JORDAN GROUPS AND AUTOMORPHISM GROUPS OF ALGEBRAIC VARIETIES

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Abstract. The first section of this paper is focused on Jordan groups in abstract setting, the second on that in the settings of automorphisms groups and groups of birational self-maps of algebraic varieties. The appendix contains formulations of some open problems and the relevant comments.

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This is the expanded version of my talk, based on [Po2010, Sect. 2], at the workshop *Groups of Automorphisms in Birational and Affine Geometry*, October 29–November 3, 2012, Levico Terme, Italy. The appendix is the expanded version of my notes on open problems posted on the site of this workshop [Po2012$_2$].

Below k is an algebraically closed field of characteristic zero. Variety means algebraic variety over k in the sense of Serre (so algebraic group means algebraic group over k). We use without explanation standard notation and conventions of [Bo1991] and [Sp1998]. In particular, $k(X)$ denotes the field of rational functions of an irreducible variety X. $\mathrm{Bir}(X)$ denotes the group of birational self-maps of an irreducible variety X, and Cr_n denotes the Cremona group over k of rank n, cf. [Po2011], [Po2012$_1$].

1. Jordan groups

1.1. **Main definition.** The notion of Jordan group was introduced in [Po2010]:

Definition 1 ([Po2010, Def. 2.1]). A group G is called a *Jordan group* if there exists a positive integer d, depending only on G, such that every finite

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subgroup K of G contains a normal abelian subgroup whose index in K is at most d. The minimal such d is called the *Jordan constant of* G and is denoted by J_G.

Informally, this means that all finite subgroups of G are “almost” abelian in the sense that they are extensions of abelian groups by finite groups taken from a finite list.

Actually, one obtains the same class of groups if the assumption of normality in Definition 1 is dropped. Indeed, for any group P containing a subgroup Q of finite index, there is a normal subgroup N of P such that $[P : N] \leqslant [P : Q]!$ and $N \subseteq Q$ (see, e.g., [La1965, Exer. 12 to Chap. I]).

1.2. **Examples.**

1.2.1. *Jordan's Theorem.* The first example that led to Definition 1 justifies the coined name. It is given by the classical Jordan's theorem [Jo1878] (see, e.g., [CR1962, §36] for a modern exposition). In terms of Definition 1 the latter can be reformulated as follows:

Theorem 1 (C. Jordan, 1878)**.** *The group* $\mathbf{GL}_n(k)$ *is Jordan for every* n*.*

Since the symmetric group Sym_{n+1} admits a faithful n-dimensional representation and the alternating group Alt_{n+1} is the only non-identity proper normal subgroup of Sym_{n+1} for $n \geqslant 2$, $n \neq 3$, Definition 1 yields the lower bound

$$(n+1)! \leqslant J_{\mathbf{GL}_n(k)} \quad \text{for } n \geqslant 4. \tag{1}$$

Frobenius, Schur, and Blichfeldt initiated exploration of the upper bounds for $J_{\mathbf{GL}_n(k)}$. In 2007, using the classification of finite simple groups, M. J. Collins [Co2007] gave optimal upper bounds and thereby found the precise values of $J_{\mathbf{GL}_n(k)}$ for all n. In particular, in [Co2007] is proved that

(i) the equality in (1) holds for all $n \geqslant 71$ and $n = 63, 65, 67, 69$;
(ii) $J_{\mathbf{GL}_n(k)} = 60^r r!$ if $n = 2r$ or $2r+1$ and either $20 \leqslant n \leqslant 62$ or $n = 64, 66, 68, 70$;
(iii) $J_{\mathbf{GL}_n(k)} = 60, 360, 25920, 25920, 6531840$ resp., for $n = 2, 3, 4, 5, 6$.

The values of $J_{\mathbf{GL}_n(k)}$ for $7 \leqslant n \leqslant 19$ see in [Co2007].

1.2.2. *Affine algebraic groups.* Since any subgroup of a Jordan groups is Jordan, Theorem 1 yields

Corollary 1. *Every linear group is Jordan.*

Since every affine algebraic group is linear [Sp1998, 2.3.7], this, in turn, yields the following generalization of Theorem 1:

Theorem 2. *Every affine algebraic group is Jordan.*

1.2.3. *Nonlinear Jordan groups.* Are there nonlinear Jordan groups? The next example, together with Theorem 1, convinced me that Definition 1 singles out an interesting class of groups and therefore deserves to be introduced.

Example 1. By [Se2009$_1$, Thm. 5.3], [Se2008, Thm. 3.1], the planar Cremona group Cr_2 is Jordan. On the other hand, by [CD2009, Prop. 5.1] (see also [Co2007, Prop. 2.2]), Cr_2 is not linear. Note that in [Se2009$_1$, Thm. 5.3] one also finds a "multiplicative" upper bound for J_{Cr_2}: as is specified there, a crude computation shows that every finite subgroup G of Cr_2 contains a normal abelian subgroup A of rank $\leqslant 2$ with $[G:A]$ dividing $2^{10}\cdot 3^4\cdot 5^2\cdot 7$ (it is also mentioned that the exponents of 2 and 3 can be somewhat lowered, but those of 5 and 7 cannot). $\square$

Example 2. Let F_d be a free group with d free generators and let F_d^n be its normal subgroup generated by the nth powers of all elements. As is known (see, e.g., [Ad2011, Thm. 2]), the group $B(d,n) := F_d/F_d^n$ is infinite for $d \geqslant 2$ and odd $n \geqslant 665$ (recently S. Adian announced in [Ad2013] that 665 may be replaced by 100). On the other hand, by I. Schur, finitely generated linear torsion groups are finite (see, e.g., [CR1962, Thm. 36.2]). Hence infinite $B(d,n)$ is nonlinear. On the other hand, for $d \geqslant 2$ and odd $n \geqslant 665$, every finite subgroup in $B(d,n)$ is cyclic (see [Ad2011, Thm. 8]); hence $B(d,n)$ is Jordan and $J_{B(d,n)} = 1$. $\square$

1.2.4. *Diffeomorphism groups of smooth manifolds.* Let M be a compact connected n-dimensional smooth manifold. Assume that M admits an unramified covering $\widetilde{M} \to M$ such that $H^1(\widetilde{M}, \mathbf{Z})$ contains the cohomology classes $\alpha_1, \ldots, \alpha_n$ satisfying $\alpha_1 \cup \cdots \cup \alpha_n \neq 0$. Then, by [MiR2010, Thm. 1.4(1)], the group $\mathrm{Diff}(M)$ is Jordan. This result is applicable to $\mathbf{T}^n$, the product of n circles, and, more generally, to the connected sum of $N\sharp\mathbf{T}^n$, where N is any compact connected orientable smooth manifold. (I thank I. Mundet i Riera who informed me about his paper [MiR2010], from which I also learned about [Fi2011] and [Pu2007].)

1.2.5. *Non-Jordan groups.* Are there non-Jordan groups?

Example 3. The group Sym_∞ of all permutations of $\mathbf{Z}$ contains the alternating group Alt_n for every n. Hence Sym_∞ is non-Jordan because Alt_n is simple for $n \geqslant 5$ and $|\mathrm{Alt}_n| = n!/2$. $\square$

Using Example 3 one obtains a finitely generated non-Jordan group:

Example 4. Let $\mathcal{N}$ be the subgroup of Sym_∞ generated by the transposition $\sigma := (1,2)$ and the "translation" δ defined by the condition

$$\delta(i) = i + 1 \quad \text{for every } i \in \mathbf{Z}.$$

Then $\delta^m\sigma\delta^{-m}$ is the transposition $(m+1, m+2)$ for every m. Since the set of transpositions $(1,2), (2,3), \ldots, (n-1,n)$ generates the symmetric group Sym_n, this shows that $\mathcal{N}$ contains Alt_n for every n; whence $\mathcal{N}$ is non-Jordan. $\square$

1.3. **General properties.**

1.3.1. *Subgroups, quotient groups, and products.* Exploring whether a group is Jordan or not leads to the questions on the connections between Jordaness of a group, its subgroup, and its quotient group.

Theorem 3 ([Po2010, Lemmas 2.6, 2.7, 2.8]).

(1) *Let H be a subgroup of a group G.*
   (i) *If G is Jordan, then H is Jordan and $J_H \leqslant J_G$.*
   (ii) *If G is Jordan and H is normal in G, then G/H is Jordan and $J_{G/H} \leqslant J_G$ in either of the cases:*
      (a) *H is finite;*
      (b) *the extension $1 \to H \to G \to G/H \to 1$ splits.*
   (iii) *If H is torsion-free, normal in G, and G/H is Jordan, then G is Jordan and $J_G \leqslant J_{G/H}$.*

(2) *Let G_1 and G_2 be two groups. Then $G_1 \times G_2$ is Jordan if and only if G_1 and G_2 are. In this case, $J_{G_i} \leqslant J_{G_1 \times G_2} \leqslant J_{G_1} J_{G_2}$ for every i.*

Proof. (1)(i). This follows from Definition 1.

If H is normal in G, let $\pi\colon G \to G/H$ be the natural projection.

(1)(ii)(a). Let F be a finite subgroup of G/H. Since H is finite, $\pi^{-1}(F)$ is finite. Since G is Jordan, $\pi^{-1}(F)$ contains a normal abelian subgroup A whose index is at most J_G. Hence $\pi(A)$ is a normal abelian subgroup of F whose index in F is at most J_G.

(1)(ii)(b). By the condition, there is a subgroup S in G such that $\pi|_S$: $S \to G/H$ is an isomorphism; whence the claim by (1)(i).

(1)(iii). Let F be a finite subgroup of G. Since H is torsion free, $F \cap H = \{1\}$; whence $\pi|_F\colon S \to \pi(F)$ is an isomorphism. Therefore, as G/H is Jordan, F contains a normal abelian subgroup whose index in F is at most $J_{G/H}$.

(2) If $G := G_1 \times G_2$ is Jordan, then (1)(i) implies that G_1 and G_2 are Jordan and $J_{G_i} \leqslant J_G$ for every i. Conversely, let G_1 and G_2 be Jordan. Let $\pi_i\colon G \to G_i$ be the natural projection. Take a finite subgroup F of G. Then $F_i := \pi_i(F)$ contains an abelian normal subgroup A_i such that

$$[F_i : A_i] \leqslant J_{G_i}. \tag{2}$$

The subgroup $\widetilde{A}_i := \pi_i^{-1}(A_i) \cap F$ is normal in F and $F/\widetilde{A}_i$ is isomorphic to F_i/A_i. From (2) we then conclude that

$$[F : \widetilde{A}_i] \leqslant J_{G_i}. \tag{3}$$

Since $A := \widetilde{A}_1 \cap \widetilde{A}_2$ is the kernel of the diagonal homomorphism

$$F \longrightarrow F/\widetilde{A}_1 \times F/\widetilde{A}_2$$

determined by the canonical projection $F \to F/\widetilde{A}_i$, we infer from (3) that

$$[F : A] = |F/A| \leqslant |F/\widetilde{A}_1 \times F/\widetilde{A}_2| = |F_1/A_1||F_2/A_2| \leqslant J_{G_1} J_{G_2}. \tag{4}$$

By construction, $A \subseteq A_1 \times A_2$ and A_i is abelian. Hence A is abelian as well. Since A is normal in F, the claim then follows from (4). $\square$

1.3.2. *Counterexample.* For a normal subgroup H of G, it is not true, in general, that G is Jordan if H and G/H are.

Example 5. For every integer $n > 0$ fix a finite group G_n with the properties:

(i) G_n has an abelian normal subgroup H_n such that G_n/H_n is abelian;

(ii) there is a subgroup Q_n of G_n such that the index in Q_n of every abelian subgroup of Q_n is greater or equal than n.

Such a G_n exists, see below. Now take $G := \prod_n G_n$ and $H := \prod_n H_n$. Then H and G/H are abelian by (i), hence Jordan, but G is not Jordan by (ii).

The following construction from [Za2010, Sect. 3] proves the existence of such a G_n. Let K be a finite commutative group of order n written additively and let $\widehat{K} := \mathrm{Hom}(K, k^*)$ be the group of characters of K written multiplicatively. The formula

$$(\alpha, g, \ell)(\alpha', g', \ell') := (\alpha\alpha'\ell'(g), g + g', \ell\ell') \tag{5}$$

endows the set $k^* \times K \times \widehat{K}$ with the group structure. Denote by G_K the obtained group. It is embedded in the exact sequence of groups

$$\{1\} \to k^* \xrightarrow{\iota} G_K \xrightarrow{\pi} K \times \widehat{K} \to \{(0,1)\},$$
$$\text{where} \quad \iota(\alpha) := (\alpha, 0, 1) \quad \text{and} \quad \pi((\alpha, g, \ell)) := (g, \ell).$$

Thus, if one takes $G_n := G_K$ and $H_n := \iota(k^*)$, then property (i) holds. Let μ_n be the subgroup of all nth roots of unity in k^*. From (5) and $|K| = n$ we infer that the subset $Q_K := \mu_n \times K \times \widehat{K}$ is a subgroup of G_K. In [Za2010, Sect. 3] is proved that for $Q_n = Q_K$ property (ii) holds. $\square$

1.3.3. *Bounded groups.* However, under certain conditions, G is Jordan if and only if H and G/H are. An example of such a condition is given in Theorem 4 below; it is based on Definition 2 below introduced in [Po2010].

Given a group G, put

$$b_G := \sup_F |F|,$$

where F runs over all finite subgroups of G.

Definition 2 ([Po2010, Def. 2.9])**.** A group G is called *bounded* if $b_G \neq \infty$.

Example 6. Finite groups and torsion free groups are bounded. $\square$

Example 7. It is immediate from Definition 2 that every extension of a bounded group by bounded is bounded. $\square$

Example 8. By the classical Minkowski's theorem $\mathbf{GL}_n(\mathbf{Z})$ is bounded (see, e.g., [Hu1998, Thm. 39.4]). Since every finite subgroup of $\mathbf{GL}_n(\mathbf{Q})$ is conjugate to a subgroup of $\mathbf{GL}_n(\mathbf{Z})$ (see, e.g., [CR1962, Thm. 73.5]), this implies that $\mathbf{GL}_n(\mathbf{Q})$ is bounded and $b_{\mathbf{GL}_n(\mathbf{Q})} = b_{\mathbf{GL}_n(\mathbf{Z})}$. H. Minkowski and I. Schur obtained the following upper bound for $b_{\mathbf{GL}_n(\mathbf{Z})}$, see, e.g., [Hu1998, §39]. Let $\mathcal{P}(n)$ be the set of all primes $p \in \mathbf{N}$ such that $[n/(p-1)] > 0$. Then

$$b_{\mathbf{GL}_n(\mathbf{Z})} \leqslant \prod_{p \in \mathcal{P}(n)} p^{d_p}, \quad \text{where} \quad d_p = \sum_{i=0}^{\infty} \left[\frac{n}{p^i(p-1)}\right]. \tag{6}$$

In particular, the right-hand side of the inequality in (6) is

$$2, 24, 48, 5760, 11520, 2903040 \quad \text{resp., for } n = 1, 2, 3, 4, 5, 6. \qquad \square$$

Example 9. Maintain the notation and assumption of Subsection 1.2.4. If $\chi(M) \neq 0$, then by [MiR2010, Thm. 1.4(2)], the group $\mathrm{Diff}(M)$ is bounded. Further information on smooth manifolds with bounded diffeomorphism groups is contained in [Pu2007]. $\square$

Example 10. Every bounded group G is Jordan with $J_G \leqslant b_G$, and there are non-bounded Jordan groups (e.g., $\mathbf{GL}_n(k)$). $\square$

Theorem 4 ([Po2010, Lemma 2.11])**.** *Let H be a normal subgroup of a group G such that G/H is bounded. Then G is Jordan if and only if H is Jordan, and in this case*

$$J_G \leqslant b_{G/H} J_H^{b_{G/H}}.$$

Proof. A proof is needed only for the sufficiency. So let H be Jordan and let F be a finite subgroup of G. By Definition 1

$$L := F \cap H \tag{7}$$

contains a normal abelian subgroup A such that

$$[L : A] \leqslant J_H. \tag{8}$$

Let g be an element of F. Since L is a normal subgroup of F, we infer that gAg^{-1} is a normal abelian subgroup of L and

$$[L : A] = [L : gAg^{-1}]. \tag{9}$$

The abelian subgroup

$$M := \bigcap_{g \in F} gAg^{-1}. \tag{10}$$

is normal in F. We intend to prove that $[F : M]$ is upper bounded by a constant not depending on F. To this end, fix the representatives $g_1, \ldots, g_{|F/L|}$ of all cosets of L in F. Then (10) and normality of A in L imply that

$$M = \bigcap_{i=1}^{|F/L|} g_i A g_i^{-1}. \tag{11}$$

From (11) we deduce that M is the kernel of the diagonal homomorphism

$$L \longrightarrow \prod_{i=1}^{|F/L|} L/g_i A g_i^{-1}$$

determined by the canonical projections $L \to L/g_i A g_i^{-1}$. This, (9), and (8) yield

$$[L : M] \leqslant [L : A]^{|F/L|} \leqslant J_H^{|F/L|}. \tag{12}$$

Let $\pi\colon G \to G/H$ be the canonical projection. By (7) the finite subgroup $\pi(F)$ of G/H is isomorphic to F/L. Since G/H is bounded, this yields $|F/L| \leqslant b_{G/H}$. We then deduce from (12) and $[F : M] = [F : L][L : M]$ that

$$[F : M] \leqslant b_{G/H} J_H^{b_{G/H}};$$

whence the claim. $\square$

The following corollary should be compared with statement (1)(ii)(a) of Theorem 3:

Corollary 2. *Let H be a finite normal subgroup of a group G such that the center of H is trivial. If G/H is Jordan, then G is Jordan and*

$$J_G \leqslant |\mathrm{Aut}(H)| J_{G/H}^{|\mathrm{Aut}(H)|}.$$

Proof. Let $\varphi\colon G \to \mathrm{Aut}(H)$ be the homomorphism determined by the conjugating action of G on H. Triviality of the center of H yields $H \cap \ker\varphi = \{1\}$. Hence the restriction of the natural projection $G \to G/H$ to $\ker\varphi$ is an embedding $\ker\varphi \hookrightarrow G/H$. Therefore, $\ker\varphi$ is Jordan since G/H is. But $G/\ker\varphi$ is finite since it is isomorphic to a subgroup of $\mathrm{Aut}(H)$ for the finite group H. By Theorem 4 this implies the claim. $\square$

2. When are $\mathrm{Aut}(X)$ and $\mathrm{Bir}(X)$ Jordan?

2.1. **Problems A and B.** In [Po2010, Sect. 2] were posed the following two problems:

Problem A. Describe algebraic varieties X for which $\mathrm{Aut}(X)$ is Jordan.
Problem B. The same with $\mathrm{Aut}(X)$ replaced by $\mathrm{Bir}(X)$.

Note that for rational varieties X Problem B means finding n such that the Cremona group Cr_n is Jordan; in this case, it was essentially posed in [Se2009$_1$, 6.1].

Describing finite subgroups of the groups $\mathrm{Aut}(X)$ and $\mathrm{Bir}(X)$ for various varieties X is a classical research direction, currently flourishing. Understanding which of these groups are Jordan sheds a light on the structure of these subgroups.

2.2. **Groups $\mathrm{Aut}(X)$.** In this subsection we shall consider Problem A.

Lemma 1. *Let $X_1, \ldots, X_n$ be all the irreducible components of a variety X. If every $\mathrm{Aut}(X_i)$ is Jordan, then $\mathrm{Aut}(X)$ is Jordan.*

Proof. Define the homomorphism $\pi\colon \mathrm{Aut}(X) \to \mathrm{Sym}_n$ by $g \cdot X_i = X_{\pi(g)}$ for $g \in \mathrm{Aut}(X)$. Then $g \cdot X_i = X_i$ for every $g \in \mathrm{Ker}(\pi)$ and i, so the homomorphism $\pi_i\colon \mathrm{Ker}(\pi) \to \mathrm{Aut}(X_i)$, $g \mapsto g|_{X_i}$, arises. The definition implies that $\pi_1 \times \ldots \times \pi_n\colon \mathrm{Ker}(\pi) \to \prod_{i=1}^n \mathrm{Aut}(X_i)$ is an injection; whence $\mathrm{Ker}(\pi)$ is Jordan by Theorem 3(2). Therefore, $\mathrm{Aut}(X)$ is Jordan by Theorem 4. $\square$

At this writing (July 2013), not a single variety X with non-Jordan $\mathrm{Aut}(X)$ is known (to me).

Question 1 ([Po2010, Quest. 2.30 and 2.14])**.** Is there an irreducible variety X such that $\mathrm{Aut}(X)$ is non-Jordan? Is there an irreducible affine variety X with this property?

Remark 1. This may be compared with the question originally posed by É. Ghys, see [Fi2011, Quest. 13.1]. Using Definition 1, it is reformulated as follows: Is the diffemorphism group of any compact smooth manifold Jordan?

On the other hand, in many cases it can be proven that $\mathrm{Aut}(X)$ is Jordan. Below are described several extensive classes of X with this property.

2.2.1. *Toral varieties.* First, consider the wide class of affine varieties singled out by the following

Definition 3 ([Po2010, Def. 1.13])**.** A variety is called *toral* if it is isomorphic to a closed subvariety of some $\mathbf{A}^n \setminus \bigcup_{i=1}^n H_i$, where H_i is the set of zeros of the ith standard coordinate function x_i on $\mathbf{A}^n$.

Remark 2. $\mathbf{A}^n \setminus \bigcup_{i=1}^n H_i$ is the group variety of the n-dimensional affine torus; whence the terminology. Warning: "toral" does not imply "affine toric" in the sense of [Fu1993].

The class of toral varieties is closed with respect to taking products and closed subvarieties.

Lemma 2 ([Po2010, Lemma 1.14(a)])**.** *The following properties of an affine variety X are equivalent:*

(i) *X is toral;*
(ii) *$k[X]$ is generated by $k[X]^*$, the group of units of $k[X]$.*

Proof. If X is closed in $\mathbf{A}^n \setminus \bigcup_{i=1}^n H_i$, then the restriction of functions is an epimorphism $k\big[\mathbf{A}^n \setminus \bigcup_{i=1}^n H_i\big] \to k[X]$. Since $k\big[\mathbf{A}^n \setminus \bigcup_{i=1}^n H_i\big] = k[x_1, \ldots, x_n, 1/x_1, \ldots 1/x_n]$, this proves (i) $\Rightarrow$ (ii).

Conversely, assume that (ii) holds and let

$$k[X] = k[f_1, \ldots, f_n] \tag{13}$$

for some $f_1, \ldots, f_n \in k[X]^*$. Since X is affine, (13) implies that $\iota\colon X \to \mathbf{A}^n$, $x \mapsto (f_1(x), \ldots, f_n(x))$, is a closed embedding. The standard coordinate functions on $\mathbf{A}^n$ do not vanish on $\iota(X)$ since every f_i does not vanish on X. Hence $\iota(X) \subseteq \mathbf{A}^n \setminus \bigcup_{i=1}^n H_i$. This proves (ii)$\Rightarrow$(i). $\square$

Lemma 3. *Any quasiprojective variety X endowed with a finite automorphism group G is covered by G-stable toral open subsets.*

Proof. First, any point $x \in X$ is contained in a G-stable affine open subset of X. Indeed, since the orbit $G \cdot x$ is finite and X is quasiprojective, there is an affine open subset U of X containing $G \cdot x$. Hence $V := \bigcap_{g\in G} g \cdot U$ is a G-stable open subset containing x, and, since every $g \cdot U$ is affine, V is affine as well, see, e.g., [Sp1998, Prop. 1.6.12(i)].

Thus, the problem is reduced to the case where X is affine. Assume then that X is affine, and let $k[X] = k[h_1, \ldots, h_s]$. Replacing h_i by $h_i + \alpha_i$ for an appropriate $\alpha_i \in k$, we may (and shall) assume that every h_i vanishes nowhere on the $G \cdot x$. Expanding the set $\{h_1, \ldots, h_s\}$ by including $g \cdot h_i$ for every i and $g \in G$, we may (and shall) assume that $\{h_1, \ldots, h_s\}$ is G-stable. Then $h := h_1 \cdots h_s \in k[X]^G$. Hence the affine open set $X_h := \{z \in X \mid h(z) \neq 0\}$ is G-stable and contains $G \cdot x$. Since $k[X_h] = k[h_1, \ldots, h_s, 1/h]$ and $h_1, \ldots, h_s, 1/h \in k[X_h]^*$, the variety X_h is toral by Lemma 2. $\square$

Remark 3. Lemma 3 and its proof remain true for any variety X such that every G-orbit is contained in an affine open subset; whence the following

Corollary 3. *Every variety is covered by open toral subsets.*

For irreducible toral varieties the following was proved in [Po2010, Thm. 2.16].

Theorem 5. *The automorphism group of every toral variety is Jordan.*

Proof. By Theorem 1 it suffices to prove this for irreducible toral varieties.

By [Ro1957], for any irreducible variety X,

$$\Gamma := k[X]^*/k^*$$

is a free abelian group of finite rank. Let X be toral and let H be the kernel of the natural action of $\mathrm{Aut}(X)$ on Γ. We claim that H is abelian. Indeed, for every function $f \in k[X]^*$, the line in $k[X]$ spanned over k by f is H-stable. Since $\mathbf{GL}_1$ is abelian, this yields that

$$h_1h_2\cdot f = h_2h_1\cdot f \quad \text{for any elements} \quad h_1, h_2 \in H. \tag{14}$$

As X is toral, $k[X]^*$ generates the k-algebra $k[X]$ by Lemma 2. Hence (14) holds for every $f \in k[X]$. Since X is affine, the automorphisms of X coincide if and only if they induce the same automorphisms of $k[X]$. Therefore, H is abelian, as claimed.

Let n be the rank of Γ. Then $\mathrm{Aut}(\Gamma)$ is isomorphic to $\mathbf{GL}_n(\mathbf{Z})$. By the definition of H, the natural action of $\mathrm{Aut}(X)$ on Γ induces an embedding of $\mathrm{Aut}(X)/H$ into $\mathrm{Aut}(\Gamma)$. Hence $\mathrm{Aut}(X)/H$ is isomorphic to a subgroup of $\mathbf{GL}_n(\mathbf{Z})$ and therefore is bounded by Example 6(2). Thus, $\mathrm{Aut}(X)$ is an extension of a bounded group by an abelian group, hence Jordan by Theorem 4. This completes the proof. □

Remark 4. Maintain the notation of the proof of Theorem 5 and assume that X is irreducible. Let $f_1, \ldots f_n$ be a basis of Γ. There are the homomorphisms $\lambda_i \colon H \to k^*$, $i = 1, \ldots, n$, such that $g \cdot f_i = \lambda(g) f_i$ for every $g \in H$ and i. Since $k[X]^*$ generates $k[X]$, the diagonal map $H \to (k^*)^n$, $h \mapsto (\lambda_1(g), \ldots, \lambda_n(g))$, is injective. This and the proof of Theorem 5 show that for any irreducible toral variety X with $\mathrm{rk}\, k[X]^*/k^* = n$, there is an exact sequence

$$\{1\} \to D \to \mathrm{Aut}(X) \to B \to \{1\},$$

where D is a subgroup of the torus $(k^*)^n$ and B is a subgroup of $\mathbf{GL}_n(\mathbf{Z})$.

Combining Theorem 5 with Corollary of Lemma 3, we get the following:

Theorem 6. *Any point of any variety has an open neighborhood U such that* $\mathrm{Aut}(U)$ *is Jordan.*

2.2.2. *Affine spaces.* Next, consider the fundamental objects of algebraic geometry, the affine spaces $\mathbf{A}^n$. The group $\mathrm{Aut}(\mathbf{A}^n)$ is the "*affine Cremona group of rank n*".

Since $\mathrm{Aut}(\mathbf{A}^1)$ is the affine algebraic group Aff_1, it is Jordan by Theorem 2.

Since $\mathrm{Aut}(\mathbf{A}^2)$ is the subgroup of Cr_2, it is Jordan by Example 1. Another proof: By [Ig1977] every finite subgroup of $\mathrm{Aut}(\mathbf{A}^2)$ is conjugate to a subgroup of $\mathbf{GL}_2(k)$, so the claim follows from Theorem 1.

The group $\mathrm{Aut}(\mathbf{A}^3)$ is Jordan being the subgroup of Cr_3 that is Jordan by Corollary 13 below.

At this writing (July 2013) is unknown whether $\mathrm{Aut}(\mathbf{A}^n)$ is Jordan for $n \geqslant 4$ or not. By Theorem 14, if the so-called BAB Conjecture (see Subsection 2.3.5 below) holds true in dimension n, then Cr_n is Jordan, hence $\mathrm{Aut}(\mathbf{A}^n)$ is Jordan as well.

2.2.3. *Fixed points and Jordaness.* The following method of proving Jordaness of $\mathrm{Aut}(X)$ was suggested in [Po2010, Sect. 2] and yields extensive classes of X with Jordan $\mathrm{Aut}(X)$. It is based on the use of the following known fact:

Lemma 4. *Let X be an irreducible variety, let G be a finite subgroup of* $\mathrm{Aut}(X)$*, and let $x \in X$ be a fixed point of G. Then the natural action of G on $\mathrm{T}_{x,X}$, the tangent space of X at x, is faithful.*

Proof. Let $\mathfrak{m}_{x,X}$ be the maximal ideal of $\mathcal{O}_{x,X}$, the local ring of X at x. Being finite, G is reductive. Since $\operatorname{char} k = 0$, this implies that $\mathfrak{m}_{x,X} = L \oplus \mathfrak{m}_{x,X}^2$ for some submodule L of the G-module $\mathfrak{m}_{x,X}$. Let K be the kernel of the action of G on L and let L^d be the k-linear span in $\mathfrak{m}_{x,X}$ of the dth powers of all the elements of L. By the Nakayama's Lemma, the restriction to L^d of the natural projection $\mathfrak{m}_{x,X} \to \mathfrak{m}_{x,X}/\mathfrak{m}_{x,X}^{d+1}$ is surjective. Hence K acts trivially on $\mathfrak{m}_{x,X}/\mathfrak{m}_{x,X}^{d+1}$ for every d.

Take an element $f \in \mathfrak{m}_{x,X}$. Since G is finite, the k-linear span $\langle K \cdot f\rangle$ of the K-orbit of f in $\mathfrak{m}_{x,X}$ is finite-dimensional. This and $\bigcap_s \mathfrak{m}_{x,X}^s = \{0\}$ (see, e.g., [AM1969, Cor. 10.18]) implies that $\langle K \cdot f\rangle \cap \mathfrak{m}_{x,X}^{d+1} = \{0\}$ for some d. Since $f - g\cdot f \in \mathfrak{m}_{x,X}^{d+1}$ for every element $g \in K$, we conclude that $f = g \cdot f$, i.e., f is K-invariant. Thus, K acts trivially on $\mathfrak{m}_{x,X}$, hence on $\mathcal{O}_{x,X}$ as well. Since $k(X)$ is the field of fractions of $\mathcal{O}_{x,X}$, K acts trivially on $k(X)$, and therefore, on X. But K acts on X faithfully because $K \subseteq \mathrm{Aut}(X)$. This proves that K is trivial. Since L is the dual of the G-module $\mathrm{T}_{x,X}$, this completes the proof. ☐

The idea of the method is to use the fact that if a finite subgroup G of $\mathrm{Aut}(X)$ has a fixed point $x \in X$, then, by Lemma 4 and Theorem 1, there is a normal abelian subgroup of G whose index in G is at most $J_{\mathbf{GL}_n(k)}$ for $n = \dim \mathrm{T}_{x,X}$.

This yields the following:

Theorem 7. *Let X be an irreducible variety and let G be a finite subgroup of* $\mathrm{Aut}(X)$*. If G has a fixed point in X, then there is a normal abelian subgroup of G whose index in G is at most $J_{\mathbf{GL}_m(k)}$, where*

$$m = \max_x \dim \mathrm{T}_{x,X}. \tag{15}$$

Corollary 4. *If every finite automorphism group of an irreducible variety X has a fixed point in X, then* $\mathrm{Aut}(X)$ *is Jordan and*

$$J_{\mathrm{Aut}(X)} \leqslant J_{\mathbf{GL}_m(k)},$$

where m is defined by (15)*.*

Corollary 5. *Let p be a prime number. Then every finite p-subgroup G of* $\mathrm{Aut}(\mathbf{A}^n)$ *contains an abelian normal subgroup whose index in G is at most* $J_{\mathbf{GL}_n(k)}$.

Proof. This follows from Theorem 7 since in this case $(\mathbf{A}^n)^G \neq \varnothing$, see [Se2009$_2$, Thm. 1.2]. $\square$

Remark 5. At this writing (July 2013), it is unknown whether or not $(\mathbf{A}^n)^G \neq \varnothing$ for every finite subgroup G of $\mathrm{Aut}(\mathbf{A}^n)$. By Theorem 7 the affirmative answer would imply that $\mathrm{Aut}(\mathbf{A}^n)$ is Jordan (cf. Subsection (2.2.2)).

Remark 6. The statement of Corollary 5 remains true if $\mathbf{A}^n$ is replaced by any p-acyclic variety X, and n in $J_{\mathbf{GL}_n(k)}$ is replaced by m (see (15)). This is because in this case $X^G \neq \varnothing$ for every finite p-subgroup G of $\mathrm{Aut}(X)$, see [Se2009$_2$, Sect. 7–8].

The following applications are obtained by combining the above idea with Theorem 4.

Theorem 8. *Let X be an irreducible variety. Consider an* $\mathrm{Aut}(X)$*-stable equivalence relation $\sim$ on the set its points. If there is a finite equivalence class C of $\sim$, then* $\mathrm{Aut}(X)$ *is Jordan and*

$$J_{\mathrm{Aut}(X)} \leqslant |C|!J^{|C|!}_{\mathbf{GL}_m(k)},$$

where m is defined by (15).

Proof. By the assumption, every equivalence class of $\sim$ is $\mathrm{Aut}(X)$-stable. The kernel K of the action of $\mathrm{Aut}(X)$ on C is a normal subgroup of $\mathrm{Aut}(X)$ and, since the elements of $\mathrm{Aut}(X)$ induce permutations of C,

$$[\mathrm{Aut}(X){:}K] \leqslant |C|!. \tag{16}$$

By Theorem 4, Jordaness of $\mathrm{Aut}(X)$ follows from that of K. To prove that the latter holds, take a point of $x \in C$. Since x is fixed by every finite subgroup of K, Theorem 7 implies that K is Jordan and $J_K \leqslant J_{\mathbf{GL}_m(k)}$. By Theorem 4, this and (16) imply the claim. $\square$

Example 11. Below are several examples of $\mathrm{Aut}(X)$-stable equivalence relations on an irreducible variety X:

(i) $x \sim y \Longleftrightarrow \mathcal{O}_{x,X}$ and $\mathcal{O}_{y,X}$ are k-isomorphic;
(ii) $x \sim y \Longleftrightarrow \dim \mathrm{T}_{x,X} = \dim \mathrm{T}_{y,X}$;
(iii) $x \sim y \Longleftrightarrow$ the tangent cones of X at x and y are isomorphic. $\square$

Corollary 6. *If an irreducible variety X has a point x such that the set*

$$\{y \in X \mid \mathcal{O}_{x,X} \text{ and } \mathcal{O}_{y,X} \text{ are } k\text{-isomorphic}\}$$

is finite, then $\mathrm{Aut}(X)$ *is Jordan.*

Call a point $x \in X$ a *vertex* of X if

$$\dim \mathrm{T}_{x,X} \geqslant \dim \mathrm{T}_{y,X} \ \text{ for every point } \ y \in X.$$

Thus every point of X is a vertex of X if and only if X is smooth.

Corollary 7. *The automorphism group of every irreducible variety with only finitely many vertices is Jordan.*

Corollary 8. *The automorphism group of every nonsmooth irreducible variety with only finitely many singular points is Jordan.*

Corollary 9. *Let $X \subset \mathbf{A}^n$ be the affine cone of a smooth closed proper irreducible subvariety Z of $\mathbf{P}^{n-1}$ that does not lie in any hyperplane. Then* $\mathrm{Aut}(X)$ *is Jordan.*

Proof. The assumptions imply that the singular locus of X consists of a single point, the origin; whence the claim by Corollary 8. □

Corollary 10. *If an irreducible variety X has a point x such that there are only finitely many points $y \in X$ for which the tangent cones of X at x and at y are isomorphic, then* $\mathrm{Aut}(X)$ *is Jordan.*

Remark 7. Smoothness in Corollary 9 may be replaced by the assumption that Z is not a cone. Indeed, in this case the origin constitutes a single equivalence class of equivalence relation (iii) in Example 11; whence the claim by Corollary 10.

2.2.4. *The Koras–Russell threefolds.* Let $X = X_{d,s,l}$ be the so-called Koras–Russell threefold of the first kind [M-J2011], i.e., the smooth hypersurface in $\mathbf{A}^4$ defined by the equation

$$x_1^d x_2 + x_3^s + x_4^l + x_1 = 0,$$

where $d \geqslant 2$ and $2 \leqslant s \leqslant l$ with s and l relatively prime; the case $d = s = 2$ and $l = 3$ is the famous Koras–Russell cubic. According to [M-J2011, Cor. 6.1], every element of $\mathrm{Aut}(X)$ fixes the origin $(0, 0, 0, 0) \in X$. By Corollary 4 and item (iii) of Subsection 1.2.1 this implies that $\mathrm{Aut}(X)$ is Jordan and

$$J_{\mathrm{Aut}(X)} \leqslant 360.$$

Actually, during the conference I learned from L. Moser-Jauslin that X contains a line ℓ passing through the origin, stable with respect to $\mathrm{Aut}(X)$, and such that every element of $\mathrm{Aut}(X)$ fixing ℓ pointwise has infinite order. This implies that every finite subgroup of $\mathrm{Aut}(X)$ is cyclic and hence

$$J_{\mathrm{Aut}(X)} = 1.$$

2.2.5. *Small dimensions.* Since $\mathrm{Aut}(X)$ is a subgroup of $\mathrm{Bir}(X)$, Jordaness of $\mathrm{Bir}(X)$ implies that of $\mathrm{Aut}(X)$. This and Theorem 12 below yield the following

Theorem 9. *Let X be an irreducible variety of dimension $\leqslant 2$ not birationally isomorphic to $\mathbf{P}^1 \times E$, where E is an elliptic curve. Then* $\mathrm{Aut}(X)$ *is Jordan.*

Note that if E is an elliptic curve and $X = \mathbf{P}^1 \times E$, then $\mathrm{Aut}(X) = \mathbf{PGL}_2(k) \times \mathrm{Aut}(E)$, see [Ma1971, pp. 98–99]. Fixing a point of E, endow E with a structure of abelian variety E_{ab}. Since $\mathrm{Aut}(E)$ is an extension of the finite group $\mathrm{Aut}(E_{\mathrm{ab}})$ by the abelian group E_{ab}, Theorems 2, 4, and 3(2) imply that $\mathrm{Aut}(\mathbf{P}^1 \times E)$ is Jordan.

Note also that all irreducible curves (not necessarily smooth and projective) whose automorphism group is infinite are classified in [Po1978].

2.2.6. *Non-uniruled varieties.* Again, using that Jordaness of $\mathrm{Bir}(X)$ implies that of $\mathrm{Aut}(X)$, we deduce from recent Theorem 15(i)(a) below the following

Theorem 10. $\mathrm{Aut}(X)$ *is Jordan for any irreducible non-uniruled variety* X*.*

2.3. **Groups $\mathbf{Bir}(X)$.** Now we shall consider Problem B (see Subsection 2.1). Exploring $\mathrm{Bir}(X)$, one may, maintaining this group, replace X by any variety birationally isomorphic to X. Note that by Theorem 6 one can always attain that after such a replacement $\mathrm{Aut}(X)$ becomes Jordan.

The counterpart of Question 1 is

Question 2 ([Po2010, Quest. 2.31])**.** Is there an irreducible variety X such that $\mathrm{Bir}(X)$ is non-Jordan?

In contrast to the case of Question 1, at present we know the answer to Question 2: motivated by my question, Yu. Zarhin proved in [Za2010] the following

Theorem 11 ([Za2010, Cor. 1.3])**.** *Let X be an abelian variety of positive dimension and let Z be a rational variety of positive dimension. Then* $\mathrm{Bir}(X \times Z)$ *is non-Jordan.*

Sketch of proof. By Theorem 3(1)(i), it suffices to prove that $\mathrm{Bir}(X \times \mathbf{A}^1)$ is non-Jordan. Consider an ample divisor D on X and the sheaf $L := \mathcal{O}_X(D)$. For a positive integer n, consider the following group $\Theta(L^n)$. Its elements are all pairs $(x, [f])$ where $x \in X$ is such that $L^n \cong T_x^*(L^n)$ for the translation $T_x\colon X \to X$, $z \mapsto z + x$, and $[f]$ is the automorphism of the additive group of $k(X)$ induced by the multiplication by $f \in k(X)^*$. The group structure of $\Theta(L^n)$ is defined by $(x, [f])(y, [h]) = (x + y, [T_x^* h \cdot f])$. One proves that $\Theta(L^n)$ enjoys the properties: (i) $\varphi\colon \Theta(L^n) \to \mathrm{Bir}(X \times \mathbf{A}^1)$, $\varphi(x, [f])(y, t) = (x+y, f(y)t)$, is a group embedding; (ii) $\Theta(L^n)$ is isomorphic to a group G_K from Example 5 with $|K| \geqslant n$. This implies the claim (see Example 5). $\square$

Below Problem B is solved for varieties of small dimensions ($\leqslant 2$).

2.3.1. *Curves.* If X is a curve, then the answer to Question 2 is negative.

Proving this, we may assume that X is smooth and projective; whence $\mathrm{Bir}(X) = \mathrm{Aut}(X)$.

If $g(X)$, the genus of X, is 0, then $X = \mathbf{P}^1$, so $\mathrm{Aut}(X) = \mathbf{PGL}_2(k)$. Hence $\mathrm{Aut}(X)$ is Jordan by Theorem 2.

If $g(X) = 1$, then X is an elliptic curve, hence $\mathrm{Aut}(X)$ is Jordan (see the penultimate paragraph in Subsection 2.2.5).

If $g(X) \geqslant 2$, then, being finite, $\mathrm{Aut}(X)$ is Jordan.

2.3.2. *Surfaces.* Answering Question 2 for surfaces X, we may assume that X is a smooth projective minimal model.

If X is of general type, then by Matsumura's theorem $\mathrm{Bir}(X)$ is finite, hence Jordan.

If X is rational, then $\mathrm{Bir}(X)$ is Cr_2, hence Jordan, see Example 1.

If X is a nonrational ruled surface, it is birationally isomorphic to $\mathbf{P}^1 \times B$ where B is a smooth projective curve such that $g(B) > 0$; we may then take $X = \mathbf{P}^1 \times B$. Since $g(B) > 0$, there are no dominant rational maps $\mathbf{P}^1 \dashrightarrow B$; whence the elements of $\mathrm{Bir}(X)$ permute fibers of the natural projection $\mathbf{P}^1 \times B \to B$. The set of elements inducing trivial permutation is a normal subgroup $\mathrm{Bir}_B(X)$ of $\mathrm{Bir}(X)$. The definition implies that $\mathrm{Bir}_B(X) = \mathbf{PGL}_2(k(B))$, hence $\mathrm{Bir}_B(X)$ is Jordan by Theorem 2. Identifying $\mathrm{Aut}(B)$ with the subgroup of $\mathrm{Bir}(X)$ in the natural way, we get the decomposition

$$\mathrm{Bir}(X) = \mathrm{Bir}_B(X) \rtimes \mathrm{Aut}(B). \tag{17}$$

If $g(B) \geqslant 2$, then $\mathrm{Aut}(B)$ is finite; whence $\mathrm{Bir}(X)$ is Jordan by virtue of (17) and Theorem 4. If $g(B)=1$, then $\mathrm{Bir}(X)$ is non-Jordan by Theorem 11.

The canonical class of all the other surfaces X is numerically effective, so, for them, $\mathrm{Bir}(X) = \mathrm{Aut}(X)$, cf. [IS1996, Sect. 7.1, Thm. 1 and Sect. 7.3, Thm. 2].

Let X be such a surface. The group $\mathrm{Aut}(X)$ has a structure of a locally algebraic group with finite or countably many components, see [Ma1958], i.e., there is a normal subgroup $\mathrm{Aut}(X)^0$ in $\mathrm{Aut}(X)$ such that

(i) $\mathrm{Aut}(X)^0$ is a connected algebraic group,
(ii) $\mathrm{Aut}(X)/\mathrm{Aut}(X)^0$ is either a finite or a countable group,

By (i) and the structure theorem on algebraic groups [Ba1955], [Ro1956] there is a normal connected affine algebraic subgroup L of $\mathrm{Aut}(X)^0$ such that $\mathrm{Aut}(X)^0/L$ is an abelian variety. By [Ma1963, Cor. 1] nontriviality of L would imply that X is ruled. Since we assumed that X is not ruled, this means that $\mathrm{Aut}(X)^0$ is an abelian variety. Hence $\mathrm{Aut}(X)^0$ is abelian and, a fortiori, Jordan.

By (i) the group $\mathrm{Aut}(X)^0$ is contained in the kernel of the natural action of $\mathrm{Aut}(X)$ on $H^2(X, \mathbf{Q})$ (we may assume that $k = \mathbf{C}$). Therefore, this action defines a homomorphism $\mathrm{Aut}(X)/\mathrm{Aut}(X)^0 \to \mathbf{GL}(H^2(X, \mathbf{Q}))$. The kernel of this homomorphism is finite by [Do1986, Prop. 1], and the image is bounded by Example 8. By Examples 6, 7 this yields that $\mathrm{Aut}(X)/\mathrm{Aut}(X)^0$ is bounded. In turn, since $\mathrm{Aut}(X)^0$ is Jordan, by Theorem 4 this implies that $\mathrm{Aut}(X)$ is Jordan.

2.3.3. *The upshot.* The upshot of the last two subsections is

Theorem 12 ([Po2010, Thm. 2.32])**.** *Let X be an irreducible variety of dimension $\leqslant 2$. Then the following two properties are equivalent*:

(a) *the group* $\mathrm{Bir}(X)$ *is Jordan;*
(b) *the variety X is not birationally isomorphic to $\mathbf{P}^1 \times B$, where B is an elliptic curve.*

2.3.4. *Finite and connected algebraic subgroups of* $\mathrm{Bir}(X)$ *and* $\mathrm{Aut}(X)$. Recall that the notions of algebraic subgroup of $\mathrm{Bir}(X)$ and $\mathrm{Aut}(X)$ make sense, and every algebraic subgroup of $\mathrm{Aut}(X)$ is that of $\mathrm{Bir}(X)$, see, e.g., [Po2011, Sect. 1]. The following reveals a relation between embeddability of finite subgroups of $\mathrm{Bir}(X)$ in connected affine algebraic subgroups of $\mathrm{Bir}(X)$ and Jordaness of $\mathrm{Bir}(X)$ (and the same holds for $\mathrm{Aut}(X)$).

For every integer $n > 0$, consider the set of all isomorphism classes of connected reductive algebraic groups of rank $\leqslant n$, and fix a group in every class. The obtained set of groups $\mathcal{R}_n$ is finite. Therefore,

$$J_{\leqslant n} := \sup_{G \in \mathcal{R}_n} J_G \tag{18}$$

is a positive integer.

Theorem 13. *Let X be an irreducible variety of dimension n. Then every finite subgroup G of every connected affine algebraic subgroup of* $\mathrm{Bir}(X)$ *has a normal abelian subgroup whose index in G is at most $J_{\leqslant n}$.*

Proof. Let L be a connected affine algebraic subgroup of $\mathrm{Bir}(X)$ containing G. Being finite, G is reductive. Let R be a maximal reductive subgroup of L containing G. Then L is a semidirect product of R and the unipotent radical of L, see [Mo1956, Thm. 7.1]. Therefore, R is connected because L is. Faithfulness of the action R acts on X yields that $\mathrm{rk}\, R \leqslant \dim X$, see, e.g., [Po2011, Lemma 2.4]. The claim then follows from (18), Theorem 2, and Definition 1. □

Theorem 13 and Definition 1 imply

Corollary 11. *Let X be an irreducible variety of dimension n such that* $\mathrm{Bir}(X)$ (*resp.* $\mathrm{Aut}(X)$) *is non-Jordan. Then for every integer $d > J_{\leqslant n}$, there is a finite subgroup G of* $\mathrm{Bir}(X)$ (*resp.* $\mathrm{Aut}(X)$) *with the properties:*

(i) *G does not lie in any connected affine algebraic subgroup of* $\mathrm{Bir}(X)$ (*resp.* $\mathrm{Aut}(X)$)*;*
(ii) *G contains an abelian normal subgroup whose index in G is $\geqslant d$.*

Corollary 12. *If* Cr_n (*resp.* $\mathrm{Aut}(\mathbf{A}^n)$) *is non-Jordan, then for every integer $d > J_{\leqslant n}$, there is a finite subgroup G of* Cr_n (*resp.* $\mathrm{Aut}(\mathbf{A}^n)$) *with the properties:*

(i) *the action of G on $\mathbf{A}^n$ is nonlinearizable;*
(ii) *G contains an abelian normal subgroup whose index in G is $\geqslant d$.*

Proof. This follows from Corollary 11 because $\mathbf{GL}_n(k)$ is a connected affine algebraic subgroup of $\mathrm{Aut}(\mathbf{A}^n)$ and nonlinearizability of the action of G on $\mathbf{A}^n$ means that G is not contained in a subgroup of Cr_n (resp. $\mathrm{Aut}(\mathbf{A}^n)$) conjugate to $\mathbf{GL}_n(k)$. □

2.3.5. *Recent developments.* The initiated in [Po2010] line of research of Jordaness of $\mathrm{Aut}(X)$ and $\mathrm{Bir}(X)$ for algebraic varieties X has generated interest of algebraic geometers in Moscow among whom I have promoted it, and led to a further progress in Problem B (hence A as well) in papers [Za2010], [PS2013$_1$], [PS2013$_2$]. In [Za2010], the earliest of them, the examples of non-Jordan groups $\mathrm{Bir}(X)$ only known to date (July 2013) have been constructed (see Theorem 11 above). Below are formulated the results obtained in [PS2013$_1$], [PS2013$_2$]. Some of them are conditional, valid under the assumption that the following general conjecture by A. Borisov, V. Alexeev, and L. Borisov holds true:

BAB Conjecture. All Fano varieties of a fixed dimension n and with terminal singularities are contained in a finite number of algebraic families.

Theorem 14 ([PS2013$_1$, Thm. 1.8]). *If the BAB Conjecture holds true in dimension n, then, for every rationally connected n-dimensional variety X, the group $\mathrm{Bir}(X)$ is Jordan and, moreover, $J_{\mathrm{Bir}(X)} \leqslant u_n$ for a number u_n depending only on n.*

Since for $n = 3$ the BAB Conjecture is proved [KMMT2000], this yields

Corollary 13 ([PS2013$_1$, Cor. 1.9]). *The space Cremona group Cr_3 is Jordan.*

Proposition 1 ([PS2013$_1$, Prop. 1.11]). $u_3 \leqslant (25920 \cdot 20736)^{20736}$.

The pivotal idea of the proof of Theorem 14 is to use a technically refined form of the "fixed-point method" described in Subsection 2.2.3.

Theorem 15 ([PS2013$_2$, Thm. 1.8]). *Let X be an irreducible n-dimensional variety.*

(i) *The group $\mathrm{Bir}(X)$ is Jordan in either of the cases:*
   (a) *X is non-uniruled;*
   (b) *the BAB Conjecture holds true in dimension n and the irregularity of X is 0.*

(ii) *If X is non-uniruled and its irregularity is 0, then the group $\mathrm{Bir}(X)$ is bounded (see Definition 2).*

3. Appendix: Problems

Below I add a few additional problems to those which have already been formulated above (Problems A and B in Subsection 2.1, and Questions 1, 2).

3.1. $\mathbf{Cr}_n$-conjugacy of finite subgroups of $\mathbf{GL}_n(k)$.

Below $\mathbf{GL}_n(k)$ is identified in the standard way with the subgroup of Cr_n, which, in turn, is identified with the subgroup of Cr_m for any $m = n+1, n+2, \ldots, \infty$ (cf. [Po2011, Sect. 1] or [Po2012$_1$, Sect. 1]).

Question 3. Consider the following properties of two finite subgroups A and B of $\mathbf{GL}_n(k)$:

(i) A and B are isomorphic,

(ii) A and B are conjugate in Cr_n.

Does (i) imply (ii)?

Comments.

1. Direct verification based on the classification in [DI2009] shows that the answer is affirmative for $n \leqslant 2$.

2. By [Po2012$_1$, Cor. 5], if A and B are abelian, then the answer is affirmative for every n.

3. If A and B are isomorphic, then they are conjugate in Cr_{2n}. This is the corollary of the following stronger statement:

Proposition 2. *For any finite group G and any injective homomorphisms*

$$\begin{tikzcd} G \arrow[r, bend left, "\alpha_1"] \arrow[r, bend right, "\alpha_2"'] & \mathbf{GL}_n(k), \end{tikzcd} \tag{19}$$

there exists an element $\varphi \in \mathrm{Cr}_{2n}$ such that $\alpha_1 = \mathrm{Int}(\varphi) \circ \alpha_2$.

Proof. Every element $g \in \mathbf{GL}_n(k)$ is a linear transformation $x \mapsto g \cdot x$ of $\mathbf{A}^n$ (with respect to the standard structure of k-linear space on $\mathbf{A}^n$). The injections α_1 and α_2 determine two faithful linear actions of G on $\mathbf{A}^n$: the ith action ($i = 1, 2$) maps $(g, x) \in G \times \mathbf{A}^n$ to $\alpha_i(g) \cdot x$. Consider the product of these actions, i.e., the action of G on $\mathbf{A}^n \times \mathbf{A}^n$ defined by

$$G \times \mathbf{A}^n \times \mathbf{A}^n \to \mathbf{A}^n \times \mathbf{A}^n, \quad (g, x, y) \mapsto (\alpha_1(g) \cdot x, \alpha_2(g) \cdot y). \tag{20}$$

The natural projection of $\mathbf{A}^n \times \mathbf{A}^n \to \mathbf{A}^n$ to the ith factor is G-equivariant. By classical Speiser's Lemma (see [LPR2006, Lemma 2.12] and references therein), this implies that $\mathbf{A}^n \times \mathbf{A}^n$ endowed with G-action (20) is G-equivariantly birationally isomorphic to $\mathbf{A}^n \times \mathbf{A}^n$ endowed with the G-action via the ith factor by means of α_i. Therefore, $\mathbf{A}^n \times \mathbf{A}^n$ endowed with the G-action via the first factor by means of α_1 is G-equivariantly birationally isomorphic to $\mathbf{A}^n \times \mathbf{A}^n$ endowed with the G-action via the second factor by means of α_2; whence the claim. □

Remark 8. In general, it is impossible to replace Cr_{2n} by Cr_n in Proposition 2. Indeed, in [RY2002] one finds the examples of finite abelian groups G and embeddings (19) such that $\alpha_1 \notin \mathrm{Int}(\mathrm{Cr}_n) \circ \alpha_2$. However, since the images of these embeddings are isomorphic finite abelian subgroups of $\mathbf{GL}_n(k)$, by [Po2012$_1$, Cor. 5] they are conjugate in Cr_n.

3.2. **Torsion primes.** Let X be an irreducible variety. The following definition is based on the fact that the notion of algebraic torus in $\mathrm{Bir}(X)$ makes sense.

Definition 4 ([Po2012$_1$, Sect. 8])**.** Let G be a subgroup of $\mathrm{Bir}(X)$. A prime integer p is called a *torsion number group* of G if there exists a finite abelian p-subgroup of G that does not lie in any torus of G.

Let $\mathrm{Tors}(G)$ be the set of all torsion primes of G. If G is a connected reductive algebraic subgroup of $\mathrm{Bir}(X)$, this set coincides with that of the torsion primes of algebraic group G in the sense of classical definition, cf., e.g., [Se2000, 1.3].

Question 4 ([Po2012$_1$, Quest. 3])**.** What are, explicitly,

$$\mathrm{Tors}(\mathrm{Cr}_n), \ \ \mathrm{Tors}(\mathrm{Aut}\, \mathbf{A}^n), \ \ \mathrm{Tors}(\mathrm{Aut}^* \mathbf{A}^n), \quad n = 3, 4, \ldots, \infty$$

where $\mathrm{Aut}^* \mathbf{A}^n := \{g \in \mathrm{Aut}\, \mathbf{A}^n \mid \mathrm{Jac}\,(g) = 1\}$?

Comments. By [Po2012$_1$, Sect. 8],

$$
\begin{aligned}
\mathrm{Tors}(\mathrm{Cr}_1) &= \{2\},\\
\mathrm{Tors}(\mathrm{Cr}_2) &= \{2,3,5\} \quad \text{(this coincides with } \mathrm{Tors}(E_8)),\\
\mathrm{Tors}(\mathrm{Cr}_n) &\supseteq \{2,3\} \quad \text{for any } n \geqslant 3,\\
\mathrm{Tors}(\mathrm{Aut}\,\mathbf{A}^n) = \mathrm{Tors}(\mathrm{Aut}^*\mathbf{A}^n) &= \varnothing \quad \text{for } n \leqslant 2.
\end{aligned}
$$

Question 5 ([Po2012$_1$, Quest. 4])**.** What is the minimal n such that 7 lies in one of the sets $\mathrm{Tors}(\mathrm{Cr}_n)$, $\mathrm{Tors}(\mathrm{Aut}\mathbf{A}^n)$, $\mathrm{Tors}(\mathrm{Aut}^*\mathbf{A}^n)$?

Question 6 ([Po2012$_1$, Quest. 5])**.** Are these sets finite?

Question 7. Are the sets

$$\textstyle\bigcup_{n\geqslant 1}\mathrm{Tors}(\mathrm{Cr}_n), \quad \bigcup_{n\geqslant 1}\mathrm{Tors}(\mathrm{Aut}\mathbf{A}^n), \quad \bigcup_{n\geqslant 1}\mathrm{Tors}(\mathrm{Aut}^*\mathbf{A}^n)$$

finite?

3.3. **Embeddability in Bir(X).** Not every group G is embeddable in $\mathrm{Bir}(X)$ for some X. For instance, by [Co2013$_2$, Thm. 1.2], if G is finitely generated, its embeddability in $\mathrm{Bir}(X)$ implies that G has a solvable word problem. Another example: by [Ca2012], $\mathbf{PGL}_\infty(k)$ is not embeddable in $\mathrm{Bir}(X)$ for $k = \mathbf{C}$ (I thank S. Cantat who informed me about these examples).

If $\mathrm{Bir}(X)$ is Jordan, then by Example 4 and Theorem 3(1)(i), $\mathcal{N}$ is not embeddable in $\mathrm{Bir}(X)$. Hence, by Theorem 15(i)(a), $\mathcal{N}$ is not embeddable in $\mathrm{Bir}(X)$ for any non-uniruled X.

Conjecture 1. The finitely generated group $\mathcal{N}$ defined in Example 4 is not embeddable in $\mathrm{Bir}(X)$ for every irreducible variety X.

3.4. **Contractions.** Developing the classical line of research, in recent years were growing activities aimed at description of finite subgroups of $\mathrm{Bir}(X)$ for various X; the case of rational X (i.e., that of the Cremona group $\mathrm{Bir}(X)$) was, probably, most actively explored with culmination in the classification of finite subgroups of Cr_2, [DI2009]. In these studies, all the classified groups appear in the corresponding lists on equal footing. However, in fact, some of them are "more basic" than the others because the latter may be obtained from the former by a certain standard construction. Given this, it is natural to pose the problem of describing these "basic" groups.

Namely, let X_1 and X_2 be the irreducible varieties and let $G_i \subset \mathrm{Bir}(X_i)$, $i = 1, 2$, be the subgroups isomorphic to a finite group G. Assume that fixing the isomorphisms $G \to G_i$, $i = 1, 2$, we obtain the rational actions of G on X_1 and X_2 such that there is a G-equivariant rational dominant map $\varphi\colon X_1 \dashrightarrow X_2$. Let $\pi_{X_i}\colon X_i \dashrightarrow X_i/G$, $i = 1, 2$ be the rational quotients (see, e.g. [Po2011, Sect. 1]) and let $\varphi_G\colon X_1/G \dashrightarrow X_2/G$ be the dominant rational map induced by φ. Then the following holds (see, e.g. [Re2000, Sect. 2.6]):

(1) The appearing commutative diagram

$$\begin{tikzcd} X_1 \arrow[r, dashed, "\varphi"] \arrow[d, dashed, "\pi_{X_1}"'] & X_2 \arrow[d, dashed, "\pi_{X_2}"] \\ X_1/G \arrow[r, dashed, "\varphi_G"] & X_2/G \end{tikzcd} \tag{21}$$

is, in fact, cartesian, i.e., $\pi_{X_1}\colon X_1 \dashrightarrow X_1/G$ is obtained from $\pi_{X_2}\colon X_2 \dashrightarrow X_2/G$ by the base change φ_G. In particular, X_1 is birationally G-isomorphic to $X_2 \times_{X_2/G} (X_1/G) := \{(x, y) \in X_2 \times (X_1/G) \mid \pi_{X_2}(x) = \varphi_G(y)\}$.

(2) For every irreducible variety Z and every dominant rational map $\beta\colon Z \to X_2/G$ such that $X_2 \times_{X_2/G} Z$ is irreducible, the variety $X_2 \times_{X_2/G} Z$ inherits via X_2 a faithful rational action of G such that one obtains commutative diagram (21) with $X_1 = X_2 \times_{X_2/G} Z$, $\varphi_G = \beta$, and $\varphi = \mathrm{pr}_1$.

If such a φ exists, we say that G_1 is *induced* from G_2 *by a base change*. The latter is called *trivial* if φ is a birational isomorphism. If a finite subgroup G of $\mathrm{Bir}(X)$ is not induced by a nontrivial base change, we say that G is *incompressible*.

Example 12. The standard embedding $\mathrm{Cr}_n \hookrightarrow \mathrm{Cr}_{n+1}$ permits to consider the finite subgroups of Cr_n as that of Cr_{n+1}. Every finite subgroup of Cr_{n+1} obtained this way is induced by the nontrivial base change determined by the projection $\mathbf{A}^{n+1} \to \mathbf{A}^n$, $(a_1, \ldots, a_n, a_{n+1}) \mapsto (a_1, \ldots, a_n, a_{n+1})$. $\square$

Example 13. This is Example 6 in [Re2004]. Let G be a finite group that does not embed in $\mathrm{Bir}(Z)$ for any curve Z of genus $\leqslant 1$ (for instance, $G = \mathrm{Sym}_5$) and let X be a smooth projective curve of minimal possible genus such that G is isomorphic to a subgroup of $\mathrm{Aut}(X)$. Then this subgroup of $\mathrm{Bir}(X)$ is incompressible.[1]

Example 14. Consider two rational actions of $G := \mathrm{Sym}_3 \times \mathbb{Z}/2\mathbb{Z}$ on $\mathbf{A}^3$. The subgroup Sym_3 acts by natural permuting the coordinates in both cases. The nontrivial element of $Z/2\mathbb{Z}$ acts by $(a_1, a_2, a_3) \mapsto (-a_1, -a_2, -a_3)$ in the first case and by $(a_1, a_2, a_3) \mapsto (a_1^{-1}, a_2^{-1}, a_3^{-1})$ in the second. The surfaces

$$\begin{aligned} P &:= \{(a_1, a_2, a_3) \in \mathbf{A}^3 \mid a_1 + a_2 + a_3 = 0\},\\ T &:= \{(a_1, a_2, a_3) \in \mathbf{A}^3 \mid a_1a_2a_3 = 1\} \end{aligned}$$

are G-stable in, resp., the first and the second case. Since P and T are rational, these actions of G on P and T determine, up to conjugacy, resp., the subgroups G_P and G_T of Cr_2, both isomorphic to G. By [Is2003] (see

[1] The proof in [Re2004] should be corrected as follows. Assume that there is a faithful action of G of a smooth projective curve Y and a dominant G-equivariant morphism $\varphi\colon X \to Y$ of degree $n > 1$. By the construction, X and Y have the same genus $g > 1$, and the Hurwitz formula yields that the number of branch points of φ (counted with positive multiplicities) is the integer $(n-1)(2-2g)$. But the latter is negative, — a contradiction.

also [LPR2006], [LPR2007]), these subgroups are not conjugate in Cr_2. However, by [LPR2007, Sect. 5], G_T is induced from G_P by a nontrivial base change (of degree 2). $\square$

In fact, Example 14 is a special case (related to the simple algebraic group $\mathbf{G}_2$) of the following

Example 15. Let G be a connected reductive algebraic group. Fix a maximal torus T of G and consider the natural actions of the Weyl group $W = N_G(T)/T$ on T and on $\mathfrak{t} := \mathrm{Lie}(T)$. Since these actions are faithful and T and $\mathfrak{t}$ are rational varieties, this determines, up to conjugacy, two embeddings of W in Cr_r, where $r = \dim T$. Let W_T and $W_{\mathfrak{t}}$ be the images of these embeddings. By [LPR2006, Lemma 3.5(a) and Sect. 1.5], if G is not Cayley and W has no outer autormorphisms, then W_T and $W_{\mathfrak{t}}$ are not conjugate in Cr_r. On the other hand, by [LPR2006, Lemma 10.3], W_T is induced from $W_{\mathfrak{t}}$ by a (nontrivial) base change (see also Lemma 5 below).

This yields, for arbitrary n, the examples of isomorphic nonconjugate finite subgroups of Cr_n one of which is induced from the other by a nontrivial base change. For instance, if $G = \mathbf{SL}_{n+1}$, then $r = n$ and $W = \mathrm{Sym}_n$. Since, by [LPR2006, Thm. 1.31], $\mathbf{SL}_{n+1}$ is not Cayley for $n \geqslant 3$ and Sym_n has no outer automorphisms for $n \neq 6$, the above construction yields for these n two nonconjugate subgroups of Cr_n isomorphic to Sym_n, one of which is induced from the other by a nontrivial base change. $\square$

The following gives a general way of constructing two finite subgroups of Cr_n one of which is induced from the other by a base change.

Consider an n-dimensional irreducible nonsingular variety X and a finite subgroup G of $\mathrm{Aut}(X)$. Suppose that $x \in X$ is a fixed point of G. By Lemma 4, the induced action of G on the tangent space of X at x is faithful. Therefore this action determines, up to conjugacy, a subgroup G_1 of Cr_n isomorphic to G. On the other hand, if X is rational, the action of G on X determines, up to conjugacy, another subgroup G_2 of Cr_n isomorphic to G.

Lemma 5. *G_2 is induced from G_1 by a base change.*

Proof. By Lemma 3 we may assume that X is affine, in which case the claim follows from [LPR2006, Lemma 10.3]. $\square$

Corollary 14. *Let X be a nonrational irreducible variety and let G be an incompressible finite subgroup of* $\mathrm{Aut}(X)$. *Then $X^G = \varnothing$.*

Question 8. Which finite subgroups of Cr_2 are incompressible?

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