this paper we give a bound for the Brauer p-dimension of F in terms of the p-rank of κ. As an application, we show that if κ is a perfect field of characteristic 2, then any quadratic form over F in at least 9 variables is isotropic. If κ is a finite field, this is a result of Heath-Brown/Leep ([HB], [Le]).

Let K be a field. For a central simple algebra A over K, the *period* of A is the order of its class in the Brauer group of K and the *index* of A is the degree of the division algebra Brauer equivalent to A. The index of A is denoted by $\mathrm{ind}(A)$ and period of A by $\mathrm{per}(A)$. Let K be a p-adic field and F the function field of a curve over K. The question whether the index of a central simple algebra over F divides the square of its period has remained open for a while. For indices which are coprime to p, an affirmative answer to this question is a theorem of Saltman ([S1]). To complete the answer, one needs to understand the relationship between the period and the index for algebras of period p over F. One of the main results proved in this paper is the following

Theorem 1. *Let K be a p-adic field and F a function field of a curve over K. Then the index of any central simple algebra over F divides the square of its period.*

Let K be any field. For a prime p, we define the *Brauer p-dimension* of K, denoted by $\mathrm{Br}_p\mathrm{dim}(K)$, to be the smallest integer $d \geq 0$ such that for every finite extension L of K and for every central simple algebra A over L of period a power of p, $\mathrm{ind}(A)$ divides $\mathrm{per}(A)^d$. The *Brauer dimension* of K, denoted by $\mathrm{Brdim}(K)$, is defined as the maximum of the Brauer p-dimension of K as p varies over all primes. Suppose the characteristic of K is $p > 0$. If $[K : K^p] = p^n$, then n is called the *p-rank* of K. A field of characteristic $p > 0$ is perfect if and only if its p-rank is 0. A theorem of Albert asserts that the Brauer p-dimension of a field K of characteristic $p > 0$ is at most the p-rank of K (cf. (1.2)).

In this paper, we discuss more generally the Brauer p-dimension of function fields of curves over a complete discretely valued field of characteristic 0 with residue field of characteristic $p > 0$.

We begin by bounding the Brauer dimension of complete discretely valued fields. Let K be a complete discretely valued field and κ its residue field. Suppose that $\mathrm{char}(K) = 0$ and $\mathrm{char}(\kappa) = p > 0$. Let ℓ be a prime. Suppose that $\mathrm{Br}_\ell\mathrm{dim}(\kappa) \leq d$. If $\ell \neq p$, then it is well known that $\mathrm{Br}_\ell\mathrm{dim}(K) \leq d+1$ (cf. [GS], Corollary 7.1.10). There seems to be no good connections between the Brauer p-dimension of K and the Brauer p-dimension of κ. For any $n \geq 0$, we give an example of a complete discretely valued field K with $\mathrm{Br}_p\mathrm{dim}(K) \geq n$ and $\mathrm{Br}_p\mathrm{dim}(\kappa) = 0$. However there are bounds for the Brauer p-dimension of K in terms of the p-rank of κ and we prove the following

Theorem 2. *Let K be a complete discretely valued field with residue field κ. Suppose that char$(\kappa) = p > 0$ and the p-rank of κ is n. Then $Br_p dim(K) \leq 1$ if $n = 0$ and $\frac{n}{2} \leq Br_p dim(K) \leq 2n$ if $n \geq 1$.*

Let F be the function field of a curve over K. Let ℓ be a prime. Suppose that there exists d such that $\mathrm{Br}_\ell\mathrm{dim}(\kappa) \leq d$ and $\mathrm{Br}_\ell\mathrm{dim}(\kappa(C)) \leq d+1$ for every curve C over κ. It was proved in [HHK1] that if char$(\kappa) \neq \ell$, then $\mathrm{Br}_\ell\mathrm{dim}(F) \leq d+2$. For $\ell =$char(κ), we prove the following

Theorem 3. *Let K be a complete discretely valued field with residue field κ. Suppose that char$(K) = 0$ and char$(\kappa) = p > 0$. Let F be the function field of a curve over K. If the p-rank of κ is n, then $Br_p dim(F) \leq 2n + 2$.*

We use the description of the Brauer group of a complete discretely valued field in the mixed characteristic case due to Kato ([K], [CT]) and the patching techniques of Harbater-Hartman-Krashen ([HHK1]) to prove our main results.

In the last section, we derive some consequences for the u-invariant of fields. The u-invariant of a field L is the maximum dimension of anisotropic quadratic forms over L. Let K be a complete discretely valued field with residue field κ. It is a theorem of Springer that $u(K) = 2u(\kappa)$. Let F be a function field of a curve over K. Suppose that char$(\kappa) \neq 2$. If $u(L) \leq d$ for every finite extension L of κ and $u(\kappa(C)) \leq 2d$ for every function field $\kappa(C)$ of a curve C over κ, then in ([HHK1]), it is proved that $u(F) \leq 4d$. For a p-adic field K, this was proved in ([PS2]). Suppose κ is a field of characteristic 2 with $[\kappa : \kappa^2] = n$. Then $u(\kappa) \leq 2n$ ([MMW], Corollary 1). Let L be a finite extension of κ. Since $[L : L^2] = n$, we have $u(L) \leq 2n$. If C is a curve over κ, then $[\kappa(C) : \kappa(C)^2] = 2n$ and hence $u(\kappa(C)) \leq 4n$. If char$(K) = 2$, $[F : F^{*2}] = 4n$ and hence $u(F) \leq 8n$ ([MMW]. Corollary 1). Suppose that char$(K) = 0$. If κ is a finite field, then results of Heath-Brown ([HB) and Leep ([Le]) lead to $u(F) = 8$. However very little is known about $u(F)$ for general κ. We prove the following

Theorem 4. *Let K be a complete discretely valued field with residue field κ. Suppose that char$(K) = 0$ and char$(\kappa) = 2$. Let F be a function field of a curve over K. If κ is a perfect field, then $u(F) \leq 8$.*

This leads us to the following

Conjecture. *Suppose K is a complete discretely valued field of characteristic 0 with residue field κ of characteristic 2. If F is a function field of a curve over K, then $u(F) \leq 8[\kappa : \kappa^2]$.*

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1. Module of differentials and Milnor k-groups

We begin by recalling two well-known results (1.1, 1.2) concerning the Brauer ℓ-dimension of a field. Lemma 1.1 reduces the computation of the Brauer ℓ-dimension to bounding indices of prime exponent algebras. Corollary 1.2 computes the Brauer p-dimension for fields of characteristic $p > 0$.

Lemma 1.1. *Let k be any field and ℓ a prime. If for every central simple algebra A of period ℓ over a finite extension K of k, ind(A) divides ℓ^d, then $Br_\ell dim(k) \leq d$.*

Proof. Let k' be a finite extension of k and A a central simple algebra over k' of period ℓ^n for some n. We prove by induction on n that ind(A) divides $(\ell^n)^d$. The case $n = 1$ is the given hypothesis. Suppose that the lemma holds for $n-1$. Let $A' = A^{\otimes \ell}$. Then per$(A') = \ell^{n-1}$. By the induction hypothesis ind(A') divides $(\ell^{n-1})^d$. Thus there exists a finite extension K of k' of degree dividing $(\ell^{n-1})^d$ such that $A' \otimes_{k'} K$ is a matrix algebra. In particular per$(A \otimes_{k'} K) = \ell$ and by the hypothesis ind$(A \otimes_{k'} K)$ divides ℓ^d. Thus there exists a finite extension L of K of degree dividing ℓ^d such that $A \otimes_{k'} L$ is a matrix algebra. Since $[L : k'] = [L : K][K : k']$ divides $\ell^d(\ell^{n-1})^d = (\ell^n)^d$, ind$(A)$ divides $(\ell^n)^d$. □

Corollary 1.2. *(Albert) Let κ be a field of characteristic $p > 0$. Suppose that the p-rank of κ is n. Then $Br_p dim(\kappa) \leq n$.*

Proof. Let k' be a finite extension of k and A be a central simple algebra over k' of period p. By (1.1), it is enough to show that ind(A) divides p^n. By ([A], Chap. VII. Theorem 32), $A \otimes_{k'} k'^{1/p}$ is a matrix algebra and hence ind(A) divides $[k'^{1/p} : k']$. Since $[k'^{1/p} : k'] = [k' : k'^p] = [k : k^p] = p^n$ ([B], A.V.135, Corollary 3), ind(A) divides p^n. □

Let κ be a field of characteristic $p > 0$. Let Ω^1_κ be the module of differentials on κ. Then the dimension of Ω^1_κ as a κ-vector space is equal to the p-rank of κ. Let Ω^2_κ be the second exterior power of Ω^1_κ. Let $K_2(\kappa)$ be the Milnor K-group and $k_2(\kappa) = K_2(\kappa)/pK_2(\kappa)$. Then there is an injective homomorphism (cf., [CT], 3.0)

$$h^2_p : k_2(\kappa) \to \Omega^2_\kappa$$

given by

$$(a, b) \mapsto \frac{da}{a} \wedge \frac{db}{b}.$$

Suppose $\kappa = \kappa^p(a_1, \cdots, a_n)$. Then every element in Ω^2_κ is a linear combination of elements $da_i \wedge da_j$. In fact if $a_1, \cdots, a_n$ is a p-basis of κ, then $da_i \wedge da_j$, $1 \leq i < j \leq n$ is a basis of Ω^2_κ over κ.

We now record a few facts about Ω^2_κ and $k_2(\kappa)$.

Lemma 1.3. *Let $a, b \in \kappa^*$ be p-dependent. Then $(a, b) = 0 \in k_2(\kappa)$.*

Proof. If a is a p^{th} power in κ, then $da = 0$. Suppose $a = \sum \lambda_i^p b^i$ for some $\lambda_i \in \kappa$. Then $da = (\sum \lambda_i^p i b^{i-1})db$ and $da \wedge db = 0$. In particular $\frac{da}{a} \wedge \frac{db}{b} = 0 \in \Omega^2_\kappa$. Since $h^2_p((a, b)) = \frac{da}{a} \wedge \frac{db}{b}$ and h^2_p is injective, we have $(a, b) = 0 \in k_2(\kappa)$. □

Lemma 1.4. *Suppose that $\kappa = \kappa^p(a_1, \cdots a_n)$. Then the natural homomorphism $k_2(\kappa) \to k_2(\kappa(\sqrt[p]{a_1}, \cdots, \sqrt[p]{a_{n-1}}))$ is zero.*

Proof. Let $(a, b) \in k_2(\kappa)$. Let $\kappa' = \kappa(\sqrt[p]{a_1}, \cdots, \sqrt[p]{a_{n-1}})$. If a is a p^{th} power in κ', then the image of $(a, b) \in k_2(\kappa')$ is zero. Suppose that a is not a p^{th} power in κ'. Since $\kappa'^p = \kappa^p(a_1, \cdots, a_{n-1})$, $\kappa = \kappa'^p(a_n)$ and hence $[\kappa : \kappa'^p] \leq p$. Since $a \notin \kappa'^p$, we have $\kappa = \kappa'^p(a) = \kappa^p(a_1, \cdots, a_{n-1}, a)$. In particular a and b are p-dependent over κ' and hence, by (1.3), $(a, b) = 0 \in k_2(\kappa')$. Since every element in $k_2(\kappa)$ is a sum of elements of the form (a, b), the image of $k_2(\kappa)$ in $k_2(\kappa')$ is zero. □

Lemma 1.5. *Let $a_1, \cdots, a_n \in \kappa^*$ be p-independent over κ and $\kappa' = \kappa(\sqrt[p^{r_1}]{a_1}, \cdots, \sqrt[p^{r_n}]{a_n})$. Then every element in the kernel of the natural homomorphism $\Omega^2_\kappa \to \Omega^2_{\kappa'}$ is of the form $da_1 \wedge f_1 + \cdots + da_n \wedge f_n$ for some $f_i \in \Omega^1_\kappa$.*

Proof. Let $B \subset \kappa^*$ be such that $B \cap \{a_1, \cdots a_n\} = \emptyset$ and $B \cup \{a_1, \cdots, a_n\}$ is a p-basis of κ. Then $B \cup \{\sqrt[p^{r_1}]{a_1}, \cdots, \sqrt[p^{r_n}]{a_n}\}$ is a p-basis of κ'. Let $\alpha \in \Omega^2_\kappa$ be in the kernel of $\Omega^2_\kappa \to \Omega^2_{\kappa'}$. Then $\alpha = \sum \lambda_{ij} da_i \wedge da_j$ with $1 \leq i < j \leq m$ and $a_{n+1}, \cdots, a_m \in B$, $\lambda_i \in \kappa$. Since the image of $da_i \wedge da_j$ in $\Omega^2_{\kappa'}$ is zero for $1 \leq i \leq n$ and the image of α in $\Omega^2_{\kappa'}$ is zero, the image of $\sum \lambda_{ij} da_i \wedge da_j$, $n+1 \leq i < j \leq m$, in $\Omega^2_{\kappa'}$ is zero. Since B is p-independent over κ' and $a_{n+1}, \cdots, a_m \in B$, $da_i \wedge da_j$, $n+1 \leq i < j \leq m$ are linearly independent over κ' and hence $\lambda_{ij} = 0$ for $n+1 \leq i < j \leq m$. Thus $\alpha = \sum \lambda_{ij} da_i \wedge da_j$ with $1 \leq i < j \leq n$. □

Lemma 1.6. *Let $a_1, \cdots, a_{2n} \in \kappa^*$ be p-independent in κ. Let κ' be an extension of κ of degree d and $\lambda_1, \cdots, \lambda_n \in \kappa^*$. If $d < p^n$, then the image of $\lambda_1 da_1 \wedge da_2 + \cdots + \lambda_n da_{2n-1} \wedge da_{2n}$ in $\Omega^2_{\kappa'}$ is non-zero.*

Proof. Since $\Omega^2_\kappa \to \Omega^2_{\kappa_1}$ is injective for any separable extension κ_1 of κ, by replacing κ by the separable closure of κ in κ', we assume that κ' is purely inseparable over κ. Then $\kappa' = \kappa(\sqrt[p^{r_1}]{b_1}, \cdots, \sqrt[p^{r_n}]{b_m})$ for some $b_i \in \kappa^*$ with $\{b_1, \cdots, b_m\}$ p-independent over κ. Since the kernels of the homomorphisms $\Omega^2_\kappa \to \Omega^2_{\kappa'}$ and $\Omega^2_\kappa \to \Omega^2_{\kappa(\sqrt[p]{b_1}, \cdots, \sqrt[p]{b_m})}$ are equal by (1.5), we assume that $\kappa' = \kappa(\sqrt[p]{b_1}, \cdots, \sqrt[p]{b_m})$. Since $[\kappa' : \kappa] = p^m < p^n$, we have $m \leq n-1$. Without loss of generality we assume that $m = n-1$.

Suppose that $\{a_1, \cdots, a_{2n}\}$ is a p-basis of κ. Let r be the maximum such that $\{b_1, \cdots, b_{n-1}, a_{i_1}, a_{i_2} \cdots, a_{i_r}\}$ is p-independent with no two a_{i_s} in $\{a_{2j-1}, a_{2j}\}$. By reindexing, we assume that $\{b_1, \cdots, b_{n-1}, a_1, a_3, \cdots, a_{2r+1}\}$ is p-independent. Then, for each $i \geq 2r+3$, $\{b_1, \cdots, b_{n-1}, a_1, a_3, \cdots, a_{2r+1}, a_i\}$ is p-dependent. Since $\{a_1, \cdots, a_{2n}\}$ is p-basis of κ, there exists $1 \leq t_1 < t_2 < \cdots < t_q \leq r+1$ such that $\{b_1, \cdots, b_{n-1}, a_1, a_3, \cdots, a_{2r+1}, a_{2t_1}, \cdots, a_{2t_q}\}$ is a p-basis of κ. After reshuffling the indices, we assume that $t_1 = 1, \cdots, t_q = q$ and $B = \{b_1, \cdots, b_{n-1}, a_1, a_3, \cdots, a_{2r+1}, a_2, a_4, \cdots a_{2q},\}$ is a p-basis of κ with $q \leq r+1$. Then $\{\sqrt[p]{b_1}, \cdots, \sqrt[p]{b_{n-1}}, a_1, a_3, \cdots, a_{2r+1}, a_2, a_4, \cdots, a_{2q}\}$ is a p-basis of κ'

Since B is a p-basis of κ, every element of Ω^2_κ can be written as a linear combination of $dx \wedge dy$, $x, y \in B$. We now compute the coefficient of $da_1 \wedge da_2$ in the expansion of $da_{2i+1} \wedge da_{2i+2}$ as a linear combination of $dx \wedge dy$, $x, y \in B$. Let $1 \leq i \leq r$. Since $a_{2i+1} \in B$, the coefficient of $da_1 \wedge da_2$ in the expansion of $da_{2i+1} \wedge da_{2i+2}$ is zero. Let $i > r$. Since a_{2i+1} and a_{2i+2} are p-dependent over $\{b_1, \cdots, b_{n-1}, a_1, a_3, \cdots, a_{2r+1}\}$, in the expansion of da_{2i+1} and da_{2i+2} there is no da_2 term. Hence, there is no $da_1 \wedge da_2$ term in the expansion of $da_{2i+1} \wedge da_{2i+2}$.

Thus, the coefficient of $da_1 \wedge da_2$ in the expansion of $\alpha = \lambda_1 da_1 \wedge da_2 + \cdots + \lambda_n da_{2n-1} \wedge da_{2n}$ as a linear combination of $dx \wedge dy$ with $x, y \in B$ is λ_1. Since $\{\sqrt[p]{b_1}, \cdots, \sqrt[p]{b_{n-1}}, a_1, a_3, \cdots, a_{2r+1}, a_2, a_4, \cdots, a_{2q}\}$ is a p-basis of κ', the image of α in $\Omega^2_{\kappa'}$ is non-zero.

Let $\{a_1, \cdots, a_{2n}\}$ be any p-independent subset of κ. Let $B' \subset \kappa$ be such that $B' \cup \{a_1, \cdots, a_{2n}\}$ is a p-basis of κ and $B' \cap \{a_1, \cdots, a_{2n}\} = \emptyset$. Let κ_1 be the extension of κ obtained by adjoining $\sqrt[p^d]{b}$ for all $b \in B'$ and $d \geq 1$. Then $\{a_1, \cdots, a_{2n}\}$ is a p-basis of κ_1. Then $\kappa_1 \kappa'$ is an extension of κ_1 of degree $< p^n$. Hence the image of $\lambda_1 dc_1 \wedge dc_2 + \cdots + \lambda_n dc_{2n-1} \wedge dc_{2n}$ in $\Omega^2_{\kappa'\kappa_1}$ is non-zero. In particular, the image of $\lambda_1 dc_1 \wedge dc_2 + \cdots + \lambda_n dc_{2n-1} \wedge dc_{2n}$ in $\Omega^2_{\kappa'}$ is non-zero. □

2. Brauer p-dimension of a complete discretely valued field

In the section we give a bound for the Brauer p-dimension of a complete discrete valued field of characteristic zero with residue of characteristic $p > 0$, in terms of the p-rank of the residue field.

Let R be a complete discrete valuation ring with field of fractions K and residue field κ. Let ν be the discrete valuation on K given by R and π a parameter in R. Suppose that $\mathrm{char}(K) = 0$ and $\mathrm{char}(\kappa) = p > 0$ and that K contains a primitive p^{th} root of unity. Fixing a primitive p^{th} root of unity $\zeta \in K^*$, for $a, b \in K^*$, let $(a, b) \in {}_p\mathrm{Br}(K)$ be the class of the cyclic K-algebra defined by $x^p = a, y^p = b$ and $xy = \zeta yx$. Let $N = \nu(p)p/(p-1)$. Let $\mathrm{br}(K)_0 = {}_p\mathrm{Br}(K)$. For $i \geq 1$, let $U_i = \{u \in R^* \mid u \equiv 1 \bmod (\pi)^i\}$ and $\mathrm{br}(K)_i$ be the subgroup of ${}_p\mathrm{Br}(K)$ generated by cyclic algebras (u, a) with $u \in U_i$ and $a \in K^*$. Since R is complete, for $n > N$, every element in U_n is a p^{th} power and $\mathrm{br}(K)_n = 0$.

Let Ω^1_κ be the module of differentiales on κ. For any $c \in \kappa$, let $\tilde{c} \in R$ be a lift of c. For $i \geq 1$, the maps

$$\Omega^1_\kappa \to \mathrm{br}(K)_i/\,\mathrm{br}(K)_{i+1}$$

given by $x\frac{dy}{y} \mapsto (1 + \tilde{x}\pi^i, \tilde{y})$ and

$$\kappa \to \mathrm{br}(K)_i/\,\mathrm{br}(K)_{i+1}$$

given by $z \mapsto (\pi, 1 + \tilde{z}\pi^i)$ yield a surjective homomorphism

$$\rho_i : \Omega^1_\kappa \oplus \kappa \to \mathrm{br}(K)_i/\,\mathrm{br}(K)_{i+1}$$

([K], Thm. 2, cf. [CT], 4.3.1).

Let $K_2(\kappa)$ be the Milnor K-group and $k_2(\kappa) = K_2(\kappa)/pK_2(\kappa)$. The maps

$$k_2(\kappa) \to \mathrm{br}(K)_0/\,\mathrm{br}(K)_1$$

given by $(x, y) \mapsto (\tilde{x}, \tilde{y})$ and

$$\kappa^*/\kappa^{*p} \to \mathrm{br}(K)_0/\,\mathrm{br}(K)_1$$

given by $(z) \mapsto (\pi, \tilde{z})$ yield an isomorphism

$$\rho_0 : k_2(\kappa) \oplus \kappa^*/\kappa^{*p} \to \mathrm{br}(K)_0/\,\mathrm{br}(K)_1$$

([K], Thm.2, cf. [CT], 4.3.1).

Lemma 2.1. *Let R, K and κ be as above. Suppose that $\kappa = \kappa^p(a_1, \cdots, a_n)$ for some $a_1, \cdots, a_n \in \kappa$. Let $u_1, \cdots, u_n \in R$ be lifts of $a_1, \cdots, a_n$. Let $\alpha \in {}_p\mathrm{Br}(K)$. Then, there exists $u \in R^*$ such that $(\alpha - (\pi, u)) \otimes K(\sqrt[p]{u_1}, \cdots, \sqrt[p]{u_{n-1}}) \in \mathrm{br}(K(\sqrt[p]{u_1}, \cdots, \sqrt[p]{u_{n-1}}))_1$.*

Proof. Since ρ_0 is surjective, there exists $x_i, y_i, a \in \kappa^*$ such that $\rho_0(\sum_i(x_i, y_i) - (a)) = \alpha \in \mathrm{br}(K)_0/\,\mathrm{br}(K)_1$. In particular, if u is a lift of a in R,

$$\rho_0(\sum_i (x_i, y_i)) = \alpha - (\pi, u) \in \mathrm{br}(K)_0/\,\mathrm{br}(K)_1.$$

Let $K' = K(\sqrt[p]{u_1}, \cdots, \sqrt[p]{u_n - 1})$. Then K' is also a complete discretely valued field with residue field $\kappa' = \kappa(\sqrt[p]{a_1}, \cdots, \sqrt[p]{a_{n-1}})$. By the functoriality of the map ρ_0, we have $\rho_0(\sum(x_i, y_i)) = \alpha - (\pi, u) \in \mathrm{br}(K')_0/\,\mathrm{br}(K')_1$. By (1.4), the image of $\sum_i(x_i, y_i)$ is zero in $k_2(\kappa')$. Hence $\alpha - (\pi, u) = \rho_0(\sum(x_i, y_i)) = 0 \in \mathrm{br}(K')_0/\,\mathrm{br}(K')_1$. In particular, $\alpha - (\pi, u) \in \mathrm{br}(K')_1$. ∎

Proposition 2.2. *Let R, K, κ and π be as above. Suppose that $\kappa = \kappa^p(a_1, \cdots, a_n)$ for some $a_1, \cdots, a_n \in \kappa$. Let $u_1, \cdots, u_n \in R$ be lifts of $a_1, \cdots, a_n$. Let $\alpha \in \mathrm{br}(K)_1$. Then there exist $\lambda, \lambda_1, \cdots, \lambda_n \in R^*$ such that*

$$\alpha = (\lambda_1, u_1) + \cdots + (\lambda_n, u_n) + (\pi, \lambda).$$

Proof. Let $\alpha \in \mathrm{br}(K)_1$. First we show, by induction on i, that for each $i \geq 0$, there exist $x_{i1}, \cdots, x_{in}, x_i \in R^*$ such that $\alpha - (x_{i1}, u_1) - \cdots (x_{in}, u_n) - (\pi, x_i) \in \mathrm{br}(K)_{i+1}$. Since $\alpha \in \mathrm{br}(K)_1$, we take $x_{0j} = x_0 = 1$, $1 \leq j \leq n$. Suppose that $i \geq 1$ and there exist $x_{i1}, \cdots, x_{in}, x_i \in R^*$ such that $\alpha - (x_{i1}, u_1) - \cdots (x_{in}, u_n) - (\pi, x_i) \in \mathrm{br}(K)_{i+1}$. Since the homomorphism $\rho_{i+1} : \Omega^1_\kappa \oplus \kappa \to \mathrm{br}(K)_{i+1}/\,\mathrm{br}(K)_{i+2}$ is surjective, there exist $w \in \Omega^1_\kappa$ and $a \in \kappa$ such that

$$\rho_{i+1}(w, a) = \alpha - (x_{i1}, u_1) - \cdots (x_{in}, u_n) - (\pi, x_i) \in \mathrm{br}(K)_{i+1}/\,\mathrm{br}(K)_{i+2}.$$

Since $\kappa = \kappa^p(a_1, \cdots, a_n)$, Ω_κ is generated by $\frac{da_i}{a_i}$, $1 \leq i \leq n$ and hence $w = \sum_i b_i \frac{da_i}{a_i}$ for some $b_i \in \kappa$. Thus

$$\rho_{i+1}(w, a) = (1 + \tilde{b}_1\pi^{i+1}, u_1) + \cdots (1 + \tilde{b}_n\pi^{i+1}, u_n) + (\pi, 1 + \tilde{a}\pi^{i+1}).$$

In particular, $\alpha - (x_{i1}, u_1) - \cdots - (x_{in}, u_n) - (\pi, x_i) - (1 + \tilde{b}_1\pi^{i+1}, u_1) - \cdots - (1 + \tilde{b}_n\pi^{i+1}, u_n) - (\pi, 1 + \tilde{a}\pi^{i+1}) \in \mathrm{br}(K)_{i+2}$. Let $x_{(i+1)j} = x_{ij}(1 + \tilde{b}_j\pi^{i+1})$ for $1 \leq j \leq n$ and $x_{i+1} = x_i(1 + \tilde{a}\pi^{i+1})$. Since $(x, yz) = (x, y) + (x, z) \in {}_p\mathrm{Br}(K)$, we have $\alpha - (x_{(i+1)1}, u_1) - \cdots - (x_{(i+1)n}, u_n) - (\pi, x_{i+1}) \in \mathrm{br}(K)_{i+2}$.

Since $\mathrm{br}(K)_i = 0$ for $i > N$, we have $\alpha = (x_{(N+1)1}, u_1) + \cdots + (x_{(N+1)n}, u_n) + (\pi, x_{N+1})$. $\square$

Corollary 2.3. *Let K and κ be as above. Suppose that the p-rank of κ is n. Let D be a central simple algebra over K of period p. If D represents an element in $\mathrm{br}(K)_1$, then ind(D) divides p^{n+1}.*

Proof. Let $\alpha \in \mathrm{br}(K)_1$ be the class of D. By (2.2), there exist $\lambda, \lambda_1, \cdots, \lambda_n \in R^*$ such that $\alpha = (\lambda_1, u_1) + \cdots + (\lambda_n, u_n) + (\pi, \lambda)$. Hence $\alpha \otimes K(\sqrt[p]{u_1}, \cdots \sqrt[p]{u_n}, \sqrt[p]{\pi}) = 0$ and the index of D divides p^{n+1}. $\square$

Theorem 2.4. *Let K be a complete discretely valued field with residue field κ. Let R be the valuation ring of K and $\pi \in R$ be a parameter. Suppose that char$(K) = 0$, char$(\kappa) = p$ and the p-rank of κ is n. Let $a_1, \cdots, a_n \in \kappa$ be such that $\kappa = \kappa^p(a_1, \cdots, a_n)$ and $u_1, \cdots, u_n \in R$ be lifts of $a_1, \cdots, a_n$. Let D be a central simple algebra over K of period p. If $n = 0$, then $D \otimes K(\sqrt[p]{\pi})$ is a matrix algebra and if $n \geq 1$, then $D \otimes K(\sqrt[p^2]{u_1}, \cdots \sqrt[p^2]{u_{n-1}}, \sqrt[p]{u_n}, \sqrt[p]{\pi})$ is a matrix algebra.*

Proof. Let ζ be a primitive p^{th} root of unity and $K' = K(\zeta)$. Since $[K' : K]$ is coprime to p, ind(D) = ind$(D \otimes K')$. Since K' is finite extension of a complete discretely valued field K, K' is also a complete discrete valued field with residue field κ' a finite extension of κ. In particular, p-rank(κ') = p-rank(κ). Thus, by replacing, K by K', we assume that K contains a primtiive p^{th} root of unity. Let $\alpha \in {}_p\mathrm{Br}(K)$ be the class of D.

Suppose $n = 0$. Then $\kappa = \kappa^p$ and $k_2(\kappa) = 0$. Since $\rho_0 : k_2(\kappa) \oplus \kappa^*/\kappa^{*p} \to \mathrm{br}(K)_0/\,\mathrm{br}(K)_1$ is an isomorphism, ${}_p\mathrm{Br}(K) = \mathrm{br}(K)_1$. Thus, by (2.2), $\alpha = (\pi, u)$. In particular $D \otimes K(\sqrt[p]{\pi})$ is a matrix algebra.

Suppose that $n \geq 1$. Since p-rank$(\kappa) = n$, there exist $a_1, \cdots, a_n \in \kappa^*$ such that $\kappa = \kappa^p(a_1, \cdots, a_n)$. Let $u_1, \cdots, u_n \in R$ be lifts of $a_1, \cdots, a_n$ and $\pi \in R$ a parameter. Let $K_1 = K(\sqrt[p]{u_1}, \cdots, \sqrt[p]{u_{n-1}})$. Then K_1 is also a complete discrete valued field with residue field $\kappa_1 = \kappa(\sqrt[p]{a_1}, \cdots, \sqrt[p]{a_{n-1}})$. Let R_1 be the valuation ring of K_1. Then π is a parameter in R_1. By (2.1), there exists $u \in R^*$ such that $(\alpha - (\pi, u)) \otimes K_1 \in \mathrm{br}(K_1)_1$. Since $\kappa_1^p = \kappa^p(a_1, \cdots, a_{n-1})$, we have $\kappa_1 = \kappa_1^p(\sqrt[p]{a_1}, \cdots, \sqrt[p]{a_{n-1}}, a_n)$. Since $\alpha - (\pi, u) \in \mathrm{br}(K_1)_1$, by (2.2), there exist $\lambda_1, \cdots, \lambda_n, \lambda \in R_1$ such that

$$\alpha - (\pi, u) = (\lambda_1, \sqrt[p]{u_1}) + \cdots + (\lambda_{n-1}, \sqrt[p]{u_{n-1}}) + (\lambda_n, u_n) + (\pi, \lambda).$$

Hence

$$\alpha = (\lambda_1, \sqrt[p]{u_1}) + \cdots + (\lambda_{n-1}, \sqrt[p]{u_{n-1}}) + (\lambda_n, u_n) + (\pi, u\lambda).$$

In particular $D \otimes K(\sqrt[p^2]{u_1}, \cdots \sqrt[p^2]{u_{n-1}}, \sqrt[p]{u_n}, \sqrt[p]{\pi})$ is a matrix algebra. □

Corollary 2.5. *Let K, κ and n be as in (2.4). Then $Br_p dim(K)$ is 1 if $n = 0$ and $Br_p dim(K) \leq 2n$ if $n \geq 1$.*

Proof. Let K' be a finite extension of K. Let D be a central simple algebra over K' of period p. Since K' is also a complete discretely valued field with the p-rank of the residue field equal to n, corollary follows by (2.4) and (1.6). □

Lemma 2.6. *Let K be a complete discretely valued field with residue field κ. Suppose that $char(K) = 0$, $char(\kappa) = p > 0$ and $[\kappa : \kappa^p] \geq 2n$. Then $Br_p dim(K) \geq n$.*

Proof. Let $a_1, \cdots, a_{2n} \in \kappa^*$ be p-independent. Let $u_1, \cdots, u_{2n} \in K^*$ be the lifts of $a_1, \cdots, a_{2n}$. Let $D = (u_1, u_2) + \cdots + (u_{2n-1}, u_{2n})$. We claim that $\mathrm{ind}(D) = p^n$. This would show that $\mathrm{Br}_p\mathrm{dim}(K) \geq n$.

Let K_1 be an extension of K of degree at most p^{n-1}. Since K is a complete discretely valued field, K_1 is also a complete discretely valued field with residue field κ_1 and $[\kappa_1 : \kappa] \leq [L : K] \leq p^{n-1}$. Then, by (1.6), the image of $da_1 \wedge da_2 + \cdots + da_{2n-1} \wedge da_{2n}$ in $\Omega^2_{\kappa_1}$ is non-zero. Since $h^2_p((a_1, a_2) + \cdots + (a_{2n-1}, a_{2n})) = da_1 \wedge da_2 + \cdots + da_{2n-1} \wedge da_{2n}$ is nonzero in $\Omega^2_{\kappa_1}$, $(a_1, a_2) + \cdots + (a_{2n-1}, a_{2n})$ is non-zero in $k_2(\kappa_1)$. Since $\rho_0((a_1, a_2) + \cdots + (a_{2n-1}, a_{2n}))$ is the class of $D \otimes_K K_1$ in $\mathrm{br}(K_1)_0 / \mathrm{br}(K)_1$ and ρ_0 is injective, $D \otimes_K K_1$ is non-trivial in ${}_p\mathrm{Br}(K_1)$. Hence $\mathrm{ind}(D) \geq p^n$. Since D is a product of n cyclic algebras, $\mathrm{ind}(D) = p^n$. □

Combining (2.4) and (2.6), we have the following

Theorem 2.7. *Let K be a complete discretely valued field with residue field κ. Suppose that $char(K) = 0$, $char(\kappa) = p > 0$ and the p-rank of κ is n. If $n = 0$, then $Br_p dim(K) \leq 1$ and if $n \geq 1$, then $\frac{n}{2} \leq Br_p dim(K) \leq 2n$.*

Example 2.8. Let k be a purely transcendental extension of the finite field $\mathbf{F}_p$ of transcendence degree 2n and κ the separable closure of k. Let K be a complete discretely valued field of characteristic 0 with residue field κ. Then the Brauer p-dimension of κ is 0 ([A], Ch.IV, §7, Theorem 18) and $\mathrm{Br}_p\mathrm{dim}(K) \geq n$ (2.6). Note that the p-rank of κ is $2n$. Thus in the mixed characteristic case, the bound on the Brauer dimension of the residue field should be replaced by the bound on the p-rank of the residue field in order to get a good bound on the Brauer dimension of a complete discretely valued field.

3. The main theorem

Let R be an integral domain and K its field of fractions. Let A be a central simple algebra over K. We say that A is *unramified on* R if there exists an Azumaya algebra $\mathscr{A}$ over R such that $\mathscr{A} \otimes_R K$ is Brauer equivalent to A. If P is a prime ideal of R and A is unramified on R_P, then we say that A is *unramified at* P. If ν is a discrete valuation of K with R as the valuation ring at ν and A is unramified on R, then we also say that A is *unramified at* ν.

Let $\mathscr{X}$ be a regular integral scheme with function field K and A a central simple algebra over K. Let $x \in \mathscr{X}$ be a point. If A is unramified on the local ring $\mathscr{O}_{\mathscr{X},x}$ at x, then we say that A is *unramified at* x. If A is not unramified at x, then we say that A is *ramified at* x. The *ramification divisor* of A on $\mathscr{X}$ is the divisor $\sum x$, where sum is taken over the codimension one points x of $\mathscr{X}$ with A ramified at x. The *support* of the ramification divisor of A is simply the union of codimension one points of $\mathscr{X}$ where A is ramified.

Let T be a complete discrete valuation ring with field of fractions K and $t \in T$ a parameter. Let $\mathscr{X}$ be an excellent regular, integral, proper scheme over Spec(T) of dimension 2 with function field F and reduced special fibre Y. For a closed point P of $\mathscr{X}$, let $\mathscr{O}_{\mathscr{X},P}$ denote the local ring at P, $\hat{\mathscr{O}}_{\mathscr{X},P}$ the completion of the regular local ring $\mathscr{O}_{\mathscr{X},P}$ at its maximal ideal and F_P the field of fractions of $\hat{\mathscr{O}}_{\mathscr{X},P}$. For an open subset U of an irreducible component of Y, let R_U be the ring consisting of elements in F which are regular on U. Then $T \subset R_U$. Let $\hat{R}_U$ be the (t)-adic completion of R_U and F_U the field of fractions of $\hat{R}_U$ (cf. [HHK1]). In this section we give a bound for the Brauer p-dimension of F in terms of the p-rank of the residue field of T.

We begin with the following results (3.1, 3.2, 3.3 and 3.4) which are well-known and we include them for the sake of completeness.

Lemma 3.1. *Let B be a regular local ring with field of fractions K, residue field κ and maximal ideal m. Let n be a natural number and $u \in B$ a unit such that $[\kappa(\sqrt[n]{\overline{u}}) : \kappa] = n$. Then $B[\sqrt[n]{u}]$ is a regular local ring with residue field $\kappa(\sqrt[n]{\overline{u}})$.*

Proof. Since $[\kappa(\sqrt[n]{\overline{u}}) : \kappa] = n$, $B[\sqrt[n]{u}]/mB[\sqrt[n]{u}] \simeq \kappa(\sqrt[n]{\overline{u}})$ is a field. Thus m generates the maximal ideal of $B[\sqrt[n]{u}]$. Since the dimension of $B[\sqrt[n]{u}]$ is equal to the dimension of B, $B[\sqrt[n]{u}]$ is a regular local ring. □

Lemma 3.2. *Let B be a regular local ring with field of fractions K, residue field κ and maximal ideal m. Let $\pi \in m$ be a regular prime and n a natural number. Then $B[\sqrt[n]{\pi}]$ is a regular local ring with residue field κ.*

Proof. Since B is a regular and $\pi \in m$ is a regular prime, there exist $\pi_2, \cdots, \pi_d \in m$ such that $m = (\pi, \pi_2, \cdots, \pi_d)$, where d is the dimension of B. Let $\tilde{m} = (\sqrt[n]{\pi}, \pi_2, \cdots, \pi_d)$ $\subset B[\sqrt[n]{\pi}]$. Then $\tilde{m}$ is the maximal ideal of $B[\sqrt[n]{\pi}]$ and $B[\sqrt[n]{\pi}]/\tilde{m} \simeq \kappa$. Since the dimension of $B[\sqrt[n]{\pi}]$ is n, $B[\sqrt[n]{\pi}]$ is a regular local ring. □

Corollary 3.3. *Let B be a regular local ring with field of fractions K, residue field κ and maximal ideal m. Let $n_1, \cdots, n_r$ be natural numbers and $u_1, \cdots, u_r \in B$ units such that $[\kappa(\sqrt[n_1]{\overline{u}_1}, \cdots, \sqrt[n_r]{\overline{u}_r}) : \kappa] = n_1 \cdots n_r$. Let $\pi_1, \cdots, \pi_s \in m$ be a system of regular parameters in B and $d_1, \cdots, d_s$ be natural numbers. Then $B[\sqrt[n_1]{u_1}, \cdots, \sqrt[n_r]{u_r}, \sqrt[d_1]{\pi_1}, \cdots, \sqrt[d_s]{\pi_s}]$ is a regular local ring with residue field $\kappa(\sqrt[n_1]{\overline{u}_1}, \cdots, \sqrt[n_r]{\overline{u}_r})$.*

Proof. Proof follows by induction using (3.1) and (3.2). □

Lemma 3.4. *(cf. [LPS], 2.4) Let R be a discrete valuation ring with field of fractions K. Let $\hat{R}$ be the completion of R at the discrete valuation and $\hat{K}$ the field of fractions of $\hat{R}$. Then a central simple algebra D over K is unramified at R if and only if $D \otimes_K \hat{K}$ is unramified at $\hat{R}$.*

Proposition 3.5. *Let A be a complete regular local ring of dimension 2 with field of fractions F, residue field κ and maximal ideal $m = (\pi, \delta)$. Suppose that char$(F) = 0$ and char$(\kappa) = p > 0$ with p-rank$(\kappa) = n$. Let $a_1, \cdots, a_n \in \kappa$ be a p-basis of κ and $u_1, \cdots, u_n \in A$ be lifts of $a_1, \cdots, a_n$. Suppose that F contains a primitive p^{th} root of unity. Let D be a central simple algebra over F of period p. Suppose that D is ramified on A at most at (π) and (δ). Then $D \otimes_F F(\sqrt[p^2]{u_1}, \cdots, \sqrt[p^2]{u_n}, \sqrt[p]{\pi}, \sqrt[p]{\delta})$ is a matrix algebra. In particular, index(D) divides p^{2n+2}.*

Proof. Let
$$E = F(\sqrt[p^2]{u_1}, \cdots, \sqrt[p^2]{u_n}, \sqrt[p]{\pi}, \sqrt[p]{\delta})$$
and
$$B = A[\sqrt[p^2]{u_1}, \cdots, \sqrt[p^2]{u_n}, \sqrt[p]{\pi}, \sqrt[p]{\delta}] \subset E.$$
By (3.3), B is a complete regular local ring of dimension 2 with field of fractions E and residue field $\kappa(\sqrt[p^2]{a_1}, \cdots \sqrt[p^2]{a_n})$.

We first show that $D \otimes_F E$ is unramified on B. Since B is a regular local ring of dimension 2, it is enough to show that $D \otimes_F E$ is unramified at every height one prime ideal of B ([AG], 7.4). Let Q be a height one prime ideal of B and $P = Q \cap A$. Since B is integral over A, the height of P is 1. If $P \neq (\pi)$ or (δ), then D is unramified at P and hence $D \otimes_F E$ is unramified at Q. Suppose that $P = (\pi)$. Then $Q = (\sqrt[p]{\pi})$.

Suppose that char$(A/P) \neq p$. Since E/F is ramified at P and char$(\kappa(P)) \neq p$, $D \otimes_F E$ is unramified at Q. Suppose that char$(A/P) = p$. Since A is complete regular local ring with maximal ideal $m = (\pi, \delta)$, $A/(\pi)$ is a complete discrete valuation ring with residue field κ and char$(A/P) =$ char$(\kappa) = p$. In particular, $A/(\pi) \simeq \kappa[[\overline{\delta}]]$, where $\overline{\delta}$ is the image of δ in $A/(\pi)$. Let $\kappa(P)$ be the field of fractions of A/P. Then $\kappa(P) \simeq \kappa((\overline{\delta}))$. Since $a_1, \cdots, a_n$ is a p-basis of κ and $u_1, \cdots, u_n \in A$ are lifts of $a_1, \cdots, a_n$, the images of $u_1, \cdots, u_n, \delta$ in $\kappa(P)$ is a p-basis of $\kappa(P)$. Let F_P be the completion of F at P and E_Q the completion of E at Q. Since $E_Q \simeq F_P(\sqrt[p^2]{u_1}, \cdots, \sqrt[p^2]{u_n}, \sqrt[p]{\delta}, \sqrt[p]{\pi})$ and the residue field of F_P is $\kappa(P)$, by (2.4), $D \otimes_F E_Q$ is split and hence unramified. Thus, by (3.4), $D \otimes_F E$ is unramified at Q.

By ([AG], 7.4), there exists an Azumaya B-algebra $\mathscr{D}$ such that $\mathscr{D} \otimes_B E \simeq D \otimes_F E$. Since $D \otimes E_Q$ is split and $\hat{B}_Q$ is a discrete valuation ring, $\mathscr{D} \otimes_B \hat{B}_Q$ is zero in the Br$(\hat{B}_Q)$ ([AG], 7.2). In particular the image $\mathscr{D} \otimes_B \kappa(Q)$ of $\mathscr{D} \otimes_B \hat{B}_Q$ in Br$(\kappa(Q))$ is zero. Since $\kappa(Q)$ is the field of fractions of regular local ring B/Q, by ([AG], 7.2), $\mathscr{D} \otimes_B B/Q$ is zero in Br(B/Q). Hence $\mathscr{D} \otimes_B B/\tilde{m}$ is zero in Br$(B/\tilde{m})$, where $\tilde{m}$ is the maximal ideal of B. Since B is a complete regular local ring, $\mathscr{D} = 0 \in$ Br(B) ([C], [KOS]). In particular $\mathscr{D} \otimes_B E \simeq D \otimes_F E$ is zero and index(D) divides $[E : F] = p^{2n+2}$. □

Theorem 3.6. *Let K be a complete discretely valued field with residue field κ. Suppose that char$(K) = 0$, char$(\kappa) = p > 0$ and p-rank$(\kappa) = n$. Let F be a finitely generated field extension of K of transcendence degree 1 and D a central simple algebra over F of period p. Then ind(D) divides p^{2n+2}.*

Proof. As in the proof of (2.4), we assume without loss of generality that F contains a primitive p^{th} root of unity. Let K' be a finite extension of K. Then K' is also a complete discretely valued field with the p-rank of the residue field is n. Thus, replacing K by a finite extension of K, we assume that F is the function field of a geometrically integral smooth projective curve X over K.

We choose a proper regular model $\mathscr{X}$ of F over T such that the support of the ramification divisor of D and the components of the reduced special fibre are a union of regular curves with normal crossings on $\mathscr{X}$. Let Y be the special fibre of $\mathscr{X}$.

Let η be a generic point of an irreducible component of Y and F_η the completion of F at the discrete valuation given by η. Then the residue field $\kappa(\eta)$ of F_η is function field of transcendence degree one over κ. Since $[\kappa : \kappa^p] = p^n$, we have $[\kappa(\eta) : \kappa(\eta)^p] = p^{n+1}$. By (2.4), $\mathrm{ind}(D \otimes_F F_\eta)$ divides p^{2n+2}. By ([HHK2], 5.8 and [KMRT], 1.17), there exists an irreducible open set U_η of Y containing η such that $\mathrm{ind}(D \otimes_F F_{U_\eta}) = \mathrm{ind}(D \otimes_F F_\eta)$. In particular $\mathrm{ind}(D \otimes_F F_{U_\eta})$ divides p^{2n+2}.

Let S_0 be a finite set of closed points of $\mathscr{X}$ containing all the points of intersection of the components of Y and the support of the ramification divisor of D. Let S be a finite set of closed points of $\mathscr{X}$ containing S_0 and $Y \setminus (\cup\, U_\eta)$, where η varies over generic points of Y. Then, by ([HHK1], 5.1),

$$\mathrm{ind}(D) = l.c.m\{\mathrm{ind}(D \otimes F_\zeta\},$$

where ζ running over S and irreducible components of $Y \setminus S$.

Suppose $\zeta = U$ for some irreducible component U of $Y \setminus S$. Let η be the generic point of U. Then $U \subset U_\eta$ and $R_{U_\eta} \subset R_U$. Since $F_{U_\eta} \subset F_U$, $\mathrm{ind}(D \otimes_F F_U)$ divides p^{2d+2}.

Suppose $\zeta = P \in S$. Let A_P be the regular local ring at P. Then, by the choice of $\mathscr{X}$, the maximal ideal m_P of A_P is generated by π and δ such that A is ramified on A_P at most possibly at (π) and (δ). Since the residue field $\kappa(P)$ at P is a finite extension of κ, we have p-rank$(\kappa(P)) = p$-rank$(\kappa) = p^n$. Thus, by (3.5), $\mathrm{ind}(D \otimes_F F_P)$ divides p^{2n+2}. Hence $\mathrm{ind}(D)$ divides p^{2n+2}. $\square$

Corollary 3.7. *Let F and n be as in (3.6). Then $Br_p dim(F) \leq 2n+2$.*

Proof. Let F' be a finite extension of F and D a central simple algebra of period p. Since the transcendence degree of F' over K is 1, by (3.6), $\mathrm{ind}(D)$ divides p^{2n+2}. Corollary follows from (1.6). $\square$

Corollary 3.8. *Let K be a complete discretely valued field with residue field κ. Suppose that κ is finitely generated field of transcendence degree n over a perfect field of characteristic $p > 0$. If F is a function field of a curve over K, then the Brauer p-dimension of F is at most $2n+2$.*

Proof. Since κ is a finitely generated field of transcendence degree d over a perfect field, we have $[\kappa : \kappa^p] = p^n$ ([B], A.V.135, Corollary 3). Hence the result follows from (3.6). $\square$

Let K be a p-adic field and F the function field of curve over K. Let A be a central simple algebra over F . If the period of A is coprime to p, then a theorem of Saltman ([S1]) asserts that $\mathrm{ind}(A)$ divides $\mathrm{per}(A)^2$. If the period of A is a power of p, then it is proved in ([LPS]) that the $\mathrm{ind}(A)$ divides $\mathrm{per}(A)^3$. We have the following

Corollary 3.9. *Let F be the function field of a curve over a p-adic field K. Then for every central simple algebra over F, the index divides the square of the period.*

Proof. Let A be a central simple algebra over F of period a power of p. Since the residue field κ of K is a finite field, $[\kappa : \kappa^p] = 1$. Thus, by (3.6), $\mathrm{ind}(A)$ divides $\mathrm{per}(A)^2$. $\square$

4. u-invariant

Let K be a complete discretely valued field with residue field κ and F the function field of a curve over K. In this section we compute the u-invariant of F when κ is a perfect field of characteristic 2 and $\mathrm{char}(K) = 0$.

For any field L of characteristic not equal to 2, let $W(L)$ be the Witt ring of quadratic forms over L and $I^n(L)$ be the n^{th} power of the fundamental ideal $I(L)$ of $W(L)$.

Let R be an integral domain with field of fractions F. A quadratic form q over R is *non-singular* if the associated quadric is smooth over R. We say that a quadratic form q over F is *defined over* R if there exists a non-singular quadratic form q' over R such that $q' \otimes_R F \simeq q$.

In the rest of this section, until (4.7), A denotes a complete regular local ring of dimension two with field of fractions F and residue field κ. Suppose that $\mathrm{char}(F) = 0$, $\mathrm{char}(\kappa) = 2$ and κ is a perfect field. Suppose that the maximal ideal $m = (\pi, \delta)$ and $2 = u_0\pi^i\delta^j$ for some $u_0 \in A^*$ and $i, j \geq 0$.

Lemma 4.1. *Let A, F, κ, $m = (\pi, \delta)$ as above. Let $\alpha \in H^2(F, \mu_2)$. If α is unramified on A except possibly at (π) and (δ). Then $\alpha = (uc, \pi) + (vc\pi^\epsilon, \delta)$ for some units $u, v \in A$, $c \in A$ not divisible by π, δ and $\epsilon = 0$ or 1.*

Proof. Since α is unramified except at (π) and (δ) and κ is perfect, by (3.5), $\alpha \otimes F(\sqrt{\pi}, \sqrt{\delta})$ is zero. In particular, by a theorem of Albert, $\alpha = (a, \pi) + (b, \delta)$ for some $a, b \in F^*$. Without loss of generality we assume that $a, b \in A$ and square free. Since $(-d, d) = 0$ for any $d \in F^*$, we assume that π does not divide a and δ does not divide b. Since A is a regular local ring, it is a unique factorisation domain ([AB]). We write $a = ca_1\delta^{\epsilon_1}$ and $b = cb_1\pi^{\epsilon_2}$ with $c, a_1, b_1 \in A$ square free, a_1 and b_1 are coprime, π and δ do not divide ca_1b_1 and $0 \leq \epsilon_1, \epsilon_2 \leq 1$.

Let θ be a prime in A which divides a_1. Write $a_1 = \theta a_2$. Then θ does not divide $cb_1\pi\delta$. In particular, the characteristic of the residue field $\kappa(\theta)$ at θ is not equal to 2 and α is unramified at θ. Since the residue of α at θ is the image $\overline{\pi}$ of π in $\kappa(\theta)/\kappa(\theta)^{*2}$, $\overline{\pi}$ is a square in $\kappa(\theta)$. Let $L = F[\sqrt{\pi}]$ and $B = A[\sqrt{\pi}]$. Then B is a regular local ring of dimension 2 (cf. (3.2)) and hence a unique factorisation domain ([AB]). Since $\overline{\pi}$ is a square in $\kappa(\theta)$ and $\mathrm{char}(\kappa(\theta)) \neq 2$, we have $\theta B = Q_1Q_2$ with Q_1 and Q_2 two distinct prime ideals of B. In particular $N_{L/F}(Q_1) = \theta A$. Since B is a unique factorisation domain, $Q_1 = (\eta)$ for some $\eta \in B$ and hence there exists a unit $u \in A$ such that $N_{L/F}(\eta) = u\theta$. We have

$$\begin{aligned}(a, \pi) &= (au\theta, \pi)\\ &= (ca_1\delta^{\epsilon_1}u\theta, \pi)\\ &= (c\theta a_2\delta^{\epsilon_1}u\theta, \pi)\\ &= (ca_2\delta^{\epsilon_1}u, \pi).\end{aligned}$$

Thus by induction on the number of primes dividing a_1, we conclude that $(a, \pi) = (uc\delta^{\epsilon_1}, \pi)$ for some unit $u \in A$. Similarly $(b, \delta) = (vc\pi^{\epsilon_2}, \delta)$ for some unit $v \in A$. Thus we have $\alpha = (uc\delta^{\epsilon_1}, \pi) + (vc\pi^{\epsilon_2}, \delta)$. Suppose that $\epsilon_1 = 1$. Then

$$\begin{aligned}\alpha &= (uc\delta, \pi) + (vc\pi^{\epsilon_2}, \delta)\\ &= (uc, \pi) + (\delta, \pi) + (vc\pi^{\epsilon_2}, \delta)\\ &= (uc, \pi) + (vc\pi^{\epsilon_2+1}, \delta)\\ &= (uc, \pi) + (vc\pi^{\epsilon}, \delta),\end{aligned}$$

where $\epsilon = \epsilon_2 + 1 \pmod 2$. □

For any field K and $i \geq 1$, let $H_2^i(F)$ be the Kato cohomology groups ([K2], §0). If $\mathrm{char}(K) \neq 2$, we have $H_2^i(F) = H^i(F, \mu_2)$. If $\mathrm{char}(K) = 2$, we have $H_2^2(K) = {}_2\mathrm{Br}(K)$ and $H_2^1(K) = H^1(K, \mathbf{Z}/2\mathbf{Z})$. For $a \in K^*$, let $[a) \in H_2^1(K)$ be the element defined by $K[X]/(X^2 + X + a)$. Note that $[a)$ is $K \times K$ or a separable extension of K of degree 2. Let $b \in K$. Let $[a) \cdot (b)$ be the quaternion algebra over K generated by i and j with $i^2 + i + a = 0$, $j^2 = b$ and $ji = -(1 + i)j$.

Let A, F, κ be as above. Let $\theta \in A$ be a prime. Suppose θ does not divide $2 = u_0\pi^i\delta^j$. Then the characteristic of the residue field $\kappa(\theta)$ at θ is 0. Suppose θ divides 2. Then $(\theta) = (\pi)$ or $(\theta) = (\delta)$ and $A/(\theta)$ is a complete discrete valuation ring. Since the residue field κ of A is a perfect field, we have $[\kappa(\theta) : \kappa(\theta)^2] = 2$. By ([K2], §1), we have residue homomorphisms $\partial_\theta : H^3(F, \mu_2) \to H_2^2(\kappa(\theta)) \simeq {}_2\mathrm{Br}(\kappa(\theta))$ and $\partial : H_2^2(\kappa(\theta)) \to H_2^1(\kappa)$.

Lemma 4.2. *(cf. [Su], 1.1) Let A, F, κ, $m = (\pi, \delta)$ be as above. Then, for any unit $u \in A^*$, $\partial_\pi([u) \cdot (\delta) \cdot (\pi)) = [\overline{u}) \cdot (\overline{\delta})$ and $\partial([\overline{u}) \cdot (\overline{\delta})) = [\overline{\overline{u}})$, where for any $a \in A$, $\overline{a}$ denotes the image modulo π and $\overline{\overline{a}}$ denotes the image modulo m.*

Proof. Suppose that $[\overline{\overline{u}})$ is trivial in $H_2^1(\kappa)$. Since u is a unit in A and A is complete, $[u)$ is trivial in $H^1(F, \mu_2)$. In particular $[u) \cdot (\pi) \cdot (\delta)$ and $[\overline{u}) \cdot (\overline{\delta})$ are trivial.

Suppose that $[\overline{\overline{u}})$ is non-trivial. Let $\kappa' = \kappa[X]/(X^2 + x + \overline{\overline{u}})$. Then κ' is a separable quadratic extension of κ and $[\overline{\overline{u}})$ is the only non-trivial element of the kernel of the restriction homomorphism from $H_2^1(\kappa)$ to $H_2^1(\kappa')$.

Let $\kappa(\pi)' = \kappa(\pi)[X]/(X^2 + X + \overline{u})$. Then $\kappa(\pi)'$ is a complete discretely valued field with residue field κ' and $\overline{\delta}$ as a parameter. Thus $\kappa(\pi)'/\kappa(\pi)$ is unramified and $\overline{\delta} \in \kappa(\pi)$ is not a norm from $\kappa(\pi)'$ and hence $[\overline{u}) \cdot (\overline{\delta})$ is non-trivial. Since ∂ is an isomorphism ([K2], Lemma 1.4(3)), $\partial([\overline{u}) \cdot (\overline{\delta}))$ is non-trivial in $H_2^1(\kappa)$. Since $[\overline{u}) \cdot (\overline{\delta})$ is trivial over $\kappa(\pi)'$, by the functoriality of ∂, the image of $\partial([\overline{u}) \cdot (\overline{\delta}))$ in $H_2^1(\kappa')$ is trivial. Since the only non-trivial element of the kernel of the restriction homomorphism from $H_2^1(\kappa)$ to $H_2^1(\kappa')$ is $[\overline{\overline{u}})$, $\partial([\overline{u}) \cdot (\overline{\delta})) = [\overline{\overline{u}})$.

Let F_π be the completion of F at π. Since u and δ are units at π, $[u) \cdot (\delta)$ is a quaternion algebra defined over $A_{(\pi)}$. If π is a reduced norm from $[u) \cdot (\delta)$ over F_π, $[\overline{u}) \cdot (\overline{\delta})$ is a split algebra over $\kappa(\pi)$, contradicting the non-triviality of $[\overline{u}) \cdot (\overline{\delta})$ in $H_2^2(\kappa(\pi))$. Hence π is not a reduced norm form of the quaternion algebra $([u) \cdot (\delta)) \otimes_F F_\pi$ and $[u) \cdot (\delta) \cdot (\pi)$ is non-trivial in $H^3(F_\pi, \mu_2)$. Let $L = F[X]/(X^2 + X + u)$. Let B be the integral closure of A in L. Then B is a complete regular local ring with maximal ideal (π, δ) and residue field κ'. Since the image of $[u) \cdot (\delta) \cdot (\pi)$ in $H^3(L, \mu_2)$ is zero, by the functoriality of the residue homomorphisms, the image of $\partial(\partial_\pi([u) \cdot (\delta) \cdot (\pi)))$ in $H_2^1(\kappa')$ is zero. Since $[u) \cdot (\delta) \cdot (\pi)$ is non-trivial in $H^3(F_\pi, \mu_2)$ and $\partial_\pi : H^3(F_\pi, \mu_2) \to H_2^2(\kappa(\pi))$ and $\partial : H_2^2(\kappa(\pi)) \to H_2^1(\kappa)$ are isomorphisms ([K2], Lemma 1.4(3)), $\partial(\partial_\pi([u) \cdot (\delta) \cdot (\pi))$ is non-trivial and hence equal to $[\overline{\overline{u}})$. Since $\partial([\overline{u}) \cdot (\overline{\delta}) = [\overline{\overline{u}})$ and ∂ is an isomorphism, we have $\partial_\pi([u) \cdot (\delta) \cdot (\pi)) = [\overline{u}) \cdot (\overline{\delta})$. $\square$

The following is a result of Kato ([K2], 1.7)

Proposition 4.3. *Let A, F and κ be as above. Then*

$$H^3(F, \mu_2) \overset{\oplus\partial_\gamma}{\to} \oplus_{\gamma \in \mathrm{Spec}(A)^{(1)}} H_2^2(\kappa(\gamma)) \overset{\sum \partial_\gamma}{\to} H_2^1(\kappa)$$

is a complex.

We define $H^3_{nr}(F/A, \mu_2)$ to be the kernel of residue homomorphism

$$H^3(F, \mu_2) \stackrel{\oplus\partial_\gamma}{\to} \oplus_{\gamma\in Spec(A)^{(1)}} H^2_2(\kappa(\gamma))).$$

Proposition 4.4. *Let A, F and κ be as above. Then $H^3_{nr}(F/A, \mu_2) = 0$.*

Proof. Since $\mathrm{cd}_2(F) \leq 3$ ([GO]), $I^4(F) = 0$ ([AEJ], Cor.2. p.653) and $e_3 : I^3(F) \to H^3(F, \mu_2)$ is an isomorphism ([AEJ], Thm. 2. p.653). Let $\zeta \in H^3_{nr}(F/A, \mu_2)$. Suppose that $\zeta \neq 0$. Let q be an anisotropic quadratic form over F such that $q \in I^3(F)$ and $e_3(q) = \zeta$. Let $\theta \in A$ be a prime and F_θ be the completion of F at θ. Since $A/(\theta)$ is a complete local ring of dimension one with residue field perfect of characteristic 2, $H^3_2(\kappa(\theta)) = 0$ ([GO]). Suppose that $\mathrm{char}(\kappa(\theta)) \neq 2$. Since $H^3(\kappa(\theta), \mu_2) = H^3_2(\kappa(\theta)) = 0$, $\partial_\theta : H^3(F_\theta, \mu_2) \to H^2_2(\kappa(\theta))$ is an isomorphism. Suppose that $\mathrm{char}(\kappa(\theta)) = 2$. Since the 2-rank of $\kappa(\theta)$ is 1, by ([K2], Lemma 1.4(3)), $\partial : H^3(F_\theta, \mu_2) \to H^2_2(\kappa(\theta))$ is an isomorphism. Since $\zeta \in H^3_{nr}(F/A, \mu_2)$, the image of ζ in $H^3(F_\theta, \mu_2)$ is zero. In particular, q is hyperbolic over F_θ. Thus q comes from a non-singular quadratic form over the localisation $A_{(\theta)}$ of A at the prime ideal (θ) (cf. [O], Thm,8). Since A is a two dimensional regular ring, there exists a non-singular quadratic forn q' over A such that $q' \otimes_A F \simeq q$ ([CTS], Cor.2.5, cf. [APS], 4.2).

Since $q \in I^3(F)$ and q is anisotropic, the rank of q, and hence the rank of q', is at least 8. Since κ is a perfect field, $q' \otimes_A \kappa$ is isotropic ([MMW], Corollary 1). Since A is a complete regular local ring and q' is a non-singular quadratic form over A with $q' \otimes_A \kappa$ is isotropic, q' is isotropic ([Gr], Theorem 18.5.17). Thus q is isotropic, leading to a contradiction. □

Lemma 4.5. *Let A, F, κ and $m = (\pi, \delta)$ be as be above. Let $\zeta \in H^3(F, \mu_2)$. Suppose that ζ is ramified at most along (π) and (δ). Then $\zeta = [u] \cdot (\pi) \cdot (\delta)$ for some unit u in A.*

Proof. Let $\alpha = \partial_\pi(\zeta) \in H^2_2(\kappa(\pi))$ and $\beta = \partial_\delta(\zeta) \in H^2_2(\kappa(\delta))$. Then, by (4.3), $\partial(\alpha) = \partial(\beta) \in H^1_2(\kappa)$. Let $a \in \kappa^*$ be such that $[a] = \partial(\alpha) = \partial(\beta) \in H^1_2(\kappa)$. Let $u \in A^*$ be a lift of a. Since $\partial(\alpha) = [a] = \partial([u]\cdot(\overline{\delta}))$ (cf. 4.2) and ∂ is an isomorphism, we have $\alpha = [\overline{u}] \cdot (\overline{\delta})$. and $\beta = [\overline{u}] \cdot (\overline{\pi})$. Let $\zeta' = [u] \cdot (\pi) \cdot (\delta) \in H^3(F, \mu_2)$. Then ζ' is unramified on A except at π and δ. By (4.2), $\partial_\pi(\zeta') = \partial_\pi(\zeta)$ and $\partial_\delta(\zeta') = \partial_\delta(\zeta)$. Since ζ is unramified on A except at π and δ, $\zeta - \zeta' \in H^3_{nr}(F, \mu_2)$. Since $H^3_{nr}(F, \mu_2) = 0$ by (4.4), we have $\zeta = \zeta' = [u] \cdot (\pi) \cdot (\delta)$. □

Proposition 4.6. *Let A, F, κ and $m = (\pi, \delta)$ be as above. Let $q =< a_1, \cdots, a_9 >$ be a quadratic form over F of rank 9 with only prime factors of $a_1a_2\cdots a_9$ are at most π and δ. Then q is isotropic.*

Proof. Let $c(q) \in H^2(F, \mu_2)$ be the Clifford invariant of q. Since the prime factors of $a_1a_2\cdots a_9$ are at most π and δ, $c(q)$ is unramified on A except possibly at (π) and (δ). By (4.1), we have $c(q) = (uc, \pi) + (vc\pi^\epsilon, \delta)$ for some units $u, v \in A$, $c \in A$ not divisible by π and δ, and $\epsilon = 0$ or 1. Let $q_1 =< 1, uc\pi, -\pi, -uc\delta, uv\pi^\epsilon\delta >$. Since $-ucq_1$ is a rank five subform of the Albert form associated to $c(q) = (uc, \pi)+(vc\pi^\epsilon, \delta)$, $c(q_1) = c(q)$ (cf. [L], p. 118). Since q is isotropic if and only if λq is isotropic for any $\lambda \in F^*$, by scaling q we assue that $d(q) = d(q_1)$. We note that we only need to scale by $\lambda \in A$ with prime factors at most π and δ. Hence, after scaling, we still have $q =< a_1, \cdots, a_9 >$ with only prime factors of $a_1a_2\cdots a_9$ at most π and δ. Since the dimension of q is odd, we have $c(\lambda q) = c(q)$. Thus, after scaling, we have $c(q) = c(q_1)$

and $d(q) = d(q_1)$. Since the rank of $q \perp -q_1$ is 14, it follows that $q - q_1 \in I^3(F)$ ([M]).

Let $\zeta = e_3(q - q_1) \in H^3(F, \mu_2)$. Let $\theta \in A$ be a prime. Suppose that θ does not divide $\pi\delta$. Then char$(\kappa(\theta))$ is 0. Hence we have the second residue homomorphism $\partial_\theta^2 : W(F) \to W(\kappa(\theta))$ with $\partial_\theta^2(I^3(F)) \subset I^2(\kappa(\theta))$. Since $q =< a_1, \cdots, a_9 >$ with $a_1 a_2 \cdots a_9$ having only π and δ as possible prime factors and θ does not divide $\pi\delta$, $\partial_\theta^2(q) = 0$. Since $q_1 =< 1, uc\pi, -\pi, -uc\delta, uv\pi^\epsilon\delta >$, the rank of $\partial_\theta^2(q_1)$ is at most two. Since $\partial_\theta^2(q - q_1) \in I^2(\kappa(\theta))$ and is of rank at most 2, $\partial_\theta^2(q - q_1) = 0$. In particular $q - q_1$ is unramified at θ and hence $\zeta = e_3(q - q_1)$ is unramified at θ. Thus, by (4.5), we have $\zeta = [w] \cdot (\pi) \cdot (\delta)$ for some unit $w \in A$. Since $[w] \cdot (w')$ is unramified on A for any unit $w' \in A$, we have $[w] \cdot (w') = 0$. In particular, we have $\zeta = [w] \cdot (\pi) \cdot (w'\delta)$ for any unit w' in A.

Suppose that $\epsilon = 0$. Since uv is a unit, we have $\zeta = [w] \cdot (\pi) \cdot (-uv\pi^\epsilon\delta)$. Suppose that $\epsilon = 1$. Since $\zeta = [w] \cdot (\pi) \cdot (-uv\delta)$ and $(\pi) \cdot (-\pi) = 1$, we have $\zeta = [w] \cdot (\pi) \cdot (-uv\pi\delta)$. Thus in either case, we have $\zeta = e_3(q - q_1) = [w] \cdot (\pi) \cdot (-uv\pi^\epsilon\delta)$.

Since char$(F) = 0$, we have $[w] = (w')$ for some unit $w' \in A$. Let $q_2 =< 1, -w' >< 1, -\pi >< 1, uv\pi^\epsilon\delta > \in I^3(F)$. Then $e_3(-q_2) = e_3(q_2) = (w') \cdot (\pi) \cdot (-uv\pi^\epsilon\delta) = [w] \cdot (\pi) \cdot (-uv\pi^\epsilon\delta) = e_3(q - q_1)$. Since $H^4(F, \mu_2) = 0$ ([AEJ], Cor.2. p.653), we have $I^4(F, \mu_2) = 0$ and e_3 is an isomorphism ([AEJ], Thm. 2. p.653). Hence

$$q - q_1 = - < 1, -w' >< 1, -\pi >< 1, uv\pi^\epsilon\delta > .$$

In particular,

$$q = q_1 - < 1, -w' >< 1, -\pi >< 1, uv\pi^\epsilon\delta > .$$

Since $< 1, -\pi, uv\pi^\epsilon\delta >$ is a subform of both q_1 and $< 1, -w' >< 1, -\pi >< 1, uv\pi^\epsilon\delta >$, the anisotropic rank of $q_1 - < 1, -w' >< 1, -\pi >< 1, uv\pi^\epsilon\delta >$ is at most 7. Since the rank of q is 9, q is isotropic. □

Theorem 4.7. *Let K be a complete discretely valued field with residue field κ and F a function field of a curve over K. If char$(K) = 0$ and κ is a perfect field of characteristic 2, then $u(F) \leq 8$.*

Proof. Let $q =< a_1, \cdots, a_9 >$ be a quadratic form over F rank 9. Let $\mathscr{X}$ be a regular proper scheme over the valuation ring of K with function field F and the support of the principle divisor $(2a_1 \cdots a_9)$ on $\mathscr{X}$ is a union of regular curves with normal crossings. Let $C_1, \cdots, C_r$ be the irreducible components of the special fibre of $\mathscr{X}$ and let $\nu_1, \cdots, \nu_r$ be the corresponding discrete valuations on F. Let F_{ν_i} be the completion F at ν_i and the residue field $\kappa(\nu_i)$. Then char$(\kappa(\nu_i)) = 2$ and 2-rank$(\kappa(\nu_i)) = 1$. Hence, by ([MMW], Corollar 1), $u(\kappa(\nu_i)) \leq 4$ and by ([Sp]), $u(F_{\nu_i}) \leq 8$. In particular q is isotropic over F_{ν_i}. By ([HHK2], 5.8), there exists an affine open subset U_i of C_i such that U_i does not intersect C_j for $j \neq i$ and q is isotropic over F_{U_i}.

Let $\mathscr{P}$ be a finite set of closed points of $\mathscr{X}$ containing all those points which are not in U_i for any i. Let $P \in \mathscr{P}$. Then $\hat{A}_P$ is a complete two dimensional local ring with residue field perfect of characteristic 2. By the choice of $\mathscr{X}$ and (4.6), q is isotropic over F_P. By ([HHK1], 4.2), q is isotropic over F and $u(F) \leq 8$. □

Corollary 4.8. *([Le]) Let K be a 2-adic field and F the function field of a curve over K. Then $u(K) = 8$.*

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