ORTHOGONAL PFISTER INVOLUTIONS IN CHARACTERISTIC TWO

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Abstract. We show that over a field of characteristic 2 a central simple algebra with orthogonal involution that decomposes into a product of quaternion algebras with involution is either anisotropic or metabolic. We use this to define an invariant of such orthogonal involutions in characteristic 2 that completely determines the isotropy behaviour of the involution. We also give an example of a non-totally decomposable algebra with orthogonal involution that becomes totally decomposable over every splitting field of the algebra.

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1. Introduction

Over fields of characteristic different from 2 it is well-known that a quadratic form of dimension equal to a power of two is anisotropic or hyperbolic over any field extension if and only it is similar to a Pfister form. Using the correspondence between quadratic and bilinear forms over fields of characteristic different from 2, we can associate an orthogonal involution on a split central simple algebra to every quadratic form. It is then natural to consider whether there are central simple algebras with involution with analogous properties to Pfister forms. One potential type of algebra with involution to consider is the algebras with involution that are isomorphic to a product of quaternion algebras with involution. That is, totally decomposable algebras with involution.

Let (A, σ) be a central simple algebra of degree equal to a power of two with orthogonal involution over a field F. We denote the central simple algebra (resp. the algebra with involution) obtained by extending scalars over a field extension K/F as A_K (resp. $(A, \sigma)_K$).

Assuming that the characteristic of F is different from 2, in [2] it is asked whether the following are equivalent:

(1) (A, σ) is totally decomposable.
(2) For all field extensions K/F such that A_K is split, there exists a Pfister form π over K such that $(A, \sigma)_K$ is isomorphic to the adjoint algebra with involution of π.
(3) For any field extension K/F, $(A, \sigma)_K$ is either anisotropic or hyperbolic.

The implication (1) $\Rightarrow$ (2) is known as the Pfister Factor Conjecture, and was proven in [3]. The implication (1) $\Rightarrow$ (3), and the equivalence (2) $\Leftrightarrow$ (3), follows from the Pfister Factor Conjecture and the non-hyperbolic splitting result of [8]. The converse implication, (2) or (3) $\Rightarrow$ (1), is still open in general.

Analogous questions may be asked when we consider fields of characteristic 2. Firstly, we may replace (A, σ) with a quadratic pair (see [11, Section 5]). The analogous result to the Pfister Factor Conjecture is shown in [6]. Otherwise the problem is open.

Alternatively we may formulate the question in terms of algebras with orthogonal involution and symmetric bilinear forms over a field of characteristic 2. The theory of symmetric bilinear forms in characteristic 2 has several features that mean we must be slightly more careful in our formulation of the analogous question to that posed in [2]. Over fields of characteristic different from 2, all 2-dimensional isotropic symmetric bilinear forms are isometric to the hyperbolic plane. This is not true in characteristic 2, and the wider variety of isotropic 2-dimensional forms means that we must use the weaker property of metabolicity rather than hyperbolicity in the formulation of our problem and be more restrictive with our statement. For example, there exist metabolic bilinear forms of dimension equal to a power of 2 that are not similar to bilinear Pfister forms (see (2.6)). That is, they are not isometric to a tensor product of 2-dimensional bilinear forms. Metabolicity for algebras with involution is studied in [4], and we recall the definitions and basic results that we use in Section 4.

Conversely however, the isotropy behaviour of symmetric bilinear forms over quadratic separable extensions is particularly simple (see (3.1)). We can often exploit this to investigate symmetric bilinear forms over fields of characteristic 2 with much simpler methods than those needed over fields of characteristic different from 2.

We therefore ask the following question, in analogy with the implication (1) $\Rightarrow$ (3) above. Let F be a field of characteistic 2 and let (A, σ) be a totally decomposable F–algebra with orthogonal involution. For every field extension K/F, is $(A, \sigma)_K$ either anisotropic or metabolic? In (5.2) we shall show that this question has a positive answer.

We shall also consider potential analogues of the other implications from the quesion asked in [2]. That a split totally decomposable F–algebra with orthogonal involution is adjoint to a bilinear Pfister form in charactersitic 2, an analogue of (1) $\Rightarrow$ (2), can be shown with relatively simple arguments (see (6.7)). We shall give another slightly less direct proof of this result in (6.5) which allows us to determine the bilinear Pfister form in the statement explicitly. This result has also been independently shown using different methods in [15].

Natural analgoues of the equivalence (2) $\Leftrightarrow$ (3) and the implication (2) $\Rightarrow$ (1) do not hold in general in characteristic 2, and we discuss this in Section 7. In (8.4) we give an explicit example in characteristic 2 of an F–algebra with orthogonal involution that is not totally decomposable, but that becomes totally decomposable over every splitting field of the algebra.

2. Bilinear forms

In this section we recall the basic terminology and results we use from bilinear form theory. We refer to [7, Chapter 1] as a general reference on bilinear forms.

Let F be a field. Let char(F) denote the characteristic of F. A *bilinear form over* F is a pair (V, b) where V is a F–vector space and b is a F-bilinear map $b : V \times V \to F$. We say that a bilinear form (V, b) is *symmetric* if $b(x, y) = b(y, x)$ for all $x, y \in V$. We call a bilinear form (V, b) *alternating* if $b(x, x) = 0$ for all $x \in V$. If (V, b) is an alternating form then we have that $b(x, y) = -b(y, x)$ for all $x, y \in V$, that is, (V, b) is *skew-symmetric*. In particular every alternating form over a field of characteristic 2 is symmetric. We say (V, b) is *nondegenerate* if $b(x, y) = 0$ for all $y \in V$ implies that $x = 0$.

An *isometry of bilinear forms over* F is a map $\phi : (V, b_1) \to (W, b_2)$, where (V, b_1) and (W, b_2) are bilinear forms over F, such that $\phi : V \to W$ is F–linear bijective F–vector space homomorphism and $b_1(v, w) = b_2(\phi(v_1), \phi(v_2))$ for all $v_1, v_2 \in V$. If such an isometry exists, we say (V, b_1) are (W, b_2) are *isometric as bilinear forms* and we write $(V, b_1) \simeq (W, b_2)$. The *orthogonal sum of the symmetric or alternating bilinear forms* (V, b_1) *and* (W, b_2) is defined to be the map $b : (V \oplus W) \times (V \oplus W) \to F$ given by $b(v_1 \oplus w_1, v_2 \oplus w_2) = b_1(v_1, v_2) + b_2(w_1, w_2)$ for all $v_1, v_2 \in V$ and $w_1, w_2 \in W$, and we write $(V \oplus W, b) = (V, b_1) \perp (W, b_2)$. The *tensor product of the symmetric or alternating bilinear forms* (V, b_1) *and* (W, b_2) is defined to be the map $b' : (V \otimes W) \times (V \otimes W) \to F$ given by $b'(v_1 \otimes w_1, v_2 \otimes w_2) = b_1(v_1, v_2) \cdot b_2(w_1, w_2)$ for all $v_1, v_2 \in V$ and $w_1, w_2 \in W$ and we write $(V \otimes W, b') = (V, b_1) \otimes (W, b_2)$.

We say a bilinear form (V, b) *represents and element* $a \in F$ if there exists and $x \in V \backslash \{0\}$ such that $b(x, x) = a$. A bilinear form (V, b) is said to be *isotropic* if it represents 0, and *anisotropic* otherwise. Given a nondegenerate bilinear form (V, b) we call a subspace $W \subset V$ *totally isotropic* (with respect to b) if $b|_W = 0$. We call (V, b) *metabolic* if it has a totally isotropic subspace W with $\dim_F(W) = \frac{1}{2}\dim_F(V)$. Note that an alternating form is always metabolic. We say two bilinear forms (V, b) and (W, b') are similar if there exists an $a \in F^\times$ such that $(V, b) \simeq (W, ab')$.

We now fix (V, b) to be a nondegenerate bilinear form over a field F for the rest of this section. Note that (V, b) can be decomposed as $(V, b) \simeq (W_1, b_1) \perp (W_2, b_2)$ with (W_1, b_1) anisotropic and (W_2, b_2) metabolic. In this decomposition (W_1, b_1) is uniquely determined up to isometry (see [7, (1.27)]), whereas (W_2, b_2) is not in general. We call (W_1, b_1) *the anisotropic part of* (V, b), which we denote by $(V, b)_{\mathrm{an}}$.

We denote $\mathbb{H} = (F^2, h)$ where

$$h : F^2 \times F^2 \to F \text{ is given by } (x, y) \mapsto x^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} y$$

and call this the *hyperbolic plane* over F. We call a bilinear form (V, b) that is isometric to an orthogonal sum of hyperbolic planes *hyperbolic*. Over a field of characteristic 2, a bilinear form is hyperbolic if and only if it is alternating (see [7, (1.8)]).

For $a_1, \ldots, a_n \in F^\times$ we denote by $\langle a_1, \ldots, a_n \rangle$ the symmetric bilinear form (F^n, b) where

$$b : F^n \times F^n \to F \text{ is given by } (x, y) \mapsto \sum_{i=1}^{n} x_i a_i y_i.$$

We call such a form a *diagonal form.* A symmetric bilinear space that is isometric to a diagonal form is called *diagonalisable.* By [7, (1.17)], a nondegenerate symmetric bilinear form is diagonalisable if and only if it is not alternating. For a nondegenerate symmetric bilinear form φ over F we denote the metabolic bilinear form $\varphi \perp \varphi$ by $\mathbb{M}(\varphi)$.

We frequently use the following isometry.

Lemma 2.1. *Assume that* $\mathrm{char}(F) = 2$. *Take* $a \in F^\times$. *Then* $\langle a \rangle \perp \mathbb{H} \simeq \langle a, a, a \rangle$.

Proposition 2.2. *Assume that* $\mathrm{char}(F) = 2$. *Let* φ *be a nondegenerate symmetric bilinear form over* F. *Then there exist* $n, m \in \mathbb{N}$ *and elements* $a_1, \ldots, a_n \in F^\times$

such that

$$\varphi \simeq \varphi_{\mathrm{an}} \perp \mathbb{M}(\langle a_1, \ldots, a_n\rangle) \perp m \times \mathbb{H}$$

with the condition that $\varphi_{\mathrm{an}} \perp \langle a_1, \ldots, a_n\rangle$ *is anisotropic. Further* $\varphi_{\mathrm{an}} \perp \mathbb{M}(\langle a_1, \ldots, a_n\rangle)$ *is unique up to isometry and* n *is uniquely determined.*

Proof. See [13, (2.1)]. □

We call the integer m in (2.2) *the hyperbolicity index of* φ.

For $a \in F^\times$ be denote the bilinear form $\langle 1, -a\rangle$ over F by $\langle\langle a\rangle\rangle$. For $a_1, \ldots, a_n \in F^\times$, we denote the $\langle\langle a_1\rangle\rangle \otimes \ldots \otimes \langle\langle a_n\rangle\rangle$ by $\langle\langle a_1, \ldots, a_n\rangle\rangle$. We call a bilinear form over F isometric to some $\langle\langle a_1, \ldots, a_n\rangle\rangle$ for some $a_1, \ldots, a_n \in F^\times$, an *$n$-fold Pfister form.*

Lemma 2.3. *Let* π *be a Pfister form over* F *and* $a \in F^\times$ *an element represented by* π. *Then* $\pi \simeq a\pi$.

Proof. See [7, (6.2)]. □

The following result is well known, but we include a proof for convenience.

Corollary 2.4. *Let* φ *be a symmetric bilinear form over* F. *If* φ *represents* 1 *and is similar to a Pfister form* π *then* $\varphi \simeq \pi$.

Proof. Let $a \in F$ be such that $a\varphi \simeq \pi$. Then since φ represents 1, we have that π represents a. Hence $\varphi \simeq a^2\varphi \simeq a\pi \simeq \pi$ by (2.3). □

Proposition 2.5. *Let* π *be a nondegenerate symmetric bilinear form over* F. *If* π *is similar to a Pfister form then* π_K *is either anisotropic or metabolic for every field extension* K/F. *The converse holds if* π *is non-metabolic.*

Proof. For the first statement see [7, (6.3)]. For the converse see [12, (5.5)] if $\mathrm{char}(F) = 2$ and [7, (23.4)] otherwise. □

Lemma 2.6. *Assume that* $\mathrm{char}(F) = 2$. *Let* φ *be a nondegenerate metabolic symmetric bilinear form over* F. *Then* φ *is a Pfister form if and only if there exists and anisotropic Pfister form* π *such that*

$$\varphi \simeq \mathbb{M}(\pi) \perp n \times \mathbb{H} \simeq \pi \otimes \langle\langle 1, \ldots, 1\rangle\rangle$$

for some integer n *such that* $2\mathrm{dim}_F\pi + 2n = 2^m$ *for some integer* m.

Proof. For a Pfister form π over F, the isomorphism $\mathbb{M}(\pi) \perp n \times \mathbb{H} \simeq \pi \otimes \langle\langle 1, \ldots, 1\rangle\rangle$ follows from repeated use of (2.1). Therefore that φ is a Pfister form if $\varphi \simeq \mathbb{M}(\pi) \perp n \times \mathbb{H}$ is clear. For the converse, see [1, (A.8)] and the comments following it. □

By a *quadratic form over* F we mean a pair (V, q) of a finite-dimensional F-vector space V and a map $q : V \to F$ such that

- $q(\lambda x) = \lambda^2\varphi(x)$ for all $x \in V$ and $\lambda \in F$,
- $b_q : V \times V \to F, (x, y) \longmapsto q(x+y) - q(x) - q(y)$ is an bilinear form over F,
- if $v \in V$ is such that $q(v) = 0$ and $b_q(v, w) = 0$ for all $w \in V$, then $v = 0$.

We say φ is *isotropic* if $q(x) = 0$ for some $x \in V \setminus \{0\}$ and *anisotropic* otherwise. For any bilinear form (V, b) over F we call the quadratic form consisting of the F–vector space V and a map $q_b : V \to F$ where $q_b(x) = b(x, x)$ for all $x \in V$ the *associated quadratic form to* (V, b). We denote this pair by $\widetilde{(V, b)}$.

Let (V,q) and (W,q') be quadratic forms over F. We denote the *orthogonal sum* of (V,q) and (W,q') by $(V,q)\perp(W,q')$. By an *isometry of quadratic forms* $\phi:(V,q)\to(W,q')$ we mean an isomorphism of F–vector spaces $\phi:V\longrightarrow W$ such that $q=q'\circ\phi$. If such an isometry exists, we say (V,q) and (W,q') are *isometric* and write $(V,q)\simeq(W,q')$.

3. Bilinear forms and separable extensions

In this section we recall some results on the behaviour of bilinear forms over separable extensions in characteristic 2 and derive some corollaries that we need.

Throughout, assume that $\mathrm{char}(F)=2$. Let K/F be a field extension. Then we write $(V,b)_K=(V\otimes_F K, b_K)$ where b_K is the extension of b is a bilinear form on $V\otimes_F K$ given by $b_K(x\otimes k_1, y\otimes k_2)=k_1k_2b(x,y)$ for all $x,y\in V$ and $k_1,k_2\in K$.

Proposition 3.1. *Let L/F be a separable extension. Let φ be a nondegenerate symmetric bilinear form over F. Then φ_L is isotropic if and only if φ is isotropic.*

Proof. See [9, (10.2.1)]. □

Corollary 3.2. *Let L/F be a separable extension. Let φ and ψ be nondegenerate symmetric bilinear forms over F that are either anisotropic or metabolic. If $\varphi_L\simeq\psi_L$ then $\varphi\simeq\psi$.*

Proof. Assume that first that φ and ψ are anisotropic. Then $(\varphi\perp\psi)_L$ is metabolic, and hence so is $\varphi\perp\psi$ by (3.1) and therefore $\varphi\simeq\psi$ by Witt Cancellation (see [7, (1.28)]).

Now assume that φ and ψ are metabolic. By (2.2) we have $\varphi\simeq\mathbb{M}(\beta_1)\perp n\times\mathbb{H}$ and $\psi\simeq\mathbb{M}(\beta_2)\perp m\times\mathbb{H}$ for anisotropic bilinear forms β_1 and β_2 over F and $n,m\in\mathbb{N}$. Since $(\beta_1)_L$ and $(\beta_2)_L$ are anisotropic by (3.1), it immediately follows from $\varphi_L\simeq\psi_L$ that $n=m$ by (2.2). Further, we have that $\mathbb{M}(\beta_1)_L\simeq\mathbb{M}(\beta_2)_L$ by (2.2).

Let $x=\dim_F\beta_1=\dim_F\beta_2$. Using (2.1), adding β_1 to each side of $\mathbb{M}(\beta_1)_L\simeq\mathbb{M}(\beta_2)_L$ gives

$$(\beta_1\perp\mathbb{M}(\beta_1))_L\simeq(\beta_1)_L\perp x\times\mathbb{H}\simeq(\beta_1)_L\perp\mathbb{M}(\beta_2)_L.$$

If follows from (2.2) that $(\beta_1\perp\beta_2)_L$ is isotropic, and hence that $(\beta_1)_L$ and $(\beta_2)_L$ represent a comment element $a\in L^\times$. That is $(\beta_1)_L\simeq\langle a\rangle\perp\beta_1'$ and $\beta_2\simeq\langle a\rangle\perp\beta_2'$ for some anisotropic bilinear forms β_1' and β_2' over L. By (2.1) this gives

$$(\beta_1\perp\mathbb{M}(\beta_2))_L\simeq\beta_1'\perp\langle a\rangle\perp\mathbb{M}(\beta_2'\perp\langle a\rangle)\simeq\beta_1'\perp\langle a\rangle\perp\mathbb{M}(\beta_2')\perp\mathbb{H}.$$

It follows from (2.2) that we can cancel hyperbolic planes, and hence

$$(\beta_1)_L\perp(x-1)\times\mathbb{H}\simeq\beta_1'\perp\mathbb{M}(\beta_2').$$

Inducting on x now gives that $(\beta_1\perp\beta_2)_L$ is metabolic. It follows that $\beta_1\simeq\beta_2$ as in the anisotropic case above, and hence the result. □

Lemma 3.3. *Let L/F be a separable field extension and φ be a symmetric bilinear form over F. If $\varphi_L\simeq\pi$, where π is a Pfister form over L, then φ is similar to a Pfister form over F.*

Proof. First assume that φ is anisotropic and let be any field extension K/F such that φ_K is isotropic. Let K' be a separable extension of K such that L is a subfield of K'. It follows that $\pi_{K'}\simeq\varphi_{K'}$ and hence $\varphi_{K'}$ is metabolic by (2.5). Therefore φ_K is metabolic by (3.1). Therefore φ is similar to a Pfister form by (2.5).

Now assume that φ is isotropic. Since φ is isotropic, it follows from (2.5) that π is metabolic. It then follows that φ is metabolic by (3.1). By (2.1) we have that $\pi \simeq \mathbb{M}(\beta)\perp n \times \mathbb{H}$ for some natural number n and some anisotropic Pfister form β over L.

By (2.2), there exists an anisotropic symmetric bilinear form α and natural number m such that $\varphi \simeq \mathbb{M}(\alpha)\perp m \times \mathbb{H}$. We may scale φ to assume that that α represents 1. Take $a_1, \ldots, a_l \in L^\times$ such that $\alpha \simeq \langle 1, a_1, \ldots, a_l\rangle$. It follows from (3.1) that α_L is anisotropic and the uniqueness of the decomposition in (2.2) implies that $m = n$ and $\mathbb{M}(\alpha_L) \simeq \mathbb{M}(\beta)$. In particular $\dim_F(\alpha) = \dim_F(\beta)$.

It follows from (2.2) and [13, (2.6)] that $\widetilde{\alpha_L} \simeq \widetilde{\beta}$. Consider the field $L(\alpha) = F(X_1, \ldots, X_l)(\sqrt{a_1X_1^2 + \ldots + a_lX_l^2})$, where $X_1, \ldots, X_l$ are indeterminates. Then $\beta_{L(\alpha)}$ is isotropic as $\widetilde{\alpha_L} \simeq \widetilde{\beta}$ and clearly $\alpha_{L(\alpha)}$ is isotropic, and hence $\beta_{L(\alpha)}$ metabolic by (2.5). It then follows from [12, (5.3)] that $a\alpha_L \simeq \beta$ for some $a \in L^\times$ as $\dim_F(\alpha) = \dim_F(\beta)$. Since α represents 1, this implies that $\alpha_L \simeq \beta$ by (2.4) .

Hence α_L is isometric to a Pfister form, and hence α is similar to a Pfister form over F by the anisotropic case above. Since $m = n$ and $\dim_F(\alpha) = \dim_F(\beta)$ it follows from (2.1) that $\varphi \simeq \langle\langle 1, \ldots, 1\rangle\rangle \otimes \alpha$. □

4. Algebras with involution

In this section we recall the basic definitions and results we use on central simple algebras with involution. We refer to [16] for a general reference on central simple algebras.

Let A be a finite-dimensional F–algebra. If A is simple (i.e. it has no non-trivial two sided ideals) and E is the centre of A, we can view A as an E–algebra and by Wedderburn's Theorem (see [11, (1.1)]) we have that $A \simeq \mathrm{End}_D(V)$ for an F–division algebra D with centre E and a right D–vector space V. In this case $\dim_E(A)$ is a square, and the positive root of this integer is called the *degree of A* and is denoted $\deg(A)$. The degree of D is called the *index of A* and is denoted $\mathrm{ind}(A)$. We call A *split* if $\mathrm{ind}(A) = 1$. If $E = F$, then we call the F–algebra A *central simple*. An *F–quaternion algebra* is a central simple F-algebra of degree 2. For any field extension K/F we will use the notation $A_K = A \otimes_F K$. We call a field extension K/F a *splitting field of A* if A_K is split.

Lemma 4.1. *If F is separably closed, then all central simple F–algebras are split.*

Proof. See [5, (9.2)]. □

For an F–algebra A and $b \in A^\times$ we denote by $\mathrm{Int}(b)$ the *inner automorphism* $A \to A$ given by $c \mapsto bcb^{-1}$.

An *F–involution* on A is an F–linear map $\sigma : A \to A$ such that $\sigma(xy) = \sigma(y)\sigma(x)$ for all $x, y \in A$ and $\sigma^2 = \mathrm{id}_A$. We call an F–involution *of the first kind* if $F = E$, the centre of A. We do not consider F–involutions of the second kind here (see [11, Section 2.B] for more details on this kind of involution). By an *F–algebra with involution* we mean a pair (A, σ) of a finite-dimensional F–algebra A and an F–involution σ on A of the first kind.

A *homomorphism of F–algebras with involution* is a map $\varphi : (A, \sigma) \to (B, \tau)$, where (A, σ) are (B, τ) F–algebras with involution, such that $\varphi : A \to B$ is an F–algebra homomorphism and $\varphi\circ\sigma = \tau\circ\varphi$; if φ is bijective then this is an isomorphism. If an isomorphism $\varphi : (A, \sigma) \to (B, \tau)$ exists, then we say we say (A, σ) and (B, τ)

are *isomorphic as F–algebras with involution* and we write $(A,\sigma) \simeq (B,\tau)$. For any field extension K/F we will use the notations $\sigma_K = \sigma \otimes \mathrm{id}_K$ and $(A,\sigma)_K = (A_K, \sigma_K)$.

We call an F-algebra with involution *totally decomposable* if is it isomorphic to the tensor product of F–quaternion algebras with involution.

To every nondegenerate symmetric or alternating bilinear form (V,b) over F we can associate an algebra with involution in the following way. Let $A = \mathrm{End}_F(V)$. Then there is a unique involution σ on A such that

$$b(x, f(y)) = b(\sigma(f)(x), y) \quad \text{for all } x, y \in V \text{ and all } f \in A.$$

We call (A,σ) the *adjoint involution to* (V,b) and we denote it by $\mathrm{Ad}(V,b)$. Moreover, for every split F–algebra with involution (A,σ), there exists a nondegenerate symmetric or alternating bilinear form (V,b) such that $\mathrm{Ad}(V,b) \simeq (A,\sigma)$ (see [11, (2.1)]).

Proposition 4.2. *Let (V,b) and (V',b') be nondegenerate symmetric bilinear forms over F. Then $\mathrm{Ad}(V,b) \simeq \mathrm{Ad}(V',b')$ if and only if there exists an $\lambda \in F^\times$ such that $(V,\lambda b) \simeq (V',b')$.*

Proof. See [11, (4.2)]. □

Let (A,σ) be an F–algebra with involution. Then it is well known (see [11, (2.1)]) that in the case where the algebra A is split, that is $A \cong \mathrm{End}_F(V)$ for some F-vector space V, each F-involution on A is adjoint to a nondegenerate symmetric or alternating bilinear space on V. An F–algebra with involution of the first kind is said to be *symplectic* if it becomes adjoint to an alternating bilinear form over a splitting field of the F–algebra, and *orthogonal* otherwise. Note that this does not depend on the choice of splitting field (see [11, (2.5)]).

Let (A,σ) be an F–algebra with involution. We call (A,σ) *isotropic* if there exists $a \in A\backslash\{0\}$ such that $\sigma(a)a = 0$, and *anisotropic* otherwise. We call an element $e \in A$ *idempotent* if $e^2 = e$. An idempotent $e \in A$ is called *metabolic with respect to* σ if $\sigma(e)e = 0$ and $\dim_F eA = \frac{1}{2}\dim_F A$; by [4, (4.3)], we may substitute the condition $\dim_F eA = \frac{1}{2}\dim_F A$ for the condition that $(1-e)(1-\sigma(e)) = 0$ in this definition. We call (A,σ) *metabolic* if A contains a metabolic idempotent element with respect to σ.

Proposition 4.3. *Let (V,b) be a nondegenerate alternating or symmetric bilinear form over F. Then (V,b) is isotropic (resp. metabolic) if and only if $\mathrm{Ad}(V,b)$ is isotropic (resp. metabolic).*

Proof. See [4, (4.8)]. □

Note that, in particular, it follows from the definition of symplectic involutions that any split F–algebra with involutions (A,σ) is isomorphic to $\mathrm{Ad}(n \times \mathbb{H})$ for some integer n.

Let (A,σ) be an F–algebra with involution. Let

$$\mathrm{Sym}(A,\sigma) = \{a \in A \mid \sigma(a) = a\} \quad \text{and} \quad \mathrm{Alt}(A,\sigma) = \{a - \sigma(a) \mid a \in A\}.$$

These are F–linear subspaces of A.

An F–algebra with involution (B,τ) is called a *part of* (A,σ) if there exists an idempotent $e \in \mathrm{Sym}(A,\sigma)$ and an F–algebra isomorphism $\varphi : B \to eAe$ such that $\varphi \circ \tau = \sigma \circ \varphi$. In this case, we say that the idempotent $e \in \mathrm{Sym}(A,\sigma)$ *defines the part*

(B,τ). Let (B,τ) be a part of of (A,σ) defined by an idempotent $e \in \mathrm{Sym}(A,\sigma)$. Then we call the part of (A,σ) defined by the idempotent $1-e \in \mathrm{Sym}(A,\sigma)$ a *counterpart of* (B,τ) *in* (A,σ) and we say that (A,σ) is an orthogonal sum of (B,τ) and its counterpart. Note however that the counterpart of a part (B,τ) of an algebra with involution is not uniquely determined by (B,τ) (see [5, (4.2)])

Parts of F–algebras with involution correspond to subforms of hermitian forms whose adjoint involution is isomorphic to those F–algebras with involution (see [11, (4.2)]). We state this more precisely for the split case.

Lemma 4.4. *Let φ be a nondegenerate symmetric or alternating bilinear form over F. An F–algebra with involution (B,τ) is a part of $\mathrm{Ad}(\varphi)$ if and only if there exist bilinear forms φ' and φ'' over F such that $\varphi' \perp \varphi'' \simeq \varphi$ and $(B,\tau) \simeq \mathrm{Ad}(\varphi')$. Let $e \in \mathrm{Sym}(A,\sigma)$ be the idempotent that defines (B,τ) in (A,σ). Then the idempotent $(1-e) \in \mathrm{Sym}(A,\sigma)$ defines a part of $\mathrm{Ad}(\varphi)$ that is isomorphic to $\mathrm{Ad}(\varphi'')$.*

Proof. See [5, (4.1)]. □

If $\mathrm{char}(F) \neq 2$, then a part of an F–algebra with involution (A,σ) must be of the same type as (A,σ). This is not the case if $\mathrm{char}(F) = 2$ (see [5, (4.6)]).

Lemma 4.5. *Assume that $\mathrm{char}(F) = 2$. Let (A,σ) be an orthogonal F–algebra with involution. Let (B,τ) be a part of (A,σ). Then if (B,τ) is symplectic, any counterpart of (B,τ) is orthogonal.*

Proof. Let K be a separable closure of F and let (C,γ) be a counterpart of (B,τ). Then A_K, B_K and C_K are split by (4.1) and hence there exists a bilinear form φ over F such that $(A,\sigma)_K \simeq \mathrm{Ad}(\varphi)$. By (4.4) there exists forms φ_1 and φ_2 such that $\varphi \simeq \varphi_1 \perp \varphi_2$ and $\mathrm{Ad}(\varphi_1) \simeq (B,\tau)_K$ and $\mathrm{Ad}(\varphi_2) \simeq (C,\gamma)_K$. By definition, (B,τ) symplectic implies that φ_1 is hyperbolic. Since (A,σ) is orthogonal φ is not hyperbolic, and therefore we must have that φ_2 is not hyperbolic. Hence (C,γ) is not symplectic. □

We call a part (B,τ) of (A,σ) *the anisotropic part of* (A,σ) if (B,τ) is direct and its counterpart is metabolic. This part is unique up to isomorphism by [4, (3.6)] and we denote it by $(A,\sigma)_{\mathrm{an}}$.

Assume that $\mathrm{char}(F) = 2$. We say (A,σ) is *direct* if for all $a \in A$ such that $\sigma(a)a \in \mathrm{Alt}(A,\sigma)$, it follows that $a = 0$. If A is split, then an algebra with involution is direct if and only if it is anisotropic (see [5, (7.1)]). We call a part (C,γ) of $(A,\sigma)_{\mathrm{an}}$ *the direct part of* (A,σ) if (C,γ) is direct and its counterpart in $(A,\sigma)_{\mathrm{an}}$ is symplectic. This part is unique up to isomorphism by [5, (7.5)], as is its counterpart, which we call *the even part of* (A,σ), denoted $(A,\sigma)_{\mathrm{ev}}$.

Lemma 4.6. *Assume that $\mathrm{char}(F) = 2$. Let L be the separable closure of F. Let (A,σ) be an F–algebra with involution and let φ be a nondegenerate symmetric bilinear form over L such that $(A,\sigma)_L \simeq \mathrm{Ad}(\varphi)$. If (A,σ) has a non-trivial even part, then the hyperbolicity index of φ is strictly positive.*

Proof. This follows from (4.4) since split algebras with symplectic involutions are adjoint to hyperbolic bilinear forms by definition. □

In characteristic 2, the direct part of an F–algebra with involution classifies the isotropy behaviour of the involution when extending scalars to a separable extension of F that splits the algebra.

Proposition 4.7. *Assume that* $\mathrm{char}(F) = 2$. *Let* (A, σ) *be an* F*–algebra with involution and let* L/F *be a separable algebraic field extension such that* A_L *is split. Then* $((A, \sigma)_L)_{\mathrm{an}} \simeq ((A, \sigma)_{\mathrm{dir}})_L$.

Proof. See [5, (9.3)]. □

5. Isotropy of totally decomposable orthogonal involutions

In this section we prove our main result. That a totally decomposable orthogonal involution is anisotropic or metabolic over every field extension. We assume throughout this section that $\mathrm{char}(F) = 2$.

Lemma 5.1. *Let* (A, σ) *be a totally decomposable orthogonal* F*–algebra with involution. Then* (A, σ) *is anisotropic if and only if it is direct.*

Proof. That (A, σ) is anisotropic if it is direct is clear as $0 \in \mathrm{Alt}(A, \sigma)$.

Assume that (A, σ) is anisotropic. Over the separable closure L of F all F–quaternion algebras are split by (4.1). Therefore $(A, \sigma)_L$ isomorphic to $\mathrm{Ad}(\pi)$ for some Pfister form π over L. Hence by (4.3) and (2.5), $(A, \sigma)_L$ is either metabolic or anisotropic. If $(A, \sigma)_L$ is anisotropic, then (A, σ) is direct by (4.7).

Otherwise, $(A, \sigma)_{\mathrm{an}}$ is symplectic by (4.7). However (A, σ) is orthogonal, and hence in this case must have a non-trivial metabolic part. □

Theorem 5.2. *Let* (A, σ) *be an orthogonal* F*–algebra with involution such that* $(A, \sigma) \simeq \bigotimes_{i=1}^{n}(Q_i, \sigma_i)$, *where* (Q_i, σ_i) *are* F*–algebras with involution for all* $i \in \{1, \ldots, n\}$. *If* (A, σ) *is isotropic then it is metabolic.*

Proof. By [11, (2.23)] we have that the (Q_i, σ_i) are all orthogonal as (A, σ) is orthogonal. We proceed by induction. The result is trivial for $n = 1$. Assume that it is true for $n - 1$.

Let $(A, \sigma) \simeq (B, \tau) \otimes (Q_n, \sigma_n)$ be an isotropic orthogonal F–algebra with involution such that (Q_i, σ_i) are orthogonal F–algebras with involution for $i = 1, \ldots, n$ and $(B, \tau) = \bigotimes_{i=1}^{n-1}(Q_i, \sigma_i)$. By the inductive assumption, we may assume that (B, τ) and (Q_n, σ_n) are anisotropic, as if one of them is metabolic then (A, σ) is metabolic. In particular (B, τ) is direct by (5.1).

Let L be the separable closure of F. By (2.5), $(A, \sigma)_L$ is metabolic and hence by [5, (9.4)], (A, σ) is some orthogonal sum of an anisotropic symplectic F–algebra with involution (C, γ) and a metabolic F–algebra with involution. If (C, γ) is trivial, then we are done.

By (4.1) all L–algebras are split, hence we may find π and $\psi \simeq \langle 1, a\rangle$, Pfister forms over L such that $\mathrm{Ad}(\pi) \simeq (B, \tau)$ and $\mathrm{Ad}(\psi) \simeq (Q_n, \sigma_n)$. By (4.7), $(B, \tau)_L$ is anisotropic and hence so is π. Since $(A, \sigma)_L \simeq \mathrm{Ad}(\pi \otimes \psi)$ is metabolic, we have

$$\pi \otimes \psi \simeq \mathbb{M}(\beta) \perp n \times \mathbb{H}$$

for some anisotropic bilinear form β over L. Hence $\pi \perp a\pi$ is metabolic, and in particular it represents 0. As π is anisotropic, it follows that π represents a and therefore $a\pi \simeq \pi$ by (2.3). Hence $\pi \otimes \psi \simeq \mathbb{M}(\pi)$ and we must have that $\mathbb{M}(\pi) \simeq \mathbb{M}(\beta)$ by (2.2). In particular, we have that $n = 0$. Hence (C, γ) is trivial by (4.6), as required. □

Lemma 5.3. *A totally decomposable orthogonal* F*-algebra with involution is either direct or metabolic. In particular, the even part of a totally decomposable orthogonal* F*–algebra with involution is trivial.*

Proof. If an orthogonal F-algebra with involution is anisotropic then it is direct by (5.1). Otherwise it is metabolic by (5.2) and has trivial anisotropic part. □

Corollary 5.4. *Let L be the separable closure of F. A totally decomposable orthogonal F-algebra with involution (A,σ) is metabolic if and only if $(A,\sigma)_L$ is metabolic. Otherwise (A,σ) is anisotropic.*

Proof. This follows immediately from (5.3) and (4.7). □

Question 5.5. Let (A,σ) be a non-metabolic F–algebra with involution. Assume that for every field extension K/F we have $(A,\sigma)_K$ is either anisotropic or metabolic. Is (A,σ) totally decomposable?

6. An invariant of totally decomposable algebras

In this section we use (5.2) to define an invariant on totally decomposable involutions and give an application of this result.

Let (A,σ) be an F–algebra with orthogonal involution. As in [11, Section 7] the *determinant of* (A,σ), denoted $\Delta(A,\sigma)$, is the square class of the reduced norm of any alternating unit. That is

$$\Delta(A,\sigma)=\mathrm{Nrd}_A(a)\cdot F^{\times 2}\in F^{\times}/F^{\times 2}\quad \text{for } a\in \mathrm{Alt}(A,\sigma)\cap A^{\times}.$$

This does not depend on the choice of $a\in \mathrm{Alt}(A,\sigma)\cap A^{\times}$ (see [11, (7.1)]).

Lemma 6.1. *Let (Q,σ) be a split orthogonal F–quaternion algebra with involution and let $d\in F$ be a representative of the class of $\Delta(Q,\sigma)$ in $F^{\times}/F^{\times 2}$. Then $(Q,\sigma)\simeq \mathrm{Ad}(\langle 1,b\rangle)$.*

Proof. This follows from [11, (7.3), (3)] and the fact that the determinant of a 2–dimensional bilinear form classifies it up to similarity. □

We now assume throughout the rest of this section that $\mathrm{char}(F)=2$.

Lemma 6.2. *Let (A,σ) be an orthogonal F–algebra with involution such that $(A,\sigma)\simeq\bigotimes_{i=1}^n(Q_i,\sigma_i)\simeq\bigotimes_{i=1}^n(Q_i',\sigma_i')$ where (Q_i,σ_i) and (Q_i',σ_i') are orthogonal F–algebras with involution for all $i\in\{1,\dots,n\}$. Let $\Delta_i=\Delta(Q_i,\sigma_i)$ and $\Delta_i'=\Delta(Q_i',\sigma_i')$. Then $\langle\langle\Delta_1,\dots,\Delta_n\rangle\rangle\simeq\langle\langle\Delta_1',\dots,\Delta_n'\rangle\rangle$.*

Proof. Let $\pi=\langle\langle\Delta_1,\dots,\Delta_n\rangle\rangle$ and $\pi'=\langle\langle\Delta_1',\dots,\Delta_n'\rangle\rangle$. Let L be the separable closure of F. Since $(Q_i)_L$ and $(Q_i')_L$ are split for all $i\in\{1,\dots,n\}$ by (4.1), it follows from (6.1) that $(A,\sigma)\simeq\mathrm{Ad}(\pi)\simeq\mathrm{Ad}(\pi')$. Therefore by (4.2) there exists a $\lambda\in L$ such that $\lambda\pi_L\simeq\pi_L'$. However, since π and π' are Pfister forms, it follows from (2.4) that $\pi_L\simeq\pi_L'$. That $\pi\simeq\pi'$ now follows from (3.2). □

Theorem 6.3. *Let (Q_i,σ_i) be orthogonal F–algebras with involution for all $i\in\{1,\dots,n\}$ and let $\Delta_i=\Delta(Q_i,\sigma_i)$. Then the map that associates $\langle\langle\Delta_1,\dots,\Delta_n\rangle\rangle$ to the F–algebra with involution $\bigotimes_{i=1}^n(Q_i,\sigma_i)$ induces a map from the set of isomorphism classes of totally decomposable orthogonal F–algebras with involution of degree 2^n to the set of isometry classes of n–fold Pfister forms over F. This map is compatible with scalar extension.*

Proof. That the map induced by associating $\langle\langle\Delta_1,\dots,\Delta_n\rangle\rangle$ with $\bigotimes_{i=1}^n(Q_i,\sigma_i)$ is well defined follows directly from (6.2). Let K/F be a field extension and let π be the Pfister form over K associated with $(A,\sigma)_K$ via the map in the statement. That $\pi\simeq\langle\langle\Delta_1,\dots,\Delta_n\rangle\rangle_K$ follows using the same argument as in (6.2). □

With notation as in (6.3), the Pfister form $\langle\langle \Delta_1, \ldots, \Delta_n \rangle\rangle$ associated to a totally decomposable F–algebra with orthogonal involution (A, σ) is uniquely determined by (A, σ) up to isometry. We call this Pfister form *the Pfister invariant of* (A, σ). We denote it by $\mathfrak{Pf}(A, \sigma)$.

Question 6.4. For totally decomposable orthogonal F–algebras with involution (A, σ) and (A, τ), does $\mathfrak{Pf}(A, \sigma) \simeq \mathfrak{Pf}(A, \tau)$ imply that $(A, \sigma) \simeq (A, \tau)$?

Corollary 6.5. *Let (A, σ) be a totally decomposable orthogonal F–algebra with involution. Then*

(1) *(A, σ) is metabolic if and only if $\mathfrak{Pf}(A, \sigma)$ is metabolic.*
(2) *If $\mathfrak{Pf}(A, \sigma)$ is not metabolic, then (A, σ) is anisotropic.*
(3) *For any field K such that A_k is split, $(A, \sigma)_K \simeq \mathrm{Ad}(\mathfrak{Pf}(A, \sigma)_K)$.*

Proof. Let L be the separable closure of F. Then we clearly have that $(A, \sigma)_L \simeq \mathrm{Ad}(\mathfrak{Pf}(A, \sigma)_L)$. Hence $(A, \sigma)_L$ is anisotropic (resp. metabolic) if and only if $\mathrm{Ad}(\mathfrak{Pf}(A, \sigma)_L)$ is anisotropic (resp. metabolic) by (4.3). This is equivalent to (1) and (2) by (3.1) and (5.4).

Now let φ be a symmetric bilinear form over K such that $(A, \sigma)_K \simeq \mathrm{Ad}(\varphi)$. By scaling we may assume that φ represents 1. Then by (4.2), there exists a $\lambda \in L^\times$ such that $\lambda\varphi_L \simeq \pi_L$. Note that this implies that φ_L is either anisotropic or metabolic and hence so is φ by (3.1). Then $\varphi_L \simeq \pi_L$ by (2.4). That $\varphi \simeq \pi_K$ now follows from (3.2). □

Remark 6.6. Let k be a field of characteristic different from 2. It is shown in [2, (3.9)] that in general one cannot associate a bilinear form over k to any F–algebra with involution (A, σ) so that φ shares its anisotropy behaviour with (A, σ) as the Pfister invariant is shown to do in (6.5). Moreover, they show that for a field extension K/k such that A_K is split, there does not always exist a bilinear form φ over k such that $(A, \sigma)_K \simeq \mathrm{Ad}(\varphi_K)$. This gives (3).

Remark 6.7. Part (2) of (6.5) can be thought of as one characteristic 2 version of the Pfister Factor Conjecture (see [3]). That is, any totally decomposable orthogonal involution over a split algebra is adjoint to a Pfister form.

This can be shown more directly as follows. Let (A, σ) be a totally decomposable F–algebra with orthogonal involution such that A is split. Then $(A, \sigma) \simeq \mathrm{Ad}(\varphi)$ for some bilinear form φ over F which we may assume represents 1. Let L be a separable closure of F. Then, since every L-quaternion algebra is split by (4.1), we must have that $(A, \sigma)_L$ is isomorphic to a product of split L–quaternion algebras with involution, and hence $(A, \sigma)_L \simeq \mathrm{Ad}(\pi)$ for some Pfister form π over L. That $\varphi_L \simeq \pi$ now follows from (4.2) and (2.4), and hence φ is a Pfister form by (3.3) and (2.4).

However (6.5) not only shows that these split totally decomposable algebras with involution are adjoint to Pfister forms, but gives an explicit description of the Pfister forms. This result was also independently obtained in [15, (4.6)] with different methods that do not rely on the non-split behaviour of totally decomposable involutions.

Corollary 6.8. *Let (Q_1, σ_1) and (Q_2, σ_2) be orthogonal F–quaternion algebras with involution such that $\Delta(Q_1, \sigma_1) = \Delta(Q_2, \sigma_2)$. Then $(Q_1, \sigma_1) \otimes (Q_2, \sigma_2)$ is metabolic.*

Proof. It is clear that $\mathfrak{Pf}((Q_1, \sigma_1) \otimes (Q_2, \sigma_2))$ is metabolic, and hence so is $(Q_1, \sigma_1) \otimes (Q_2, \sigma_2)$ by (6.5). □

Remark 6.9. Let k be a field of arbitrary characteristic. Let (Q_1, σ_1) and (Q_2, σ_2) be split orthogonal k–quaternion algebras with involution such that $\Delta = \Delta(Q_1, \sigma_1) = -\Delta(Q_2, \sigma_2)$. Then by (6.1) we have that $(Q_1, \sigma_1) \simeq \mathrm{Ad}(\langle 1, \Delta\rangle)$ and $(Q_2, \sigma_2) \simeq \mathrm{Ad}(\langle 1, -\Delta\rangle)$. Moreover, since we have $(Q_1, \sigma_1) \otimes (Q_2, \sigma_2) \simeq \mathrm{Ad}(\langle 1, \Delta\rangle \otimes \langle 1, -\Delta\rangle)$ it is clear that $(Q_1, \sigma_1) \otimes (Q_2, \sigma_2)$ is metabolic.

(6.8) shows that this behaviour extends to the non-split case when $\mathrm{char}(k) = 2$. Note that is is not the case when $\mathrm{char}(k) \neq 2$, where $\Delta(Q_1, \sigma_1) = -\Delta(Q_2, \sigma_2)$ does not even imply that $Q_1 \otimes Q_2$ is not division in general, as (6.11) shows.

Let k be a field with $\mathrm{char}(k) \neq 2$. For $a, b \in k^\times$, let $(1, u, v, w)$ be basis elements of a k-vector space such that $u^2 = a$, $v^2 = b$ and $w = uv = -vu$. Then $(1, u, v, w)$ generate an k–quaternion algebra which we denote $(a, b)_k$. There is a unique symplectic k–involution γ on $(a, b)_k$ given by $\gamma(u) = -u$ and $\gamma(v) = -v$ (see [11, (2.21)]). We call this k–involution the *canonical involution on* $(a, b)_F$.

Lemma 6.10. *Let k be a field with $\mathrm{char}(k) \neq 2$. Let $K = k(X, Y, Z)$ where X, Y and Z are indeterminates. Then we have*

$$(X, Y)_K \otimes (-1, Z)_K \simeq (X, YZ)_K \otimes (-X, Z)_K.$$

Proof. Let $\varphi \simeq \langle -X, -Y, XY, 1, -Z, Z\rangle$ and $\psi \simeq \langle -X, -YZ, XYZ, X, -Z, -XZ\rangle$. Direct computation shows that $\varphi \perp (-\psi) \simeq \langle\langle X, Y, Z\rangle\rangle \perp 2 \times \mathbb{H}$. By [14, Chapter XII, (2.13)], it then follows that $(X, Y)_K \otimes (-1, Z)_K \simeq (X, YZ)_K \otimes (-X, Z)_K$. □

Example 6.11. Let k be a field with $\mathrm{char}(k) \neq 2$. Further assume that $-1 \notin k^2$. Let $K = k(X, Y, Z)$ where X, Y and Z are indeterminates. Then the K–algebra $A = (X, Y)_K \otimes (-1, Z)_K$ is division by [16, (19.6, Corollary c)].

By (6.10) we have $A \simeq (X, YZ)_K \otimes (-X, Z)_K$. Let $(1, u_1, v_1, w_1)$ and $(1, u_2, v_2, w_2)$ be K–bases of $Q_1 = (X, YZ)_K$ and $Q_2 = (-X, Z)_K$ respectively. Let γ_1 and γ_2 be the canonical involutions of Q_1 and Q_2 respectively. Then $(Q_1, \mathrm{Int}(u_1) \circ \gamma_1)$ and $(Q_2, \mathrm{Int}(u_2) \circ \gamma_2)$ are orthogonal K–algebras with involution by [11, (2.21)]. Further by [11, (7.4)] $\Delta(Q_1, \mathrm{Int}(i_1) \circ \gamma_1) = X$ and $\Delta(Q_2, \mathrm{Int}(i_2) \circ \gamma_2) = -X$.

However, every F–involution on A must be anisotropic, as for all $a \in A^\times$ we have $\sigma(a)a \neq 0$ since A is division. In particular, $(Q_1, \mathrm{Int}(i_1) \circ \gamma_1) \otimes (Q_2, \mathrm{Int}(i_2) \circ \gamma_2)$ is not metabolic.

7. Involutions adjoint to Pfister forms after splitting

In this section we show that a non-metabolic algebra with involution that is anisotropic or metabolic over every field extension is adjoint to a Pfister form over any splitting field, but that the converse fails in general. This result also shows that over fields of characteristic 2, there exist non-totally decomposable algebras with involution that become totally decomposable over some field extension. We construct an explicit example in Section 8. We assume throughout this section that $\mathrm{char}(F) = 2$.

Proposition 7.1. *Let (A, σ) be a split anisotropic orthogonal F–algebra with involution. Assume that for all field extensions K/F, $(A, \sigma)_K$ is either anisotropic or metabolic. Then there exists a Pfister form over F such that $(A, \sigma) \simeq \mathrm{Ad}(\pi)$.*

Proof. By hypothesis (A, σ) must be anisotropic. Let φ be a bilinear form over F such that $(A, \sigma) \simeq \mathrm{Ad}(\varphi)$. We may assume φ represents 1. Then φ is anisotropic and φ_K is either anisotropic or metabolic over every field extension K/F by (4.3). Hence φ is a Pfister form by (2.5) and (2.4). □

Proposition 7.2. *Let (A,σ) be an anisotropic F–algebra with involution. Assume that for all field extensions K/F, $(A,\sigma)_K$ is either anisotropic or metabolic. Let L be the separable closure of F. Then $(A,\sigma)_L$ is anisotropic and there exists a Pfister form π over L such that $(A,\sigma)_L \simeq \mathrm{Ad}(\pi)$.*

Proof. By the hypothesis, (A,σ) is anisotropic and not symplectic. If $(A,\sigma)_{\mathrm{an}} = (A,\sigma)$ is not isomorphic to $(A,\sigma)_{\mathrm{dir}}$ then it follows from (4.6) and (4.7) that $(A,\sigma)_L$ is neither anisotropic nor metabolic. Hence (A,σ) is direct. Therefore, by (4.7), $(A,\sigma)_L$ is anisotropic and by the hypothesis, either anisotropic or metabolic over every field extension of L. Therefore, there exists a Pfister form π over L such that $(A,\sigma)_L \simeq \mathrm{Ad}(\pi)$ by (7.1). □

Theorem 7.3. *Let (A,σ) be an anisotropic orthogonal F–algebra with involution. Assume that for all field extensions K'/F, $(A,\sigma)_{K'}$ is either anisotropic or metabolic. Then for every field extension K/F such that A_K is split, there exists a Pfister form over K such that $(A,\sigma)_K \simeq \mathrm{Ad}(\pi)$.*

Proof. Let L be a separable closure of F. By (7.2) there exists a Pfister form π over L such that $(A,\sigma)_L \simeq \mathrm{Ad}(\pi)$.

Let K/F be a field extension such that A_K is split, and let ψ be a bilinear form over K such that $(A,\sigma)_K \simeq \mathrm{Ad}(\psi)$. We may assume that ψ represents 1. Let K' be a separable extension of K such that L is a subfield of K'. It follows that $(A,\sigma)_{K'} \simeq \mathrm{Ad}(\varphi_{K'})$ and hence $\pi_{K'} \simeq \psi_{K'}$ by (4.2) and (2.4). Hence ψ is similar to a Pfister form by (2.5). □

Question 7.4. In the situation of (7.3), can we find a Pfister form π' over F such that $\pi'_K \simeq \pi$?

Note that the converse to (7.3) does not hold in general as we now show.

Lemma 7.5. *Let (A,σ) be an orthogonal F–algebra with involution with $\deg(A) = 2^n$ for some integer n such that $(A,\sigma)_{\mathrm{an}} \simeq (A,\sigma)_{\mathrm{ev}}$ and the counterpart of $(A,\sigma)_{\mathrm{an}}$ is totally decomposable. Let K/F be a field extension such that A_K is split. Then there exists a Pfister form over K such that $(A,\sigma)_K \simeq \mathrm{Ad}(\pi)$.*

Proof. Denote the counterpart of $(A,\sigma)_{\mathrm{an}}$ by (B,τ). Note that (B,τ) must be orthogonal by (4.5). Since A_K is split there exists some symmetric bilinear form φ over K such that $(A,\sigma)_K \simeq \mathrm{Ad}(\varphi)$.

Then by (4.4) it follows from the hypothesis that $\varphi \simeq \mathbb{M}(\pi) \perp n \times \mathbb{H}$ for some bilinear form π over K and integer n with $2\dim_F(\pi) + 2n = 2^m$ for some integer m such that $((A,\sigma)_{\mathrm{ev}})_K \simeq \mathrm{Ad}(n \times \mathbb{H})$ and $(B,\tau)_K \simeq \mathrm{Ad}(\mathbb{M}(\pi))$. Since (B,τ) is totally decomposable, it follows from (6.5) that π is a Pfister form and hence so is φ by (2.6). □

Proposition 7.6. *Let (A,σ) be an orthogonal F–algebra with involution with $\deg(A) = 2^n$ for some integer n such that $(A,\sigma)_{\mathrm{an}} \simeq (A,\sigma)_{\mathrm{ev}}$ is non-trivial and a counterpart of $(A,\sigma)_{\mathrm{an}}$ is totally decomposable and non-trivial. Then the following hold:*

- (a) *For any field extension K/F such that A_K is split $(A,\sigma)_K$ is totally decomposable.*
- (b) *(A,σ) is isotropic but not metabolic.*
- (c) *(A,σ) is not totally decomposable.*

Proof. Statement (a) follows directly from (7.5). Statement (b) is clear as $(A,\sigma)_{\mathrm{an}}$ is non-trivial but not isomorphic to (A,σ). It is also clear that (A,σ) is neither direct nor metabolic, hence (5.3) gives (c). $\square$

8. A non-totally decomposable involution that becomes totally decomposable after splitting

We now use hermitian forms to construct an explicit example of the type of algebra with involution described in (7.6). Throughout this section, assume that $\mathrm{char}(F)=2$.

Let (D,θ) be an F–division algebra with involution of the first kind. A *hermitian form* over (D,θ) is a pair (V,h) where V is a finite-dimensional right D-vector space and h is a non-degenerate bi-additive map $h:V\times V\to D$ such that

$$h(x,yd)=h(x,y)d \quad \text{and} \quad h(y,x)=\theta(h(x,y))$$

holds for all $x,y\in V$ and $d\in D$. We say (V,h) is *non-degenerate* if $h(x,y)=0$ for all $y\in V$ implies that $x=0$. We say (V,h) *represents an element* $a\in D$ if $h(x,x)=a$ for some $x\in V\backslash\{0\}$. We call a hermitian form (V,h) *isotropic* it represents 0, and *anisotropic* otherwise. We call a hermitian form (V,h) *metabolic* if there exists a subspace $W\subset V$ such that $h|_W=0$ and $\dim_F(W)=\frac{1}{2}\dim_F(V)$.

For $a_1,\ldots,a_n\in D^\times\cap\mathrm{Sym}(D,\theta)$, we denote by $\langle a_1,\ldots,a_n\rangle_\theta$ the hermitian form (D^n,h) where

$$h:D^n\times D^n\to D, \quad \text{is given by} \quad (x,y)\mapsto\sum_{i=1}^n\theta(x_i)a_iy_i.$$

There is a well known correspondence between non-degenerate hermitian forms on V and F–involutions on A, generalising the correspondence between bilinear forms and involutions on a split algebra:

Proposition 8.1. *Let (D,θ) be an F–division algebra with involution, V a right D–vector space and let $A=\mathrm{End}_D(V)$. For every non-degenerate hermitian form (V,h), there is a unique F–involution σ on A such that*

$$h(f(x),y)=h(x,\sigma(f)(y)) \text{ for all } x,y\in V \text{ and } f\in A.$$

Proof. See [11, (4.1)]. $\square$

In the situation of (8.1), we call (A,σ) the *F–algebra with involution adjoint to* (V,h) and we write $\mathrm{Ad}(V,h)=(\mathrm{End}_D(V),\sigma)$.

Proposition 8.2. *Let (V,h) be a nondegenerate hermitian form over and F–division algebra with involution (D,θ). Then (V,h) is isotropic (resp. metabolic) if and only if* $\mathrm{Ad}(V,h)$ *is isotropic (resp. metabolic).*

Proof. See [4, (4.8)]. $\square$

Let $a\in F$ be such that $-4a\neq 1$ and let $b\in F^\times$. Let $K=F(\alpha)$ where $\alpha^2+\alpha=a$ and let τ be the non-trivial F–automorphism of K. Let $(1,u,v,w)$ be basis elements of the F-vector space $K\oplus vK$ such that $u^2=u+a$, $v^2=b$ and $w=uv=\tau(u)$. Then $(1,u,v,w)$ generate an F–quaternion algebra which we denote by $[a,b)_F$.

Let $Q=[a,b)_F$ for $a\in F$ and $b\in F^\times$ as above. Let γ be the F–involution on Q given by $\gamma(u)=\tau(u)$ and $\gamma(v)=v$. This is the unique symplectic involution on Q (see [11, (2.21)]). Let $\mathrm{Alt}(Q,\gamma)=\{\gamma(s)+s\mid s\in Q\}$. Then direct computation gives $\mathrm{Alt}(Q,\gamma) = F$ and that for $s = t_1 + t_2u + t_3v + t_4w$ where $t_1, \dots, t_4 \in F$ we have

$$\gamma(s)s = t_1^2 + t_1t_2 + t_2^2a + (t_3^2 + t_3t_4 + t_4^2a)b.$$

If we consider Q as a 4-dimensional F–vector space, Q together with the map $n_Q : Q \to F$, $s \mapsto \gamma(s)s$ for $s \in Q$ can be considered as a quadratic form over F.

Lemma 8.3. *Let k be a field of arbitrary characteristic and let $F = k(X,Y)$, where X, Y are indeterminates. Let $Q = [X,Y)_F$. Then Q is an F–division algebra.*

Proof. By [7, (12.5)], Q is division if and only if (Q, n_Q) is anisotropic.

By [7, (17.4)], (Q, n_Q) is anisotropic if and only if

$$\rho = (k(Y)^2, (t_1,t_2) \mapsto t_1^2 + t_1t_2 + t_2^2Y)$$

is anisotropic. By [7, (17.3)], ρ is anisotropic if and only if there exists an element $a \in k[Y]$ such that $1 + a + a^2Y = 0$.

Assume $a = a_nY^n + \ldots + a_1Y + a_0$ for $n > 0$ and $a_0, \dots, a_n \in k$ with $a_n \neq 0$. Substituting this expression for a into $1+a+a^2Y = 0$ gives $a_n^2Y^{2n+1} + b = 0$ where $b \in k[Y]$ such that $\deg_Y(b) < 2n + 1$. This contradicts $a_n \neq 0$. Therefore we must have that $\deg_Y(a) = 0$. However, since $Y \notin k$, it follows that $a = 0$, and hence ρ is anisotropic. □

Example 8.4. Let k be a field of characteristic 2 and let $F = k(X,Y,Z)$ where X, Y and Z are indeterminates. Let $Q = [X,Y)_F$, which is an F–divison algebra by (8.3).

Let ψ be the hermitian form $\langle 1, Z, v, v\rangle_\gamma$ over (Q,γ) and let $(A,\sigma) \simeq \mathrm{Ad}(\psi)$. We have that $\mathrm{Ad}(\langle 1, Z\rangle_\gamma)$ is symplectic and that $\mathrm{Ad}(\langle v, v\rangle_\gamma)$ and (A,σ) are orthogonal by [11, (4.2)].

Since $\langle v, v\rangle_\gamma$ is clearly metabolic, by (8.2) we have that $\mathrm{Ad}(\langle v, v\rangle_\gamma)$ is metabolic. We have that $\langle 1, Z\rangle_\gamma$ is isotropic if and only if $\gamma(a_1)a_1 + Z\gamma(a_2)a_2 = 0$ for some $(a_1,a_2) \neq (0,0) \in Q \times Q$. That is, if and only if $(Q,n_Q)\perp Z(Q,n_Q)$ is isotropic. That $(Q,n_Q)\perp Z(Q,n_Q)$ is anisotropic follows from [7, (17.14)] using an argument similar to that in (8.3). Hence $\langle 1, Z\rangle_\gamma$ is anisotropic, and hence so is $\mathrm{Ad}(\langle 1, Z\rangle_\gamma)$ by (8.2). It follows that ψ is not metabolic by [10, Chapter 1, (6.1.1)] and hence (A,σ) is not metabolic by (8.2).

Finally, $\mathrm{Ad}(\langle v, v\rangle_\gamma) \simeq \mathrm{Ad}(\langle 1, 1\rangle) \otimes \mathrm{Ad}(\langle v\rangle_\gamma)$ and $\mathrm{Ad}(\langle v\rangle_\gamma) \simeq (Q,\sigma')$ for some orthogonal F–involution σ'. That is, $\mathrm{Ad}(\langle v, v\rangle_\gamma)$ is totally decomposable. Therefore (A,σ) satisfies the conditions of (7.6). That is, (A,σ) is a non-totally decomposable orthogonal F–algebra with involution such that $(A,\sigma)_L$ is totally decomposable for any field extension L/F such that A_L is split.

Question 8.5. Let (A,σ) be an orthogonal F–algebra with involution. Assume that there exists a field extension L/F such that $(A,\sigma)_L$ is anisotropic and A_L is split. That is, by (4.7), (A,σ) is direct. Assume further that for all field extensions K/L, $(A,\sigma)_K$ is either anisotropic or metabolic. Does it follow that for every field extension K'/F, $(A,\sigma)_{K'}$ is either anisotropic or metabolic?

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