Graded Rings and Graded Grothendieck Groups

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... inside of a book is there whether you read it or not. Even if nobody ever reads it, it's there, doing its work.

The Ground Beneath Her Feet
Salman Rushdie

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Introduction

A bird's eye view of the graded module theory over a graded ring gives an impression of the module theory with the added adjective "graded" to all its statements. Once the grading is considered to be trivial, the graded theory reduces to the usual module theory. So from this perspective, the graded module theory can be considered as an extension of the module theory. However, one aspect that could be easily missed from such a panoramic view is that, the graded module theory comes equipped with a shifting, thanks to being able to partition the structures and rearranging these partitions. This adds an extra layer of structure (and complexity) to the theory. An sparkling example of this is the theory of graded Grothendieck groups which is the main focus of this monograph.

The construction of the Grothendieck group of a ring is easy. Starting from a ring A with identity, the isomorphism classes of finitely generated projective (right) A-modules equipped with the direct sum forms an abelian monoid, denoted by $\mathcal{V}(A)$. The group completion of this monoid is denoted by $K_0(A)$ and is called the Grothendieck group of A. For a Γ-graded ring A, the graded isomorphism classes of graded finitely generated projective modules equipped with the direct sum forms an abelian monoid, denoted by $\mathcal{V}^{\mathrm{gr}}(A)$. The shifting of graded modules (see §1.2.1) induces a Γ-module structure on $\mathcal{V}^{\mathrm{gr}}(A)$. The group completion of this monoid is denoted by $K_0^{\mathrm{gr}}(A)$ and is called the graded Grothendieck group of A which has, consequently, a natural $\mathbb{Z}[\Gamma]$-module structure. As we will see throughout this note, this extra structure carries a substantial information about the graded ring A.

Starting from a graded ring A, the relation between the category of graded A-modules, Gr-A, the category of A-modules, Mod-A, and the category of modules over the ring of the zero homogeneous part, Mod-A_0, is one of the main forces to study the graded theory. In the same way, the relations between $K_0^{\mathrm{gr}}(A)$, $K_0(A)$ and $K_0(A_0)$ are of interest.

The motivation to write this note came from the recent activities which adopt the graded Grothendieck group as an invariant to classify the Leavitt path algebras [46, 47, 48, 76]. Surprisingly, not much is recorded about the graded version of the Grothendieck group in the literature, despite the fact that K_0 has been used in many occasions as a crucial invariant, and there is a substantial amount of information about the graded version of other invariants such as (co)homology groups, Brauer groups, etc. The other surge of interest on this group stems from the recent activities on (graded) representation theory of Hecke algebras. In particular for a quiver Hecke algebra, its graded Grothendieck group is closely related to its corresponding quantised enveloping algebra. For this line of research see the survey [56].

This note tries to fill this gap, by systematically developing the theory of graded Grothendieck groups. In order to do this, we have to carry over and work out the details of known results in non-graded case to the graded setting, and gather important results scattered in research papers on the graded theory. The emphasis is on using the graded Grothendieck groups as invariants for classifications.

The group K_0 has been successfully used in the theory of C^*-algebras to classify certain class of C^*-algebras. Building on the work of Brattelli, Elliott in [34] used the the pointed ordered K_0-groups (called dimension groups) as a complete invariant for AF C^*-algebras. Another cornerstone of using K-groups for the classifications of a wider range of C^*-algebras was the work of Kirchberg and Phillips [77], who showed that K_0 and K_1-groups together are complete invariant for a certain type of C^*-algebras. The Grothendieck group considered as a module induced by a group action was used by Handelman and Rossmann [43] to give a complete invariant for the class of direct limits of finite dimensional, representable dynamical systems. Krieger [57] introduced (past) dimension groups as a complete invariant for the shift equivalence of topological Markov chains (shift of finite types) in symbolic dynamics. Surprisingly as we will see, Krieger's groups can be naturally expressed by graded Grothendieck groups (§3.9).

For two reasons we develop the theory for rings graded by abelian groups rather than an arbitrary groups, although most of the results could be carried over to non-abelian graded rings. One reason is that, by using the abelian grading, the presentation and proofs are much more transparent. Furthermore, in most applications of graded K_0, the ring has an abelian grading (often times $\mathbb{Z}$-grading).

A brief outline. Let Γ be an (abelian) group and A be a Γ-graded ring. For any subgroup $\Omega \subseteq \Gamma$, one can consider the natural Γ/Ω-grading on A (see Example 1.1.8). In the same manner, any Γ-graded A-module can be considered as Γ/Ω-graded module. These induce functors (see §1.2.8)

$$U : \mathrm{Gr}^{\Gamma}\text{-}A \longrightarrow \mathrm{Gr}^{\Gamma/\Omega}\text{-}A,$$
$$(-)_{\Omega} : \mathrm{Gr}^{\Gamma}\text{-}A \longrightarrow \mathrm{Mod}\text{-}\, A_{\Omega},$$

where $\mathrm{Gr}^{\Gamma}\text{-}A$ is the category of Γ-graded (right) A-modules, $\mathrm{Gr}^{\Gamma/\Omega}\text{-}A$ is the category of Γ/Ω-graded A-modules and $\mathrm{Mod}\text{-}\,A_{\Omega}$ is the category of A_{Ω}-modules.

One aspect of the graded theory of rings is to investigate how these categories are related, and what properties can be lifted from one category to another. In particular when the subgroup Ω happen to be Γ or trivial group, respectively, we obtain the (forgetful) functors

$$U : \mathrm{Gr}^{\Gamma}\text{-}A \longrightarrow \mathrm{Mod}\text{-}\,A,$$
$$(-)_0 : \mathrm{Gr}^{\Gamma}\text{-}A \longrightarrow \mathrm{Mod}\text{-}\,A_0.$$

One aspect of this note is to study the interrelation between these categories. This has been done in Chapters 1 and 2.

The rest of the note is devoted to the Grothendieck group, denoted by K_0, of these category. The category of graded finitely generated projective A-modules, $\mathrm{Pgr}^{\Gamma}\text{-}A$, is an exact category. Thus using Quillen's machinery of K-theory [79], one can define graded K-groups,

$$K_i^{\mathrm{gr}}(A) := K_i(\mathrm{Pgr}^{\Gamma}\text{-}A),$$

for $i \in \mathbb{N}$. Since the shifting of modules induces an auto-equivalence functor on $\mathrm{Gr}^{\Gamma}\text{-}A$ (§1.2.2), these K-groups carry a Γ-module structure. This extra structure is an emphasis of this note. In many important examples, in fact this shifting is all the difference between the graded Grothendieck group and the usual Grothendieck group, i.e.,

$$K_0^{\mathrm{gr}}(A)/\langle [P] - [P(\alpha)] \rangle \cong K_0(A),$$

where P is a graded projective A-module, $\alpha \in \Gamma$ and $P(\alpha)$ is a shift module (§6, see Corollary 6.3.2).

In Chapter 3 we compute K_0^{gr} for certain graded rings, such as graded local rings and (Leavitt) path algebras. We study the pre-ordering available on K_0^{gr} and determine the action of Γ on this group. In Chapter 4 the graded Picard groups are studied and in Chapter 5 we prove that for the so called graded ultramatricial algebras, the graded Grothendieck group is a complete invariant. Finally in Chapter 6, we explore the relations between (higher) K_i^{gr} and K_i, for the class of $\mathbb{Z}$-graded rings. We describe a generalisation of the Quillen and van den Bergh theorems. The latter theorem uses the techniques employed in the proof of the fundamental theorem of K-theory, where the graded K-theory appears. For this reason we present a proof of the fundamental theorem in this chapter.

Conventions. Throughout this note, unless it is explicitly stated, all rings have identities, homomorphisms preserve the identity and all modules are unitary. Furthermore, all modules are considered right modules. For a set Γ by $\bigoplus_{\Gamma} \mathbb{Z}$ or $\mathbb{Z}^{\Gamma}$, we mean $\bigoplus_{\gamma \in \Gamma} \mathbb{Z}_{\gamma}$, where $\mathbb{Z}_{\gamma} = \mathbb{Z}$ for each $\gamma \in \Gamma$. We denote the cyclic group $\mathbb{Z}/n\mathbb{Z}$ with n elements by $\mathbb{Z}_n$.

Acknowledgement. I learned about the graded techniques in algebra from Adrian Wadsworth. Judith Millar worked with me to study the graded K-theory of Azumaya algebras. Gene Abrams was a source of encouragement that the graded techniques would be fruitful in the study of Leavitt path algebras. Andrew Mathas told me how the graded Grothendieck groups are relevant in the representation theory and pointed me to the relevant literature. To all of them, I am grateful.

Chapter 1

Graded Rings And Graded Modules

This note deals with graded rings and the category of graded modules over a graded ring. This category is an abelian category (in fact a Grothendieck category). Many of the classical invariants constructed for the category of modules can be constructed, *mutatis mutandis*, starting from the category of graded modules. The general viewpoint of this note is, once a ring has a natural graded structure, the graded invariants would capture more information than the non-graded counterparts.

In this chapter we give a quick introduction to the graded ring theory. We introduce grading on matrices, study graded division rings and introduce grading on graph algebras which would be a source of many interesting examples.

1.1 Graded rings

1.1.1

A ring A is called a Γ-*graded ring*, or simply a *graded ring*, if $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$, where Γ is an (abelian) group, each A_γ is an additive subgroup of A and $A_\gamma A_\delta \subseteq A_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$.

The set $A^h = \bigcup_{\gamma \in \Gamma} A_\gamma$ is called the set of *homogeneous elements* of A. The additive group A_γ is called the γ-*component* of A and the non-zero elements of A_γ are called *homogeneous of degree* γ. We write $\deg(a) = \gamma$ if $a \in A_\gamma \backslash \{0\}$. We call the set

$$\boxed{\Gamma_A = \{\gamma \in \Gamma \mid A_\gamma \neq 0\}}$$

the *support* of A. We say the Γ-graded ring A has a *trivial grading*, or A is *concentrated in degree zero* if the support of A is the trivial group, i.e., $A_0 = A$ and $A_\gamma = 0$ for $\gamma \in \Gamma \backslash \{0\}$.

For Γ-graded rings A and B, a Γ-*graded ring homomorphism* $f : A \to B$ is a ring homomorphism such that $f(A_\gamma) \subseteq B_\gamma$ for all $\gamma \in \Gamma$. A graded homomorphism f is called a *graded isomorphism* if f is bijective and, when such a graded isomorphism exists, we write $A \cong_{\mathrm{gr}} B$. Notice that if f is a graded ring homomorphism which is bijective, then its inverse f^{-1} is also a graded ring homomorphism.

Proposition 1.1.1. *Let $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ be a Γ-graded ring. Then*

(1) 1_A *is homogeneous of degree* 0*;*

(2) A_0 *is a subring of* A*;*

(3) *Each A_γ is an A_0-bimodule;*

(4) *For an invertible element $a \in A_\gamma$, its inverse a^{-1} is homogeneous of degree $-\gamma$, i.e., $a^{-1} \in A_{-\gamma}$.*

Proof. (1) Suppose $1_A = \sum_{\gamma\in\Gamma} a_\gamma$ for $a_\gamma \in A_\gamma$. Let $b \in A_\delta$, $\delta \in \Gamma$, be an arbitrary non-zero homogeneous element. Then $b = b1_A = \sum_{\gamma\in\Gamma} ba_\gamma$, where $ba_\gamma \in A_{\delta+\gamma}$ for all $\gamma \in \Gamma$. Since the decomposition is unique, $ba_\gamma = 0$ for all $\gamma \in \Gamma$ with $\gamma \neq 0$. But as b was arbitrary, this holds for all $b \in A$ (not necessarily homogeneous), and in particular $1_A a_\gamma = a_\gamma = 0$ if $\gamma \neq 0$. Thus $1_A = a_0 \in A_0$.

(2) This follows since A_0 is an additive subgroup of A with $A_0A_0 \subseteq A_0$ and $1 \in A_0$.

(3) This is immediate.

(4) Let $b = \sum_{\delta\in\Gamma} b_\delta$, with $\deg(b_\delta) = \delta$, be the inverse of $a \in A_\gamma$, so that $1 = ab = \sum_{\delta\in\Gamma} ab_\delta$, where $ab_\delta \in A_{\gamma+\delta}$. By (1), since 1 is homogeneous of degree 0 and the decomposition is unique, it follows that $ab_\delta = 0$ for all $\delta \neq -\gamma$. Since a is invertible, $b_{-\gamma} \neq 0$, so $b = b_{-\gamma} \in A_{-\gamma}$ as required. $\square$

The proof of Proposition 1.1.1(4), in fact, shows that if $a \in A_\gamma$ has a left (or right) inverse then that inverse is in $A_{-\gamma}$. In Theorem 1.6.8, we characterise $\mathbb{Z}$-graded rings such that A_1 has a left (or right) invertible element.

Example 1.1.2. <u>Group rings</u>

For a group Γ, the group ring $\mathbb{Z}[\Gamma]$ has a natural Γ-grading

$$\mathbb{Z}[\Gamma] = \bigoplus_{\gamma\in\Gamma} \mathbb{Z}[\Gamma]_\gamma, \text{ where } \mathbb{Z}[\Gamma]_\gamma = \mathbb{Z}\gamma.$$

In §1.1.3, we study the crossed-products which are graded rings and are generalisations of group rings and skew groups rings.

Example 1.1.3. <u>Tensor algebras as positively graded rings</u>

Let A be a commutative ring and M be an A-module. Denote by $T_n(M)$, $n \geq 1$, the tensor product of n copies of M over A. Set $T_0(M) = A$. Then the natural A-module isomorphism $T_n(M) \otimes_A T_m(M) \to T_{n+m}(M)$, induces a ring structure on

$$T(M) := \bigoplus_{n\in\mathbb{N}} T_n(M).$$

The A-algebra $T(M)$ is called the *tensor algebra* of M. Setting $T(M)_n := T_n(M)$, makes $T(M)$ a $\mathbb{Z}$-graded ring with support $\mathbb{N}$. From the definition, we have $T(M)_0 = A$.

If M is a free A-module, then $T(M)$ is a free algebra over A, generated by a basis of M. Thus free rings are $\mathbb{Z}$-graded rings with the generators being homogeneous elements of degree 1. We will systematically study the grading of free rings in §1.6.1.

Example 1.1.4. <u>Formal matrix rings as graded rings</u>

Let R and S be rings, M be a $R-S$-bimodule and N be a $S-R$-bimodule. Consider the set T consisting of 2×2-matrices

$$\begin{pmatrix} r & m \\ n & s \end{pmatrix},$$

where $r \in R$, $s \in S$, $m \in M$ and $n \in N$. Suppose that there are bimodule homomorphisms $\phi : M \otimes_S N \to R$ and $\psi : N \otimes_R M \to S$ such that $(mn)m' = n(nm')$, where we set $\phi(m,n) = mn$

and $\psi(n,m) = nm$. One can then check that T with the matrix addition and multiplication forms a ring with identity. The ring T is called the *formal matrix ring* and denoted also by

$$T = \begin{pmatrix} R & M \\ N & S \end{pmatrix}.$$

Considering

$$T_0 = \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix}, \quad T_1 = \begin{pmatrix} 0 & M \\ N & 0 \end{pmatrix},$$

it is easy to check that T becomes a $\mathbb{Z}_2$-graded ring. On the other hand, considering further $T_i = 0$ for $i \neq 0, 1$, makes T also a $\mathbb{Z}$-graded ring. In the cases that the images of ϕ and ψ are zero, these rings have been extensively studied (see [58] and references therein). When $N = 0$, the ring T is called a *formal triangular matrix ring*. In this case there is no need of considering the homomorphisms ϕ and ψ.

One specific example of such grading on (subrings of) formal triangular matrix ring is used in representation theory. Recall that for a field K, a finite dimensional K-algebra R is called *Frobenius algebra* if $R \cong R^*$ as a right R-module, where $R^* := \mathrm{Hom}_K(R, K)$. Note that R^* has a natural R-bimodule structure.

Starting from a finite dimensional K-algebra R, one constructs the *trivial extension* of R which is a Frobenius algebra and has a natural $\mathbb{Z}$-graded structure as follows. Consider $A := R \bigoplus R^*$, with addition defined component-wise and multiplication defined as

$$(r_1, q_1)(r_2, q_2) = (r_1 r_2, r_1 q_1 + q_2 r_2),$$

where $r_1, r_2 \in R$ and $q_1, q_2 \in R^*$. Clearly A is a Frobenius algebra with identity $(1, 0)$. Furthermore, setting

$$\begin{aligned} A_0 &= R \oplus 0, \\ A_1 &= 0 \oplus R^*, \\ A_i &= 0, \text{ otherwise}, \end{aligned}$$

makes A into a $\mathbb{Z}$-graded ring with support $\{0, 1\}$. In fact this ring is a subring of formal triangular matrix ring

$$T_0 = \begin{pmatrix} R & R^* \\ 0 & R \end{pmatrix},$$

consisting of elements $\begin{pmatrix} a & q \\ 0 & a \end{pmatrix}$.

These rings appear in representation theory (see [44, §2.2]).

Example 1.1.5. <u>The graded ring A as A_0-module</u>

Let A be a Γ-graded ring. Then A can be considered as a A_0-bimodule. In many cases A is a projective A_0-module, for example in the case of group rings (Example 1.1.2) or when A is a strongly graded ring (see §1.1.2 and §1.1.2). Here is an example that this is not the case in general. Consider the ring T of formal matrix ring

$$T = \begin{pmatrix} R & M \\ 0 & 0 \end{pmatrix},$$

where M is a left R-module, which is not projective R-module. Then by Example 1.1.4, T is a $\mathbb{Z}$-graded ring with $T_0 = R$ and $T_1 = M$. Now T as a T_0-module is $R \oplus M$ as R-module. Since M is not projective, $R \oplus M$ is not R-module projective. We also get that T_1 is not T_0-projective.

Example 1.1.6. <u>Tensor product of graded rings</u>

Let A and B be a Γ-graded rings. Then $A \otimes_{\mathbb{Z}} B$ has a natural Γ-graded ring structure as follows. Since each of A_γ and B_γ, $\gamma \in \Gamma$, are $\mathbb{Z}$-module then $A \otimes_{\mathbb{Z}} B$ can be decomposed as a direct sum $A \otimes_{\mathbb{Z}} B = \bigoplus_{\gamma \in \Gamma} (A \otimes B)_\gamma$, where

$$(A \otimes B)_\gamma = \Big\{ \sum_i a_i \otimes b_i \mid a_i \in A^h, b_i \in A^h, \deg(a_i) + \deg(b_i) = \gamma \Big\}.$$

It is easy to see that with this decomposition $A \otimes_{\mathbb{Z}} B$ becomes a Γ-graded ring. We give specific examples of this construction in Example 1.1.7. One can replace $\mathbb{Z}$ by a field K, if A and B are K-algebras and A_γ, B_γ are K-modules.

Example 1.1.7. Let A be a ring with identity and Γ be a group. We consider A as Γ-graded ring concentrated in degree zero. Then by Example 1.1.6, $A[\Gamma] \cong A \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma]$ has a Γ-graded structure, i.e, $A[\Gamma] = \bigoplus_{\gamma \in \Gamma} A\gamma$. If A itself is a (nontrivial) Γ-graded ring $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$, then by Example 1.1.6, $A[\Gamma]$ has also a Γ-grading

$$A[\Gamma] = \textstyle\bigoplus_{\gamma \in \Gamma} A^\gamma, \text{ where } A^\gamma = \bigoplus_{\gamma = \zeta + \zeta'} A_\zeta \zeta'. \tag{1.1}$$

An specific example is when A is a positively graded $\mathbb{Z}$-graded ring. Then $A[x] \cong A \otimes \mathbb{Z}[x]$ is a $\mathbb{Z}$-graded ring with support $\mathbb{N}$, where

$$A[x]_n = \bigoplus_{i+j=n} A_i x^j.$$

This graded ring will be used in §6.2.2 when we prove the fundamental theorem of K-theory.

Example 1.1.8. <u>Partitioning of the graded rings</u>

Let A be a Γ-graded ring and Ω be a subgroup of Γ.

Subgroup grading: The ring $A_\Omega := \bigoplus_{\gamma \in \Omega} A_\gamma$ forms a Ω-graded ring. In particular, A_0 corresponds to the trivial subgroup of Γ.

Quotient grading: Considering

$$A = \bigoplus_{\Omega + \alpha \in \Gamma/\Omega} A_{\Omega+\alpha},$$

where

$$A_{\Omega+\alpha} := \bigoplus_{\omega \in \Omega} A_{\omega+\alpha},$$

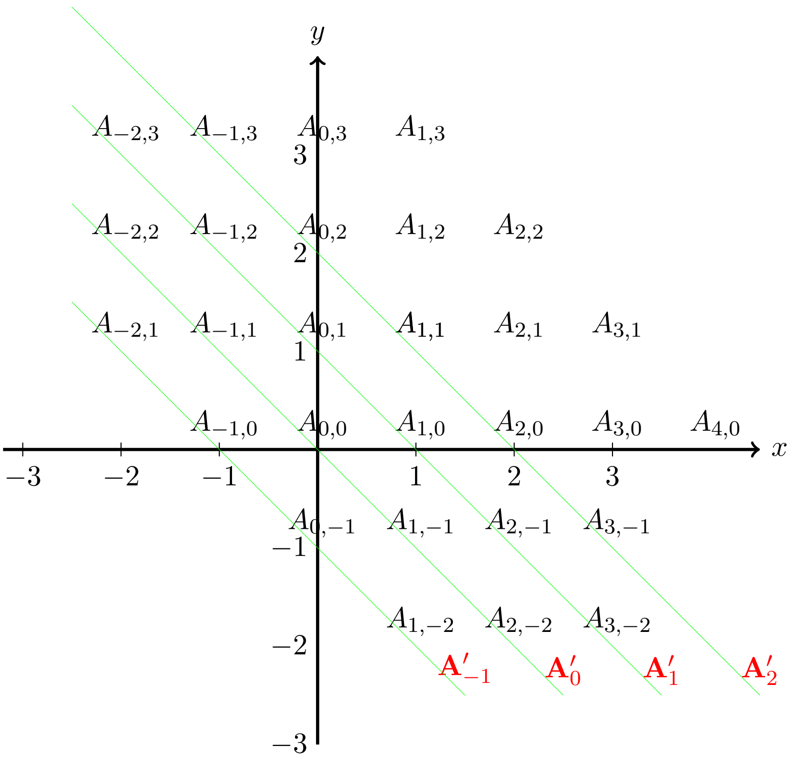
makes A a Γ/Ω-graded ring. (Note that if Γ is not abelian, then for this construction, Ω needs to be a normal subgroup.) Notice that with this grading, $A_0 = A_\Omega$. If $\Gamma_D \subseteq \Omega$, then A considered as Γ/Ω-graded ring, is concentrated in degree zero.

This construction induces a functor from the category of Γ-graded rings to the category of Γ/Ω-graded rings. If $\Omega = \Gamma$, this gives the obvious forgetful functor from the category of Γ-graded rings to the category of rings. We give a specific example of this construction in Example 1.1.9 and another in Example 1.6.1.

Example 1.1.9. Let A be a $\Gamma \times \Gamma$-graded ring. Define a Γ-grading on A as follows. For $\gamma \in \Gamma$, set

$$A'_\gamma = \sum_{\alpha \in \Gamma} A_{\gamma-\alpha,\alpha}.$$

It is easy to see that $A = \bigoplus_{\gamma\in\Gamma} A'_\gamma$ is a Γ-graded ring. When A is a $\mathbb{Z} \times \mathbb{Z}$-graded, then the $\mathbb{Z}$-grading on A is obtained from considering all the homogeneous components on a diagonal together as the following figure shows.



In fact this example follows from the general construction given in Example 1.1.8. Consider the homomorphism $\Gamma \times \Gamma \to \Gamma, (\alpha, \beta) \mapsto \alpha + \beta$. Let I be the kernel of this map. Clearly $\Gamma \times \Gamma/I \cong \Gamma$. One can check that the $\Gamma \times \Gamma/I$-graded ring A gives the graded ring constructed in this example (Remark 1.1.26).

Example 1.1.10. THE DIRECT LIMIT OF GRADED RINGS

Let A_i, $i \in I$, be a direct system of Γ-graded rings, i.e., I is a directed partially ordered set and for $i \le j$, there is a graded homomorphism $\phi_{ij} : A_i \to A_j$ which is compatible with the ordering. Then $A := \varinjlim A_i$ is a Γ-graded ring with homogeneous components $A_\alpha = \varinjlim A_{i\alpha}$. For a detailed construction of such direct limits see [22, II, §11.3, Remark 3].

As an example, the ring $A = \mathbb{Z}[x_i \mid i \in \mathbb{N}]$, where $A = \varinjlim_{i\in\mathbb{N}} \mathbb{Z}[x_1, \dots, x_i]$, with $\deg(x_i) = 1$ is a $\mathbb{Z}$-graded ring with support $\mathbb{N}$. We give another specific example of this construction in Example 1.1.11.

We will study in detail one type of these graded rings, i.e., graded ultramatricial algebras (§5, Definition 5.2.1) and will show that the graded Grothendieck group (§3) will classify these graded rings completely.

Example 1.1.11. Let $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ and $B = \bigoplus_{\gamma \in \Gamma} B_\gamma$ be Γ-graded rings. Then $A \times B$ has a natural grading given by $A \times B = \bigoplus_{\gamma \in \Gamma} (A \times B)_\gamma$ where $(A \times B)_\gamma = A_\gamma \times B_\gamma$.

Example 1.1.12. LOCALISATION OF GRADED RINGS

Let S be a central multiplicative closed subsets of Γ-graded ring A, consisting of homogeneous elements. Then $S^{-1}A$ has a natural a Γ-graded structure. Namely, for $a \in A^h$, define $\deg(a/s) = \deg(a) - \deg(s)$ and for $\gamma \in \Gamma$,

$$(S^{-1}A)_\gamma = \big\{ a/s \mid a \in A^h, \deg(a/s) = \gamma \big\}.$$

It is easy to see that this is well-defined and make $S^{-1}A$ a Γ-graded ring.

Many rings have a 'canonical' graded structure, among them, crossed products (group rings, skew group rings, twisted group rings), edge algebras, path algebras, incidence rings, etc. (see [55] for a review of these ring constructions). We will study some of these rings in this note.

Remark 1.1.13. RINGS GRADED BY A CATEGORY

There is a generalised notion of groupoid graded rings as follows. Recall that a *groupoid* is a small category with the property that all morphisms are isomorphisms. Let Γ be a groupoid and A be a ring. A is called Γ-*groupoid graded* ring, if $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$, where γ is a morphism of Γ, each A_γ is an additive subgroup of A and $A_\gamma A_\delta \subseteq A_{\gamma\delta}$ if the morphism $\gamma\delta$ is defined and $A_\gamma A_\delta = 0$, otherwise. For a group Γ, considering it as a category with one element and Γ as the set of morphisms, we recover the Γ-group graded ring A. One can develop the theory of groupoid graded rings parallel and similar to the group graded rings. See [65] for this approach. For a general notion of a ring graded by a category see [1, §2], where it is shown that the category of graded modules (graded by a category) is a Grothendieck category.

Remark 1.1.14. RINGS GRADED BY A SEMIGROUP

In the definition of a graded ring (§1.1.1), one can replace the group grading with a semigroup. With this setting, the tensor algebras of Example 1.1.3 are $\mathbb{N}$-graded rings. A number of results on the group graded rings can also be established in the more general setting of rings graded by cancellative monoids or semigroups (see for example [22, §11]). However, in this note we only consider group graded rings.

Remark 1.1.15. GRADED RINGS WITHOUT IDENTITY

For a ring without identity, one defines the concept of the graded ring exactly as when the ring has an identity. The concept of the strongly graded is defined similarly. In several occasions in this note we construct graded rings without identity. For example, the Leavitt path algebras arising from infinite graphs are graded rings without identity §1.6.3. See also §1.6.1, the graded free rings.

1.1.2

Let A be a Γ-graded ring. By Proposition 1.1.1, $1 \in A_0$. This implies $A_0 A_\gamma = A_\gamma$ and $A_\gamma A_0 = A_\gamma$ for any $\gamma \in \Gamma$. If these equality holds for any two arbitrary elements of Γ, we call the ring a strongly graded ring. Namely, a Γ-graded ring $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ is called a *strongly graded ring* if $A_\gamma A_\delta = A_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$. A graded ring A is called a *crossed-product* if there is an invertible element in every homogeneous component A_γ of A; that is, $A^* \cap A_\gamma \neq \emptyset$ for all $\gamma \in \Gamma$, where A^* is the group of all invertible elements of A. We define the support of homogeneous elements of A as

$$\boxed{\Gamma^*_A = \{\gamma \in \Gamma \mid A^*_\gamma \neq \emptyset\},} \tag{1.2}$$

where $A^*_\gamma := A^* \cap A_\gamma$. It is easy to see that Γ^*_A is a subgroup of Γ_A (see Proposition 1.1.1(4)). Clearly A is a crossed-product if and only if $\Gamma^*_A = \Gamma$.

Proposition 1.1.16. *Let $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ be a Γ-graded ring. Then*

(1) *A is strongly graded if and only if $1 \in A_\gamma A_{-\gamma}$ for any $\gamma \in \Gamma$;*

(2) *If A is strongly graded then the support of A is Γ;*

(3) *Any crossed-product ring is strongly graded.*

Proof. (1) If A is strongly graded, then $1 \in A_0 = A_\gamma A_{-\gamma}$ for any $\gamma \in \Gamma$. For the converse, the assumption $1 \in A_\gamma A_{-\gamma}$ implies that $A_0 = A_\gamma A_{-\gamma}$ for any $\gamma \in \Gamma$. Then for $\sigma, \delta \in \Gamma$,

$$A_{\sigma+\delta} = A_e A_{\sigma+\delta} = (A_\sigma A_{-\sigma})A_{\sigma+\delta} = A_\sigma(A_{-\sigma}A_{\sigma+\delta}) \subseteq A_\sigma A_\delta \subseteq A_{\sigma+\delta}$$

proving $A_{\sigma\delta} = A_\sigma A_\delta$, so A is strongly graded.

(2) By (1), $1 \in A_\gamma A_{-\gamma}$ for any $\gamma \in \Gamma$. This implies $A_\gamma \neq 0$ for any γ, i.e., $\Gamma_A = \Gamma$.

(3) Let A be a crossed-product ring. By definition, for $\gamma \in \Gamma$, there exists $a \in A^* \cap A_\gamma$. So $a^{-1} \in A_{-\gamma}$ by Proposition 1.1.1(4) and $1 = aa^{-1} \in A_\gamma A_{-\gamma}$. Thus A is strongly graded by (1). $\square$

The converse of (3) in Proposition 1.1.16 does not hold. One can prove that if A is strongly graded and A_0 is a local ring, then A is a crossed-product algebra (see [73, Theorem 3.3.1]). In §1.6 we give examples of strongly graded algebra A such that A is crossed-product but A_0 is not a local ring. We also give an example such that A_0 is not local and A is not crossed-product (Example 1.6.18).

If Γ is a finitely generated group, generated by the set $\{\gamma_1, \dots, \gamma_n\}$, then (1) in Proposition 1.1.16 can be simplified to the following: A is strongly graded if and only if $1 \in A_{\gamma_i} A_{-\gamma_i}$ and $1 \in A_{-\gamma_i} A_{\gamma_i}$, where $1 \leq i \leq n$. Thus if $\Gamma = \mathbb{Z}$, in order that A to be strongly graded, we only need to have $1 \in A_1 A_{-1}$ and $1 \in A_{-1} A_1$. This will be used, for example, in Proposition 1.6.6 to show that certain corner skew Laurent polynomial rings (§1.6.2) are strongly graded.

Example 1.1.17. <u>Constructing strongly graded rings via tensor products</u>

Let A and B be Γ-graded rings. Then by Example 1.1.6, $A \otimes_{\mathbb{Z}} B$ is a Γ-graded ring. If one of the rings is strongly graded then $A \otimes_{\mathbb{Z}} B$ is strongly graded. Indeed, suppose B is strongly graded. Then by Proposition 1.1.16(1) for any $\gamma \in \Gamma$, $1 \in B_\gamma B_{-\gamma}$. So

$$1 \otimes 1 \in A_0 \otimes B_\gamma B_{-\gamma} \subseteq (A_0 \otimes B_\gamma)(A_0 \otimes B_{-\gamma}) \subseteq (A \otimes_{\mathbb{Z}} B)_\gamma (A \otimes_{\mathbb{Z}} B)_{-\gamma}.$$

Again, Proposition 1.1.16(1) implies that $A \otimes_{\mathbb{Z}} B$ is strongly graded.

As an specific case, suppose A is a $\mathbb{Z}$-graded ring. Then $A[x, x^{-1}] = A \otimes \mathbb{Z}[x, x^{-1}]$ is a strongly graded ring. Notice that with this grading, $A[x, x^{-1}]_0 \cong A$.

Example 1.1.18. <u>Strongly graded as a Γ/Ω-graded ring</u>

Let A be a Γ-graded ring. Using Proposition 1.1.16, it is easy to see that if A is a strongly Γ-graded ring, then it is also strongly Γ/Ω-graded ring, where Ω is a subgroup of Γ. However the strongly graded is not a "closed" property, i.e, if A is strongly Γ/Ω-graded ring and A_Ω is strongly Ω-graded ring, it does not follow that A is strongly Γ-graded. For example, for a field K, consider the $\mathbb{Z}$-graded ring $K[x^3, x^{-3}]$ and the subgroup $2\mathbb{Z}$.

1.1.3 Crossed-product

Natural examples of strongly graded rings are crossed-product algebras (see Proposition 1.1.16(3)). They cover, as special cases, the skew group rings and twisted groups rings. We briefly describe the construction here.

Let A be a ring, Γ a group (as usual we use the additive notation), and let $\phi : \Gamma \to \mathrm{Aut}(A)$ and $\psi : \Gamma \times \Gamma \to A^*$ be maps such that for any $\alpha, \beta, \gamma \in \Gamma$ and $a \in A$,

(i) ${}^{\alpha}({}^{\beta}a) = \psi(\alpha, \beta)\, {}^{\alpha+\beta}a\, \psi(\alpha, \beta)^{-1}$,

(ii) $\psi(\alpha, \beta)\psi(\alpha + \beta, \gamma) = {}^{\alpha}\psi(\beta, \gamma)\, \psi(\alpha, \beta + \gamma)$,

(iii) $\psi(\alpha, 0) = \psi(0, \alpha) = 1$

Here for $\alpha \in \Gamma$ and $a \in A$, $\phi(\alpha)(a)$ is denoted by ${}^{\alpha}a$. The map ψ is called a *2-cocycle map*. Denote by $A^{\phi}_{\psi}[\Gamma]$ the free A-module with the basis Γ, and define the multiplication by

$$(a\alpha)(b\beta) = a\, {}^{\alpha}b\, \psi(\alpha, \beta)(\alpha + \beta).$$

One can show that this is a ring, and is Γ-graded ring with homogeneous components $A\gamma$, $\gamma \in \Gamma$. In fact $\gamma \in A\gamma$ is invertible, so $A^{\phi}_{\psi}[\Gamma]$ is a crossed-product algebra [73, Proposition 1.4.1].

On the other hand any crossed-product algebra is of this form (see [73, §1.4]): for any $\gamma \in \Gamma$ choose $u_\gamma \in A^* \cap A_\gamma$ and define $\phi : \Gamma \to \mathrm{Aut}(A_0)$ by $\phi(\gamma)(a) = u_\gamma a u_\gamma^{-1}$ for $\gamma \in \Gamma$ and $a \in A_0$. Furthermore define the cocycle map $\psi : \Gamma \times \Gamma \to A_0^*$ by $\psi(\zeta, \eta) = u_\zeta u_\eta u^{-1}_{\zeta+\eta}$. Then $A = {A_0}^{\phi}_{\psi}[\Gamma] = \bigoplus_{\gamma \in \Gamma} A_0\gamma$ with multiplication $(a\zeta)(b\eta) = a\,{}^{\zeta}b\psi(\zeta, \eta)(\zeta + \eta)$, where ${}^{\zeta}b$ is defined as $\phi(\zeta)(b)$.

Note that when Γ is cyclic, one can choose $u_i = u_1^i$ for $u_1 \in A^* \cap A_1$ and thus the cocycle map ψ is trivial, ϕ is a homomorphism and the crossed-product is a skew group ring denoted by $A_0 \star_\phi \mathbb{Z}$. Moreover if u_1 is in the centre of A, then ϕ is the identity map and the crossed-product reduces to the group ring $A_0[\Gamma]$.

If $\psi : \Gamma \times \Gamma \to A^*$ is a trivial map, i.e., $\psi(\alpha, \beta) = 1$ for all $\alpha, \beta \in \Gamma$, then Conditions (ii) and (iii) trivially hold, and Condition (i) reduces to ${}^{\alpha}({}^{\beta}a) = {}^{\alpha+\beta}a$ which means that $\phi : \Gamma \to \mathrm{Aut}(A)$ is a group homomorphism. In this case $A^{\phi}_{\psi}[\Gamma]$, denoted by $A \star_\phi \Gamma$, is a *skew group ring* with multiplication

$$(a\alpha)(b\beta) = a\, {}^{\alpha}b\, (\alpha + \beta). \tag{1.3}$$

If $\phi : \Gamma \to \mathrm{Aut}(A)$ is trivial, i.e., $\phi(\alpha) = 1_A$ for all $\alpha \in \Gamma$, then Condition (i) implies that $\psi(\alpha, \beta) \in Z(A) \cap A^*$ for any $\alpha, \beta \in \Gamma$. In this case $A^{\phi}_{\psi}[\Gamma]$, denoted by $A_\psi[\Gamma]$, is a *twisted group ring* with multiplication

$$(a\alpha)(b\beta) = ab\psi(\alpha, \beta)(\alpha + \beta). \tag{1.4}$$

A well-known theorem in the theory of central simple algebras states that if D is a central simple F-algebra with a maximal subfield L such that L/F is a Galois extension and $[A : F] = [L : F]^2$, then D is a crossed-product, with $\Gamma = \mathrm{Gal}(L/F)$ and $A = L$ (see [33, §12, Theorem 1]).

1.1.4

Some of the graded rings we are dealing with in this note are of the form $K[x, x^{-1}]$, where K is a field. This is an example of a graded field.

A Γ-graded ring $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ is called a *graded division ring* if every non-zero homogeneous element has a multiplicative inverse. If A is also a commutative ring, then A is called a *graded field.*

Let A be a Γ-graded division ring. It follows from Proposition 1.1.1(4) that Γ_A is a group, so we can write $A = \bigoplus_{\gamma \in \Gamma_A} A_\gamma$. Then, as a Γ_A-graded ring, A is a crossed-product and it follows from Proposition 1.1.16(3) that A is strongly Γ_A-graded. Note that if $\Gamma_A \neq \Gamma$, then A is not strongly Γ-graded. Also note that if A is a graded division ring, then A_0 is a division ring.

Theorem 1.1.19. *Let $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ be a Γ-graded ring with A_0 a field. Then A is a graded division ring if and only if A is isomorphic to a twisted group ring $A_{0\psi}[\Gamma]$, where $\psi : \Gamma \times \Gamma \to A^*$ is a 2-cocycle.*

Proof. Suppose $A_{0\psi}[\Gamma]$ is a twisted group ring. For any non-zero homogeneous element $d\gamma$ of $A_{0\psi}[\Gamma]$, by (1.4), we have

$$\big(d\gamma\big)\big(d^{-1}\psi(\gamma, -\gamma)^{-1}(-\gamma)\big) = 1(0) = 1_{A_{0\psi}[\Gamma]}.$$

Thus any non-zero homogeneous element of $A_{1\psi}[\Gamma]$ is invertible. This shows that $A_{1\psi}[\Gamma]$ is a graded division ring.

Conversely, suppose A is a graded division ring. Observe that for any $\alpha \in \Gamma$, A_α is a A_0-vector space of dimension 1. For $0 \in \Gamma$ set $a_0 = 1 \in A_0$ and for any other $\alpha \in \Gamma$, fix a non-zero $a_\alpha \in A_\alpha$. Then

$$a_\alpha a_\beta = \psi(\alpha, \beta) a_{\alpha+\beta}, \tag{1.5}$$

for some $\psi(\alpha, \beta) \in A_0$. From (1.5) it easily follows that for any $\alpha \in \Gamma$, $\psi(\alpha, 0) = \psi(0, \alpha) = 1$. Furthermore, the associativity of A shows that

$$\psi(\alpha, \beta)\psi(\alpha + \beta, \gamma) = \psi(\beta, \gamma)\psi(\alpha, \beta + \gamma).$$

Thus ψ is a 2-cocycle. Define the map

$$\begin{aligned} A_{0\psi}[\Gamma] &\longrightarrow A \\ \sum_{\alpha \in \Gamma} r_\alpha \alpha &\longmapsto \sum_{\alpha \in \Gamma} r_\alpha a_\alpha. \end{aligned}$$

This map is a graded surjective homomorphism. Since $A_{0\psi}[\Gamma]$ is graded simple (i.e., has no two sided graded ideals, see §1.1.5), this map is a graded isomorphism. This completes the proof. $\square$

In the following we give some concrete examples of graded division rings.

Example 1.1.20. <u>THE VERONESE SUBRING</u>

Let $A = \bigoplus_{i \in \mathbb{Z}} A_i$ be a $\mathbb{Z}$-graded ring. The *nth-Veronese subring* of A is defined as $A^{(n)} = \bigoplus_{i \in \mathbb{Z}} A_{in}$. This is a $\mathbb{Z}$-graded ring. It is easy to see that the support of A is $\mathbb{Z}$ if and only if the support of $A^{(n)}$ is $\mathbb{Z}$. Note that if A is strongly graded, so is $A^{(n)}$.

Let D be a division ring and let $A = D[x, x^{-1}]$ be the Laurent polynomial rings. The elements of A consist of finite sums $\sum_{i \in \mathbb{Z}} a_i x^i$, where $a_i \in D$. Then A is a $\mathbb{Z}$-graded division ring with $A = \bigoplus_{i \in \mathbb{Z}} A_i$, where $A_i = \{ax^i \mid a \in D\}$. Consider the nth-Veronese subring $A^{(n)}$ which is the ring $D[x^n, x^{-n}]$. The elements of $A^{(n)}$ consist of finite sums $\sum_{i \in \mathbb{Z}} a_i x^{in}$, where $a_i \in D$. Then $A^{(n)}$ is a $\mathbb{Z}$-graded division ring, with $A^{(n)} = \bigoplus_{i \in \mathbb{Z}} A_{in}$. Here both A and $A^{(n)}$ are strongly graded rings.

There is also another way to consider the $\mathbb{Z}$-graded ring $B = D[x^n, x^{-n}]$ such that it becomes a graded subring of $A = D[x, x^{-1}]$. Namely, we define $B = \bigoplus_{i \in \mathbb{Z}} B_i$, where $B_i = Dx^i$ if $i \in n\mathbb{Z}$ and $B_i = 0$ otherwise. This way B is a graded division ring and a graded subring of A. The support of B is clearly the subgroup $n\mathbb{Z}$ of $\mathbb{Z}$.

Example 1.1.21. DIFFERENT GRADINGS ON A GRADED DIVISION RING

Let $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ be the real *quaternion algebra*, with multiplication defined by $i^2 = -1$, $j^2 = -1$ and $ij = -ji = k$. It is known that $\mathbb{H}$ is a noncommutative division ring with centre $\mathbb{R}$. We give $\mathbb{H}$ two different graded division ring structures, with grade groups $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$, respectively as follows.

$\mathbb{Z}_2$-grading: Let $\mathbb{H} = \mathbb{C}_0 \oplus \mathbb{C}_1$, where $\mathbb{C}_0 = \mathbb{R} \oplus \mathbb{R}i$ and $\mathbb{C}_1 = \mathbb{C}j = \mathbb{R}j \oplus \mathbb{R}k$. One can check that $\mathbb{C}_0\mathbb{C}_0 = \mathbb{C}_0$, $\mathbb{C}_0\mathbb{C}_1 = \mathbb{C}_1\mathbb{C}_0 = \mathbb{C}_1$ and $\mathbb{C}_1\mathbb{C}_1 = \mathbb{C}_0$. This makes $\mathbb{H}$ a strongly $\mathbb{Z}_2$-graded division ring.

$\mathbb{Z}_2 \times \mathbb{Z}_2$-grading: Let $\mathbb{H} = R_{(0,0)} \oplus R_{(1,0)} \oplus R_{(0,1)} \oplus R_{(1,1)}$, where

$$R_{(0,0)} = \mathbb{R}, \;\; R_{(1,0)} = \mathbb{R}i, \;\; R_{(0,1)} = \mathbb{R}j, \;\; R_{(1,1)} = \mathbb{R}k.$$

It is routine to check that $\mathbb{H}$ forms a strongly $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded division ring.

We give here another example, which generalises the above example of $\mathbb{H}$ as a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded ring.

Example 1.1.22. SYMBOL ALGEBRAS

Let F be a field, ξ be a primitive n-th root of unity and let a, $b \in F^*$. Let

$$A = \bigoplus_{i=0}^{n-1} \bigoplus_{j=0}^{n-1} F x^i y^j$$

be the F-algebra generated by the elements x and y, which are subject to the relations $x^n = a$, $y^n = b$ and $xy = \xi yx$. By [33, Thm. 11.1], A is an n^2-dimensional central simple algebra over F. We will show that A forms a graded division ring. Clearly A can be written as a direct sum

$$A = \bigoplus_{(i,j) \in \mathbb{Z}_n \oplus \mathbb{Z}_n} A_{(i,j)}, \quad \text{where } A_{(i,j)} = F x^i y^j$$

and each $A_{(i,j)}$ is an additive subgroup of A. Using the fact that $\xi^{-kj} x^k y^j = y^j x^k$ for each j, k, with $0 \le j, k \le n-1$, we can show that $A_{(i,j)} A_{(k,l)} \subseteq A_{([i+k],[j+l])}$, for $i, j, k, l \in \mathbb{Z}_n$. A non-zero homogeneous element $f x^i y^j \in A_{(i,j)}$ has an inverse

$$f^{-1} a^{-1} b^{-1} \xi^{-ij} x^{n-i} y^{n-j},$$

proving A is a graded division ring. Clearly the support of A is $\mathbb{Z}_n \times \mathbb{Z}_n$, so A is strongly $\mathbb{Z}_n \times \mathbb{Z}_n$-graded.

These examples can also be obtained from graded free rings (see Example 1.6.3).

Example 1.1.23. A GOOD COUNTER-EXAMPLE

In the theory of graded rings, in many instances it has been established that if the graded group Γ is finite (or in some cases, finitely generated), then the graded property descends to the non-graded setting. For example, one can prove that if a $\mathbb{Z}$-graded ring is graded Artinian (Noetherian), then the ring is Artinian (Noetherian). One good example which provides counter-examples to such phenomena is the following graded field.

Let K be a field and $A = K[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}, \dots]$ be a Laurent polynomial ring with infinite variables. This ring is a graded field with its 'canonical' $\bigoplus_\infty \mathbb{Z}$-grading.

1.1.5 Graded ideals

Let A be a Γ-graded ring. A two-sided ideal I of A is called a *graded ideal* (or *homogeneous ideal*) if

$$I = \bigoplus_{\gamma \in \Gamma} (I \cap A_\gamma).$$

There are similar notions of a *graded subring*, a *graded left* and *graded right ideal*.

When I is a graded ideal of A, the quotient ring A/I forms a graded ring, with

$$\boxed{A/I = \bigoplus_{\gamma \in \Gamma} (A/I)_\gamma, \quad \text{where} \quad (A/I)_\gamma = (A_\gamma + I)/I.} \tag{1.6}$$

With this grading $(A/I)_0 \cong A_0/I_0$, where $I_0 = A_0 \cap I$. It is not difficult to show that an ideal I of A is a graded ideal if and only if I is generated as a two-sided ideal of A by homogeneous elements. Also, for a two sided ideal I of A, if (1.6) induces a grading on A/I, then I has to be a graded ideal. It is also easy to see that if A is a strongly graded or a crossed product, so is the graded quotient ring A/I.

Example 1.1.24. SYMMETRIC ALGEBRAS AND EXTERIOR ALGEBRAS

Recall from Example 1.1.3 that for a commutative ring A and an A-module M, the tensor algebra $T(M)$ is a $\mathbb{Z}$-graded ring with support $\mathbb{N}$. The *symmetric algebra* $S(M)$ is defined as the quotient of $T(M)$ by the ideal generated by elements $x \otimes y - y \otimes x$, $x, y \in M$. Since these elements are homogeneous of degree two, $S(M)$ is a $\mathbb{Z}$-graded commutative ring.

Similarly, the *exterior algebra* of M, denoted by $\bigwedge M$, is defined as the quotient of $T(M)$ by the ideal generated by homogeneous elements $x \otimes x$, $x \in M$. So $\bigwedge M$ is a $\mathbb{Z}$-graded ring.

A nonzero graded ring A is said to be *graded simple* if the only graded two-sided ideals of A are $\{0\}$ and A. The structure of graded simple Artinian rings are known (see Remark 1.4.8).

Recall that for Γ-graded rings A and B, a Γ-graded ring homomorphism $f : A \to B$ is a ring homomorphism such that $f(A_\gamma) \subseteq B_\gamma$ for all $\gamma \in \Gamma$. It can easily be shown that $\ker(f)$ is a graded ideal of A and $\operatorname{im}(f)$ is a graded subring of B. It is also easy to see that f is injective (surjective/bijective) if and only if for any $\gamma \in \Gamma$, the restriction of f on A_γ is injective (surjective/bijective).

Note that if Γ is an abelian group, then the centre of a graded ring A, $Z(A)$, is a graded subring of A.

Example 1.1.25. THE CENTRE OF THE GRADED RING

If a group Γ is not abelian, then the centre of a Γ-graded ring may not be a graded subring. For example, let $\Gamma = S_3 = \{e, a, b, c, d, f\}$ be the symmetric group of order 3, where

$$a = (23), \quad b = (13), \quad c = (12), \quad d = (123), \quad f = (132).$$

Let A be a ring, and consider the group ring $R = A[\Gamma]$, which is a Γ-graded ring by Example 1.1.2. Let $x = 1d + 1f \in R$, where $1 = 1_A$, and we note that x is not homogeneous in R. Then $x \in Z(R)$, but the homogeneous components of x are not in the centre of R. As x is expressed uniquely as the sum of homogeneous components, we have $x \notin \bigoplus_{\gamma \in \Gamma} (Z(R) \cap R_\gamma)$.

This example can be generalised by taking a non-abelian finite group Γ with a subgroup Ω which is normal and non-central. Let A be a ring and consider the group ring $R = A[\Gamma]$ as above. Then

$x = \sum_{\omega \in \Omega} 1\omega$ is in the centre of R, but the homogeneous components of x are not all in the centre of R.

Remark 1.1.26. Let Γ and Λ be two groups. Let A be a Γ-graded ring and B be a Λ-graded ring. Suppose $f : A \to B$ is a ring homomorphism and $g : \Gamma \to \Lambda$ a group homomorphism such that for any $\gamma \in \Gamma$, $f(A_\gamma) \subseteq B_{g(\gamma)}$. Then f is called a $\Gamma - \Lambda$-graded homomorphism. In case $\Gamma = \Lambda$ and $g = \mathrm{id}$, we recover the usual definition of a Γ-graded homomorphism. For example, if Ω is a subgroup of Γ, then the identity map $1_A : A \to A$ is a $\Gamma - \Gamma/\Omega$-graded homomorphism, where A is considered as Γ and Γ/Ω-graded rings, respectively (see Example 1.1.8).

Throughout this note, we fix a given group Γ and we work with the Γ-graded category and all our considerations are within this category (See Remark 2.3.12 for references to literature where mixed grading is studied).

Graded prime and maximal ideals

A graded ideal P of Γ-graded ring A is called a *graded prime ideal* of A if $P \neq A$ and if for two graded ideals I, J, $IJ \subseteq P$, then $I \subseteq P$ or $J \subseteq P$. If A is commutative, we obtain the familiar formulation that P is a graded prime ideal if and only if for $x, y \in A^h$, $xy \in P$ implies that $x \in P$ or $y \in P$. Note that a graded prime ideal is not necessarily a prime ideal.

A *graded maximal ideal* of a Γ-graded ring A is defined to be a proper graded ideal of A which is maximal among the set of proper graded ideals of A. Using Zorn's lemma, one can show that graded maximal ideals exist, and it is not difficult to show that a graded maximal ideal is a graded prime. For a graded commutative ring, a graded ideal is maximal if and only if its quotient ring is graded field. There are similar notions of graded maximal left and right ideals.

Parallel to the non-graded setting, for a Γ-graded ring A, the *graded Jacobson radical*, $J^{\mathrm{gr}}(A)$, is defined as the intersection of all graded left maximal ideals of A. This coincides with the intersection of all graded right maximal ideals and so $J^{\mathrm{gr}}(A)$ is a two-sided ideal (see [73, Proposition 2.9.1]). We denote by $J(A)$ the usual Jacobson radical.

1.1.6 Graded von Neumann regular rings

The von Neumann regular rings constitute an important class of rings. A unital ring A is *von Neumann regular*, if for any $a \in A$, we have $a \in aAa$. There are several equivalent module theoretical definitions, such as A is von Neumann regular if and only if any module over A is flat. This gives a comparison with the class of semisimple rings, which are characterised by the property that any module is projective. Goodearl's book [38] is devoted to the class of von Neumann regular rings. The definition extends to non-unital ring in an obvious manner.

If a ring has a graded structure, one defines the graded version of regularity in a natural way: the graded ring A is called a *graded von Neumann regular*, if for any homogeneous element $a \in A$, we have $a \in aAa$. This means, for any homogeneous element $a \in A$, one can find a homogeneous element $b \in A$ such that $a = aba$. As an example, a direct sum of graded division rings is a graded von Neumann regular ring. Many of the module theoretic properties established for von Neumann regular rings can be extended to the graded setting; For example, A is graded regular if and only if any graded module is (graded) flat. We refer the reader to [72, C, I.5] for a treatment of such rings and [10, §2.2] for a concise survey. Several of the graded rings we construct in this note are graded von Neumann regular, such as Leavitt path algebras (Corollary 1.6.13) and corner skew Laurent series (Proposition 1.6.7).

In this section, we briefly give some of the properties of graded von Neumann regular rings. The following proposition is the graded version of [38, Theorem 1.1] which has a similar proof.

Proposition 1.1.27. *Let A be a Γ-graded ring. The following statements are equivalent.*

(1) *A is a graded von Neumann regular ring;*

(2) *Any finitely generated right (left) graded ideal of A is generated by one homogeneous idempotent.*

Proof. (1) $\Rightarrow$ (2) First we show that any principal graded ideal is generated by a homogeneous idempotent. So consider the principal ideal xA, where $x \in A^h$. By the assumption, there is $y \in A^h$ such that $xyx = x$. This immediately implies $xA = xyA$. Now note that xy is homogeneous idempotent.

Next we will prove the claim for graded ideals generated by two elements. The general case follows by an easy induction. So let $xA + yA$ be a graded ideal generated by two homogeneous elements x, y. By the previous paragraph, $xA = eA$ for a homogeneous idempotent e. Note that $y - ey \in A^h$ and $y - ey \in xA + yA$. Thus

$$xA + yA = eA + (y - ey)A. \tag{1.7}$$

Again, the previous paragraph gives us a homogeneous idempotent f such that $(y - ey)A = fA$. Let $g = f - fe \in A_0$. Notice that $ef = 0$ which implies that e and g are orthogonal idempotents. Furthermore, $fg = g$ and $gf = f$. It then follows $gA = fA = (y - ey)A$. Now from (1.7), we get

$$xA + yA = eA + gA = (e + g)A.$$

(2) $\Rightarrow$ (1) Let $x \in A^h$. Then $xA = eA$ for some homogeneous idempotent e. Thus $x = ea$ and $e = xy$ for some $a, y \in A^h$. Then $x = ea = eea = ex = xyx$. $\square$

Recall that a (graded) ring is called a *(graded) semi-prime* if for any (graded) ideal I in A, $I^n \subseteq A$, $n \in \mathbb{N}$, implies $I \subseteq A$.

Proposition 1.1.28. *Let A be a Γ-graded von Neumann regular ring. Then*

(1) *Any graded right (left) ideal of A is idempotent;*

(2) *Any graded ideal is graded semi-prime;*

(3) *Any finitely generated right (left) graded ideal of A is a projective module.*

Furthermore, if A is a $\mathbb{Z}$-graded regular ring then,

(4) $J(A) = J^{\mathrm{gr}}(A) = 0$.

Proof. The proofs of (1)-(3) are similar to the non-graded case [38, Corollary 1.2]. We provide the easy proofs here.

(1) Let I be a graded right ideal. For any homogeneous element $x \in I$, there is $y \in A^h$ such that $x = xyx$. Thus $x = (xy)x \in I^2$. It follows that $I^2 = I$.

(2) This follows immediately from (1).

(3) By Proposition 1.1.27, any finitely generated right ideal is generated by a homogeneous idempotent. However this latter ideal is a direct summand of the ring, and so is a projective module.

(4) By Bergman's observation, for a $\mathbb{Z}$-graded ring A, $J(A)$ is a graded ideal and $J(A) \subseteq J^{\mathrm{gr}}(A)$ (see [18]). By Proposition 1.1.27, $J^{\mathrm{gr}}(A)$ contains an idempotent, which then forces $J^{\mathrm{gr}}(A) = 0$. $\square$

Later in the note, in Corollary 1.5.9, we show that if A is a strongly graded ring, then A is graded von Neumann regular if and only if A_0 is a von Neumann regular ring. The proof uses the equivalence of suitable categories over the rings A and A_0. An element-wise proof of this fact can also be found in [95, Theorem 3].

1.2 Graded modules

1.2.1

Let A be a Γ-graded ring. A *graded right A-module M* is defined to be a right A-module M with a direct sum decomposition $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$, where each M_γ is an additive subgroup of M such that $M_\lambda \cdot A_\gamma \subseteq M_{\gamma+\lambda}$ for all $\gamma, \lambda \in \Gamma$.

For Γ-graded right A-modules M and N, a *Γ-graded module homomorphism* $f : M \to N$ is a module homomorphism such that $f(M_\gamma) \subseteq N_\gamma$ for all $\gamma \in \Gamma$. A graded homomorphism f is called a *graded module isomorphism* if f is bijective and, when such a graded isomorphism exists, we write $M \cong_{\mathrm{gr}} N$. Notice that if f is a graded module homomorphism which is bijective, then its inverse f^{-1} is also a graded module homomorphism.

1.2.2 Shifting of modules

For $\delta \in \Gamma$, we define the *δ-suspended*, or *δ-shifted module*, A-module, $M(\delta)$ as

$$\boxed{M(\delta) = \bigoplus_{\gamma \in \Gamma} M(\delta)_\gamma, \text{ where } M(\delta)_\gamma = M_{\gamma+\delta}.}$$

This shifting plays a pivotal role in the theory of graded rings. For example, if M is a $\mathbb{Z}$-graded A-module, then the following table shows how the shifting like "the tick of the clock" moves the homogeneous components of M to the left.

	degrees	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**
M				M_{-1}	M_0	M_1	M_2	
$M(1)$			M_{-1}	M_0	M_1	M_2		
$M(2)$		M_{-1}	M_0	M_1	M_2			

Let M be a Γ-graded right A-module. A submodule N of M is called a *graded submodule* if

$$N = \bigoplus_{\gamma \in \Gamma} (N \cap M_\gamma).$$

Example 1.2.1. Let A be a Γ-graded ring and $a \in A$ be a homogeneous element of degree ω. Then aA is a graded right A-module with $\alpha \in \Gamma$ homogeneous component defined as $(aA)_\alpha := aA_{\alpha-\omega} \subseteq A_\alpha$. With this grading aA is a graded submodule (and graded right ideal) of A.

However, note that defining the grading on aA as $(aA)_\alpha := aA_\alpha \subseteq A_{\alpha+\omega}$ makes aA a graded submodule of $A(\omega)$.

There are similar notions of graded left and graded bi-submodules (see §1.2.5). When N is a graded submodule of M, the factor module M/N forms a graded A-module, with

$$\boxed{M/N = \bigoplus_{\gamma\in\Gamma}(M/N)_\gamma, \quad \text{where} \quad (M/N)_\gamma = (M_\gamma + N)/N.} \tag{1.8}$$

Example 1.2.2. Let A be a Γ-graded ring. Define a grading on the matrix ring $\mathbb{M}_n(A)$ as follows. For $\alpha \in \Gamma$, $\mathbb{M}_n(A)_\alpha = \mathbb{M}_n(A_\alpha)$ (for a general theory of grading on matrix rings see §1.3). Let $e_{ii} \in \mathbb{M}_n(A)$, $1 \leq 1 \leq n$, be a matrix with 1 in (i,i) position and zero everywhere else, and consider $e_{ii}\,\mathbb{M}_n(A)$. By Example 1.2.1, $e_{ii}\,\mathbb{M}_n(A)$ is a graded $\mathbb{M}_n(A)$-module and $\bigoplus_{i=1}^n e_{ii}\,\mathbb{M}_n(A) = \mathbb{M}_n(A)$. This shows that the graded module $e_{ii}\,\mathbb{M}_n(A)$ is a projective module. This is an example of a graded projective module (see §1.2.9).

Example 1.2.3. Let A be ring. Consider the matrix ring $\mathbb{M}_n(A)$ as a $\mathbb{Z}$-graded ring concentrated in degree zero. Furthermore, consider $\mathbb{M}_n(A)$ as a graded $\mathbb{M}_n(A)$-module with the grading defined as follows: $\mathbb{M}_n(A)_i = e_{ii}\,\mathbb{M}_n(A)$ for $1 \leq i \leq n$ and zero otherwise. Note that all non-zero homogeneous elements of this module are zero-divisiors. We will use this example to show that a free module which is graded is not necessarily a graded free module (§1.2.4).

1.2.3 The Hom group

For two graded right A-modules M and N, a *graded A-module homomorphism of degree δ* is an A-module homomorphism $f : M \to N$, such that $f(M_\gamma) \subseteq N_{\gamma+\delta}$ for any $\gamma \in \Gamma$. Let $\operatorname{Hom}_A(M,N)_\delta$ denote the subgroup of $\operatorname{Hom}_A(M,N)$ consisting of all graded A-module homomorphisms of degree δ, i.e.,

$$\boxed{\operatorname{Hom}_A(M,N)_\delta = \{f \in \operatorname{Hom}_A(M,N) \mid f(M_\gamma) \subseteq N_{\gamma+\delta}, \gamma \in \Gamma\}.}$$

Clearly a graded module homomorphism defined in §1.2.1 is a graded homomorphism of degree 0. By Gr-A (or Gr^Γ-A to emphasis the graded group of A), we denote a category consists of Γ-graded right A-modules as objects and graded homomorphisms as the morphisms, i.e.,

$$\boxed{\operatorname{Hom}_{\text{Gr-}A}(M,N) = \operatorname{Hom}_A(M,N)_0.}$$

Furthermore, for $\alpha \in \Gamma$, one can write

$$\operatorname{Hom}_{\text{Gr-}A}\big(M(-\alpha),N\big) = \operatorname{Hom}_{\text{Gr-}A}\big(M,N(\alpha)\big) = \operatorname{Hom}_A(M,N)_\alpha. \tag{1.9}$$

A full subcategory of Gr-A, consisted of graded finitely generated A-modules is denoted by gr-A.

For $\alpha \in \Gamma$, the *α-suspension functor* or *shift functor*

$$\begin{aligned} \mathcal{T}_\alpha : \text{Gr-}A &\longrightarrow \text{Gr-}A, \\ M &\longmapsto M(\alpha), \end{aligned}$$

is an isomorphism with the property $\mathcal{T}_\alpha\mathcal{T}_\beta = \mathcal{T}_{\alpha+\beta}$, $\alpha, \beta \in \Gamma$.

Remark 1.2.4. Let A be a Γ-graded ring and Ω be a subgroup of Γ such that $\Gamma_A \subseteq \Omega \subseteq \Gamma$. Then the ring A can be considered naturally as a Ω-graded ring. Similarly, if A, B are Γ-graded rings and $f : A \to B$ is a Γ-graded homomorphism and $\Gamma_A, \Gamma_B \subseteq \Omega \subseteq \Gamma$, then the homomorphism f can be naturally considered as a Ω-graded homomorphism. In this case, to make a distinction, we write Gr^{Γ}-A for the category of Γ-graded A-modules and Gr^{Ω}-A for the category of Ω-graded A-modules.

Theorem 1.2.5. *For graded right A-modules M and N, if M is finitely generated, then* $\operatorname{Hom}_A(M,N)$ *has a natural decomposition*

$$\operatorname{Hom}_A(M,N) = \bigoplus_{\gamma\in\Gamma} \operatorname{Hom}_A(M,N)_\gamma. \tag{1.10}$$

Furthermore, $\operatorname{Hom}_A(M,M)$ *is a Γ-graded ring.*

Proof. Let $f \in \operatorname{Hom}_A(M,N)$. For $\lambda \in \Gamma$, define $f_\lambda \in \operatorname{Hom}_A(M,N)$ as follows: for $m \in M$,

$$f_\lambda(m) = \sum_{\gamma\in\Gamma} f(m_{\gamma-\lambda})_\gamma, \tag{1.11}$$

where $m = \sum_{\gamma\in\Gamma} m_\gamma$. If $m \in M_\alpha$, $\alpha \in \Gamma$, then (1.11) reduces to $f_\lambda(m) = f(m)_{\alpha+\lambda} \subseteq M_{\alpha+\lambda}$. This shows that $f_\lambda \in \operatorname{Hom}_A(M,N)_\lambda$. Also, $f_\lambda(m)$ is zero for all but a finite number of $\lambda \in \Gamma$. Furthermore, $\sum_\lambda f_\lambda(m) = \sum_\lambda f(m)_{\alpha+\lambda} = f(m)$. Now since M is finitely generated, there is a finite number of homogeneous elements which generate any $m \in M$. The above argument shows only a finite number of $f_\lambda(m)$ are not zero and $f = \sum_\lambda f_\lambda$. This shows $\operatorname{Hom}_A(M,N) = \sum_{\gamma\in\Gamma} \operatorname{Hom}_A(M,N)_\gamma$. Finally, it is easy to see that $\operatorname{Hom}_A(M,N)_\gamma$, $\gamma \in \Gamma$, constitute a direct sum.

For the second part, replacing N by M in (1.10), we get $\operatorname{Hom}_A(M,M) = \bigoplus_{\gamma\in\Gamma} \operatorname{Hom}_A(M,M)_\gamma$. Furthermore, if $f \in \operatorname{Hom}_A(M,M)_\gamma$ and $g \in \operatorname{Hom}_A(M,M)_\lambda$ then $fg \in \operatorname{Hom}_A(M,M)_{\gamma+\lambda}$. This shows that when M is finitely generated, the ring $\operatorname{Hom}_A(M,M)$ is a Γ-graded ring. This complete the proof. ☐

Let M be a graded finitely generated right A-module. Then the usual *dual* of M, i.e., $M^* = \operatorname{Hom}_A(M,A)$, is a left A-module. Furthermore, using Theorem 1.2.5, one can check that M^* is a graded left A-module. Since

$$\operatorname{Hom}(M,N)(\alpha) = \operatorname{Hom}(M(-\alpha),N) = \operatorname{Hom}(M,N(\alpha)),$$

we have

$$M(\alpha)^* = M^*(-\alpha). \tag{1.12}$$

This should also make sense as if we "push forward" M with α, then we "pull back" the dual M^* by α.

When M is a free module, $\operatorname{Hom}_A(M,M)$ can be represented by a matrix ring over A. Next we define graded free modules. In §1.3 we will see that if M is a graded free module, the graded ring $\operatorname{Hom}_A(M,M)$ can be represented by matrix ring over A with a very concrete grading.

1.2.4 Graded free modules

A graded (right) A-module F is called a *graded free A-module* if F is a free right A-module with a homogeneous base. Clearly a graded free module is a free module but the converse is not correct (see Example 1.2.3).

Consider a Γ-graded A-module $\bigoplus_{i\in I} A(\delta_i)$, where I is an indexing set and $\delta_i \in \Gamma$. Note that for each $i \in I$, the element e_i of the standard basis (i.e., 1 in the ith component and zero elsewhere) is homogeneous of degree $-\delta_i$. The set $\{e_i\}_{i\in I}$ forms a base for $\bigoplus_{i\in I} A(\delta_i)$, which by definition makes this a graded free A-module. On the other hand, a graded free A-module F with a homogeneous base $\{b_i\}_{i\in I}$, where $\deg(b_i) = -\delta_i$ is graded isomorphic to $\bigoplus_{i\in I} A(\delta_i)$. Indeed one can easily observe that the map induced by

$$\begin{aligned} \varphi : \bigoplus\nolimits_{i\in I} A(\delta_i) &\longrightarrow F \\ e_i &\longmapsto b_i \end{aligned}$$

is a graded A-module isomorphism.

1.2.5 Graded bimodules

The notion of the graded *left* A-modules is developed similarly. The category of graded left A-modules with graded homomorphisms is denoted by A-Gr. In a similar manner for Γ-graded rings A and B, we can consider the *graded* $A-B$ *bimodule* M. Namely, M is a $A-B$-bimodule and additionally $M = \bigoplus_{\gamma\in\Gamma} M_\gamma$ is a graded left A-module and a graded right B-module, i.e., $A_\alpha M_\gamma B_\beta \subseteq M_{\alpha+\gamma+\beta}$, $\alpha, \gamma, \beta \in \Gamma$. The category of graded A-bimodules is denoted by Gr-A-Gr.

1.2.6 Tensor product of graded modules

Let A be a Γ-graded ring and M_i, $i \in I$, be a direct system of Γ-graded A-modules, i.e., I is a directed partially ordered set and for $i \le j$, there is a graded A-homomorphism $\phi_{ij} : M_i \to M_j$ which is compatible with the ordering. Then $M := \varinjlim M_i$ is a Γ-graded A-module with homogeneous components $M_\alpha = \varinjlim M_{i\alpha}$ (see Example 1.1.10 for the similar construction for rings).

In particular, let $\{M_i \mid i \in I\}$ be Γ-graded right A-modules. Then $\bigoplus_{i\in I} M_i$ has a natural graded A-module given by $(\bigoplus_{i\in I} M_i)_\alpha = \bigoplus_{i\in I} M_{i\alpha}$, $\alpha \in \Gamma$.

Let M be a graded right A-module and N be a graded left A-modules. We will observe that the tensor product $M \otimes_A N$ has a natural graded $\mathbb{Z}$-module structure. Since each of M_γ, $\gamma \in \Gamma$, is a right A_0-module and similarly N_γ, $\gamma \in \Gamma$, is a left A_0-module, then $M \otimes_{A_0} N$ can be decomposed as a direct sum $M \otimes_{A_0} N = \bigoplus_{\gamma\in\Gamma} (M \otimes N)_\gamma$, where

$$\boxed{(M \otimes N)_\gamma = \Big\{ \sum_i m_i \otimes n_i \mid m_i \in M^h, n_i \in N^h, \deg(m_i) + \deg(n_i) = \gamma \Big\}.}$$

Now note that $M \otimes_A N \cong (M \otimes_{A_0} N)/J$, where J is a subgroup of $M \otimes_{A_0} N$ generated by the homogeneous elements $\{ma \otimes n - m \otimes an \mid m \in M^h, n \in N^h, a \in A^h\}$. This shows that $M \otimes_A N$ is also a graded module. It is easy to check that, for example, if N is a graded A-bimodule, then $M \otimes_A N$ is a graded right A-module.

Observe that for a graded right A-module M, the map

$$\begin{aligned} M \otimes_A A(\alpha) &\longrightarrow M(\alpha), \\ m \otimes a &\longmapsto ma, \end{aligned} \tag{1.13}$$

is a graded isomorphism. In particular, for any $\alpha, \beta \in \Gamma$, there is a graded A-bimodule isomorphism

$$\boxed{A(\alpha) \otimes_A A(\beta) \cong_{\mathrm{gr}} A(\alpha + \beta).}$$

Example 1.2.6. GRADED FORMAL MATRIX RINGS

The construction of formal matrix rings (Example 1.1.4) can be carried over to the graded setting as follows. Let A and B be Γ-graded rings, M be a graded $R-S$-bimodule and N be a graded $S-R$-bimodule. Suppose that there are graded bimodule homomorphisms $\phi : M \otimes_S N \to R$ and $\psi : N \otimes_R M \to S$ such that $(mn)m' = n(nm')$, where we set $\phi(m, n) = mn$ and $\psi(n, m) = nm$. Consider the ring

$$T = \begin{pmatrix} R & M \\ N & S \end{pmatrix},$$

and define, for any $\gamma \in \Gamma$,

$$T_\gamma = \begin{pmatrix} R_\gamma & M_\gamma \\ N_\gamma & S_\gamma \end{pmatrix}.$$

One can easily check that T is a Γ-graded ring, called a *graded formal matrix ring*. One specific type of such rings is a Morita ring which appears in graded Morita theory (§2.3).

1.2.7 Forgetting the grading

Most forgetful functors in algebra tend to have left adjoints, which have a 'free' constructions. One such example is the forgetful functor from the category of abelian groups to abelian monoids that we will study in §3 in relation with the Grothendieck groups. However, some of the forgetful functors in the graded setting naturally have right adjoints as we will see below.

Consider the *forgetful* functor

$$U : \text{Gr-}A \to \text{Mod-}A, \tag{1.14}$$

which simply assigns to any graded module M in Gr-A, its underlying module M in Mod-A, ignoring the grading. Similarly, the graded homomorphisms are sent to the same homomorphisms, disregarding their graded compatibilities.

There is a functor $F : \text{Mod-}A \to \text{Gr-}A$ which is a right adjoint to U. The construction is as follows: let M be an A-module. Consider the abelian group $F(M) := \bigoplus_{\gamma\in\Gamma} M_\gamma$, where M_γ is a copy of M. Furthermore, for $a \in A_\alpha$ and $m \in M_\gamma$ define $m.a = ma \in M_{\alpha+\gamma}$. This defines a graded A-module structure on $F(M)$ and makes F an exact functor from Mod-A to Gr-A. One can prove that for any $M \in \text{Gr-}A$ and $N \in \text{Mod-}A$, we have a bijective map

$$\begin{aligned} \operatorname{Hom}_{\text{Mod-}A}\big(U(M), N\big) &\xrightarrow{\phi} \operatorname{Hom}_{\text{Gr-}A}\big(M, F(M)\big), \\ f &\longmapsto \phi_f, \end{aligned}$$

where $\phi_f(m_\alpha) = f(m_\alpha) \in N_\alpha$.

Remark 1.2.7. One can prove that for any $M \in \text{Gr-}A$, $FU(M) \cong_{\text{gr}} \bigoplus_{\gamma\in\Gamma} M(\gamma)$. We also note that if Γ is finite, then F is also a left adjoint functor of U. Further, if U has a left adjoint functor, then one can prove that Γ is finite (see [73, §2.5] for details).

1.2.8 Partitioning of the graded modules

Let Γ be a group and Ω a subgroup of Γ. Recall from Example 1.1.8 that the functor from the category of Γ-graded rings to the category of Γ/Ω-graded rings which gives the natural forgetful functor when $\Omega = \Gamma$. This functor has also a right adjoint functor (see [73, Proposition 1.2.2]).

There is a canonical 'forgetful' functor $U : \mathrm{Gr}^{\Gamma}\text{-}A \to \mathrm{Gr}^{\Gamma/\Omega}\text{-}A$, such that when $\Omega = \Gamma$, it gives the functor 1.14. The construction is as follows. Let $M = \bigoplus_{\alpha\in\Gamma} M_\alpha$ be a Γ-graded A-module. Write

$$M = \bigoplus_{\Omega+\alpha\in\Gamma/\Omega} M_{\Omega+\alpha}, \tag{1.15}$$

where

$$M_{\Omega+\alpha} := \bigoplus_{\omega\in\Omega} M_{\omega+\alpha}. \tag{1.16}$$

One can easily check that M is a Γ/Ω-graded A-module. Furthermore, $U(M(\alpha)) = M(\Omega+\alpha)$. We will use this functor to relate the Grothendieck groups of these categories in Example 3.1.10.

Let A be a Γ-graded ring and Ω be a subgroup of Γ. The above construction motivates the following which establishes a relation between the categories $\mathrm{Gr}^{\Gamma}\text{-}A$ and $\mathrm{Gr}^{\Omega}\text{-}A$.

Consider the quotient group Γ/Ω and fix a complete set of coset representative $\{\alpha_i\}_{i\in I}$. Let $\beta\in\Gamma$ and consider the permutation map ρ_β

$$\begin{aligned}\rho_\beta : \Gamma/\Omega &\longrightarrow \Gamma/\Omega\\ \Omega+\alpha_i &\longmapsto \Omega+\alpha_i+\beta = \Omega+\alpha_j.\end{aligned}$$

This defines a bijective map (called ρ_β again) $\rho_\beta : \{\alpha_i\}_{i\in I} \to \{\alpha_i\}_{i\in I}$. Furthermore, for any α_i, since

$$\Omega+\alpha_i+\beta = \Omega+\alpha_j = \Omega+\rho_\beta(\alpha_i),$$

there is a unique $w_i\in\Omega$ such that

$$\alpha_i+\beta = \omega_i+\rho_\beta(\alpha_i). \tag{1.17}$$

Define the functor

$$\boxed{\begin{aligned}\mathcal{P} : \mathrm{Gr}^{\Gamma}\text{-}A &\longrightarrow \bigoplus_{\Gamma/\Omega} \mathrm{Gr}^{\Omega}\text{-}A_\Omega,\\ M &\longmapsto \bigoplus_{\Omega+\alpha_i\in\Gamma/\Omega} M_{\Omega+\alpha_i}\end{aligned}} \tag{1.18}$$

where

$$\boxed{M_{\Omega+\alpha_i} = \bigoplus_{\omega\in\Omega} M_{\omega+\alpha_i}.}$$

Since $M_{\Omega+\alpha}$, $\alpha\in\Gamma$, defined in (1.16), can be naturally considered as Ω-graded A_Ω-module, where

$$(M_{\Omega+\alpha})_\omega = M_{\omega+\alpha}, \tag{1.19}$$

the functor $\mathcal{P}$ defined in 1.18 is well-defined. Note that the homogeneous components defined in (1.19), depend on the coset representation, thus choosing another complete set of coset representative gives a different functor between these categories.

For any $\beta\in\Gamma$, define a shift functor

$$\begin{aligned}\overline{\rho}_\beta : \bigoplus_{\Gamma/\Omega} \mathrm{Gr}^{\Omega}\text{-}A_\Omega &\longrightarrow \bigoplus_{\Gamma/\Omega} \mathrm{Gr}^{\Omega}\text{-}A_\Omega,\\ \bigoplus_{\Omega+\alpha_i\in\Gamma/\Omega} M_{\Omega+\alpha_i} &\longmapsto \bigoplus_{\Omega+\alpha_i\in\Gamma/\Omega} M(\omega_i)_{\Omega+\rho_\beta(\alpha_i)},\end{aligned} \tag{1.20}$$

where $\rho_\beta(\alpha_i)$ and ω_i are defined in (1.17). The $\overline{\rho}_\beta$ for morphisms are defined accordingly.

We are in a position to prove the next theorem.

Theorem 1.2.8. *Let A be a Γ-graded ring and Ω a subgroup of Γ. Then for any $\beta \in \Gamma$, the following diagram is commutative,*

$$\begin{tikzcd}
\mathrm{Gr}^{\Gamma}\text{-}A \arrow[r, "\mathcal{P}"] \arrow[d, "\mathcal{T}_\beta"'] & \bigoplus_{\Gamma/\Omega} \mathrm{Gr}^{\Omega}\text{-}A_\Omega \arrow[d, "\overline{\rho}_\beta"] \\
\mathrm{Gr}^{\Gamma}\text{-}A \arrow[r, "\mathcal{P}"] & \bigoplus_{\Gamma/\Omega} \mathrm{Gr}^{\Omega}\text{-}A_\Omega,
\end{tikzcd} \tag{1.21}$$

where the functor $\mathcal{P}$ is defined in (1.18). Furthermore, if $\Gamma_A \subseteq \Omega$, then the functor $\mathcal{P}$ induces an equivalence of categories.

Proof. We first show that Diagram 1.21 is commutative. Let $\beta \in \Gamma$ and M a Γ-graded A-module. As in (1.17), let $\{\alpha_i\}$ be a fixed complete set of coset representative and

$$\alpha_i + \beta = \omega_i + \rho_\beta(\alpha_i).$$

Then

$$\mathcal{P}(\mathcal{T}_\beta(M)) = \mathcal{P}(M(\beta)) = \bigoplus_{\Omega+\alpha_i \in \Gamma/\Omega} M(\beta)_{\Omega+\alpha_i}. \tag{1.22}$$

But

$$M(\beta)_{\Omega+\alpha_i} = \bigoplus_{\omega\in\Omega} M(\beta)_{\omega+\alpha_i} = \bigoplus_{\omega\in\Omega} M_{\omega+\alpha_i+\beta} = \bigoplus_{\omega\in\Omega} M_{\omega+\omega_i+\rho_\beta(\alpha_i)} =$$
$$\bigoplus_{\omega\in\Omega} M(\omega_i)_{\omega+\rho_\beta(\alpha_i)} = M(\omega_i)_{\Omega+\rho_\beta(\alpha_i)}.$$

Replacing this into equation 1.22 we have

$$\mathcal{P}(\mathcal{T}_\beta(M)) = \bigoplus_{\Omega+\alpha_i \in \Gamma/\Omega} M(\omega_i)_{\Omega+\rho_\beta(\alpha_i)}. \tag{1.23}$$

On the other hand by (1.20),

$$\overline{\rho}_\beta \mathcal{P}(M) = \overline{\rho}_\beta\Big(\bigoplus_{\Omega+\alpha_i \in \Gamma/\Omega} M_{\Omega+\alpha_i}\Big) = \bigoplus_{\Omega+\alpha_i \in \Gamma/\Omega} M(\omega_i)_{\Omega+\rho_\beta(\alpha_i)}. \tag{1.24}$$

Comparing (1.23) and (1.24) shows that the Diagram 1.21 is commutative.

For the last part, suppose $\Gamma_A \subseteq \Omega$. We construct a functor $\mathcal{P}' : \bigoplus_{\Gamma/\Omega} \mathrm{Gr}^{\Omega}\text{-}A_\Omega \to \mathrm{Gr}^{\Gamma}\text{-}A$, which depends on the coset representative $\{\alpha_i\}_{i\in I}$ of Γ/Ω. First note that any $\alpha \in \Gamma$ can be written uniquely as $\alpha = \alpha_i + \omega$, where $\omega \in \Omega$. Now let

$$\bigoplus_{\Omega+\alpha_i \in \Gamma/\Omega} N_{\Omega+\alpha_i} \in \bigoplus_{\Gamma/\Omega} \mathrm{Gr}^{\Omega}\text{-}A_\Omega,$$

where $N_{\Omega+\alpha_i}$ is Ω-graded A_Ω-module. Define a Γ-graded A-module N as follows: $N = \bigoplus_{\alpha\in\Gamma} N_\alpha$, where $N_\alpha := (N_{\Omega+\alpha_i})_\omega$ and $\alpha = \alpha_i + \omega$. We check that N is a Γ-graded A-module, i.e, $N_\alpha A_\gamma \subseteq N_{\alpha+\gamma}$, for $\alpha, \gamma \in \Gamma$. If $\gamma \notin \Omega$, since $\Gamma_A \subseteq \Omega$, $A_\gamma = 0$ and thus $0 = N_\alpha A_\gamma \subseteq N_{\alpha+\gamma}$. Let $\gamma \in \Omega$. Then

$$N_\alpha A_\gamma = (N_{\Omega+\alpha_i})_\omega A_\gamma \subseteq (N_{\Omega+\alpha_i})_{\omega+\gamma} = N_{\alpha+\gamma},$$

as $\alpha+\gamma=\alpha_i+\omega+\gamma$.

We define $\mathcal{P}'(\bigoplus_{\Omega+\alpha_i\in\Gamma/\Omega} N_{\Omega+\alpha_i}) = N$ and accordingly for the morphisms. It is now not difficult to check that $\mathcal{P}'$ is an inverse of the functor $\mathcal{P}$. This finishes the proof. □

The above theorem will be used to compare the graded K-theories with respect to Γ and Ω (see Example 3.1.11).

Corollary 1.2.9. *Let A be a Γ-graded ring concentrated in degree zero. Then*

$$\text{Gr-}A \approx \bigoplus_{\Gamma} \text{Mod-}A.$$

Proof. This follows by replacing Ω by a trivial group in Theorem 1.2.8. □

The following corollary, which is a more general case of Corollary 1.2.9 with a similar proof, will be used in the proof of Lemma 6.1.6.

Corollary 1.2.10. *Let A be a $\Gamma\times\Omega$ graded ring which is concentrated in Ω. Then*

$$\text{Gr}^{\Gamma\times\Omega}\text{-}A \cong \bigoplus_{\Gamma} \text{Gr}^{\Omega}\text{-}A.$$

The action of $\Gamma\times\Omega$ on $\bigoplus_\Gamma \text{Gr}^\Omega\text{-}A$ described in (1.20) reduces to the following: for $(\gamma,\omega)\in\Gamma\times\Omega$,

$$\overline{\rho}_{(\gamma,\omega)}\Big(\bigoplus_{\alpha\in\Gamma} M_\alpha\Big) = \bigoplus_{\alpha\in\Gamma} M_{\alpha+\gamma}(\omega).$$

1.2.9 Graded projective modules

In this note, the graded projective modules play a crucial role. They will appear in the graded Morita theory §2 and will be used to define the graded Grothendieck groups §3. Furthermore, the graded higher K-theory is constructed from the exact category consisted of graded finitely generated projective modules §6. In this section we define the graded projective modules and give several equivalent criteria for a module to be graded projective. As before, unless stated otherwise, we work in the category of (graded) right modules.

A graded A-module P is called a *graded projective module* if it is a projective object in the abelian category Gr-A. More concretely, P is graded projective if for any diagram of graded A-module homomorphisms

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow{\scriptstyle j} & & \\ M & \xrightarrow[g]{} & N & \longrightarrow & 0, \end{array}$$

there is a graded A-module homomorphism $h: P\to M$ with $gh=j$.

In Proposition 1.2.12 we give some equivalent characterisations of graded projective modules, including the one that shows an A-module is graded projective if and only if it is graded and projective as an A-module. By Pgr-A (or Pgr^Γ-A to emphasis the graded group of A) we denote a full subcategory of Gr-A, consisting of graded finitely generated projective right A-modules. This is the primary category we are interested in. The graded Grothendieck group (§3) and higher K-groups (§6) are constructed from this category.

We need the following lemma, which says if a graded map factors into two maps, with one being graded, then one can replace the other one with a graded map as well.

Lemma 1.2.11. *Let P, M, N be graded A-modules, with A-module homomorphisms f, g, h*

$$\begin{tikzcd} & M \arrow[dr, "g"] & \\ P \arrow[ur, "h"] \arrow[rr, "f"'] & & N \end{tikzcd}$$

such that $f = gh$, where f is a graded A-module homomorphism. If g (resp. h) is a graded A-homomorphism then there exists a graded homomorphism $h' : P \to M$ (resp. $g' : M \to N$) such that $f = gh'$ (resp. $f = g'h$).

Proof. Suppose $g : M \to N$ is a graded A-module homomorphism. Define $h' : P \to M$ as follows: for $p \in P_\alpha$, $\alpha \in \Gamma$, let $h'(p) = h(p)_\alpha$ and extend this linearly to all elements of P. One can easily see that $h' : P \to M$ is a graded A-module homomorphism. Furthermore, for $p \in P_\alpha$, $\alpha \in \Gamma$, we have

$$f(p) = gh(p) = g\big(\sum_{\gamma \in \Gamma} h(p)_\gamma\big) = \sum_{\gamma \in \Gamma} g\big(h(p)_\gamma\big).$$

Since f and g are graded homomorphisms, comparing the degrees of the homogeneous elements of each side of the equation, we get $f(p) = g(h(p)_\alpha) = gh'(p)$. Using the linearity of f, g, h' we get that $f = gh'$ which proves the lemma for the case g. The other case is similar. $\square$

We are in a position to give equivalent characterisations of graded projective modules.

Proposition 1.2.12. *Let A be a Γ-graded ring and P be a graded A-module. Then the following are equivalent:*

(1) *P is graded and projective;*

(2) *P is graded projective;*

(3) $\operatorname{Hom}_{\text{Gr-}A}(P, -)$ *is an exact functor in* Gr-A*;*

(4) *Every short exact sequence of graded A-module homomorphisms*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} P \longrightarrow 0$$

splits via a graded map;

(5) *P is graded isomorphic to a direct summand of a graded free A-module.*

Proof. (1) $\Rightarrow$ (2) Consider the diagram

$$\begin{tikzcd} & P \arrow[d, "j"] & \\ M \arrow[r, "g"'] & N \arrow[r] & 0, \end{tikzcd}$$

where g and j are graded homomorphisms and g is surjective. Since P is projective, there is an A-module homomorphism $h : P \to M$ with $gh = j$. By Lemma 1.2.11, there is a graded A-module homomorphism $h' : P \to M$ with $gh' = j$. This gives that P is a graded projective module.

(2) $\Rightarrow$ (3) In exactly the same way as the non-graded setting (see [52, Thm. IV.4.2]) we can show that $\operatorname{Hom}_{\text{Gr-}A}(P, -)$ is left exact. Then it follows immediately from (2) that it is right exact.

(3) $\Rightarrow$ (4) Let

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} P \longrightarrow 0 \tag{1.25}$$

be a short exact sequence. Since $\operatorname{Hom}_{\text{Gr-}A}(P, -)$ is exact,

$$\begin{aligned} \operatorname{Hom}_{\text{Gr-}A}(P, M) &\longrightarrow \operatorname{Hom}_{\text{Gr-}A}(P, P) \\ h &\longmapsto gh \end{aligned}$$

is an epimorphism. In particular, there is graded homomorphism h such that $gh = 1$, i.e., the short exact sequence (1.25) is spilt.

(4) $\Rightarrow$ (5) First note that P is a homomorphic image of a graded free A-module as follows: Let $\{p_i\}_{i\in I}$ be a homogeneous generating set for P, where $\deg(p_i) = \delta_i$. Let $\bigoplus_{i\in I} A(-\delta_i)$ be the graded free A-module with standard homogeneous basis $\{e_i\}_{i\in I}$ where $\deg(e_i) = \delta_i$. Then there is an exact sequence

$$0 \longrightarrow \ker(g) \xrightarrow{\subseteq} \bigoplus_{i\in I} A(-\delta_i) \xrightarrow{g} P \longrightarrow 0, \tag{1.26}$$

since the map $g : \bigoplus_{i\in I} A(-\delta_i) \to P$; $e_i \mapsto p_i$ is a surjective graded A-module homomorphism. By the assumption, there is a graded A-module homomorphism $h : P \to \bigoplus_{i\in I} A(-\delta_i)$ such that $gh = \mathrm{id}_P$.

Since the exact sequence 1.26 is, in particular, a split exact sequence of A-modules, we know from the non-graded setting [66, Prop. 2.5] that there is an A-module isomorphism

$$\begin{aligned} \theta : P \oplus \ker(g) &\longrightarrow \bigoplus\nolimits_{i\in I} A(-\delta_i) \\ (p, q) &\longmapsto h(p) + q. \end{aligned}$$

Clearly this map is also a graded A-module homomorphism, so $P \oplus \ker(g) \cong_{\text{gr}} \bigoplus_{i\in I} A(-\delta_i)$.

(5) $\Rightarrow$ (1) Graded free modules are free, so P is isomorphic to a direct summand of a free A-module. From the non-graded setting, we know that P is projective. $\square$

The proof of Proposition 1.2.12 (see in particular (4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (1)) shows that a graded A-module P is a graded finitely generated projective A-module if and only if

$$\boxed{P \oplus Q \cong_{\text{gr}} A^n(\overline{\alpha}),} \tag{1.27}$$

for some $\overline{\alpha} = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in \Gamma$. This fact will be used frequently throughout this note.

Theorem 1.2.13 (The dual basis lemma). *Let A be a Γ-graded ring and P be a graded A-module. Then P is graded projective if and only if there exists $p_i \in P^h$ with $\deg(p_i) = \delta_i$ and $f_i \in \operatorname{Hom}_{\text{Gr-}A}(P, A(-\delta_i))$, for some indexing set I, such that*

(1) *for every $p \in P$, $f_i(p) = 0$ for all but a finite subset of $i \in I$,*

(2) *for every* $p \in P$, $\sum_{i\in I} f_i(p)p_i = p$.

Proof. Since P is graded projective, by Proposition 1.2.12(5), there is a graded module Q such that $P \oplus Q \cong_{\mathrm{gr}} \bigoplus_i A(-\delta_i)$. This gives two graded maps $\phi : P \to \bigoplus_i A(-\delta_i)$ and $\pi : \bigoplus_i A(-\delta_i) \to P$, such that $\pi\phi = 1_P$. Let $\pi_i : \bigoplus_i A(-\delta_i) \to A(-\delta_i)$, $\{a_i\}_{i\in I} \mapsto a_i$ be the projection on the ith component. So if $a = \{a_i\}_{i\in I} \in \bigoplus_i A(-\delta_i)$ then $\sum_i \pi_i(a)e_i = a$, where $\{e_i\}_{i\in I}$ are the standard homogeneous basis of $\bigoplus_i A(-\delta_i)$. Now let $p_i = \pi(e_i)$ and $f_i = \pi_i\phi$. Note that $\deg(p_i) = \delta_i$ and $f_i \in \mathrm{Hom}_{\mathrm{Gr}\text{-}A}(P, A(-\delta_i))$. Clearly $f_i(p) = \pi_i\phi(p)$ is zero for all but a finite number of $i \in I$. This gives (1). Furthermore,

$$\sum_i p_i f_i(p) = \sum_i p_i\pi_i\phi(p) = \sum \pi(e_i)\pi_i\phi(p) = \pi\big(\sum_i e_i\pi_i\phi(p)\big) = \pi\phi(p) = p.$$

This gives (2).

Conversely, suppose that there exists a dual basis $\{p_i, f_i \mid i \in I\}$. Consider the maps $\phi : P \to \bigoplus_i A(-\delta_i)$, $p \mapsto \{f_i(p)\}_{i\in I}$ and $\pi : \bigoplus_i A(-\delta_i) \to P$, $\{a_i\}_{i\in I} \mapsto \sum_i p_i a_i$. One can easily see that ϕ and π are graded right A-module homomorphism and $\pi\phi = 1_P$. Therefore the exact sequence

$$0 \longrightarrow \ker(\pi) \longrightarrow \bigoplus_i A(-\delta_i) \overset{\pi}{\longrightarrow} P \longrightarrow 0$$

splits. Thus P is a direct summand of the graded free module $\bigoplus_i A(-\delta_i)$. By Proposition 1.2.12, P is a graded projective. $\square$

Remark 1.2.14. <u>Graded injective modules</u>

Proposition 1.2.12 shows that an A-module P is graded projective if and only if P is graded and projective. However the similar statement is not valid for graded injective modules. Recall that a graded A-module I is called a *graded injective module* if for any diagram of graded A-module homomorphisms

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "g"] \arrow[d, "j"'] & M \\ & I & \end{tikzcd}$$

there is a graded A-module homomorphism $h : M \to I$ with $hg = j$.

Using Lemma 1.2.11 one can show that a graded module which is injective, is also graded injective. However a graded injective module is not necessarily injective. The reason for this difference between projective and injective modules is that the forgetful functor U is a left adjoint functor (see Remark 1.2.7). In details, consider a graded projective module P and the diagram

$$\begin{tikzcd} & P \arrow[d, "j"] & \\ M \arrow[r, "g"'] & N \arrow[r] & 0. \end{tikzcd}$$

Since the diagram below is commutative

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{Mod}\text{-}A}(U(P), M) \arrow[r, "\cong"] \arrow[d] & \mathrm{Hom}_{\mathrm{Gr}\text{-}A}(P, F(M)) \arrow[d] \\ \mathrm{Hom}_{\mathrm{Mod}\text{-}A}(U(P), N) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathrm{Gr}\text{-}A}(P, F(N)) \end{tikzcd}$$

and there is a graded homomorphism $h' : P \to F(M)$ such that the diagram

$$\begin{tikzcd} & P \arrow[dl, "h'"'] \arrow[d, "j'"] & \\ F(M) \arrow[r, "F(g)"'] & F(N) \arrow[r] & 0, \end{tikzcd}$$

is commutative, there is a homomorphism $h : P \to M$ such that $gh = j$. So P is projective (see Proposition 1.2.12 for another proof).

If the graded group is finite, then the forgetful functor is right adjoint as well (see Remark 1.2.7) and a similar argument as above shows that a graded injective module is injective.

Example 1.2.15. <u>Graded rings associated to filter rings</u>

A ring A with identity is called a *filtered ring* if there is an ascending family $\{A_i \mid i \in \mathbb{Z}\}$ of additive subgroup of A such that $1 \in A_0$ and $A_iA_j \subseteq A_{i+j}$, for all $i, j \in \mathbb{Z}$. Let M be a right A-module where A is a filtered ring. M is called a *filtered module* if there is an ascending family $\{M_i \mid i \in \mathbb{Z}\}$ of additive subgroup of M such that $M_iA_j \subseteq M_{i+j}$, for all $i, j \in \mathbb{Z}$. An A-module homomorphism $f : M \to N$ of filtered modules M and N is called a *filtered homomorphism* if $f(M_i) \subseteq N_i$ for $i \in \mathbb{Z}$. A category consisting of filtered A-modules for objects and filtered homomorphisms as morphisms is denoted by Filt-A. If M is a filtered A-module then $\operatorname{gr} M := \bigoplus_{i\in\mathbb{Z}} M_i/M_{i-1}$ is a $\mathbb{Z}$-graded $\operatorname{gr} A := \bigoplus_{i\in\mathbb{Z}} A_i/A_{i-1}$-module. The operations here are defined naturally. This gives a functor $\operatorname{gr} :$ Filt-$A \to$ Gr-A. In Example 1.4.7 we will use a variation of this construction to associate to a valued division algebra a graded division algebra.

In the theory of filtered rings, one defines the concepts of filtered free and projective modules and under certain conditions the functor gr sends these objects to the corresponding objects in the category Gr-A. For a comprehensive study of filtered rings see [71].

1.2.10 Homogeneous idempotents calculus

The idempotents naturally arise in relation with decomposition of rings and modules. The following facts about idempotents are well known in a non-graded setting and one can check that they translate in the graded setting with the similar proofs (cf. [61, §21]). Let P_i, $1 \leq i \leq l$, be graded right ideals of A such that $A = P_1 \oplus \cdots \oplus P_l$. Then there are homogeneous orthogonal idempotents e_i (hence of degree zero) such that $1 = e_1 + \cdots + e_l$ and $e_iA = P_i$.

Let e and f be homogeneous idempotent elements in the graded ring A. (Note that, in general, there are non-homogeneous idempotents in a graded ring.) Let $\theta : eA \to fA$ be a (not necessarily) right A-module homomorphism. Then $\theta(e) = \theta(e^2) = \theta(e)e = fae$ for some $a \in A$ and for $b \in eA$, $\theta(b) = \theta(eb) = \theta(e)b$. This shows that there is a map

$$\begin{aligned} \operatorname{Hom}_A(eA, fA) &\to fAe, \\ \theta &\mapsto \theta(e) \end{aligned} \tag{1.28}$$

and one can easily check this is a group isomorphism. We have $fAe = \bigoplus_{\gamma\in\Gamma} fA_\gamma e$ and by Theorem 1.2.5,

$$\operatorname{Hom}_A(eA, fA) \cong \bigoplus_{\gamma\in\Gamma} \operatorname{Hom}_A(eA, fA)_\gamma.$$

Then one can see that the homomorphism (1.28) respects the graded decomposition.

Now if $\theta : eA \to fA(\alpha)$, where $\alpha \in \Gamma$, is a graded A-isomorphism, then $x = \theta(e) \in fA_\alpha e$ and $y = \theta^{-1}(f) \in eA_{-\alpha}f$, where x and y are homogeneous of degrees α and $-\alpha$, respectively, such that $yx = e$ and $xy = f$. Finally, for $f = e$, the map (1.28) gives a graded ring isomorphism $\operatorname{End}_A(eA) \cong_{\mathrm{gr}} eAe$. These facts will be used later in Theorem 5.1.3.

1.3 Grading on matrices

Starting from any ring A and any group Γ, one can consider a Γ-grading on the matrix ring $\mathbb{M}_n(A)$. At the first glance, this grading looks somewhat artificial. However, this type of grading on matrices appears quite naturally in the graded rings arising from graphs. In this section we study the grading on matrices. We then include a section on graph algebras (including path algebras and Leavitt path algebras, §1.6). These algebras give us a wealth of examples of graded rings and graded matrix rings.

For a free right A-module V of dimension n, there is a ring isomorphism $\operatorname{End}_A(V) \cong \mathbb{M}_n(A)$. When A is a Γ-graded ring and V is a graded free module of finite rank, by Theorem 1.2.5, $\operatorname{End}_A(V)$ has a natural Γ-grading. This induces a graded structure on the matrix ring $\mathbb{M}_n(A)$. In this section we study this grading on matrices. For an n-tuple $(\delta_1, \dots, \delta_n)$, $\delta_i \in \Gamma$, we construct a grading on the matrix ring $\mathbb{M}_n(A)$, denoted by $\mathbb{M}_n(A)(\delta_1, \dots, \delta_n)$, and we show that

$$\operatorname{End}_A\big(A(-\delta_1) \oplus A(-\delta_2) \oplus \cdots \oplus A(-\delta_n)\big) \cong_{\mathrm{gr}} \mathbb{M}_n(A)(\delta_1, \dots, \delta_n).$$

We will see that these graded structures on matrices appear very naturally when studying the graded structure of path algebras in §1.6.

1.3.1

Let A be a Γ-graded ring and let $M = M_1 \oplus \cdots \oplus M_n$, where M_i are graded finitely generated right A-modules. Then M is also a graded A-module (see §1.2.6). Let

$$\big(\operatorname{Hom}(M_j, M_i)\big)_{1 \le i,j \le n} = \begin{pmatrix} \operatorname{Hom}_A(M_1, M_1) & \operatorname{Hom}_A(M_2, M_1) & \cdots & \operatorname{Hom}_A(M_n, M_1) \\ \operatorname{Hom}_A(M_1, M_2) & \operatorname{Hom}_A(M_2, M_2) & \cdots & \operatorname{Hom}_A(M_n, M_2) \\ \vdots & \vdots & \ddots & \vdots \\ \operatorname{Hom}_A(M_1, M_n) & \operatorname{Hom}_A(M_2, M_n) & \cdots & \operatorname{Hom}_A(M_n, M_n) \end{pmatrix}. \tag{1.29}$$

It is easy to observe that $\big(\operatorname{Hom}(M_j, M_i)\big)_{1 \le i,j \le n}$ forms a ring with the component-wise addition and the matrix multiplication. Furthermore, for $\lambda \in \Gamma$, assigning the additive subgroup

$$\begin{pmatrix} \operatorname{Hom}_A(M_1, M_1)_\lambda & \operatorname{Hom}_A(M_2, M_1)_\lambda & \cdots & \operatorname{Hom}_A(M_n, M_1)_\lambda \\ \operatorname{Hom}_A(M_1, M_2)_\lambda & \operatorname{Hom}_A(M_2, M_2)_\lambda & \cdots & \operatorname{Hom}_A(M_n, M_2)_\lambda \\ \vdots & \vdots & \ddots & \vdots \\ \operatorname{Hom}_A(M_1, M_n)_\lambda & \operatorname{Hom}_A(M_2, M_n)_\lambda & \cdots & \operatorname{Hom}_A(M_n, M_n)_\lambda \end{pmatrix} \tag{1.30}$$

as a γ-homogeneous component of $\big(\operatorname{Hom}(M_j, M_i)\big)_{1 \le i,j \le n}$, using Theorem 1.2.5, one can show that $\big(\operatorname{Hom}(M_j, M_i)\big)_{1 \le i,j \le n}$ is a Γ-graded ring.

Let $\pi_j : M \to M_j$ and $\kappa_j : M_j \to M$ be the (graded) projection and injection homomorphisms. For the next theorem, we need the following identities

$$\sum_{i=1}^{n} \kappa_i \pi_i = \mathrm{id}_M \quad \text{and} \quad \pi_i \kappa_j = \delta_{ij}\, \mathrm{id}_{M_j}, \tag{1.31}$$

where δ_{ij} is the Kronecker delta.

Theorem 1.3.1. *Let A be a Γ-graded ring and $M = M_1 \oplus \cdots \oplus M_n$, where M_i are graded finitely generated right A-modules. Then there is a graded ring isomorphism*

$$\Phi : \mathrm{End}_A(M) \longrightarrow \big(\mathrm{Hom}(M_j, M_i)\big)_{1 \leq i,j \leq n}$$

defined by $f \mapsto (\pi_i f \kappa_j)$, $1 \leq i, j \leq n$.

Proof. The map Φ is clearly well-defined. Since for $f, g \in \mathrm{End}_A(M)$,

$$\Phi(f+g) = \big(\pi_i(f+g)\kappa_j\big)_{1\leq i,j\leq n} = \big(\pi_i f \kappa_j + \pi_i g \kappa_j\big)_{1\leq i,j\leq n} = \big(\pi_i f \kappa_j\big)_{1\leq i,j\leq n} + \big(\pi_i g \kappa_j\big)_{1\leq i,j\leq n} = \Phi(f) + \Phi(g)$$

and

$$\Phi(fg) = \big(\pi_i f g \kappa_j\big)_{1\leq i,j\leq n} = \Big(\pi_i f (\sum_{l=1}^{n} \kappa_l \pi_l) g \kappa_j\Big)_{1\leq i,j\leq n} = \Big(\sum_{l=1}^{n} (\pi_i f \kappa_l)(\pi_l g \kappa_j)\Big)_{1\leq i,j\leq n} = \Phi(f)\Phi(g),$$

Φ is a ring homomorphism. Furthermore, if $f \in \mathrm{End}_A(M)_\lambda$, $\lambda \in \Gamma$, then $\pi_i f \kappa_j \in \mathrm{Hom}_A(M_j, M_i)_\lambda$, for $1 \leq i, j \leq n$. This (see (1.30)) shows that Φ is a graded ring homomorphism. Define the map

$$\begin{aligned} \Psi : \big(\mathrm{Hom}(M_j, M_i)\big)_{1\leq i,j\leq n} &\longrightarrow \mathrm{End}_A(M) \\ (g_{ij})_{1\leq i,j\leq n} &\longmapsto \sum_{1\leq i,j\leq n} \kappa_i g_{ij} \pi_j \end{aligned}$$

Using the identities 1.31, one can observe that the compositions $\Psi\Phi$ and $\Phi\Psi$ give the identity maps of the corresponding rings. Thus Φ is an isomorphism. $\square$

For a graded ring A, observe that

$$\Phi_{\delta_j, \delta_i} : \mathrm{Hom}_A\big(A(\delta_i), A(\delta_j)\big) \cong_{\mathrm{gr}} A(\delta_j - \delta_i), \tag{1.32}$$

such that $\Phi_{\delta_k,\delta_i}(gf) = \Phi_{\delta_k,\delta_j}(g)\Phi_{\delta_j,\delta_i}(f)$, where $f \in \mathrm{Hom}(A(\delta_i), A(\delta_j))$ and $g \in \mathrm{Hom}(A(\delta_j), A(\delta_k))$ (see 1.10). If

$$V = A(-\delta_1) \oplus A(-\delta_2) \oplus \cdots \oplus A(-\delta_n),$$

then by Theorem 1.3.1,

$$\mathrm{End}_A(V) \cong_{\mathrm{gr}} \Big(\mathrm{Hom}\big(A(-\delta_j), A(-\delta_i)\big)\Big)_{1\leq i,j\leq n}.$$

Applying Φ defined in (1.32) to each entry, we have

$$\mathrm{End}_A(V) \cong_{\mathrm{gr}} \Big(\mathrm{Hom}\big(A(-\delta_j), A(-\delta_i)\big)\Big)_{1\leq i,j\leq n} \cong_{\mathrm{gr}} \big(A(\delta_j - \delta_i)\big)_{1\leq i,j\leq n}.$$

Denoting the graded matrix ring $\big(A(\delta_j - \delta_i)\big)_{1\leq i,j\leq n}$ by $\mathbb{M}_n(A)(\delta_1,\dots,\delta_n)$, we have

$$\mathbb{M}_n(A)(\delta_1,\dots,\delta_n) = \begin{pmatrix} A(\delta_1-\delta_1) & A(\delta_2-\delta_1) & \cdots & A(\delta_n-\delta_1) \\ A(\delta_1-\delta_2) & A(\delta_2-\delta_2) & \cdots & A(\delta_n-\delta_2) \\ \vdots & \vdots & \ddots & \vdots \\ A(\delta_1-\delta_n) & A(\delta_2-\delta_n) & \cdots & A(\delta_n-\delta_n) \end{pmatrix}. \tag{1.33}$$

By (1.30), $\mathbb{M}_n(A)(\delta_1,\dots,\delta_n)_\lambda$, the λ-homogeneous elements, are the $n\times n$-matrices over A with the degree shifted (suspended) as follows:

$$\boxed{\mathbb{M}_n(A)(\delta_1,\dots,\delta_n)_\lambda = \begin{pmatrix} A_{\lambda+\delta_1-\delta_1} & A_{\lambda+\delta_2-\delta_1} & \cdots & A_{\lambda+\delta_n-\delta_1} \\ A_{\lambda+\delta_1-\delta_2} & A_{\lambda+\delta_2-\delta_2} & \cdots & A_{\lambda+\delta_n-\delta_2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{\lambda+\delta_1-\delta_n} & A_{\lambda+\delta_2-\delta_n} & \cdots & A_{\lambda+\delta_n-\delta_n} \end{pmatrix}.} \tag{1.34}$$

This also shows that

$$\deg(e_{ij}(x)) = \deg(x) + \delta_i - \delta_j. \tag{1.35}$$

In particular the zero homogeneous component (which is a ring) is of the form

$$\boxed{\mathbb{M}_n(A)(\delta_1,\dots,\delta_n)_0 = \begin{pmatrix} A_0 & A_{\delta_2-\delta_1} & \cdots & A_{\delta_n-\delta_1} \\ A_{\delta_1-\delta_2} & A_0 & \cdots & A_{\delta_n-\delta_2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{\delta_1-\delta_n} & A_{\delta_2-\delta_n} & \cdots & A_0 \end{pmatrix}.} \tag{1.36}$$

Setting $\overline{\delta} = (\delta_1,\dots,\delta_n) \in \Gamma^n$, one denotes the graded matrix ring (1.33) by $\mathbb{M}_n(A)(\overline{\delta})$. To summarise, we have shown that there is a graded ring isomorphism

$$\boxed{\operatorname{End}_A\big(A(-\delta_1)\oplus A(-\delta_2)\oplus\cdots\oplus A(-\delta_n)\big) \cong_{\mathrm{gr}} \mathbb{M}_n(A)(\delta_1,\dots,\delta_n).} \tag{1.37}$$

Consider the graded A-bimodule $A^n(\overline{\delta}) = A(\delta_1)\oplus\cdots\oplus A(\delta_n)$. Then one can check that $A^n(\overline{\delta})$ is a graded right $\mathbb{M}_n(A)(\overline{\delta})$-module and $A^n(-\overline{\delta})$ is a graded left $\mathbb{M}_n(A)(\overline{\delta})$-module, where $-\overline{\delta} = (-\delta_1,\dots,-\delta_n)$. These will be used in the Morita theory (see Proposition 2.1.1).

One can easily check the graded ring $R = \mathbb{M}_n(A)(\overline{\delta})$, where $\overline{\delta} = (\delta_1,\dots\delta_n)$, $\delta_i\in\Gamma$, has the support

$$\Gamma_R = \bigcup_{1\leq i,j\leq n} \Gamma_A + \delta_i - \delta_j. \tag{1.38}$$

One can rearrange the shifting, without changing the graded matrix ring as the following theorem shows (see also [73, pp. 60-61]).

Theorem 1.3.2. *Let A be a Γ-graded ring and $\delta_i\in\Gamma$, $1\leq i\leq n$.*

(1) *If $\alpha\in\Gamma$, and $\pi\in S_n$ is a permutation then*

$$\mathbb{M}_n(A)(\delta_1,\dots,\delta_n) \cong_{\mathrm{gr}} \mathbb{M}_n(A)(\delta_{\pi(1)}+\alpha,\dots,\delta_{\pi(n)}+\alpha). \tag{1.39}$$

(2) *If* $\tau_1, \dots, \tau_n \in \Gamma_A^*$, *then*

$$\mathbb{M}_n(A)(\delta_1, \dots, \delta_n) \cong_{\mathrm{gr}} \mathbb{M}_n(A)(\delta_1 + \tau_1, \dots, \delta_n + \tau_n). \tag{1.40}$$

Proof. (1) Let V be a graded free module over A with a homogeneous basis $v_1, \dots, v_n$ of degree $\lambda_1, \dots, \lambda_n$, respectively. It is easy to see that

$$V \cong_{\mathrm{gr}} A(-\lambda_1) \oplus \dots \oplus A(-\lambda_n),$$

and thus $\mathrm{End}_A(V) \cong_{\mathrm{gr}} \mathbb{M}_n(A)(\lambda_1, \dots, \lambda_n)$. Now let $\pi \in S_n$. Rearranging the homogeneous basis as $v_{\pi(1)}, \dots, v_{\pi(n)}$ and defining the A-graded isomorphism $\phi : V \to V$ by $\phi(v_i) = v_{\pi^{-1}(i)}$, we get a graded isomorphism in the level of endomorphism rings

$$\mathbb{M}_n(A)(\lambda_1, \dots, \lambda_n) \cong_{\mathrm{gr}} \mathrm{End}_A(V) \xrightarrow{\phi} \mathrm{End}_A(V) \cong_{\mathrm{gr}} \mathbb{M}_n(A)(\lambda_{\pi(1)}, \dots, \lambda_{\pi(n)}).$$

Moreover, (1.34) shows that it does not make any difference adding a fixed $\alpha \in \Gamma$ to each of the entries in the shifting. This gives us (1.39).

(2) For (1.40), let $\tau_i \in \Gamma_A^*$, $1 \leq i \leq n$, that is, $\tau_i = \deg(u_i)$ for some units $u_i \in A^h$. Consider the basis $v_i u_i$, $1 \leq i \leq n$ for V. With this basis, $\mathrm{End}_A(V) \cong_{\mathrm{gr}} \mathbb{M}_n(A)(\delta_1 + \tau_1, \dots, \delta_n + \tau_n)$. Consider the A-graded isomorphism $\mathrm{id} : V \to V$, by $\mathrm{id}(v_i) = (v_i u_i) u_i^{-1}$. A similar argument as Part (1) now gives (1.40). ☐

Note that if A has a trivial Γ-grading, i.e., $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$, where $A_0 = A$ and $A_\gamma = 0$, for $0 \neq \gamma \in \Gamma$, this construction induces a *good grading* on $\mathbb{M}_n(A)$. By definition, this is a grading on $\mathbb{M}_n(A)$ such that e_{ij}, the matrix with 1 in the ij-position and zero everywhere else, is homogeneous, for $1 \leq i, j \leq n$. This particular group gradings on matrix rings have been studied by Dăscălescu et al. [32] (see Remark 1.3.8). Therefore for $x \in A$,

$$\deg(e_{ij}(x)) = \delta_i - \delta_j. \tag{1.41}$$

One can easily check that for a ring A with trivial Γ-grading, the graded ring $\mathbb{M}_n(A)(\overline{\delta})$, where $\overline{\delta} = (\delta_1, \dots \delta_n)$, $\delta_i \in \Gamma$, has the support $\{\delta_i - \delta_j \mid 1 \leq i, j \leq n\}$. (This follows also immediately from (1.38).)

The grading on matrices appears quite naturally in the graded rings arising from graphs. We will show that the graded structure Leavitt path algebras of acyclic and comet graphs are in effect the graded matrix rings as constructed above (see §1.6, in particular, Theorems 1.6.15 and 1.6.17).

Example 1.3.3. Let A be a ring, Γ be a group and A be graded trivially by Γ, i.e., A is concentrated in degree zero (see §1.1.1). Consider the Γ-graded matrix ring $R = \mathbb{M}_n(A)(0, -1, \dots, -n+1)$, where $n \in \mathbb{N}$. By (1.38) the support of R is the set $\{-n+1, -n+2, \dots, n-2, n-1\}$ and by (1.34) we have the following arrangements for the homogeneous elements of R. For $k \in \mathbb{Z}$,

$$R_k = \begin{pmatrix} A_k & A_{k-1} & \dots & A_{k+1-n} \\ A_{k+1} & A_k & \dots & A_{k+2-n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k+n-1} & A_{k+n-2} & \dots & A_k \end{pmatrix}.$$

Thus

$$R_0 = \begin{pmatrix} A & 0 & \dots & 0 \\ 0 & A & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A \end{pmatrix}, R_{-1} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ A & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & A & 0 \end{pmatrix}, \dots, R_{-n+1} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A & 0 & \dots & 0 \end{pmatrix},$$

$$R_1 = \begin{pmatrix} 0 & A & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & A \\ 0 & 0 & \dots & 0 \end{pmatrix}, \dots, R_{n-1} = \begin{pmatrix} 0 & 0 & \dots & A \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

In Chapter 2, we will see that $R = \mathbb{M}_n(A)(0, -1, \dots, -n+1)$ is graded Morita equivalent to the trivially graded ring A.

Example 1.3.4. Let R be a ring, $R[x, x^{-1}]$ the $\mathbb{Z}$-graded Laurent polynomial ring and $A = R[x^3, x^{-3}]$ the $\mathbb{Z}$-graded subring with the support $3\mathbb{Z}$ (see Example 1.1.20). Consider the $\mathbb{Z}$-graded matrix ring

$$\mathbb{M}_6(A)(0, 1, 1, 2, 2, 3).$$

By (1.34), the homogeneous elements of degree 1 have the form

$$\begin{pmatrix} A_1 & A_0 & A_0 & A_{-1} & A_{-1} & A_{-2} \\ A_2 & A_1 & A_1 & A_0 & A_0 & A_{-1} \\ A_2 & A_1 & A_1 & A_0 & A_0 & A_{-1} \\ A_3 & A_2 & A_2 & A_1 & A_1 & A_0 \\ A_3 & A_2 & A_2 & A_1 & A_1 & A_0 \\ A_4 & A_3 & A_3 & A_2 & A_2 & A_1 \end{pmatrix} = \begin{pmatrix} 0 & R & R & 0 & 0 & 0 \\ 0 & 0 & 0 & R & R & 0 \\ 0 & 0 & 0 & R & R & 0 \\ Rx^3 & 0 & 0 & 0 & 0 & R \\ Rx^3 & 0 & 0 & 0 & 0 & R \\ 0 & Rx^3 & Rx^3 & 0 & 0 & 0 \end{pmatrix}.$$

Example 1.3.5. Let K be a field. Consider the $\mathbb{Z}$-graded ring $A = \mathbb{M}_5(K)(0, 1, 2, 2, 3)$. Then the support of this ring is $\{0, \pm 1, \pm 2\}$ and by (1.36) the zero homogeneous component (which is a ring) is

$$A_0 = \begin{pmatrix} K & 0 & 0 & 0 & 0 \\ 0 & K & 0 & 0 & 0 \\ 0 & 0 & K & K & 0 \\ 0 & 0 & K & K & 0 \\ 0 & 0 & 0 & 0 & K \end{pmatrix} \cong K \oplus K \oplus \mathbb{M}_2(K) \oplus K.$$

Example 1.3.6. $\underline{\mathbb{M}_n(A)(\delta_1, \dots, \delta_n), \text{ FOR } \Gamma = \{\delta_1, \dots, \delta_n\}, \text{ IS A SKEW GROUP RING}}$

Let A be a Γ-graded ring where $\Gamma = \{\delta_1, \dots, \delta_n\}$ is a finite group. Consider $\mathbb{M}_n(A)(\delta_1, \dots, \delta_n)$, which is a Γ-graded ring with its homogeneous components described by (1.34). We will show that this graded ring is a skew group ring $\mathbb{M}_n(A)_0 \star \Gamma$. In particular, by Proposition 1.1.16(3), it is a strongly graded ring. Consider the matrix $u_\alpha \in \mathbb{M}_n(A)(\delta_1, \dots, \delta_n)_\alpha$, where in each row i, we have 1 in (i, j) position, where $\delta_j - \delta_i + \alpha = 0$ and zero everywhere else. One can easily see that u_α is a permutation matrix with exactly one 1 in each row and column. Furthermore, for $\alpha, \beta \in \Gamma$, $u_\alpha u_\beta = u_{\alpha+\beta}$. Indeed, consider the ith row of u_α, with the only 1 in jth column where $\delta_j - \delta_i + \alpha = 0$. Now, consider the jth row of u_β with a kth column such that $\delta_k - \delta_j + \beta = 0$ and so with 1 in (j, k) row. Thus multiplying u_α with u_β, we have zero everywhere in ith row except in (i, k)th position. On the other hand since $\delta_k - \delta_i + \alpha + \beta = 0$, in ith row of $u_{\alpha+\beta}$ we have zero except in (i, k)th position. Repeating this argument for each row of u_α shows that $u_\alpha u_\beta = u_{\alpha+\beta}$.

Now defining $\phi : \Gamma \to \mathrm{Aut}(\mathbb{M}_n(A)_0)$ by $\phi(\alpha)(a) = u_\alpha a {u_\alpha}^{-1}$, and setting the 2-cycle ψ trivial, by §1.1.3, $R = \mathbb{M}_n(A)_0 \star_\phi \Gamma$.

This was observed in [74], where it was proved that $\mathbb{M}_n(A)(\delta_1, \dots, \delta_n)_0$ is isomorphic to the smash product of Cohen and Montgomery [27] (see Remark 2.3.11).

Example 1.3.7. The following examples (from [32, Example 1.3]) provide two instances of $\mathbb{Z}_2$-grading on $\mathbb{M}_2(K)$, where K is a field. The first grading is a good grading, whereas the second one is not a good grading.

1. Let $R = \mathbb{M}_2(K)$ with the $\mathbb{Z}_2$-grading defined by
$$R_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in K \right\} \quad \text{and} \quad R_1 = \left\{ \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix} \mid c, d \in K \right\}.$$
Since $e_{11}, e_{22} \in R_0$ and $e_{12}, e_{21} \in R_1$, by definition, this is a good grading. Note that $R = \mathbb{M}_2(K)(0, 1)$.

2. Let $S = \mathbb{M}_2(K)$ with the $\mathbb{Z}_2$-grading defined by
$$S_0 = \left\{ \begin{pmatrix} a & b-a \\ 0 & b \end{pmatrix} \mid a, b \in K \right\} \quad \text{and} \quad S_1 = \left\{ \begin{pmatrix} d & c \\ d & -d \end{pmatrix} \mid c, d \in K \right\}.$$
Then S is a graded ring, such that the $\mathbb{Z}_2$-grading is not a good grading, since e_{11} is not homogeneous. Furthermore, comparing S_0 with (1.36), shows that the grading on S does not come from the construction given by (1.33).

   Consider the map
$$f : R \longrightarrow S; \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a+c & b+d-a-c \\ c & d-c \end{pmatrix}.$$
This map is in fact a graded ring isomorphism, and so $R \cong_{\mathrm{gr}} S$. This shows that the good grading is not preserved under graded isomorphisms.

Remark 1.3.8. <u>Good gradings on matrix algebras</u>

Let K be a field and Γ be an abelian group. One can put a Γ-grading on the ring $\mathbb{M}_n(K)$, by assigning a degree (an element of the group Γ) to each matrix $e_{i,j}$, $1 \leq i, j \leq n$. This is called a *good grading*. This grading has been studied in [32]. In particular it has been shown that a grading on $\mathbb{M}_n(K)$ is good if and only if it can be described as $\mathbb{M}_n(K)(\delta_1, \dots, \delta_n)$ for some $\delta_i \in \Gamma$. Furthermore, any grading on $\mathbb{M}_n(K)$ is a good grading if Γ is torsion free. It has also been shown that if $R = \mathbb{M}_n(K)$ has a Γ-grading such that e_{ij} is a homogeneous for some $1 \leq i, j \leq n$, then there exists a good grading on $S = \mathbb{M}_n(K)$ with a graded isomorphism $R \cong S$. It is shown that if Γ is finite, then the number of good gradings on $\mathbb{M}_n(K)$ is $|\Gamma|^{n-1}$. Furthermore, (for a finite Γ) the class of strongly graded and crossed-product good gradings of $\mathbb{M}_n(K)$ have been classified.

1.3.2 Mixed shifting

For a Γ-graded ring A, $\overline{\alpha} = (\alpha_1, \dots \alpha_m) \in \Gamma^m$ and $\overline{\delta} = (\delta_1, \dots, \delta_n) \in \Gamma^n$, set

$$\boxed{\mathbb{M}_{m \times n}(A)[\overline{\alpha}][\overline{\delta}] := \begin{pmatrix} A_{\alpha_1-\delta_1} & A_{\alpha_1-\delta_2} & \cdots & A_{\alpha_1-\delta_n} \\ A_{\alpha_2-\delta_1} & A_{\alpha_2-\delta_2} & \cdots & A_{\alpha_2-\delta_n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{\alpha_m-\delta_1} & A_{\alpha_m-\delta_2} & \cdots & A_{\alpha_m-\delta_n} \end{pmatrix}}.$$

So $\mathbb{M}_{m\times n}(A)[\overline{\alpha}][\overline{\delta}]$ consists of matrices with the ij-entry in $A_{\alpha_i-\delta_j}$. If $a \in \mathbb{M}_{m\times n}(A)[\overline{\alpha}][\overline{\delta}]$, then one can easily check that multiplying a from the left induces a graded right A-module homomorphism

$$\begin{aligned} A^n(\overline{\delta}) &\longrightarrow A^m(\overline{\alpha}), \\ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} &\longmapsto a \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}. \end{aligned} \tag{1.42}$$

Conversely, suppose $\phi : A^n(\overline{\delta}) \to A^m(\overline{\alpha})$ is graded right A-module homomorphism. Let e_j denote the standard basis element of $A^n(\overline{\delta})$ with 1 in the j-th entry and zeros elsewhere. Let $\phi(e_j) = (a_{1j}, a_{2j}, \dots, a_{mj})$, $1 \le j \le n$. Since ϕ is a graded map, comparing the grading of both sides, one can observe that $\deg(a_{ij}) = \alpha_i - \delta_j$. So that the map ϕ is represented by the left multiplication with the matrix $a = (a_{ij})_{m\times n} \in \mathbb{M}_{m\times n}(A)[\overline{\alpha}][\overline{\delta}]$.

The following simple Lemma comes in handy.

Lemma 1.3.9. *Let $a \in \mathbb{M}_{m\times n}(A)[\overline{\alpha}][\overline{\delta}]$ and $b \in \mathbb{M}_{n\times k}(A)[\overline{\delta}][\overline{\beta}]$. Then $ab \in \mathbb{M}_{m\times k}(A)[\overline{\alpha}][\overline{\beta}]$.*

Proof. Let $\phi_a : A^n(\overline{\delta}) \to A^m(\overline{\alpha})$ and $\phi_b : A^k(\overline{\beta}) \to A^n(\overline{\delta})$ be graded A-module homomorphisms induced by multiplications with a and b, respectively. Then

$$\phi_{ab} = \phi_a\phi_b : A^k(\overline{\beta}) \longrightarrow A^m(\overline{\alpha}).$$

This shows that $ab \in \mathbb{M}_{m\times k}(A)[\overline{\alpha}][\overline{\beta}]$. (This can also be checked directly, by multiplying the matrices a and b and taking into account the shifting arrangements.) $\square$

In particular $\mathbb{M}_{m\times m}(A)[\overline{\alpha}][\overline{\alpha}]$ represents $\mathrm{End}(A^m(\alpha), A^m(\alpha))_0$. Combining this with (1.37), we get

$$\mathbb{M}_{m\times m}(A)[\overline{\alpha}][\overline{\alpha}] = \mathbb{M}_m(A)(-\overline{\alpha})_0. \tag{1.43}$$

Proposition 1.3.10. *Let A be a Γ-graded ring and let $\overline{\alpha} = (\alpha_1, \dots, \alpha_m) \in \Gamma^m$, $\overline{\delta} = (\delta_1, \dots, \delta_n) \in \Gamma^n$. Then the following are equivalent:*

(1) *$A^m(\overline{\alpha}) \cong_{\mathrm{gr}} A^n(\overline{\delta})$ as graded right A-modules;*

(2) *$A^m(-\overline{\alpha}) \cong_{\mathrm{gr}} A^n(-\overline{\delta})$ as graded left A-modules;*

(3) *There exist $a = (a_{ij}) \in \mathbb{M}_{n\times m}(A)[\overline{\delta}][\overline{\alpha}]$ and $b = (b_{ij}) \in \mathbb{M}_{m\times n}(A)[\overline{\alpha}][\overline{\delta}]$ such that $ab = \mathbb{I}_n$ and $ba = \mathbb{I}_m$.*

Proof. (1) $\Rightarrow$ (3) Let $\phi : A^m(\overline{\alpha}) \to A^n(\overline{\delta})$ and $\psi : A^n(\overline{\delta}) \to A^m(\overline{\alpha})$ be graded right A-module isomorphisms such that $\phi\psi = 1$ and $\psi\phi = 1$. The paragraph prior to the Proposition shows that the map ϕ is represented by the left multiplication with a matrix $a = (a_{ij})_{n\times m} \in \mathbb{M}_{n\times m}(A)[\overline{\delta}][\overline{\alpha}]$. In the same way one can construct $b \in \mathbb{M}_{m\times n}(A)[\overline{\alpha}][\overline{\delta}]$ which induces ψ. Now $\phi\psi = 1$ and $\psi\phi = 1$ translate to $ab = \mathbb{I}_n$ and $ba = \mathbb{I}_m$.

(3) $\Rightarrow$ (1) If $a \in \mathbb{M}_{n\times m}(A)[\overline{\delta}][\overline{\alpha}]$, then multiplication from the left, induces a graded right A-module homomorphism $\phi_a : A^m(\overline{\alpha}) \longrightarrow A^n(\overline{\delta})$. Similarly b induces $\psi_b : A^n(\overline{\delta}) \longrightarrow A^m(\overline{\alpha})$. Now $ab = \mathbb{I}_n$ and $ba = \mathbb{I}_m$ translate to $\phi_a\psi_b = 1$ and $\psi_b\phi_a = 1$.

(2) $\iff$ (3) This part is similar to the previous cases by considering the matrix multiplication from the right. Namely, the graded left A-module homomorphism $\phi : A^m(-\overline{\alpha}) \to A^n(-\overline{\delta})$ represented by a matrix multiplication from the right of the form $\mathbb{M}_{m\times n}(A)[\overline{\alpha}][\overline{\delta}]$ and similarly ψ gives a matrix in $\mathbb{M}_{n\times m}(A)[\overline{\delta}][\overline{\alpha}]$. The rest follows easily. $\square$

The following corollary shows that $A(\alpha) \cong_{\mathrm{gr}} A$ as graded right A-modules if and only if $\alpha \in \Gamma_A^*$. In fact, replacing $m = n = 1$ in Proposition 1.3.10 we obtain the following.

Corollary 1.3.11. *Let A be a Γ-graded ring and $\alpha \in \Gamma$. Then the following are equivalent:*

(1) *$A(\alpha) \cong_{\mathrm{gr}} A$ as graded right A-modules;*

(2) *$A(-\alpha) \cong_{\mathrm{gr}} A$ as graded right A-modules;*

(3) *$A(\alpha) \cong_{\mathrm{gr}} A$ as graded left A-modules;*

(4) *$A(-\alpha) \cong_{\mathrm{gr}} A$ as graded left A-modules;*

(5) *There is an invertible homogeneous element of degree α.*

Proof. This follows from Proposition 1.3.10. □

Corollary 1.3.12. *Let A be a Γ-graded ring. Then the following are equivalent:*

(1) *A is a crossed-product;*

(2) *$A(\alpha) \cong_{\mathrm{gr}} A$, as graded right A-modules, for all $\alpha \in \Gamma$;*

(3) *$A(\alpha) \cong_{\mathrm{gr}} A$, as graded left A-modules, for all $\alpha \in \Gamma$;*

(4) *The shift functor $\mathcal{T}_\alpha : \text{Gr-}A \to \text{Gr-}A$ is isomorphic to identity functor, for all $\alpha \in \Gamma$.*

Proof. This follows from Corollary 1.3.11, (1.13) and the definition of the crossed-product rings (§1.1.2). □

The Corollary above will be used to show that the action of Γ on the graded Grothendieck group of a crossed-product algebra is trivial (see Example 3.1.9).

Example 1.3.13. <u>The Leavitt algebra $\mathcal{L}(1, n)$</u>

In [63] Leavitt considered the free associative ring A with coefficient in $\mathbb{Z}$ generated by symbols $\{x_i, y_i \mid 1 \leq i \leq n\}$ subject to relations

$$x_i y_j = \delta_{ij}, \text{ for all } 1 \leq i, j \leq n, \qquad (1.44)$$
$$\sum_{i=1}^{n} y_i x_i = 1,$$

where $n \geq 2$ and δ_{ij} is the Kronecker delta. The relations guarantee the right A-module homomorphism

$$\begin{aligned} \phi : A &\longrightarrow A^n \\ a &\mapsto (x_1 a, x_2 a, \dots, x_n a) \end{aligned} \qquad (1.45)$$

has an inverse

$$\begin{aligned} \psi : A^n &\longrightarrow A \\ (a_1, \dots, a_n) &\mapsto y_1 a_1 + \dots + y_n a_n, \end{aligned} \qquad (1.46)$$

so $A \cong A^n$ as a right A-module. He showed that A is universal with respect to this property, of type $(1, n-1)$ (see §1.7) and it is a simple ring.

Leavitt's algebra constructed in (1.44) has a natural grading; assigning 1 to y_i and -1 to x_i, $1 \leq i \leq n$, since the relations are homogeneous (of degree zero), the ring A is a $\mathbb{Z}$-graded ring (see §1.6.1 for a general construction of graded rings from free algebras). The isomorphism (1.45) induces a graded isomorphism

$$\begin{aligned} \phi : A &\longrightarrow A(-1)^n \\ a &\mapsto (x_1 a, x_2 a, \dots, x_n a), \end{aligned} \tag{1.47}$$

where $A(-1)$ is the suspension of A by -1. In fact letting

$$y = (y_1, \dots, y_n), \text{ and}$$
$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

we have $y \in \mathbb{M}_{1\times n}(A)[\overline{\alpha}][\overline{\delta}]$ and $x \in \mathbb{M}_{n\times 1}(A)[\overline{\delta}][\overline{\alpha}]$, where $\overline{\alpha} = (0)$ and $\overline{\delta} = (-1, \dots, -1)$. Thus by Proposition 1.3.10, $A \cong_{\text{gr}} A(-1)^n$.

Motivated by this algebra, the Leavitt path algebras were introduced in [2, 6], which associate to a direct graph a certain algebra. When the graph has one vertex and n loops, the algebra corresponds to this graph is the Leavitt algebra constructed in (1.44) and is denoted by $\mathcal{L}(1, n)$ or $\mathcal{L}_n$. The Leavitt path algebras will provide a vast array of examples of graded algebras. We will study these algebras in §1.6.3.

1.4 Graded division rings

Graded fields and its noncommutative version, graded division rings, are among the simplest graded rings. With a little effort, we can completely compute the invariants of these algebras which we are interested in, namely, the graded Grothendieck groups (§3.6) and the graded Picard groups (§4).

Recall from §1.1.4 that a Γ-graded ring $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$ is called a graded division ring if every non-zero homogeneous element has a multiplicative inverse. Throughout this section we are considering graded (right) modules over graded division rings. Note that we work with the abelian graded group, however all the results are valid for non-abelian grading as well. We first show that for graded modules over a graded division ring, there is well-defined notion of dimension. The proofs follow the standard proofs in the non-graded setting (see [52, §IV, Thms. 2.4, 2.7, 2.13]), or the graded setting (see [73, Prop. 4.6.1], [51, p. 79]).

Proposition 1.4.1. *Let A be a Γ-graded division ring and M be a graded A-module. Then M is a graded free A-module. More generally, any linearly independent subset of M consisting of homogeneous elements can be extended to a homogeneous basis of M.*

Proof. Since A is a graded division ring (i.e., all homogeneous elements are invertible), for any $m \in M^h$, $\{m\}$ is a linearly independent subset of M. This immediately gives the first statement of the theorem as a consequence of the second.

Fix a linearly independent subset X of M consisting of homogeneous elements. Let

$$F = \{Q \subseteq M^h \mid X \subseteq Q \text{ and } Q \text{ is } A\text{-linearly independent}\}.$$

This is a non-empty partially ordered set with inclusion, and every chain $Q_1 \subseteq Q_2 \subseteq \dots$ in F has an upper bound $\bigcup Q_i \in F$. By Zorn's Lemma, F has a maximal element, which we denote by P. If $\langle P \rangle \neq M$, then there is a homogeneous element $m \in M^h \setminus \langle P \rangle$. We will show that $P \cup \{m\}$ is a linearly independent set containing X, contradicting the maximality of P.

Suppose $ma + \sum p_i a_i = 0$, where $a, a_i \in A$, $p_i \in P$ with $a \neq 0$. Then there is a homogeneous component of a, say a_λ, which is also non-zero. Considering the $\lambda \deg(m)$-homogeneous component of this sum, we have $m = m a_\lambda a_\lambda^{-1} = -\sum p_i a_i' a_\lambda^{-1}$ for a_i' homogeneous, which contradicts the assumption $m \in M^h \setminus \langle P \rangle$. Hence $a = 0$, which implies each $a_i = 0$. This gives the required contradiction, so $M = \langle P \rangle$, completing the proof. □

The following proposition shows in particular that a graded division ring has graded invariant basis number (we will discuss this type of rings in §1.7).

Proposition 1.4.2. *Let A be a Γ-graded division ring and M be a graded A-module. Then any two homogeneous bases of M over A have the same cardinality.*

Proof. By [52, Thm. IV.2.6], if a module M has an infinite basis over a ring, then any other basis of M has the same cardinality. This proves the Proposition in the case the the homogeneous basis is infinite.

Now suppose that M has two finite homogeneous bases X and Y. Then $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_m\}$, for x_i, $y_i \in M^h \setminus 0$. As X is a basis for M, we can write

$$y_m = x_1 a_1 + \cdots + x_n a_n,$$

for some $a_i \in A^h$, where $\deg(y_m) = \deg(a_i)\deg(x_i)$ for each $1 \leq i \leq n$. Since $y_m \neq 0$, we have at least one $a_i \neq 0$. Let a_k be the first non-zero a_i, and we note that a_k is invertible as it is non-zero and homogeneous in A. Then

$$x_k = y_m a_k^{-1} - x_{k+1} a_{k+1} a_k^{-1} - \cdots - x_n a_n a_k^{-1},$$

and the set $X' = \{y_m, x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n\}$ spans M since X spans M. So

$$y_{m-1} = y_m b_m + x_1 c_1 + \cdots + x_{k-1} c_{k-1} + x_{k+1} c_{k+1} + \cdots + x_n c_n,$$

for b_m, $c_i \in A^h$. There is at least one non-zero c_i, since if all the c_i are zero, then either y_m and y_{m-1} are linearly dependent or y_{m-1} is zero which are not the case. Let c_j denote the first non-zero c_i. Then x_j can be written as a linear combination of y_{m-1}, y_m and those x_i with $i \neq j, k$. Therefore the set $X'' = \{y_{m-1}, y_m\} \cup \{x_i : i \neq j, k\}$ spans M since X' spans M.

Continuing this process of adding a y and removing an x gives, after the k-th step, a set which spans M consisting of $y_m, y_{m-1}, \dots, y_{m-k+1}$ and $n-k$ of the x_i. If $n < m$, then after the n-th step, we would have that the set $\{y_m, \dots, y_{m-n+1}\}$ spans M. But if $n < m$, then $m - n + 1 \geq 2$, so this set does not contain y_1, and therefore y_1 can be written as a linear combination of the elements of this set. This contradicts the linear independence of Y, so we must have $m \leq n$. Repeating a similar argument with X and Y interchanged gives $n \leq m$, so $n = m$. □

The Propositions 1.4.1 and 1.4.2 above show that for a graded module M over a graded division ring A, M has a homogeneous basis and any two homogeneous bases of M have the same cardinality. The cardinal number of any homogeneous basis of M is called the *dimension* of M over A, and it is denoted by $\dim_A(M)$ or $[M : A]$.

Proposition 1.4.3. *Let A be a Γ-graded division ring and M be a graded A-module. If N is a graded submodule of M, then*

$$\dim_A(N) + \dim_A(M/N) = \dim_A(M).$$

Proof. By Proposition 1.4.1, the submodule N is a graded free A-module with a homogeneous basis Y which can be extended to a homogeneous basis X of M. We will show that $U = \{x + N \mid x \in X \setminus Y\}$ is a homogeneous basis of M/N. Note that by (1.8), U consists of homogeneous elements. Let $t \in (M/N)^h$. Again by (1.8), $t = m + N$, where $m \in M^h$ and $m = \sum x_i a_i + \sum y_j b_j$ where a_i, $b_j \in A$, $y_j \in Y$ and $x_i \in X \setminus Y$. So $m + N = \sum (x_i + N)a_i$, which shows that U spans M/N. If $\sum (x_i + N)a_i = 0$, for $a_i \in A$, $x_i \in X \setminus Y$, then $\sum x_i a_i \in N$ which implies that $\sum x_i a_i = \sum y_k b_k$ for $b_k \in A$ and $y_k \in Y$, which implies that $a_i = 0$ and $b_k = 0$ for all i, k. Therefore U is a homogeneous basis for M/N and as we can construct a bijective map $X \setminus Y \to U$, we have $|U| = |X \setminus Y|$. Then $\dim_A M = |X| = |Y| + |X \setminus Y| = |Y| + |U| = \dim_A N + \dim_A(M/N)$. □

The following statement is the graded version of a similar statement on simple rings (see [52, §IX.1]). This is required for the proof of Theorem 1.4.5.

Proposition 1.4.4. *Let A and B be Γ-graded division rings. If*

$$\mathbb{M}_n(A)(\lambda_1, \dots, \lambda_n) \cong_{\mathrm{gr}} \mathbb{M}_m(B)(\gamma_1, \dots, \gamma_m)$$

as graded rings, where $\lambda_i, \gamma_j \in \Gamma$, $1 \leq i \leq n$, $1 \leq j \leq m$, then $n = m$ and $A \cong_{\mathrm{gr}} B$.

Proof. The proof follows the ungraded case (see [52, §IX.1]) with an extra attention given to the grading. We refer the reader to [70, §4.3] for a detailed proof. □

We can further determine the relations between the shifting $(\lambda_1, \dots, \lambda_n)$ and $(\gamma_1, \dots, \gamma_m)$ in the above proposition. For this we need to extend [25, Theorem 2.1] (see also [73, Theorem 9.2.18]) from fields (with trivial grading) to graded division algebras. The following theorem states that two graded matrix algebras over a graded division ring with two shiftings are isomorphic if and only if one can obtain one shifting from the other by applying (1.39) and (1.40).

Theorem 1.4.5. *Let A be a Γ-graded division ring. Then for $\lambda_i, \gamma_j \in \Gamma$, $1 \leq i \leq n$, $1 \leq j \leq m$,*

$$\mathbb{M}_n(A)(\lambda_1, \dots, \lambda_n) \cong_{\mathrm{gr}} \mathbb{M}_m(A)(\gamma_1, \dots, \gamma_m) \tag{1.48}$$

if and only if $n = m$ and for a suitable permutation $\pi \in S_n$, we have $\lambda_i = \gamma_{\pi(i)} + \tau_i + \sigma$, $1 \leq i \leq n$, where $\tau_i \in \Gamma_A$ and a fixed $\sigma \in \Gamma$, i.e., $(\lambda_1, \dots, \lambda_n)$ is obtained from $(\gamma_1, \dots, \gamma_m)$ by applying (1.39) and (1.40).

Proof. One direction is Theorem 1.3.2, noting that since A is a graded division ring, $\Gamma_A = \Gamma_A^*$.

We now prove the converse. That $n = m$ follows from Proposition 1.4.4. By (1.4.1) one can find $\epsilon = (\varepsilon_1, \dots, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_2, \dots, \varepsilon_k, \dots, \varepsilon_k)$ in Γ such that $\mathbb{M}_n(A)(\lambda_1, \dots, \lambda_n) \cong_{\mathrm{gr}} \mathbb{M}_n(A)(\epsilon)$ as in (1.52). Now set

$$V = A(-\varepsilon_1) \times \cdots \times A(-\varepsilon_1) \times \cdots \times A(-\varepsilon_k) \times \cdots \times A(-\varepsilon_k)$$

and pick the (standard) homogeneous basis e_i, $1 \leq i \leq n$ and define $E_{ij} \in \operatorname{End}_A(V)$ by $E_{ij}(e_t) = \delta_{j,t} e_i$, $1 \leq i, j, t \leq n$. One can easily see that E_{ij} is a A-graded homomorphism of degree $\varepsilon_{s_i} - \varepsilon_{s_j}$ where ε_{s_i} and ε_{s_j} are i-th and j-th elements in ϵ. Moreover, $\operatorname{End}_A(V) \cong_{\text{gr}} \mathbb{M}_n(A)(\epsilon)$ and E_{ij} corresponds to the matrix e_{ij} in $\mathbb{M}_n(A)(\epsilon)$. In a similar manner, one can find $\epsilon' = (\varepsilon'_1, \dots, \varepsilon'_1, \varepsilon'_2, \dots, \varepsilon'_2, \dots, \varepsilon'_{k'}, \dots, \varepsilon'_{k'})$ and a graded A-vector space W such that

$$\mathbb{M}_n(A)(\gamma_1, \dots, \gamma_n) \cong_{\text{gr}} \mathbb{M}_n(A)(\epsilon'),$$

and $\operatorname{End}_A(W) \cong_{\text{gr}} \mathbb{M}_n(A)(\epsilon')$. Therefore (1.48) provides a graded ring isomorphism $\theta : \operatorname{End}_A(V) \to \operatorname{End}_A(W)$. Define $E'_{ij} := \theta(E_{ij})$ and $E'_{ii}(W) = Q_i$, for $1 \leq i, j \leq n$. Since $\{E_{ii} \mid 1 \leq i \leq n\}$ is a complete system of orthogonal idempotents, so is $\{E'_{ii} \mid 1 \leq i \leq n\}$. It follows that

$$W \cong_{\text{gr}} \bigoplus_{1 \leq j \leq n} Q_j.$$

Furthermore, $E'_{ij} E'_{tr} = \delta_{j,t} E'_{ir}$ and E'_{ii} acts as identity on Q_i. These relations show that restricting E'_{ij} on Q_j induces an A-graded isomorphism $E'_{ij} : Q_j \to Q_i$ of degree $\varepsilon_{s_i} - \varepsilon_{s_j}$ (same degree as E_{ij}). So $Q_j \cong_{\text{gr}} Q_1(\varepsilon_{s_1} - \varepsilon_{s_j})$ for any $1 \leq j \leq n$. Therefore

$$W \cong_{\text{gr}} \bigoplus_{1 \leq j \leq n} Q_1(\varepsilon_{s_1} - \varepsilon_{s_j}).$$

By dimension count (see Proposition 1.4.3), it follows that $\dim_A Q_1 = 1$.

A similar argument for the identity map $\operatorname{id} : \operatorname{End}_A(V) \to \operatorname{End}_A(V)$ produces

$$V \cong_{\text{gr}} \bigoplus_{1 \leq j \leq n} P_1(\varepsilon_{s_1} - \varepsilon_{s_j}),$$

where $P_1 = E_{11}(V)$, with $\dim_A P_1 = 1$.

Since P_1 and Q_1 are A-graded vector spaces of dimension 1, there is $\sigma \in \Gamma$, such that $Q_1 \cong_{\text{gr}} P_1(\sigma)$. Using the fact that for an A-graded module P and $\alpha, \beta \in \Gamma$, $P(\alpha)(\beta) = P(\alpha+\beta) = P(\beta)(\alpha)$, we have

$$\begin{aligned} W \cong_{\text{gr}} \bigoplus_{1 \leq j \leq n} Q_1(\varepsilon_{s_1} - \varepsilon_{s_j}) \cong_{\text{gr}} \bigoplus_{1 \leq j \leq n} P_1(\sigma)(\varepsilon_{s_1} - \varepsilon_{s_j}) \cong_{\text{gr}} \\ \bigoplus_{1 \leq j \leq n} P_1(\varepsilon_{s_1} - \varepsilon_{s_j})(\sigma) \cong_{\text{gr}} V(\sigma). \end{aligned} \tag{1.49}$$

We denote this A-graded isomorphism with $\phi : W \to V(\sigma)$. Let e'_i, $1 \leq i \leq n$ be a (standard) homogeneous basis of degree ε'_{s_i} in W. Then $\phi(e'_i) = \sum_{1 \leq j \leq n} e_j a_j$, where $a_j \in A^h$ and e_j are homogeneous of degree $\varepsilon_{s_j} - \sigma$ in $V(\sigma)$. Since $\deg(\phi(e'_i)) = \varepsilon'_{s_i}$, all e_j's with non-zero a_j in the sum have the same degree. For if $\varepsilon_{s_j} - \sigma = \deg(e_j) \neq \deg(e_l) = \varepsilon_{s_l} - \sigma$, then since $\deg(e_j a_j) = \deg(e_l a_l) = \varepsilon'_{s_i}$ it follows that $\varepsilon_{s_j} - \varepsilon_{s_l} \in \Gamma_A$ which is a contradiction as $\Gamma_A + \varepsilon_{s_j}$ and $\Gamma_A + \varepsilon_{s_l}$ are distinct. Thus $\varepsilon'_{s_i} = \varepsilon_{s_j} + \tau_j - \sigma$ where $\tau_j = \deg(a_j) \in \Gamma_A$. In the same manner one can show that, $\varepsilon'_{s_i} = \varepsilon'_{s_{i'}}$ in ϵ' if and only if ε_{s_j} and $\varepsilon_{s_{j'}}$ assigned to them by the previous argument are also equal. This shows that ϵ' can be obtained from ϵ by applying (1.39) and (1.40). Since ϵ' and ϵ are also obtained from $\gamma_1, \dots, \gamma_n$ and $\lambda_1, \dots, \lambda_n$, respectively, by applying (1.39) and (1.40), putting these together shows that $\lambda_1, \dots, \lambda_n$ and $\gamma_1, \dots, \gamma_n$ have the similar relations, i.e., $\lambda_i = \gamma_{\pi(i)} + \tau_i + \sigma$, $1 \leq i \leq n$, where $\tau_i \in \Gamma_A$ and a fixed $\sigma \in \Gamma$. $\square$

A *graded division algebra* A is defined to be a graded division ring with centre R such that $[A : R] < \infty$. Note that since R is a graded field, by Propositions 1.4.1 and 1.4.2, A has a well-defined dimension over R. A graded division algebra A over its centre R is said to be *unramified* if $\Gamma_A = \Gamma_R$ and *totally ramified* if $A_0 = R_0$.

Let A be a graded division ring and let R be a graded subfield of A which is contained in the centre of A. It is clear that $R_0 = R \cap A_0$ is a field and A_0 is a division ring. The group of invertible homogeneous elements of A is denoted by A^{h*}, which is equal to $A^h \setminus 0$. Considering A as a graded R-module, since R is a graded field, there is a uniquely defined dimension $[A : R]$ by Theorem 1.4.1. The proposition below has been proven by Hwang, Wadsworth [50, Prop. 2.2] for two graded fields $R \subseteq S$ with a torsion-free abelian grade group.

Proposition 1.4.6. *Let A be a graded division ring and let R be a graded subfield of A which is contained in the centre of A. Then*

$$[A : R] = [A_0 : R_0]|\Gamma_A : \Gamma_R|.$$

Proof. Since A is a graded division ring, A_0 is a division ring. Furthermore, R_0 is a field. Let $\{x_i\}_{i\in I}$ be a basis for A_0 over R_0. Consider the cosets of Γ_A over Γ_R and let $\{\delta_j\}_{j\in J}$ be a coset representative, where $\delta_j \in \Gamma_A$. Take $\{y_j\}_{j\in J} \subseteq A^{h*}$ such that $\deg(y_j) = \delta_j$ for each j. We will show that $\{x_i y_j\}$ is a basis for A over F.

Consider the map

$$\begin{aligned} \psi : A^{h*} &\longrightarrow \Gamma_A/\Gamma_R, \\ a &\longmapsto \deg(a) + \Gamma_R. \end{aligned}$$

This is a group homomorphism with kernel $A_0 R^{h*}$, since for any $a \in \ker(\psi)$ there is some $r \in R^{h*}$ with $ar^{-1} \in A_0$. For $a \in A$, $a = \sum_{\gamma\in\Gamma} a_\gamma$ where $a_\gamma \in A_\gamma$ and $\psi(a_\gamma) = \gamma + \Gamma_R = \delta_j + \Gamma_F$ for some δ_j in the coset representative of Γ_A over Γ_R. Then there is some y_j with $\deg(y_j) = \delta_j$ and $a_\gamma y_j^{-1} \in \ker(\psi) = A_0 R^{h*}$. So

$$a_\gamma y_j^{-1} = (\sum_i x_i r_i) g$$

for $g \in R^{h*}$ and $r_i \in R_0$. Since R is in the centre of A, It follows $a_\gamma = \sum_i x_i y_j r_i g$. So a can be written as an R-linear combination of the elements of $\{x_i y_j\}$.

To show linear independence, suppose

$$\sum_{i=1}^{n} x_i y_i r_i = 0, \tag{1.50}$$

for $r_i \in R$. Write r_i as the sum of its homogeneous components, and then consider a homogeneous component of the sum (1.50), say $\sum_{k=1}^{m} x_k y_k r'_k$, where $\deg(x_k y_k r'_k) = \alpha$. Since $x_k \in A_0$, $\deg(r'_k) + \deg(y_k) = \alpha$ for all k, so all of the y_k are the same. This implies that $\sum_k x_k r'_k = 0$, where all of the r'_k have the same degree. If $r'_k = 0$ for all k then $r_i = 0$ for all i we are done. Otherwise, for some $r'_l \neq 0$, we have $\sum_k x_k (r'_k {r'_l}^{-1}) = 0$. Since $\{x_i\}$ forms a basis for A_0 over R_0, this implies $r'_k = 0$ for all k and thus $r_i = 0$ for all $1 \leq i \leq n$. $\square$

Example 1.4.7. <u>A graded division algebra associated to a valued division algebra</u>

Let D be a division algebra with a valuation. To this one associates a graded division algebra $\mathrm{gr}(D) = \bigoplus_{\gamma\in\Gamma_D} \mathrm{gr}(D)_\gamma$, where Γ_D is the value group of D and the summands $\mathrm{gr}(D)_\gamma$ arise from the filtration on D induced by the valuation (see details below and also Example 1.2.15). As it is illustrated in [51, 45], even though computations in the graded setting are often easier than working directly with D, it seems that not much is lost in passage from D to its corresponding graded division algebra $\mathrm{gr}(D)$. This has provided motivation to systematically study this correspondence, notably by Hwang, Tignol and Wadsworth [50, 51], and to compare certain functors defined on these objects, notably the Brauer group and the reduced Whitehead group SK_1 [45, 89]. We introduce this correspondence here and in §3 we calculate their graded Grothendieck groups (Example 3.6.5).

Let D be a division algebra finite dimensional over its centre F, with a *valuation* $v : D^* \to \Gamma$. So Γ is a totally ordered abelian group, and for any $a, b \in D^*$, v satisfies the following conditions

(i) $v(ab) = v(a) + v(b)$;

(ii) $v(a + b) \geq \min\{v(a), v(b)\} \quad (b \neq -a)$.

Let

$$\begin{aligned}
V_D &= \{a \in D^* : v(a) \geq 0\} \cup \{0\}, \text{ the valuation ring of } v;\\
M_D &= \{a \in D^* : v(a) > 0\} \cup \{0\}, \text{ the unique maximal left (and right) ideal of } V_D;\\
\overline{D} &= V_D/M_D, \text{ the residue division ring of } v \text{ on } D; \text{ and}\\
\Gamma_D &= \operatorname{im}(v), \text{ the value group of the valuation.}
\end{aligned}$$

For background on valued division algebras, see [54] or the survey paper [91]. One associates to D a graded division algebra as follows: For each $\gamma \in \Gamma_D$, let

$$\begin{aligned}
D^{\geq\gamma} &= \{d \in D^* : v(d) \geq \gamma\} \cup \{0\}, \text{ an additive subgroup of } D;\\
D^{>\gamma} &= \{d \in D^* : v(d) > \gamma\} \cup \{0\}, \text{ a subgroup of } D^{\geq\gamma}; \text{ and}\\
\operatorname{gr}(D)_\gamma &= D^{\geq\gamma}/D^{>\gamma}.
\end{aligned}$$

Then define

$$\operatorname{gr}(D) = \bigoplus_{\gamma\in\Gamma_D} \operatorname{gr}(D)_\gamma.$$

Because $D^{>\gamma}D^{\geq\delta} + D^{\geq\gamma}D^{>\delta} \subseteq D^{>(\gamma+\delta)}$ for all $\gamma, \delta \in \Gamma_D$, the multiplication on $\operatorname{gr}(D)$ induced by multiplication on D is well-defined, giving that $\operatorname{gr}(D)$ is a Γ-graded ring, called the *associated graded ring* of D. The multiplicative property (i) of the valuation v implies that $\operatorname{gr}(D)$ is a graded division ring. Clearly, we have $\operatorname{gr}(D)_0 = \overline{D}$, and $\Gamma_{\operatorname{gr}(D)} = \Gamma_D$. For $d \in D^*$, we write $\widetilde{d}$ for the image $d + D^{>v(d)}$ of d in $\operatorname{gr}(D)_{v(d)}$. Thus, the map given by $d \mapsto \widetilde{d}$ is a group epimorphism $D^* \to \operatorname{gr}(D)^*$ with kernel $1 + M_D$.

The restriction $v|_F$ of the valuation on D to its centre F is a valuation on F, which induces a corresponding graded field $\operatorname{gr}(F)$. Then it is clear that $\operatorname{gr}(D)$ is a graded $\operatorname{gr}(F)$-algebra, and one can prove that for

$$[\operatorname{gr}(D) : \operatorname{gr}(F)] = [\overline{D} : \overline{F}]\,|\Gamma_D : \Gamma_F| \leq [D : F] < \infty.$$

Now let F be a field with a henselian valuation v. Then the valuation of F extends uniquely to D (see [91, Theorem 2.1]), and with respect to this valuation, D is said to be *tame* if $Z(\overline{D})$ is separable over $\overline{F}$ and $\operatorname{char}(\overline{F}) \nmid \operatorname{ind}(D)/\big(\operatorname{ind}(\overline{D})[Z(\overline{D}) : \overline{F}]\big)$. It is known ([51, Proposition 4.3]) that D is tame if and only if $[\operatorname{gr}(D) : \operatorname{gr}(F)] = [D : F]$ and $Z(\operatorname{gr}(D)) = \operatorname{gr}(F)$.

We will compute the graded Grothendieck group and the graded Picard group of these division algebras in Examples 3.6.5 and 4.2.6.

1.4.1 Ring of zero homogeneous elements of a graded central simple ring

Let A be a Γ-graded division ring and $\mathbb{M}_n(A)(\lambda_1,\dots,\lambda_n)$ be a graded simple ring, where $\lambda_i \in \Gamma$, $1 \leq i \leq n$. Since A is a division ring, Γ_A is a subgroup of Γ. Consider the quotient group Γ/Γ_A and let $\Gamma_A+\varepsilon_1,\dots,\Gamma_A+\varepsilon_k$ be the distinct elements in Γ/Γ_A representing the cosets $\Gamma_A+\lambda_i$, $1 \leq i \leq n$, and for each ε_l, let r_l be the number of i with $\Gamma_A+\lambda_i=\Gamma_A+\varepsilon_l$. It was observed in [51, Proposition 1.4] that

$$\boxed{\mathbb{M}_n(A)_0 \cong \mathbb{M}_{r_1}(A_0)\times\cdots\times\mathbb{M}_{r_k}(A_0)} \tag{1.51}$$

and in particular $\mathbb{M}_n(A)_0$ is a simple ring if and only if $k=1$. Indeed, using (1.39) and (1.40) we get

$$\mathbb{M}_n(A)(\lambda_1,\dots,\lambda_n) \cong_{\mathrm{gr}} \mathbb{M}_n(A)(\varepsilon_1,\dots,\varepsilon_1,\varepsilon_2,\dots,\varepsilon_2,\dots,\varepsilon_k,\dots,\varepsilon_k), \tag{1.52}$$

with each ε_l occurring r_l times. Now (1.34) for $\lambda=0$ and

$$(\delta_1,\dots,\delta_n)=(\varepsilon_1,\dots\varepsilon_1,\varepsilon_2,\dots,\varepsilon_2,\dots,\varepsilon_k,\dots,\varepsilon_k)$$

immediately gives (1.51).

Remark 1.4.8. <u>The graded Artin-Wedderburn structure theorem</u>

The Artin-Wedderburn theorem shows that division rings are basic "building blocks" of ring theory. A graded version of Artin-Wedderburn structure theorem holds. We state the statement here without proof. We refer the reader to [73, 51] for proofs of these statements.

A Γ-graded ring B is isomorphic to $\mathbb{M}_n(A)(\lambda_1,\dots,\lambda_n)$, where A is a Γ-graded division ring and $\lambda_i \in \Gamma$, $1 \leq i \leq n$, if and only if B is *graded right Artinian* (i.e., a decreasing chain of graded right ideals becomes stationary) and graded simple.

A Γ-graded ring B is isomorphic to a finite product of matrix rings overs graded division rings (with suitable shiftings) if and only if B is graded right Artinian and *graded primitive* (i.e., $J^{\mathrm{gr}}(B)=0$).

1.5 Strongly graded rings and Dade's theorem

Let A be a Γ-graded ring and Ω be a subgroup of Γ. Recall from Example 1.1.8 the Ω-graded ring $A_\Omega=\bigoplus_{\gamma\in\Omega}A_\gamma$. If A is a Γ-strongly graded ring, then it is immediate that A_Ω is a Ω-strongly graded ring. In this case one can show that the category of Γ-graded A-modules, $\mathrm{Gr}^\Gamma\text{-}A$, is equivalent to the category of Ω-graded A_Ω-modules, $\mathrm{Gr}^\Omega\text{-}A_\Omega$. In fact, the equivalence

$$\mathrm{Gr}^\Gamma\text{-}A \approx \mathrm{Gr}^\Omega\text{-}A_\Omega,$$

under the given natural functors (see Theorem 1.5.7) implies that A is Γ-strongly graded ring. This was first proved by Dade [30] in the case of $\Omega=0$. We prove Dade's theorem (Theorem 1.5.1) and then state this more general case in Theorem 1.5.7.

Let A be a Γ-graded ring. For any right A_0-module N and any $\gamma\in\Gamma$, we identify the right A_0-module $N\otimes_{A_0}A_\gamma$ with its image in $N\otimes_{A_0}A$. Since $A=\bigoplus_{\gamma\in\Gamma}A_\gamma$ and A_γ are A_0-bimodules, $N\otimes_{A_0}A$ is a Γ-graded right A-module, with

$$N\otimes_{A_0}A=\bigoplus_{\gamma\in\Gamma}\big(N\otimes_{A_0}A_\gamma\big). \tag{1.53}$$

Consider the *restriction functor*

$$\boxed{\begin{aligned} \mathcal{G} := (-)_0 : \text{Gr-}A &\longrightarrow \text{Mod-}\,A_0 \\ M &\longmapsto M_0 \\ \psi &\longmapsto \psi|_{M_0}, \end{aligned}}$$

and the *induction functor* defined by

$$\boxed{\begin{aligned} \mathcal{I} := - \otimes_{A_0} A : \text{Mod-}\,A_0 &\longrightarrow \text{Gr-}A \\ N &\longmapsto N \otimes_{A_0} A \\ \phi &\longmapsto \phi \otimes \text{id}_A\,. \end{aligned}}$$

One can easily check that $\mathcal{G} \circ \mathcal{I} \cong \text{id}_{A_0}$ with the natural transformation,

$$\begin{aligned} \mathcal{G}\,\mathcal{I}(N) = \mathcal{G}(N \otimes_{A_0} A) = N \otimes_{A_0} A_0 &\longrightarrow N, \\ n \otimes a &\mapsto na. \end{aligned} \tag{1.54}$$

On the other hand, there is a natural transformation,

$$\begin{aligned} \mathcal{I}\,\mathcal{G}(M) = \mathcal{I}(M_0) = M_0 \otimes_{A_0} A &\longrightarrow M, \\ m \otimes a &\mapsto ma. \end{aligned} \tag{1.55}$$

The theorem below shows that $\mathcal{I} \circ \mathcal{G} \cong \text{id}_A$ (under (1.55)), if and only if A is a strongly graded ring. Theorem 1.5.1 was proved by Dade [30, Theorem 2.8] (see also [73, Thm. 3.1.1]).

Theorem 1.5.1 (DADE'S THEOREM)**.** *Let A be a Γ-graded ring. Then A is strongly graded if and only if the functors $(-)_0 : \text{Gr-}A \to \text{Mod-}\,A_0$ and $- \otimes_{A_0} A : \text{Mod-}\,A_0 \to \text{Gr-}A$ form mutually inverse equivalences of categories.*

Proof. Suppose A is strongly graded. One can easily check (without using the assumption that A is strongy graded) that $\mathcal{G} \circ \mathcal{I} \cong \text{id}_{A_0}$. We show that $\mathcal{I} \circ \mathcal{G} \cong \text{id}_A$.

For a graded A-module M we have $\mathcal{I} \circ \mathcal{G}(M) = M_0 \otimes_{A_0} A$. We will show that the natural homomorphism $\phi : M_0 \otimes_{A_0} A \to M$; $m \otimes a \mapsto ma$ is a graded A-module isomorphism. The map ϕ is clearly graded (see (1.53)). Since A is strongly graded, it follows that for $\gamma, \delta \in \Gamma$,

$$M_{\gamma+\delta} = M_{\gamma+\delta}A_0 = M_{\gamma+\delta}A_{-\gamma}A_\gamma \subseteq M_\delta A_\gamma \subseteq M_{\gamma+\delta}. \tag{1.56}$$

Thus $M_\delta A_\gamma = M_{\gamma+\delta}$. Therefore, $\phi(M_0 \otimes_{A_0} A_\gamma) = M_0 A_\gamma = M_\gamma$, which implies that ϕ is surjective.

Let $N = \ker(\phi)$, which is a graded A-submodule of $M_0 \otimes_{A_0} A$, so $N_0 = N \cap (M_0 \otimes_{A_0} A_0)$. However the restriction of ϕ to $M_0 \otimes_{A_0} A_0 \to M_0$ is the canonical isomorphism, so $N_0 = 0$. Since N is a graded A-module, a similar argument as (1.56) shows $N_\gamma = N_0 A_\gamma = 0$ for all $\gamma \in \Gamma$. It follows that ϕ is injective. Thus $\mathcal{I} \circ \mathcal{G}(M) = M_0 \otimes_{A_0} A \cong M$. Since all the homomorphisms involved are natural, this shows that $\mathcal{I} \circ \mathcal{G} \cong \text{id}_A$.

For the converse suppose $\mathcal{I}$ and $\mathcal{G}$ are mutually inverse (under (1.55) and (1.54)). For any graded A-module M, since $\mathcal{I}\,\mathcal{G}(M) \cong_{\text{gr}} M$, we have, for any $\alpha \in \Gamma$,

$$\begin{aligned} M_0 \otimes_{A_0} A_\alpha &\longrightarrow M_\alpha, \\ m \otimes a &\mapsto ma \end{aligned}$$

is an bijective map. This immediately implies

$$M_0 A_\alpha = M_\alpha. \tag{1.57}$$

Now for any $\beta \in \Gamma$, consider the graded A-module $A(\beta)$. Replacing M by $A(\beta)$ in (1.57), we get $A(\beta)_0 A_\alpha = A(\beta)_\alpha$, i.e., $A_\beta A_\alpha = A_{\beta+\alpha}$. This shows that A is strongly graded. ☐

Corollary 1.5.2. *Let A be a strongly Γ-graded ring. Then the functors*

$$(-)_0 : \mathrm{Gr}^{\Gamma/\Omega}\text{-}A \to \mathrm{Mod}\text{-}A_\Omega,$$

and

$$- \otimes_{A_\Omega} A : \mathrm{Mod}\text{-}A_\Omega \to \mathrm{Gr}^{\Gamma/\Omega}\text{-}A$$

form mutually inverse equivalences of categories.

Proof. Since A is a strongly Γ-graded ring, it follows that A is a Γ/Ω-graded ring (Example 1.1.18). The result now follows from Theorem 1.5.1. ☐

Remark 1.5.3. Recall that gr- A denotes the category of graded finitely generated right A-modules and Pgr-A denotes the category of graded finitely generated projective right A-modules. Note that in general the restriction functor $(-)_0 : \mathrm{Gr}\text{-}A \to \mathrm{Mod}\text{-}A_0$ does not induce a functor $(-)_0 : \mathrm{Pgr}\text{-}A \to \mathrm{Pr}\text{-}A_0$. In fact, one can easily produce a graded finitely generated projective A-module P such that P_0 is not projective A_0-module. As an example, consider the $\mathbb{Z}$-graded ring T of Example 1.1.5. Then $T(1)$ is clearly a graded finitely generated projective T-module. However $T(1)_0 = M$ is not $T_0 = R$-module.

Remark 1.5.4. The proof of Theorem 1.5.1 also shows that A is strongly graded if and only if gr- $A \cong$ mod- A_0, if and only if Pgr-$A \cong$ Pr- A_0, (see Remark 1.5.3) via the same functors $(-)_0$ and $- \otimes_{A_0} A$ of the Theorem 1.5.1.

Remark 1.5.5. <u>Strongly graded modules</u>

Let A be a Γ-graded ring and M be a graded A-module. Then M is called *strongly graded* A-module if

$$M_\alpha A_\beta = M_{\alpha+\beta}, \tag{1.58}$$

for any $\alpha, \beta \in \Gamma$. The proof of Theorem 1.5.1 shows that A is strongly graded if and only if any graded A-module is strongly graded. Indeed if A is strongly graded then (1.56) shows that any graded A-module is strongly graded. Conversely, if any graded module is strongly graded, then considering A as a graded A-module, (1.58) for $M = A$, shows that $A_\alpha A_\beta = A_{\alpha+\beta}$ for any $\alpha, \beta \in \Gamma$.

Remark 1.5.6. <u>Ideal correspondence between A_0 and A for a strongly graded ring</u>

The proof of Theorem 1.5.1 shows that there is a one-to-one correspondence between the right ideals of A_0 and the graded right ideals of A (similarly for the left ideals). However, this correspondence does not hold between two sided ideals. As an example, $A = \mathbb{M}_2(K[x^2, x^{-2}])(0,1)$, where K is a field, is a strongly $\mathbb{Z}$-graded simple ring, whereas $A_0 \cong K \otimes K$ is not a simple ring (see §1.4.1).

In the same way, the equivalence $\mathrm{Gr}\text{-}A \approx \mathrm{Mod}\text{-}A_0$ of Theorem 1.5.1 gives a correspondence between several (one-sided) properties of graded objects in A with objects over A_0. For example, one can easily show that A is graded right (left) Noetherian if and only if A_0 is right (left) Noetherian (see also Corollary 1.5.9).

Using Theorem 1.5.1, we will see that the graded Grothendieck group of a strongly graded ring coincides with the (classical) Grothendieck group of the ring of zero homogeneous part (see §3.1.3).

Theorem 1.5.7. *Let A be a Γ-graded ring and Ω be a subgroup of Γ. Then A is Γ-strongly graded ring if and only if*

$$\begin{aligned}(-)_\Omega : \mathrm{Gr}^\Gamma\text{-}A &\longrightarrow \mathrm{Gr}^\Omega\text{-}A_\Omega\\ M &\longmapsto M_\Omega\\ \psi &\longmapsto \psi|_{M_\Omega},\end{aligned}$$

and

$$\begin{aligned}-\otimes_{A_0} A : \mathrm{Gr}^\Omega\text{-}A_\Omega &\longrightarrow \mathrm{Gr}^\Gamma\text{-}A\\ N &\longmapsto N_0\otimes_{A_0} A\\ \phi &\longmapsto \phi_0\otimes \mathrm{id}_A,\end{aligned}$$

form mutually inverse equivalences of categories.

Proof. The proof is similar to the proof of Theorem 1.5.1 and it is omitted. □

Example 1.5.8. Let A be a $\Gamma\times\Omega$-strongly graded ring. Then by Theorem 1.5.7

$$\mathrm{Gr}^{\Gamma\times\Omega}\text{-}A \approx \mathrm{Gr}^\Omega\text{-}A_{(0,-)},$$

where $A_{(0,-)} = \bigoplus_{\omega\in\Omega} A_{(0,\omega)}$. This example will be used in §6.3. Compare this also with Corollary 1.2.10.

Another application of Theorem 1.5.1 is to provide a condition when a strongly graded ring is a graded von Neumann ring (§1.1.6). This will be used later in Coroallry 1.6.13 to show that the Leavitt path algebras are von Neumann regular rings.

Corollary 1.5.9. *Let A be a strongly graded ring. Then A a is graded von Neumann regular ring if and only if A_0 is a von Neumann regular ring.*

Sketch of proof. Since any (graded) flat module is a direct limit of (graded) projective modules, from the equivalence of categories $\mathrm{Gr}\text{-}A \approx_{\mathrm{gr}} \mathrm{Mod}\text{-}\,A_0$ (Theorem 1.5.1), it follows that A is graded von Neumann regular if and only if A_0 is von Neumann regular. □

Remark 1.5.10. An element-wise proof of Corollary 1.5.9 can also be found in [95, Theorem 3].

For a Γ-graded ring A, and $\alpha,\beta\in\Gamma$, one has a A_0-bimodule homomorphism

$$\begin{aligned}\phi_{\alpha,\beta} : A_\alpha\otimes_{A_0} A_\beta &\longrightarrow A_{\alpha+\beta}\\ a\otimes b &\longmapsto ab.\end{aligned} \tag{1.59}$$

The following theorem gives another characterisation for strongly graded rings.

Theorem 1.5.11. *Let A be a Γ-graded ring. Then A is a strongly graded ring if and only if for any $\gamma\in\Gamma$, the homomorphism*

$$\begin{aligned}\phi_{\gamma,-\gamma} : A_\gamma\otimes A_{-\gamma} &\longrightarrow A_0,\\ a\otimes b &\longmapsto ab,\end{aligned}$$

is an isomorphism.

Proof. Suppose that for any $\gamma \in \Gamma$, the map $\phi_{\gamma,-\gamma} : A_\gamma \otimes A_{-\gamma} \to A_0$ is an isomorphism. Thus there are $a_i \in A_\gamma$, $b_i \in A_{-\gamma}$ such that

$$\sum_i a_i b_i = \phi_{\gamma,-\gamma}\big(\sum_i a_i \otimes b_i\big) = 1.$$

So $1 \in A_\gamma A_{-\gamma}$. Now by Proposition 1.1.16(1) A is strongly graded.

Conversely, suppose A is a strongly graded ring. We prove that the homomorphism (1.59) is an isomorphism. The definition of strongly graded implies that $\phi_{\alpha,\beta}$ is surjective. Suppose

$$\phi_{\alpha,\beta}\big(\sum_i a_i \otimes b_i\big) = \sum_i a_i b_i = 0. \tag{1.60}$$

Using Proposition 1.1.16(1), write $1 = \sum_j x_j y_j$, where $x_j \in A_{-\beta}$ and $y_j \in A_\beta$. Then

$$\sum_i a_i \otimes b_i = \big(\sum_i a_i \otimes b_i\big)\big(\sum_j x_j y_j\big) = \sum_i \big(a_i \otimes \sum_j b_i x_j y_j\big) =$$
$$\sum_i \big(\sum_j (a_i b_i x_j \otimes y_j)\big) = \sum_j \sum_i (a_i b_i x_j \otimes y_j) = \sum_j \Big(\sum_i (a_i b_i) x_j \otimes y_j\Big) = 0.$$

This shows that $\phi_{\alpha,\beta}$ is injective. Now setting $\alpha = \gamma$ and $\beta = -\gamma$ finishes the proof. $\square$

1.5.1

Let A and B be rings and P be a $A-B$-bimodule. Then P is called an *invertible* $A-B$-bimodule, if there is a $B-A$-bimodule Q such that $P \otimes_B Q \cong A$ as $A-A$-bimodules and $Q \otimes_A P \cong B$ as $B-B$-bimodules and the following diagrams are commutative.

$$\begin{tikzcd} P \otimes_B Q \otimes_A P \arrow[r] \arrow[d] & A \otimes_A P \arrow[d] \\ P \otimes_B B \arrow[r] & P \end{tikzcd} \qquad \begin{tikzcd} Q \otimes_A P \otimes_B Q \arrow[r] \arrow[d] & B \otimes_B Q \arrow[d] \\ Q \otimes_A A \arrow[r] & Q \end{tikzcd}$$

One can prove that P is a finitely generated projective A and B-modules.

Now Theorem 1.5.11 shows that for a strongly Γ-graded ring A, the A_0-bimodules A_γ, $\gamma \in \Gamma$, is an invertible module and thus is a finitely generated projective A_0-module. This in return implies that A is a projective A_0-module. Note that in general, one can easily construct a graded ring A where A is not projective over A_0 (see Example 1.1.5) and A_γ is not finitely generated A_0-module, such as the $\mathbb{Z}$-graded ring $\mathbb{Z}[x_i \mid i \in \mathbb{N}]$ of Example 1.1.10.

Remark 1.5.12. <u>Other terminologies for strongly graded rings</u>

The term "strongly graded" for such rings was coined by E. Dade in [30] which is now commonly in use. Other terms for these rings are *fully graded* and *generalised crossed products*. See [31] for a history of development of such rings in literature.

1.6 Grading on graph algebras

1.6.1 Grading on free rings

Let X be a nonempty set of symbols and Γ be a group. (As always we assume the groups are abelian, although the entire theory can be written for an arbitrary group.) Let $d : X \to \Gamma$ be a map. One can extend d in a natural way to a map from the set of finite words on X to Γ, which is called d again. For example if $x, y, z \in X$ and xyz is a word, then $d(xyz) = d(x)+d(y)+d(z)$. One can easily see that if w_1, w_2 are two words, then $d(w_1w_2) = d(w_1) + d(w_2)$. If we allow an empty word, which will be the identity element in the free ring, then we assign the identity of Γ to this word.

Let R be a ring and $R(X)$ be the free ring (with or without identity) on a set X with coefficients in R. The elements of $R(X)$ are of the form $\sum_w r_w w$, where $r_w \in R$ and w stands for a word on X. The multiplication is defined by

$$\Big(\sum_w r_w w\Big)\Big(\sum_v r_v v\Big) = \sum_z \Big(\sum_{\{w,v \mid z=wv,\, r_w, r_v \neq 0\}} r_w r_v\Big) z.$$

Define $R(X)_\gamma = \{\sum_w r_w w \mid d(w) = \gamma\rangle$. One can check that $R(X) = \bigoplus_{\gamma\in\Gamma} R(X)_\gamma$. Thus $R(X)$ is a Γ-graded ring. Note that if we don't allow the empty word in the construction, then $R(X)$ is a graded ring without identity (see Remark 1.1.15). It is easy to see that $R(X)$ is never a strongly graded ring.

Example 1.6.1. Let R be a ring and $R(X)$ be the free ring on a set X with a graded structure induced by a map $d : X \to \Gamma$. Let Ω be a subgroup of Γ and consider the map

$$\begin{aligned} \overline{d} : X &\longrightarrow \Gamma/\Omega, \\ x &\longmapsto \Omega + d(x). \end{aligned}$$

The map $\overline{d}$ induces a Γ/Ω-graded structure on $R(X)$ which coincides with the general construction of quotient grading given in Example 1.1.8.

Example 1.6.2. Let $X = \{x\}$ be a set of symbols with one element and $\mathbb{Z}_n$ be the cyclic group with n elements. Assign $1 \in \mathbb{Z}_n$ to x and generate the free ring with identity on X with coefficients in a field F. This ring is the usual polynomial ring $F[x]$ which, by the above construction, is equipped by a $\mathbb{Z}_n$-grading. Namely,

$$F[x] = \bigoplus_{k\in\mathbb{Z}_n} \Big(\sum_{\substack{l\in\mathbb{N},\\ \bar{l}=k}} Fx^l\Big),$$

where $\bar{l}$ is the image of l in the group $\mathbb{Z}_n$. For $a \in F$, since the polynomial $x^n - a$ is a homogeneous element of degree zero, the ideal $\langle x^n - a\rangle$ is a graded ideal and thus the quotient ring $F[x]/\langle x^n - a\rangle$ is also a $\mathbb{Z}_n$-graded ring (see §1.1.5). In particular if $x^n - a$ is an irreducible polynomial in $F[x]$, then the field $F[x]/\langle x^n - a\rangle$ is a $\mathbb{Z}_n$-graded field as well.

Example 1.6.3. Let $\{x, y\}$ be a set of symbols. Assign $1 \in \mathbb{Z}_2$ to x and y and consider the graded free ring $\mathbb{R}(x, y)$. The ideal generated by homogeneous elements $\{x^2 + 1, y^2 + 1, xy + yx\}$ is graded and thus we retrieve the $\mathbb{Z}_2$-graded Hamiltonian quaternion algebra of Example 1.1.21 as follows:

$$\mathbb{H} \cong \mathbb{R}(x, y)/\langle x^2 + 1, y^2 + 1, xy + yx\rangle.$$

Furthermore, assigning $(1, 0) \in \mathbb{Z}_2\times\mathbb{Z}_2$ to x and $(0, 1) \in \mathbb{Z}_2\times\mathbb{Z}_2$ to y we obtained the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded quaternion algebra of Example 1.1.21.

Example 1.6.4. THE WEYL ALGEBRA

For a (commutative) ring R, the *Weyl algebra* $R(x, y)/\langle xy - yx - 1\rangle$ can be considered as a $\mathbb{Z}$-graded ring by assigning 1 to x and -1 to y.

Example 1.6.5. THE LEAVITT ALGEBRA $\mathcal{L}(n, k+1)$

Let K be a field, n and k positive integers and A be the free associative K-algebra with identity generated by symbols $\{x_{ij}, y_{ji} \mid 1 \leq i \leq n+k, 1 \leq j \leq n\}$ subject to relations (coming from)

$$Y \cdot X = I_{n,n} \qquad \text{and} \qquad X \cdot Y = I_{n+k,n+k},$$

where

$$Y = \begin{pmatrix} y_{11} & y_{12} & \dots & y_{1,n+k} \\ y_{21} & y_{22} & \dots & y_{2,n+k} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n,1} & y_{n,2} & \dots & y_{n,n+k} \end{pmatrix}, \quad X = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1,n} \\ x_{21} & x_{22} & \dots & x_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n+k,1} & x_{n+k,2} & \dots & x_{n+k,n} \end{pmatrix}. \tag{1.61}$$

In Example 1.3.13 we studied a special case of this algebra when $n = 1$ and $k = n - 1$. This algebra was studied by Leavitt in relation with its type in [63, p.130] which is shown that for arbitrary n and k the algebra is of type (n, k) (see §1.7) and when $n \geq 2$ they are domains. We denote this algebra by $\mathcal{L}(n, k+1)$. (Cohn's notation in [26] for this algebra is $V_{n,n+k}$.)

Assigning $\deg(y_{ji}) = (0, \dots, 0, 1, 0 \dots, 0)$ and $\deg(x_{ij}) = (0, \dots, 0, -1, 0 \dots, 0)$, $1 \leq i \leq n+k$, $1 \leq j \leq n$, in $\bigoplus_n \mathbb{Z}$, where 1 and -1 are in j-th entries respectively, makes the free algebra generated by x_{ij} and y_{ji} a graded ring. Furthermore, one can easily observe that the relations coming from (1.61) are all homogeneous with respect to this grading, so that the Leavitt algebra $\mathcal{L}(n, k+1)$ is a $\bigoplus_n \mathbb{Z}$-graded ring. In particular, $\mathcal{L}(1, k)$ is a $\mathbb{Z}$-graded ring (Example 1.3.13).

1.6.2 Corner skew Laurent polynomial rings

Let R be a ring with identity and p an idempotent of R. Let $\phi : R \to pRp$ be a *corner* isomorphism, i.e, a ring isomorphism such that $\phi(1) = p$. A *corner skew Laurent polynomial ring* with coefficients in R, denoted by $R[t_+, t_-, \phi]$, is a unital ring which is constructed as follows: The elements of $R[t_+, t_-, \phi]$ are formal expressions

$$t_-^j r_{-j} + t_-^{j-1} r_{-j+1} + \dots + t_- r_{-1} + r_0 + r_1 t_+ + \dots + r_i t_+^i,$$

where $r_{-n} \in p_n R$ and $r_n \in R p_n$, for all $n \geq 0$, where $p_0 = 1$ and $p_n = \phi^n(p_0)$. The addition is component-wise, and the multiplication is determined by the distribution law and the following rules:

$$t_- t_+ = 1, \qquad t_+ t_- = p, \qquad r t_- = t_- \phi(r), \qquad t_+ r = \phi(r) t_+. \tag{1.62}$$

The corner skew Laurent polynomial rings are studied in [7], where their K_1-groups are calculated. This construction is a special case of a so called fractional skew monoid rings constructed in [8]. Assigning -1 to t_- and 1 to t_+ makes $A := R[t_+, t_-, \phi]$ a $\mathbb{Z}$-graded ring with $A = \bigoplus_{i \in \mathbb{Z}} A_i$ (see [8, Proposition 1.6])

$$\begin{aligned} A_i &= R p_i t_+^i, \text{ for } i > 0, \\ A_i &= t_-^i p_{-i} R, \text{ for } i < 0, \\ A_0 &= R. \end{aligned}$$

Clearly, when $p = 1$ and ϕ is the identity map, then $R[t_+, t_-, \phi]$ reduces to the familiar ring $R[t, t^{-1}]$.

In the next two propositions we will characterise those corner skew Laurent polynomials which are strongly graded rings (§1.1.2) and graded von Neumann regular rings (§1.1.6).

Recall that an idempotent element p of the ring R is called a *full idempotent* if $RpR = R$.

Proposition 1.6.6. *Let R be a ring with identity and $A = R[t_+, t_-, \phi]$ be a corner skew Laurent polynomial ring. Then A is strongly graded if and only if $\phi(1)$ is a full idempotent.*

Proof. First note that $A_1 = R\phi(1)t_+$ and $A_{-1} = t_-\phi(1)R$. Furthermore, since $\phi(1) = p$, we have

$$r_1\phi(1)t_+t_-\phi(1)r_2 = r_1\phi(1)p\phi(1)r_2 = r_1pppr_2 = r_1\phi(1)r_2.$$

Suppose A is strongly graded. Then $1 \in A_1A_{-1}$. That is

$$1 = \sum_i \big(r_i\phi(1)t_+\big)\big(t_-\phi(1)r'_i\big) = \sum_i r_i\phi(1)r'_i, \tag{1.63}$$

where $r_i, r'_i \in R$. So $R\phi(1)R = R$, that is $\phi(1)$ is a full idempotent.

On the other hand suppose $\phi(1)$ is a full idempotent. Since $\mathbb{Z}$ is generated by 1, in order to prove that A is strongly graded, it is enough to show that $1 \in A_1A_{-1}$ and $1 \in A_{-1}A_1$ (see §1.1.2). But

$$t_-\phi(1)\phi(1)t_+ = t_{-1}\phi(1)t_+ = 1t_-t_+ = 1,$$

shows that $1 \in A_{-1}A_1$. Since $\phi(1)$ is a full idempotent, there are $r_i, r'_i \in R$, $i \in I$ such that $\sum r_i\phi(1)r'_i = 1$. Then Equation 1.63 shows that $1 \in A_1A_{-1}$. ☐

Proposition 1.6.7. *Let R be a ring with identity and $A = R[t_+, t_-, \phi]$ be a corner skew Laurent polynomial ring. Then A is a graded von Neumann regular ring if and only if R is a von Neumann regular ring.*

Proof. If a graded ring is graded von Neumann regular, then it is easy to see that its zero component ring is von Neumann regular. This proves one direction of the theorem. For the converse, suppose R is regular. Let $x \in A_i$, where $i > 0$. So $x = rp_it_+^i$, for some $r \in R$, where $p_i = \phi^i(1)$. By relations (1.62) and induction, we have $t_+^it_-^i = \phi^i(p_0) = p_i$. Since R is regular, there is an $s \in R$ such that $rp_isrp_i = rp_i$. Then choosing $y = t_-^ip_is$, we have

$$xyx = (rp_it_+^i)(t_-^ip_is)(rp_it_+^i) = (rp_it_+^it_-^ip_is)(rp_it_+^i) = rp_ip_ip_isrp_it_+^i = rp_it_+^i = x.$$

A similar argument shows that for $x \in A_i$, where $i < 0$, there is a y such that $xyx = x$. This shows that A is a graded von Neumann regular ring. ☐

Note that in a corner skew Laurent polynomial ring $R[t_+, t_-, \phi]$, t_+ is a left invertible element with a right inverse t_- (see the relations (1.62)). In fact this property characterises such rings. Namely, a graded ring $A = \bigoplus_{i\in\mathbb{Z}} A_i$ such that A_1 has a left invertible element is a corner skew Laurent polynomial ring as the following theorem shows. The following theorem (first established in [8]) will be used to realise Leavitt path algebras (§1.6.3) as corner skew Laurent polynomial rings.

Theorem 1.6.8. *Let A be a $\mathbb{Z}$-graded ring which has a left invertible element $t_+ \in A_1$. Then its right inverse $t_- \in A_{-1}$, and $A = A_0[t_+, t_-, \phi]$, where $\phi : A_0 \to t_+t_-A_0t_+t_-$, $\phi(a) = t_+at_-$.*

Proof. Since $t_-t_+ = 1$ and $t_+ \in A_1$, it easily follows that $t_- \in A_{-1}$. Furthermore $t_+t_- = t_+t_-t_+t_-$ is a homogeneous idempotent of degree zero. Observe that

$$\begin{aligned}\phi : A_0 &\longrightarrow t_+t_-A_0t_+t_-,\\ a &\longmapsto t_+at_-\end{aligned}$$

is a (unital) ring isomorphism. Consider the corner skew Laurent polynomial ring $\widetilde{A} = A_0[\widetilde{t}_+, \widetilde{t}_-, \phi]$. Since $\phi(a) = t_+ a t_-$, it follows that $t_- \phi(a) = a t_-$ and $\phi(a) t_+ = t_+ a$. Thus t_+ and t_- satisfy all the relations in (1.62). Therefore there is a well-defined map $\psi : \widetilde{A} \to A$, such that $\psi(\widetilde{t}_\pm) = t_\pm$ and the restriction of ψ on A_0 is the identity and

$$\psi\Big(\sum_{k=1}^{j} \widetilde{t}_-^k a_{-k} + a_0 + \sum_{k=1}^{i} a_i \widetilde{t}_+^i\Big) = \sum_{k=1}^{j} t_-^k a_{-k} + a_0 + \sum_{k=1}^{i} a_i t_+^i.$$

This also shows that ψ is a graded homomorphism. In order to show that ψ is an isomorphism, it suffices to show that its restriction to each homogeneous component $\psi : \widetilde{A}_i \to A_i$ is a bijection. Suppose $x \in \widetilde{A}_i$, $i > 0$ such that $\psi(x) = 0$. Then $x = d\widetilde{t}_+^i$ for some $d \in A_0 p_i$ where $p_i = \phi^i(1)$ and $\psi(x) = dt_+^i$. Note that $\phi^i(1) = t_+^i t_-^i$. Thus $d\phi^i(1) = dt_+^i t_-^i = \psi(x) t_-^i = 0$. It now follows $x = d\widetilde{t}_+^i = d\phi^i(1)\widetilde{t}_+^i = 0$ in $\widetilde{A}_i$. This shows ψ is injective. Suppose $y \in A_i$. Then $yt_-^i \in A_0$ and $yt_-^i t_+^i t_-^i = yt_-^i \phi^i(1) \in A_0 \phi^i(1) = A_0 p_i$. This shows $yt_-^i t_+^i t_-^i \widetilde{t}_+^i \in \widetilde{A}_i$. But $\psi(yt_-^i t_+^i t_-^i \widetilde{t}_+^i) = yt_-^i t_+^i t_-^i t_+^i = y$. This shows that $\psi : \widetilde{A}_i \to A_i$, $i > 0$ is a bijection. A similar argument can be written for the case of $i < 0$. The case $i = 0$ is obvious. This completes the proof. □

1.6.3 Graphs and Leavitt path algebras

In this subsection we gather some graph-theoretic definitions and recall the basics on path and Leavitt path algebras. A *directed graph* $E = (E^0, E^1, r, s)$ consists of two countable sets E^0, E^1 and maps $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 *edges*. If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the graph is called *row-finite*. In this note we will consider only row-finite graphs. In this setting, if the number of vertices, i.e., $|E^0|$, is finite, then the number of edges, i.e., $|E^1|$, is finite as well and we call E a *finite* graph.

For a graph $E = (E^0, E^1, r, s)$, a vertex v for which $s^{-1}(v)$ is empty is called a *sink*, while a vertex w for which $r^{-1}(w)$ is empty is called a *source*. An edge with the same source and range is called a *loop*. A path μ in a graph E is a sequence of edges $\mu = \mu_1 \dots \mu_k$, such that $r(\mu_i) = s(\mu_{i+1}), 1 \leq i \leq k-1$. In this case, $s(\mu) := s(\mu_1)$ is the *source* of μ, $r(\mu) := r(\mu_k)$ is the *range* of μ, and k is the *length* of μ which is denoted by $|\mu|$. We consider a vertex $v \in E^0$ as a *trivial* path of length zero with $s(v) = r(v) = v$. If μ is a nontrivial path in E, and if $v = s(\mu) = r(\mu)$, then μ is called a *closed path based at* v. If $\mu = \mu_1 \dots \mu_k$ is a closed path based at $v = s(\mu)$ and $s(\mu_i) \neq s(\mu_j)$ for every $i \neq j$, then μ is called a *cycle*.

For two vertices v and w, the existence of a path with the source v and the range w is denoted by $v \geq w$. Here we allow paths of length zero. By $v \geq_n w$, we mean there is a path of length n connecting these vertices. Therefore $v \geq_0 v$ represents the vertex v. Also, by $v > w$, we mean a path from v to w where $v \neq w$. In this note, by $v \geq w' \geq w$, it is understood that there is a path connecting v to w and going through w' (i.e., w' is on the path connecting v to w). For $n \geq 2$, we define E^n to be the set of paths of length n and $E^* = \bigcup_{n \geq 0} E^n$, the set of all paths.

For a graph E, let $n_{v,w}$ be the number of edges with the source v and range w. Then the *adjacency matrix* of the graph E is $A_E = (n_{v,w})$. Usually one orders the vertices and then writes A_E based on this ordering. Two different ordering of vertices give different adjacency matrices. However if A_E and A'_E are two adjacency matrices of E, then there is a permutation matrix P such that $A'_E = PA_E P^{-1}$.

A graph E is called *essential* if E does not have sinks and sources. Furthermore, a graph is called *irreducible* if for every ordered pair vertices v and w, there is a path from v to w.

A path algebra, with coefficients in the field K, is constructed as follows: consider a K-vector space with finite paths as the basis and define the multiplication by concatenation of paths. A path algebra has a natural graded structure by assigning paths as homogeneous elements of degree equal to their lengths. A formal definition of path algebras with coefficients in a ring R is given below.

Definition 1.6.9. Path algebras.
For a graph E and a ring R with identity, we define the *path algebra of* E, denoted by $\mathcal{P}_R(E)$, to be the algebra generated by the sets $\{v \mid v \in E^0\}$, $\{\alpha \mid \alpha \in E^1\}$ with the coefficients in R, subject to the relations

1. $v_i v_j = \delta_{ij} v_i$ for every $v_i, v_j \in E^0$.
2. $s(\alpha)\alpha = \alpha r(\alpha) = \alpha$ for all $\alpha \in E^1$.

Here the ring R commutes with the generators $\{v, \alpha \mid v \in E^0, \alpha \in E^1\}$. When the coefficient ring R is clear from the context, we simply write $\mathcal{P}(E)$ instead of $\mathcal{P}_R(E)$. When R is not commutative, then we consider $\mathcal{P}_R(E)$ as a left R-module. Using the above two relations, it is easy to see that when the number of vertices is finite, then $\mathcal{P}_R(E)$ is a ring with identity $\sum_{v\in E^0} v$.

When the graph has one vertex and n-loops, the path algebra associated to this graph is isomorphic to $R\langle x_1, \dots, x_n\rangle$, i.e., a free associative unital algebra over R with n non-commuting variables.

Setting $\deg(v) = 0$ for $v \in E^0$ and $\deg(\alpha) = 1$ for $\alpha \in E^1$, we obtain a natural $\mathbb{Z}$-grading on the free R-ring generated by $\{v, \alpha \mid v \in E^0, \alpha \in E^1\}$ (§1.6.1). Since the relations in Definition 1.6.9 are all homogeneous, the ideal generated by these relations is homogeneous and thus we have a natural $\mathbb{Z}$-grading on $\mathcal{P}_R(E)$. Note that $\mathcal{P}(E)$ is positively graded, and for any $m, n \in \mathbb{N}$,

$$\mathcal{P}(E)_m \, \mathcal{P}(E)_n = \mathcal{P}(E)_{m+n}.$$

However by Proposition 1.1.16(2), $\mathcal{P}(E)$ is not an strongly $\mathbb{Z}$-graded ring.

The theory of Leavitt path algebras were introduced in [2, 6] which associate to directed graphs certain type of algebras. These algebras were motivated by Leavitt's construction of universal non-IBN rings [63]. Leavitt path algebras are quotients of path algebras by relations resembling those in construction of algebras studied by Leavitt (see Example 1.3.13).

Definition 1.6.10. Leavitt path algebras.
For a row-finite graph E and a ring R with identity, we define the *Leavitt path algebra of* E, denoted by $\mathcal{L}_R(E)$, to be the algebra generated by the sets $\{v \mid v \in E^0\}$, $\{\alpha \mid \alpha \in E^1\}$ and $\{\alpha^* \mid \alpha \in E^1\}$ with the coefficients in R, subject to the relations

1. $v_i v_j = \delta_{ij} v_i$ for every $v_i, v_j \in E^0$.
2. $s(\alpha)\alpha = \alpha r(\alpha) = \alpha$ and $r(\alpha)\alpha^* = \alpha^* s(\alpha) = \alpha^*$ for all $\alpha \in E^1$.
3. $\alpha^*\alpha' = \delta_{\alpha\alpha'} r(\alpha)$, for all $\alpha, \alpha' \in E^1$.
4. $\sum_{\{\alpha\in E^1, s(\alpha)=v\}} \alpha\alpha^* = v$ for every $v \in E^0$ for which $s^{-1}(v)$ is non-empty.

Here the ring R commutes with the generators $\{v, \alpha, \alpha^* \mid v \in E^0, \alpha \in E^1\}$. When the coefficient ring R is clear from the context, we simply write $\mathcal{L}(E)$ instead of $\mathcal{L}_R(E)$. When R is not commutative, then we consider $\mathcal{L}_R(E)$ as a left R-module. The elements α^* for $\alpha \in E^1$ are called *ghost edges*. One can show that $\mathcal{L}_R(E)$ is a ring with identity if and only if the graph E is finite (otherwise, $\mathcal{L}_R(E)$ is a ring with local identities).

Setting $\deg(v) = 0$, for $v \in E^0$, $\deg(\alpha) = 1$ and $\deg(\alpha^*) = -1$ for $\alpha \in E^1$, we obtain a natural $\mathbb{Z}$-grading on the free R-ring generated by $\{v, \alpha, \alpha^* \mid v \in E^0, \alpha \in E^1\}$. Since the relations in Definition 1.6.10 are all homogeneous, the ideal generated by these relations is homogeneous and thus we have a natural $\mathbb{Z}$-grading on $\mathcal{L}_R(E)$.

If $\mu = \mu_1 \dots \mu_k$, where $\mu_i \in E^1$, is an element of $\mathcal{L}(E)$, then we denote by μ^* the element $\mu_k^* \dots \mu_1^* \in \mathcal{L}(E)$. Further we define $v^* = v$ for any $v \in E^0$. Since $\alpha^*\alpha' = \delta_{\alpha\alpha'} r(\alpha)$, for all $\alpha, \alpha' \in E^1$, any word in the generators $\{v, \alpha, \alpha^* \mid v \in E^0, \alpha \in E^1\}$ in $\mathcal{L}(E)$ can be written as $\mu\gamma^*$ where μ and γ are paths in E (recall that vertices were considered paths of length zero). The elements of the form $\mu\gamma^*$ are called *monomials*.

If the graph E is infinite, $\mathcal{L}_R(E)$ is a graded ring without identity (see Remark 1.1.15).

Taking the grading into account, one can write $\mathcal{L}_R(E) = \bigoplus_{k \in \mathbb{Z}} \mathcal{L}_R(E)_k$ where,

$$\boxed{\mathcal{L}_R(E)_k = \Big\{ \sum_i r_i \alpha_i \beta_i^* \mid \alpha_i, \beta_i \text{ are paths}, r_i \in R, \text{ and } |\alpha_i| - |\beta_i| = k \text{ for all } i \Big\}.}$$

For simplicity we denote $\mathcal{L}_R(E)_k$, the homogeneous elements of degree k, by $\mathcal{L}_k$. The following theorem was proved in [46] which determines the finite graphs so that their associated Leavitt path algebras are strongly graded.

Theorem 1.6.11. *Let E be a finite graph. Then $L(E)$ is strongly graded if and only if E does not have sinks.*

The proof of this theorem is quite long and does not fit for purpose of this note. However, we can realise the Leavitt path algebras of finite graphs with no source in terms of corner skew Laurent polynomial rings (see §1.6.2). This way we can provide a short proof for the above theorem when the graph has no sources.

Let E be a finite graph with no source and $E^0 = \{v_1, \dots, v_n\}$ be the set of all vertices of E. For each $1 \le i \le n$, we choose an edge e_i such that $r(e_i) = v_i$ and consider $t_+ = e_1 + \dots + e_n \in \mathcal{L}(E)_1$. Then $t_- = e_1^* + \dots + e_n^*$ is its right inverse. Thus by Theorem 1.6.8, $\mathcal{L}(E) = \mathcal{L}(E)_0[t_+, t_-, \phi]$, where $\phi : \mathcal{L}(E)_0 \to t_+t_- \mathcal{L}(E)_0 t_+t_-$, $\phi(a) = t_+ a t_-$.

Using this interpretation of Leavitt path algebras we are able to prove the following theorem.

Theorem 1.6.12. *Let E be a finite graph with no source. Then $L(E)$ is strongly graded if and only if E does not have sinks.*

Proof. Write $\mathcal{L}(E) = \mathcal{L}(E)_0[t_+, t_-, \phi]$, where $\phi(1) = t_+t_-$. The theorem now follows from an easy to prove observation that t_+t_- is a full idempotent if and only if E does not have sinks along with Proposition 1.6.6, that $\phi(1)$ is a full idempotent if and only if $\mathcal{L}(E)_0[t_+, t_-, \phi]$ is strongly graded. □

As a consequence of Theorem 1.6.11, we can show that Leavitt path algebras associated to finite graphs with no sinks are graded regular von Neumann rings (§1.1.6).

Corollary 1.6.13. *Let E be a finite graph with no sinks. Then $\mathcal{L}(E)$ is a graded von Neumann regular ring.*

Proof. Since $\mathcal{L}(E)$ is strongly graded (Theorem 1.6.11), by Corollary 1.5.9, $\mathcal{L}(E)$ is von Neumann regular if $\mathcal{L}(E)_0$ is a von Neumann regular ring. But we know that the zero component ring $\mathcal{L}(E)_0$ is an ultramatricial algebra which is von Neumann regular (see the proof of [6, Theorem 5.3]). This finishes the proof. $\square$

Example 1.6.14. <u>Leavitt path algebras are not graded unit regular rings</u>

Recall that a graded ring is *graded von Neumann unit regular* (or *graded unit regular* for short) if for any homogeneous element x, there is an invertible homogeneous element y such that $xyx = x$. Clearly any graded unit regular ring is von Neumann regular. However the converse is not the case. For example, Leavitt path algebras are not in general unit regular as the following example shows. Consider the graph:

$$E: \begin{tikzcd} \bullet \arrow[loop left, "y_1"] \arrow[loop right, "y_2"] \end{tikzcd}$$

Then it is easy to see that there is no homogeneous invertible element x such that $y_1xy_1 = y_1$ in $\mathcal{L}(E)$.

The following theorem determines the graded structure of Leavitt path algebras associated to acyclic graphs.

Theorem 1.6.15. *Let K be a field and E a finite acyclic graph with sinks $\{v_1, \dots, v_t\}$. For any sink v_s, let $R(v_s) = \{p_1^{v_s}, \dots, p_n^{v_s}\}$ denote the set of all paths ending at v_s. Then there is a $\mathbb{Z}$-graded isomorphism*

$$L_K(E) \cong_{\mathrm{gr}} \bigoplus_{s=1}^{t} M_{n(v_s)}(K)(|p_1^{v_s}|, \dots, |p_{n(v_s)}^{v_s}|). \tag{1.64}$$

Sketch of proof. Fix a sink v_s and denote $R(v_s) = \{p_1, \dots, p_n\}$. The set

$$I_{v_s} = \big\{ \sum kp_ip_j^* | k \in K, p_i, p_j \in R(v_s) \big\}$$

is an ideal of $L_K(E)$, and we have an isomorphism $\phi : I_{v_s} \to M_{n(v_s)}(K)$, $kp_ip_j^* \mapsto k(e_{ij})$, where $k \in K$, $p_i, p_j \in R(v_s)$ and e_{ij} is the standard matrix unit. Now, considering the grading on $M_{n(v_s)}(K)(|p_1^{v_s}|, \dots, |p_{n(v_s)}^{v_s}|)$, we show that ϕ is a graded isomorphism. Let $p_ip_j^* \in I_{v_s}$. Then

$$\deg(p_ip_j^*) = |p_i| - |p_j| = \deg(e_{ij}) = \deg(\phi(p_ip_j^*)).$$

So ϕ respects the grading. Hence ϕ is a graded isomorphism. One can check that

$$L_K(E) = \bigoplus_{s=1}^{t} I_{v_s} \cong_{\mathrm{gr}} \bigoplus_{s=1}^{t} M_{n(v_s)}(K)(|p_1^{v_s}|, \dots, |p_{n(v_s)}^{v_s}|). \qquad \square$$

Example 1.6.16. Theorem 1.6.15 shows that the Leavitt path algebras of the graphs E_1 and E_2 below with coefficients from the field K are graded isomorphic to $\mathbb{M}_5(K)\big(0, 1, 1, 2, 2\big)$ and thus $\mathcal{L}(E_1) \cong_{\mathrm{gr}} \mathcal{L}(E_2)$. However $\mathcal{L}(E_3) \cong_{\mathrm{gr}} \mathbb{M}_5(K)(0, 1, 2, 2, 3)$.

$$E_1: \begin{tikzcd} & & \bullet \arrow[d] \\ & & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd} \qquad E_2: \begin{tikzcd} & \bullet \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd} \qquad E_3: \begin{tikzcd} \bullet \arrow[d] & \bullet \arrow[d] & \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

Theorem 1.6.17. *Let K be a field and E a C_n-comet with the cycle C of length $n \geq 1$. Let v be a vertex on the cycle C with e be the edge in the cycle with $s(e) = v$. Eliminate the edge e and consider the set $\{p_i | 1 \leq i \leq m\}$ of all paths with end in v. Then*

$$L_K(E) \cong_{\mathrm{gr}} \mathbb{M}_m(K[x^n, x^{-n}])(|p_1|, \dots, |p_m|). \tag{1.65}$$

Sketch of proof. One can show that the set of monomials $\{p_i C^k p_j^* | 1 \leq i, j \leq n, k \in \mathbb{Z}\}$ are a basis of $L_K(E)$ as a K-vector space. Define a map

$$\phi : L_K(E) \to \mathbb{M}_m(K[x^n, x^{-n}])(|p_1|, \dots, |p_m|), \text{ by } \quad \phi(p_i C^k p_j^*) = e_{ij}(x^{kn}),$$

where $e_{ij}(x^{kn})$ is a matrix with x^{kn} in the ij-position and zero elsewhere. Extend this linearly to $L_K(E)$. We have

$$\begin{aligned} \phi((p_i C^k p_j^*)(p_r C^t p_s^*)) &= \phi(\delta_{jr} p_i C^{k+t} p_s^*) \\ &= \delta_{jr} e_{is} x^{(k+t)n} \\ &= (e_{ij} x^{kn})(e_{rs} x^{tn}) \\ &= \phi(p_i C^k p_j^*) \phi(p_r C^t p_s^*). \end{aligned}$$

Thus ϕ is a homomorphism. Also, ϕ sends the basis to the basis, so ϕ is an isomorphism.
We now need to show that ϕ is graded. We have

$$\deg(p_i C^k p_j^*) = |p_i C^k p_j^*| = nk + |p_i| - |p_j|$$

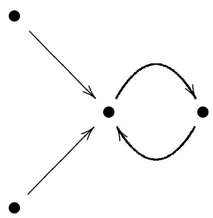
and

$$\deg(\phi(p_i C^k p_j^*)) = \deg(e_{ij}(x^{kn})) = nk + |p_i| - |p_j|.$$

Therefore ϕ respects the grading. This finishes the proof. $\square$

Example 1.6.18. By Theorem 1.6.11, the Leavitt path algebra of the following graph with coefficients in a field K,

$E :$



is strongly graded. By Theorem 1.6.17,

$$\mathcal{L}_K(E) \cong_{\mathrm{gr}} \mathbb{M}_4(K[x^2, x^{-2}])(0, 1, 1, 1). \tag{1.66}$$

However this algebra is not a crossed-product. Set $B = K[x, x^{-1}]$ with the grading $B = \bigoplus_{n \in \mathbb{Z}} K x^n$ and consider $A = K[x^2, x^{-2}]$ as a graded subring of B with $A_n = K x^n$ if $n \equiv 0 \pmod 2$, and $A_n = 0$ otherwise. Using the graded isomorphism of (1.66), by (1.34) a homogeneous element of degree 1 in $\mathcal{L}_K(E)$ has the form

$$\begin{pmatrix} A_1 & A_2 & A_2 & A_2 \\ A_0 & A_1 & A_1 & A_1 \\ A_0 & A_1 & A_1 & A_1 \\ A_0 & A_1 & A_1 & A_1 \end{pmatrix}.$$

Since $A_1 = 0$, the determinants of these matrices are zero, and thus no homogeneous element of degree 1 is invertible. Thus $\mathcal{L}_K(E)$ is not crossed-product (see §1.1.2).

Now consider the following graph,

$$E: \quad \begin{tikzcd} \bullet \arrow[r, "f"] & \bullet \arrow[r, bend left, "g"] & \bullet \arrow[l, bend left, "h"] & \bullet \arrow[l, "e"'] \end{tikzcd}$$

By Theorem 1.6.17,

$$\mathcal{L}_K(E) \cong_{\mathrm{gr}} \mathbb{M}_4(K[x^2, x^{-2}])(0,1,1,2). \tag{1.67}$$

Using the graded isomorphism of (1.67), by (1.36) homogeneous elements of degree 0 in $\mathcal{L}_K(E)$ have the form

$$\mathcal{L}_K(E)_0 = \begin{pmatrix} A_0 & A_1 & A_1 & A_2 \\ A_{-1} & A_0 & A_0 & A_1 \\ A_{-1} & A_0 & A_0 & A_1 \\ A_{-2} & A_{-1} & A_{-1} & A_0 \end{pmatrix} = \begin{pmatrix} K & 0 & 0 & Kx^2 \\ 0 & K & K & 0 \\ 0 & K & K & 0 \\ Kx^{-2} & 0 & 0 & K \end{pmatrix}.$$

In the same manner, homogeneous elements of degree one have the form,

$$\mathcal{L}_K(E)_1 = \begin{pmatrix} 0 & Kx^2 & Kx^2 & 0 \\ K & 0 & 0 & Kx^2 \\ K & 0 & 0 & Kx^2 \\ 0 & K & K & 0 \end{pmatrix}.$$

Choose

$$u = \begin{pmatrix} 0 & 0 & x^2 & 0 \\ 0 & 0 & 0 & x^2 \\ 1 & 0 & 0 & x^2 \\ 0 & 1 & 0 & 0 \end{pmatrix} \in \mathcal{L}(E)_1$$

and observe that u is invertible (this matrix corresponds to the element $g + h + fge^* + ehf^* \in \mathcal{L}_K(E)_1$).

Thus $\mathcal{L}_K(E)$ is crossed-product (and therefore a skew group ring as the grading is cyclic), i.e.,

$$\mathcal{L}_K(E) \cong_{\mathrm{gr}} \bigoplus_{i \in \mathbb{Z}} \mathcal{L}_K(E)_0 u^i$$

and a simple calculation shows that one can describe this algebra as follows:

$$\mathcal{L}_K(E)_0 \cong \mathbb{M}_2(K) \times \mathbb{M}_2(K),$$

and

$$\mathcal{L}_K(E) \cong_{\mathrm{gr}} \big(\mathbb{M}_2(K) \times \mathbb{M}_2(K)\big) \star_\tau \mathbb{Z}, \tag{1.68}$$

where,

$$\tau\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right) = \left(\begin{pmatrix} b_{22} & b_{21} \\ b_{12} & b_{11} \end{pmatrix}, \begin{pmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{pmatrix}\right).$$

Remark 1.6.19. For a graph E, the Leavitt path algebra $\mathcal{L}_K(E)$ has a $\mathbb{Z}$-graded structure. This grading was obtained by assigning 0 to vertices, 1 to edges and -1 to ghost edges. However, one can equip $\mathcal{L}_K(E)$ with other graded structures as well. Let Γ be an arbitrary group with the identity element e. Let $w : E^1 \to \Gamma$ be a *weight* map and further define $w(\alpha^*) = w(\alpha)^{-1}$, for any edge $\alpha \in E^1$ and $w(v) = e$ for $v \in E^0$. The free K-algebra generated by the vertices, edges and ghost edges is a Γ-graded K-algebra (see §1.6.1). Furthermore, the Leavitt path algebra is the quotient of this algebra by relations in Definition 1.6.10 which are all homogeneous. Thus $\mathcal{L}_K(E)$ is a Γ-graded K-algebra.

As an example, consider the graphs

$$E: \quad \begin{tikzcd} \bullet \arrow[r, "f"] & \bullet \arrow[loop right, "e"] \end{tikzcd} \qquad\qquad F: \quad \begin{tikzcd} \bullet \arrow[r, bend left, "g"] & \bullet \arrow[l, bend left, "h"] \end{tikzcd}$$

and assign 1 for the degree of f and 2 for the degree of e in E and 1 for the degrees of g and h in F. Then the proof of Theorem 1.6.17 shows that $\mathcal{L}_K(E) \cong \mathbb{M}_2(K[x^2, x^{-2}])(0, 1)$ and $\mathcal{L}_K(F) \cong \mathbb{M}_2(K[x^2, x^{-2}])(0, 1)$. So with these gradings, $\mathcal{L}_K(E) \cong_{\mathrm{gr}} \mathcal{L}_K(F)$.

1.7 The graded IBN and graded type

A ring A with identity has *invariant basis number* (IBN) or *invariant basis property* if any two bases of a free (right) A-module have the same cardinality, i.e., if $A^n \cong A^m$ as A-modules, then $n = m$. When A does not have IBN, the *type* of A is defined as a pair of positive integers (n, k) such that $A^n \cong A^{n+k}$ as A-modules and these are the smallest number with this property, that is, (n, k) is the minimum under the usual lexicographic order. This means any two bases of a free A-module have the unique cardinality if one of the basis has the cardinality less than n and further if a free module has rank n, then a free module with the smallest cardinality (other than n) isomorphic to this module is of rank $n + k$. Another way to describe a type (n, k) is that, $A^n \cong A^{n+k}$ is the first repetition in the list $A, A^2, A^3, \dots$.

It was shown that if A has type (n, k), then $A^m \cong A^{m'}$ if and only if $m = m'$ or $m, m' \geq n$ and $m \equiv m' \pmod{k}$ (see [26, p. 225], [63, Theorem 1]).

One can show that a (right) noetherian ring has IBN. Furthermore, if there is a ring homomorphism $A \to B$, (which preserves 1), and B has IBN then A has IBN as well. Indeed, if $A^m \cong A^n$, then $B^m \cong A^m \otimes_A B \cong A^n \otimes_A B \cong B^n$, so $n = m$. One can describe the type of a ring by using the monoid of isomorphism classes of finitely generated projective modules (see Example 3.1.4). For a nice discussion about these rings see [20, 26, 66].

A graded ring A has a *graded invariant basis number* (gr-IBN) if any two homogeneous bases of a graded free (right) A-module have the same cardinality, i.e., if $A^m(\overline{\alpha}) \cong_{\mathrm{gr}} A^n(\overline{\delta})$, where $\overline{\alpha} = (\alpha_1, \dots, \alpha_m)$ and $\overline{\delta} = (\delta_1, \dots, \delta_n)$, then $m = n$. Note that contrary to the non-graded case, this does not imply that two graded free modules with bases of the same cardinality are graded isomorphic (see Proposition 1.3.10). A graded ring A has *IBN in* gr-A, if $A^m \cong_{\mathrm{gr}} A^n$ then $m = n$. If A has IBN in gr-A, then A_0 has IBN. Indeed, if $A_0^m \cong A_0^n$ as A_0-modules, then $A^m \cong_{\mathrm{gr}} A_0^m \otimes_{A_0} A \cong A_0^n \otimes_{A_0} A \cong_{\mathrm{gr}} A^n$, so $n = m$ (see [73, p. 215]).

When the graded ring A does not have gr-IBN, the *graded type* of A is defined as a pair of positive integers (n, k) such that $A^n(\overline{\delta}) \cong_{\mathrm{gr}} A^{n+k}(\overline{\alpha})$ as A-modules, for some $\overline{\delta} = (\delta_1, \dots, \delta_n)$ and

$\overline{\alpha} = (\alpha_1, \dots, \alpha_{n+k})$ and these are the smallest number with this property. In Proposition 1.7.1 we show that the Leavitt algebra $\mathcal{L}(n, k+1)$ (see Example 1.6.5) has graded type (n, k).

Parallel to the non-graded setting, one can show that a graded (right) noetherian ring has gr-IBN. Furthermore, if there is a graded ring homomorphism $A \to B$, (which preserves 1), and B has gr-IBN then A has gr-IBN as well. Indeed, if $A^m(\overline{\alpha}) \cong_{\mathrm{gr}} A^n(\overline{\delta})$, where $\overline{\alpha} = (\alpha_1, \dots, \alpha_m)$ and $\overline{\delta} = (\delta_1, \dots, \delta_n)$, then

$$B^m(\overline{\alpha}) \cong_{\mathrm{gr}} A^m(\overline{\alpha}) \otimes_A B \cong A^n(\overline{\delta}) \otimes_A B \cong_{\mathrm{gr}} B^n(\overline{\delta}),$$

which implies $n = m$. Using this, one can show that any graded commutative ring has gr-IBN. For, there exists a graded maximal ideal and its quotient ring is a graded field which has gr-IBN (see §1.1.5 and Proposition 1.4.2).

Let A be a Γ-graded ring such that $A^m(\overline{\alpha}) \cong_{\mathrm{gr}} A^n(\overline{\delta})$, where $\overline{\alpha} = (\alpha_1, \dots, \alpha_m)$ and $\overline{\delta} = (\delta_1, \dots, \delta_n)$. Then there is a universal Γ-graded ring R such that $R^m(\overline{\alpha}) \cong_{\mathrm{gr}} R^n(\overline{\delta})$, and a graded ring homomorphism $R \to A$ which induces the graded isomorphism $A^m(\overline{\alpha}) \cong_{\mathrm{gr}} R^m(\overline{\alpha}) \otimes_R A \cong_{\mathrm{gr}} R^n(\overline{\delta}) \otimes_R A \cong_{\mathrm{gr}} A^n(\overline{\delta})$. Indeed, by Proposition 1.3.10, there are matrices $a = (a_{ij}) \in \mathbb{M}_{n\times m}(A)[\overline{\delta}][\overline{\alpha}]$ and $b = (b_{ij}) \in \mathbb{M}_{m\times n}(A)[\overline{\alpha}][\overline{\delta}]$ such that $ab = \mathbb{I}_n$ and $ba = \mathbb{I}_m$. The free ring generated by symbols in place of a_{ij} and b_{ij} subject to relations imposed by $ab = \mathbb{I}_n$ and $ba = \mathbb{I}_m$ is the desired universal graded ring. In detail, let F be a free ring generated by x_{ij}, $1 \leq i \leq n$, $1 \leq j \leq m$ and y_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$. Assign the degrees $\deg(x_{ij}) = \delta_i - \alpha_j$ and $\deg(y_{ij}) = \alpha_i - \delta_j$ (see §1.6.1). This makes F a Γ-graded ring. Let R be a ring F modulo the relations $\sum_{s=1}^m x_{is}y_{sk} = \delta_{ik}$, $1 \leq i, k \leq n$ and $\sum_{t=1}^n y_{it}x_{tk} = \delta_{ik}$, $1 \leq i, k \leq m$, where δ_{ik} is the Kronecker delta. Since all the relations are homogeneous, R is a Γ-graded ring. Clearly the map sending x_{ij} to a_{ij} and y_{ij} to b_{ij} induces a graded ring homomorphism $R \to A$. Again Proposition 1.3.10 shows that $R^m(\overline{\alpha}) \cong_{\mathrm{gr}} R^n(\overline{\delta})$.

Proposition 1.7.1. *Let $R = \mathcal{L}(n, k+1)$ be the Leavitt algebra of type (n, k). Then*

(1) *R is a universal $\bigoplus_n \mathbb{Z}$-graded ring which does not have gr-IBN;*

(2) *R has graded type (n, k);*

(3) *For $n = 1$, R has IBN in* gr-R.

Proof. (1) Consider the Leavitt algebra $\mathcal{L}(n, k+1)$ constructed in Example 1.6.5, which is a $\bigoplus_n \mathbb{Z}$-graded ring and is universal. Furthermore (1.61) combined by Proposition 1.3.10(3) shows that $R^n \cong_{\mathrm{gr}} R^{n+k}(\overline{\alpha})$. Here $\overline{\alpha} = (\alpha_1, \dots, \alpha_{n+k})$, where $\alpha_i = (0, \dots, 0, 1, 0 \dots, 0)$ and 1 is in i-th entry. This shows that $R = \mathcal{L}(n, k+1)$ does not have gr-IBN.

(2) By [26, Theorem 6.1], R is of type (n, k). This immediately implies the graded type of R is also (n, k).

(3) Suppose $R^n \cong_{\mathrm{gr}} R^m$ as graded R-modules. Then $R_0^n \cong R_0^m$ as R_0-modules. But R_0 is an ultramatricial algebra, i.e., direct limit of matrices over a field. Since IBN respects direct limits ([26, Theorem 2.3]), R_0 has IBN. Therefore, $n = m$. □

Remark 1.7.2. Assignment of $\deg(y_{ij}) = 1$ and $\deg(x_{ij}) = -1$, for all i, j, makes $R = \mathcal{L}(n, k+1)$ a $\mathbb{Z}$-graded algebra of graded type (n, k) with $R^n \cong_{\mathrm{gr}} R^{n+k}(1)$.

Remark 1.7.3. Let A be a Γ-graded ring. In [75, Proposition 4.4], it was shown that if Γ is finite then A has gr-IBN if and only if A has IBN.

1.8 The graded stable rank

The notion of stable rank was defined by H. Bass [12] in order to study K_1-group of rings with finite Krull dimension. For a concise introduction of the stable rank, we refer the reader to [60, 61], and for its applications to K-theory to [11, 12]. It seems the natural notion of the graded stable rank in the context of graded ring theory, has not yet been investigated in the literature. In this section we define the graded stable rank and study the important case of graded rings with graded stable rank 1. This will be used later in §3 in relation with the graded Grothendieck groups.

A row $(a_1, \dots, a_n)$ of homogeneous elements of a Γ-graded ring A is called a *graded left unimodular row* if the graded ideal generated by a_i, $1 \leq i \leq n$, is A. When $n \geq 2$, a graded left unimodular row is called *stable* if there exists homogeneous elements $r_1, \dots, r_{n-1}$ of A such that the graded left ideal generated by homogeneous elements $a_i + r_i a_n$, $1 \leq i \leq n-1$, is A.

The *graded left stable rank* of a ring A is defined to be n, denoted $\operatorname{sr}^{\mathrm{gr}}(A) = n$, if any graded unimodular row of length $n+1$ is stable, but there exists a non-stable unimodular row of length n. If such n does not exist (i.e. there are non-stable unimodular rows of arbitrary length) we say that the graded stable rank of A is infinite.

In order that this definition would be well-defined, one needs to show that if any graded unimodular row of fixed length n is stable, so is any unimodular row of a greater length. This can be proved similar to the non-graded case and we omit the proof (see for example [60, Proposition 1.3]).

When the graded group Γ is a trivial group, the above definitions reduce to the standard definitions of unimodular rows and stable ranks.

The case of graded stable rank 1 is of special importance. Suppose A is a Γ-graded with $\operatorname{sr}^{\mathrm{gr}}(A) = 1$. Then from the definition it follows that, if $a, b \in A^h$ such that $Aa + Ab = A$, then there is a homogeneous element c such that the homogeneous element $a + cb$ is left invertible. When $\operatorname{sr}^{\mathrm{gr}}(A) = 1$, any left invertible homogeneous element is in fact invertible. For, suppose $c \in A^h$ is left invertible, i.e., there is $a \in A^h$ such that $ac = 1$. Then the row $(a, 1 - ca)$ is graded left unimodular. Thus, there is a $s \in A^h$ such that $u := a + s(1 - ca)$ is left invertible. But $uc = 1$. Thus u is (left and right) invertible and consequently, c is an invertible homogeneous element.

The graded stable rank 1 is quite a strong condition. In fact if $\operatorname{sr}^{\mathrm{gr}}(A) = 1$ then $\Gamma_A = \Gamma_A^*$. For, if $a \in A_\gamma$ is a non-zero element, then since $(a, 1)$ is unimodular and $\operatorname{sr}^{\mathrm{gr}}(A) = 1$, there is $b \in A^h$, such that $a + b$ is an invertible homogeneous element, necessarily of degree γ. Thus $\gamma \in \Gamma_A^*$.

Example 1.8.1. Graded division rings have graded stable rank 1

Since any nonzero homogeneous element of a graded division ring is invertible, one can easily show that its graded stable rank is 1. Thus for a field K, $\operatorname{sr}^{\mathrm{gr}}(K[x, x^{-1}]) = 1$, whereas $\operatorname{sr}(K[x, x^{-1}]) = 2$.

Example 1.8.2. For a strongly graded ring A, $\operatorname{sr}^{\mathrm{gr}}(A) \neq \operatorname{sr}(A_0)$

Let $A = \mathcal{L}(1, 2)$ be the Leavitt algebra generated by x_1, x_2, y_1, y_2 (see Example 1.3.13). Then relations (1.44) show that y_1 is left invertible but it is not invertible. This shows that $\operatorname{sr}^{\mathrm{gr}}(A) \neq 1$. On the other hand, since A_0 is an ultramatricial algebra, $\operatorname{sr}(A_0) = 1$ (see §3.8.3, [60, Corollary 5.5] and [38]).

We have the following theorem which is a graded version of the Cancellation Theorem with a similar proof (see [61, Theorem 20.11]).

Theorem 1.8.3. Graded Cancellation Theorem

Let A be a Γ-graded ring and M, N, P be graded right A-modules, with P being finitely generated. If *the graded ring $E := \mathrm{End}_A(P)$ has graded left stable rank 1, then $P \oplus M \cong_{\mathrm{gr}} P \oplus N$ as A-modules implies $M \cong_{\mathrm{gr}} N$ as A-modules.*

Proof. Let $h : P \oplus M \to P \oplus N$ be a graded A-module isomorphism. Then the composition of the maps

$$P \xrightarrow{i_1} P \oplus M \xrightarrow{h} P \oplus N \xrightarrow{\pi_1} P,$$
$$M \xrightarrow{i_2} P \oplus M \xrightarrow{h} P \oplus N \xrightarrow{\pi_1} P$$

induces a graded split epimorphism of degree zero, denoted by $(f, g) : P \oplus M \to P$. Here $(f, g)(p, m) = f(p) + g(m)$, where $f = \pi_1 h i_1$ and $g = \pi_1 h i_2$. It is clear that $\ker(f, g) \cong_{\mathrm{gr}} N$. Let $\begin{pmatrix} f' \\ g' \end{pmatrix} : P \to P \oplus M$ be the split homomorphism. Thus

$$1 = (f, g) \begin{pmatrix} f' \\ g' \end{pmatrix} = ff' + gg'.$$

This shows that the left ideal generated by f' and gg' is E. Since E has graded stable rank 1, it follows there is $e \in E$ of degree 0 such that $u := f' + e(gg')$ is an invertible element of E. Writing $u = (1, eg) \begin{pmatrix} f' \\ g' \end{pmatrix}$ implies that both $\ker(f, g)$ and $\ker(1, eg)$ are graded isomorphic to

$$P \oplus M / \operatorname{Im} \begin{pmatrix} f' \\ g' \end{pmatrix}.$$

Thus $\ker(f, g) \cong_{\mathrm{gr}} \ker(1, eg)$. But $\ker(f, g) \cong_{\mathrm{gr}} N$ and $\ker(1, eg) \cong_{\mathrm{gr}} M$. Thus $M \cong_{\mathrm{gr}} N$. □

The following corollary will be used in §3 to show that for a graded ring with graded stable rank 1, the monoid of graded finitely generated projective modules injects into the graded Grothendieck group (see Corollary 3.1.8).

Corollary 1.8.4. *Let A be a Γ-graded ring with graded left stable rank 1 and M, N, P be graded right A-modules. If P is a graded finitely generated projective A-module, then $P \oplus M \cong_{\mathrm{gr}} P \oplus N$ as A-modules implies $M \cong_{\mathrm{gr}} N$ as A-modules.*

Proof. Suppose $P \oplus M \cong_{\mathrm{gr}} P \oplus N$ as A-modules. Since P is a graded finitely generated A-module, there is a graded A-module Q such that $P \oplus Q \cong_{\mathrm{gr}} A^n(\overline{\alpha})$ (see (1.27)). It follows that

$$A^n(\overline{\alpha}) \oplus M \cong_{\mathrm{gr}} A^n(\overline{\alpha}) \oplus N.$$

We prove that if

$$A(\alpha) \oplus M \cong_{\mathrm{gr}} A(\alpha) \oplus N, \tag{1.69}$$

then $M \cong_{\mathrm{gr}} N$. The corollary then follows by an easy induction.

By (1.37) there is a graded ring isomorphism $\mathrm{End}_A(A(\alpha)) \cong_{\mathrm{gr}} A$. Since A has graded stable rank 1, so does $\mathrm{End}_A(A(\alpha))$. Now by Theorem 1.8.3, from (1.69) it follows that $M \cong_{\mathrm{gr}} N$. This finishes the proof. □

Chapter 2

Graded Morita Theory

Starting from a right A-module P, one can construct a 6-tuple, $(A, P, P^*, B; \phi, \psi)$, where $P^* = \operatorname{Hom}_A(P, A)$, $B = \operatorname{Hom}_A(P, P)$, $\phi : P^* \otimes_B P \to A$ and $\psi : P \otimes_R P^* \to B$, which have appropriate bimodule structures. This is called the *Morita context* associated with P. When P is a progenerator (i.e., finitely generated projective and a generator), then one can show that $\text{Mod-}A \approx \text{Mod-}B$, i.e, the category of (right) A-modules is equivalent to the category of (right) B-modules. Conversely, if for two rings A and B, $\text{Mod-}A \approx \text{Mod-}B$, then one can construct a A-progenerator P such that $B \cong \operatorname{End}_A(P)$. The feature of this theory, called the Morita theory, is that the whole process involves working with Hom and tensor functors, both of which respect the grading. Thus starting from a graded ring A and a graded progenerator P, and carrying out the Morita theory, one can naturally extending the equivalence from $\text{Gr-}A \approx \text{Gr-}B$ to $\text{Mod-}A \approx \text{Mod-}B$ and vice versa.

Extending the equivalence from the (sub)categories of graded modules to the categories of modules is not at the first glance obvious. Recall that two categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if and only if there is a functor $\phi : \mathcal{C} \to \mathcal{D}$ which is fully faithful and for any $D \in \mathcal{D}$, there is $C \in \mathcal{C}$ such that $\phi(C) = D$.

Suppose $\phi : \text{Gr-}A \to \text{Gr-}B$ is a graded equivalent (see Definition 2.3.2). Then for two graded A-modules, M and N, (where M is finitely generated) we have

$$\operatorname{Hom}_A(M, N)_0 = \operatorname{Hom}_{\text{Gr-}A}(M, N) \cong \operatorname{Hom}_{\text{Gr-}B}(\phi(M), \phi(N)) = \operatorname{Hom}_B(\phi(M), \phi(N))_0.$$

The fact that, this can be extended to

$$\operatorname{Hom}_A(M, N) \cong \operatorname{Hom}_B(\phi(M), \phi(N)),$$

is not immediate. We will show that this is indeed the case.

In this chapter we study the equivalences between the categories of graded modules and their relations with the categories of modules. The main theorem of this chapter shows that for two graded rings A and B, if $\text{Gr-}A \approx_{\text{gr}} \text{Gr-}B$ then $\text{Mod-}A \approx \text{Mod-}B$ (Theorem 2.3.5). This was first studied by Gordon and Green [40] in the setting of $\mathbb{Z}$-graded rings.

Throughout this chapter, A is a Γ-graded ring unless otherwise stated. Furthermore, all functors are *additive* functors. For the theory of (non-graded) Morita theory we refer the reader to [62] and [4]

2.1 First instance of the graded Morita equivalence

Before describing the general graded Morita equivalence, we treat a special case of matrix algebras, which, in the word of T.Y. Lam, is the "first instance" of equivalence between module categories [62,

§17B]. This special case is quite explicit and will help us in calculating the graded Grothendieck group of matrix algebras. Recall from §1.3 that for a Γ-graded ring A, $A^n(\overline{\delta}) = A(\delta_1)\oplus\cdots\oplus A(\delta_n)$ is a graded free A-bimodule. Furthermore, $A^n(\overline{\delta})$ is a graded right $\mathbb{M}_n(A)(\overline{\delta})$-module and $A^n(-\overline{\delta})$ is a graded left $\mathbb{M}_n(A)(\overline{\delta})$-module, where $-\overline{\delta} = (-\delta_1,\dots,-\delta_n)$.

Proposition 2.1.1. *Let A be a Γ-graded ring and let $\overline{\delta} = (\delta_1,\dots,\delta_n)$, where $\delta_i \in \Gamma$, $1 \leq i \leq n$. Then the functors*

$$\begin{aligned}\psi : \text{Gr-}\,\mathbb{M}_n(A)(\overline{\delta}) &\longrightarrow \text{Gr-}A,\\ P &\longmapsto P \otimes_{\mathbb{M}_n(A)(\overline{\delta})} A^n(-\overline{\delta})\end{aligned}$$

and

$$\begin{aligned}\varphi : \text{Gr-}A &\longrightarrow \text{Gr-}\,\mathbb{M}_n(A)(\overline{\delta}),\\ Q &\longmapsto Q \otimes_A A^n(\overline{\delta})\end{aligned}$$

form equivalences of categories and commute with suspensions, i.e, $\psi\mathcal{T}_\alpha = \mathcal{T}_\alpha\psi$, $\alpha\in\Gamma$.

Proof. One can check that there is a Γ-graded A-bimodule isomorphism

$$\begin{aligned}f:\ A^n(\overline{\delta}) \otimes_{\mathbb{M}_n(A)(\overline{\delta})} A^n(-\overline{\delta}) &\longrightarrow A,\\ (a_1,\dots,a_n)\otimes\begin{pmatrix}b_1\\ \vdots\\ b_n\end{pmatrix} &\longmapsto a_1b_1+\cdots+a_nb_n\end{aligned} \tag{2.1}$$

with

$$\begin{aligned}f^{-1} : A &\longrightarrow A^n(\overline{\delta}) \otimes_{\mathbb{M}_n(A)(\overline{\delta})} A^n(-\overline{\delta}),\\ a &\longmapsto (a,0,\dots,0)\otimes\begin{pmatrix}1\\ \vdots\\ 0\end{pmatrix}.\end{aligned}$$

Furthermore, there is a Γ-graded $\mathbb{M}_n(A)(\overline{\delta})$-bimodule isomorphism

$$\begin{aligned}g:\ A^n(-\overline{\delta}) \otimes_A A^n(\overline{\delta}) &\longrightarrow \mathbb{M}_n(A)(\overline{\delta}),\\ \begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}\otimes(b_1,\dots,b_n) &\longmapsto \begin{pmatrix}a_1b_1 & \cdots & a_1b_n\\ \vdots & & \vdots\\ a_nb_1 & \cdots & a_nb_n\end{pmatrix}\end{aligned} \tag{2.2}$$

with

$$\begin{aligned}g^{-1}:\ \mathbb{M}_n(A)(\overline{\delta}) &\longrightarrow A^n(-\overline{\delta}) \otimes_A A^n(-\overline{\delta}),\\ (a_{i,j}) &\longmapsto \begin{pmatrix}a_{1,1}\\ a_{2,1}\\ \vdots\\ a_{n,1}\end{pmatrix}\otimes(1,0,\dots,0)+\cdots+\begin{pmatrix}a_{1,n}\\ \vdots\\ a_{n-1,n}\\ a_{n,n}\end{pmatrix}\otimes(1,0,\dots,0).\end{aligned}$$

Now using (2.1) and (2.2), it easily follows that $\varphi\psi$ and $\psi\varphi$ are equivalent to identity functors. The general fact that for $\alpha \in \Gamma$,

$$(P \otimes Q)(\alpha) = P(\alpha) \otimes Q = P \otimes Q(\alpha),$$

(see §1.2.6) shows that the suspension functor commutes with ψ and ϕ. $\square$

Since the functors ϕ and ψ, being tensor functors, preserve the projectivity, and send finitely generated modules to finitely generated modules, we immediately get the following corollary which we will use it to compute graded K_0 of matrix algebras.

Corollary 2.1.2. *Let A be a Γ-graded ring and let $\overline{\delta} = (\delta_1, \dots, \delta_n)$, where $\delta_i \in \Gamma$, $1 \le i \le n$. Then the functors*

$$\begin{aligned} \psi : \text{Pgr-}\,\mathbb{M}_n(A)(\overline{\delta}) &\longrightarrow \text{Pgr-}A, \\ P &\longmapsto P \otimes_{\mathbb{M}_n(A)(\overline{\delta})} A^n(-\overline{\delta}) \end{aligned}$$

and

$$\begin{aligned} \varphi : \text{Pgr-}A &\longrightarrow \text{Pgr-}\,\mathbb{M}_n(A)(\overline{\delta}), \\ Q &\longmapsto Q \otimes_A A^n(\overline{\delta}) \end{aligned}$$

form equivalences of categories and commute with suspensions, i.e, $\psi\mathcal{T}_\alpha = \mathcal{T}_\alpha\psi$, $\alpha \in \Gamma$.

Example 2.1.3. <u>Strongly graded is not a Morita invariant property</u>

Let K be a field and consider the $\mathbb{Z}$-graded ring $K[x^2, x^{-2}]$ which has the support $2\mathbb{Z}$. By Proposition 2.1.1, $A = \mathbb{M}_2(K[x^2, x^{-2}])(0, 1)$ is graded Morita equivalent to $B = K[x^2, x^{-2}]$. However, one can easily see that A is strongly graded whereas B is not.

One can also observe that although A and B are graded Morita equivalent, the support set of A is $\mathbb{Z}$, whereas the support of B is $2\mathbb{Z}$ (see (1.38)).

Remark 2.1.4. Let A be a Γ-graded ring and let $\overline{\delta} = (\delta_1, \dots, \delta_n)$, where $\delta_i \in \Gamma$, $1 \le i \le n$. Suppose Ω is a subgroup of Γ such that $\Gamma_A \subseteq \Omega$ and $\{\delta_1, \dots, \delta_n\} \subseteq \Omega$. Since the support of $\mathbb{M}_n(A)(\overline{\delta})$ is a subset of Ω (see (1.38)), we can naturally consider A and $\mathbb{M}_n(A)(\overline{\delta})$ as Ω-graded rings (see Remark 1.2.4). Since the Γ-isomorphisms (2.1) and (2.2) in the proof of Proposition 2.1.1 can be considered as Ω-isomorphism, we can easily see that $\text{Gr}^{\Omega}\text{-}\,\mathbb{M}_n(A)(\overline{\delta})$ is equivalent to $\text{Gr}^{\Omega}\text{-}A$ as well. The converse of this statement is valid for the general graded Morita Theory, see Remark 2.3.9.

2.2 Graded generators

We start with a categorical definition of graded generators. In this section, as usual, the modules are (graded) right A-modules.

Definition 2.2.1. Let A be a Γ-graded ring. A graded A-module P is called a *graded generator* if $f : M \to N$ is a nonzero graded A-module homomorphism, then there exists $\alpha \in \Gamma$ and a graded homomorphism $g : P(\alpha) \to M$ such that $fg : P(\alpha) \to M \to N$ is not zero map.

Let A be a Γ-graded ring and P be a graded right A-module. Define the *graded trace ideal* of P as follows

$$\boxed{\operatorname{Tr}^{\text{gr}}(P) := \Big\{ \sum_i f_i(p_i) \mid f_i \in \operatorname{Hom}(P, A)_\alpha, \alpha \in \Gamma,\, p_i \in P^h \Big\}.}$$

One can check that $\operatorname{Tr}^{\mathrm{gr}}(P)$ is a graded two sided ideal of A. The following theorem provides a set theoretic way to define a graded generator.

Theorem 2.2.2. *For any graded A-module P, the following are equivalent:*

(1) *P is a graded generator;*

(2) $\operatorname{Tr}^{\mathrm{gr}}(P) = A$*;*

(3) *A is a direct summand of a finite direct sum $\bigoplus_i P(\alpha_i)$, where $\alpha_i \in \Gamma$;*

(4) *A is a direct summand of a direct sum $\bigoplus_i P(\alpha_i)$, where $\alpha_i \in \Gamma$;*

(5) *Every graded A-module M is a homomorphic image of $\bigoplus_i P(\alpha_i)$, where $\alpha_i \in \Gamma$.*

Proof. (1) $\Rightarrow$ (2) First note that

$$\operatorname{Tr}^{\mathrm{gr}}(P) = \Big\{ \sum_i f_i(P(\alpha)) \mid f_i \in \operatorname{Hom}_{A\text{-Gr}}\big(P(\alpha), A\big), \alpha \in \Gamma \Big\}.$$

Suppose that $\operatorname{Tr}^{\mathrm{gr}}(P) \neq A$. Then the graded canonical projection $f : A \to A/\operatorname{Tr}^{\mathrm{gr}}(P)$ is not a zero map. Since P is graded generator, there is $g \in \operatorname{Hom}_{A\text{-Gr}}(P(\alpha), A)$ such that fg is not zero. But this implies $g(P(\alpha)) \not\subseteq \operatorname{Tr}^{\mathrm{gr}}(P)$ which is a contradiction.

(2) $\Rightarrow$ (3) Since $\operatorname{Tr}^{\mathrm{gr}}(P) = A$, one can find $g_i \in \operatorname{Hom}_{A\text{-Gr}}(P(\alpha_i), A)$, $\alpha_i \in \Gamma$, $1 \leq i \leq n$ such that $\sum_{i=1}^n g_i(P(\alpha_i)) = A$. Consider the graded A-module epimorphism $\bigoplus_{i=1}^n P(\alpha_i) \to A$, $(p_1, \dots, p_n) \mapsto \sum_{i=1}^n g_i(p_i)$. Since A is graded projective, this map splits. Thus $\bigoplus_{i=1}^n P(\alpha_i) \cong_{\mathrm{gr}} A \oplus Q$, for some graded A-module Q. This gives (3).

(3) $\Rightarrow$ (4) This is immediate.

(4) $\Rightarrow$ (5) Since any module is a homomorphic image of a graded free A-module, there is a graded epimorphism $\bigoplus_j A(\alpha'_j) \to M$. By (3) there is a graded epimorphism $\bigoplus_i P(\alpha_i) \to A$, so an epimorphism, $\bigoplus_i P(\alpha_i + \alpha'_j) \to A(\alpha'_j)$. Thus $\bigoplus_j \bigoplus_i P(\alpha_i + \alpha'_j) \to \oplus_j A(\alpha'_j) \to M$ is a graded epimorphism.

(5) $\Rightarrow$ (1) Let $f : M \to N$ be a nonzero graded homomorphism. By (4) there is an epimorphism $\bigoplus_i P(\alpha_i) \to M$. So the composition $\bigoplus_i P(\alpha_i) \to M \to N$ is not zero. This immediately implies there is an i such that the composition $P(\alpha_i) \to M \to N$ is not zero. This gives (1). $\square$

Example 2.2.3. Let A be a graded simple ring and P be a graded projective A-module. Since $\operatorname{Tr}^{\mathrm{gr}}(P)$ is a graded two sided ideal, one can easily show, using Proposition 2.2.2(2), that P is a graded generator.

Recall that in the category of A-right modules, Mod-A, a generator is defined as in Definition 2.2.1 by dropping all the graded adjective. Furthermore a similar theorem as in 2.2.2 can be written in the non-graded case, by considering Γ to be the trivial group (see [62, Theorem 18.8]). In particular P is a generator if and only if $\operatorname{Tr}(P) = A$, where

$$\operatorname{Tr}(P) := \Big\{ \sum_i f_i(p_i) \mid f_i \in \operatorname{Hom}_A(P, A),\, p_i \in P \Big\}.$$

Remark 2.2.4. Considering Gr-A as a subcategory of Mod-A (which is clearly not a full subcategory), one can define generators for Gr-A. In this case one can easily see that $\bigoplus_{\gamma \in \Gamma} A(\gamma)$ is a generator for the category Gr-A. Recall that, in comparison, A is a generator for Mod-A and A is a graded generator for Gr-A.

An A-module P is called a *progenerator* if it is both a finitely generated projective and a generator, i.e., there is an $n \in \mathbb{N}$ such that $A^n \cong P \oplus K$ and $P^n \cong A \oplus L$, where K and L are A-modules. Similarly, a graded A-module is called a *graded progenerator* if it is both a graded finitely generated projective and a graded generator.

Reminiscent of the case of graded projective modules (Proposition 1.2.12), we have the following relation between generators and graded generators.

Theorem 2.2.5. *Let P be a finitely generated A-module. Then P is a graded generator if and only if P is graded and is a generator.*

Proof. Suppose P is a graded generator. By Theorem 2.2.2, $\operatorname{Tr}^{\mathrm{gr}}(P) = A$ which implies that there are graded homomorphisms f_i (of possibly different degrees) in $\operatorname{Hom}_{A\text{-}\mathrm{Gr}}(P(\alpha_i), A)$ and $p_i \in P^h$, such that $\sum_i f_i(p_i) = 1$. This immediately implies $\operatorname{Tr}(P)$, being an ideal, is A. Thus P is a generator.

Conversely, suppose P is graded and a generator. Thus there are homomorphisms f_i in $\operatorname{Hom}(P, A)$ and $p_i \in P$ such that $\sum_i f_i(p_i) = 1$. Since P is finitely generated, by Theorem 1.2.5, f_i can be written as a sum of graded homomorphisms, and p_i as sum of homogeneous elements in P. This shows $1 \in \operatorname{Tr}^{\mathrm{gr}}(P)$. Since $\operatorname{Tr}^{\mathrm{gr}}(P)$ is an ideal, $\operatorname{Tr}^{\mathrm{gr}}(P) = A$ and so P is a graded generator by Theorem 2.2.2. $\square$

2.3 General graded Morita equivalence

Let P be a right A-module. Consider the ring $B = \operatorname{Hom}_A(P, P)$. Then P has a natural $B - A$-bimodule structure. The actions of A and B on P are defined by $p.a = pa$ and $g.p = g(p)$, respectively, where $g \in B$, $p \in P$ and $a \in A$. Consider the dual $P^* = \operatorname{Hom}_A(P, A)$. Then P^* has a natural $A - B$-bimodule structure. The actions of A and B on P^* defined by $(a.q)(p) = aq(p)$ and $q.g = q \circ g$, respectively, where $g \in B$, $q \in P^*$, $p \in P$ and $a \in A$. Furthermore one defines

$$\begin{aligned} \phi : P^* \otimes_B P &\longrightarrow A, \\ q \otimes p &\longmapsto q(p), \end{aligned}$$

and

$$\begin{aligned} \psi : P \otimes_A P^* &\longrightarrow B \\ p \otimes q &\longmapsto pq, \end{aligned}$$

where $pq(p') = p(q(p'))$. One can check that ϕ is a $A - A$-bimodule homomorphism and ψ is a $B - B$-bimodule homomorphism. We leave it to the reader to check these and that with the actions introduced above, P has a $B - A$-bimodule structure and P^* has a $A - B$-bimodule structure (see [62, §18C]). These compatibility conditions amount to the fact that the *Morita ring*

$$M = \begin{pmatrix} A & P^* \\ P & B \end{pmatrix}, \tag{2.3}$$

with the matrix multiplication has the associativity property. In fact M is a formal matrix ring as defined in Example 1.1.4.

As part of the Morita theory, one proves that when P is a generator, then ϕ is an isomorphism. Similarly, if P is finitely generated and projective, then ψ is an isomorphism (see [62, §18C]).

Putting these facts together, it is an easy observation that

$$-\otimes_A P^* : \text{Mod-}A \to \text{Mod-}B, \qquad \text{and} \qquad -\otimes_B P : \text{Mod-}B \to \text{Mod-}A, \tag{2.4}$$

are inverse of each other and so these two categories are (Morita) equivalent.

If P is a graded finitely generated right A-module, then by Theorem 1.2.5, $B = \operatorname{End}_A(P, P)$ is also a graded ring and P^* a graded left A-module. In fact, one can easily check that with the actions defined above, P is a graded $B - A$-bimodule, P^* is a graded $A - B$-module and similarly ϕ and ψ are graded $A - A$ and $B - B$-module homomorphisms, respectively. The Morita ring M of (2.3) is a graded formal matrix ring (see Example 1.2.6), with

$$M_\alpha = \begin{pmatrix} A_\alpha & P^*_\alpha \\ P_\alpha & B_\alpha \end{pmatrix} = \begin{pmatrix} A_\alpha & \operatorname{Hom}_{\text{Gr-}A}(P, A(\alpha)) \\ P_\alpha & \operatorname{Hom}_{\text{Gr-}A}(P, P(\alpha)) \end{pmatrix}.$$

We demonstrate here that P^* is a graded $A - B$-bimodule and leave the others which are similar and routine to the reader. Recall that $P^* = \bigoplus_{\alpha\in\Gamma} \operatorname{Hom}_A(P, A)_\alpha$. Let $a \in A_\alpha$ and $q \in P^*_\beta = \operatorname{Hom}_A(P, A)_\beta$. Then $a.q \in P^*$, where $(a.q)(p) = aq(p)$. If $p \in P_\gamma$, then one easily sees that $(a.q)(p) \in A_{\alpha+\beta+\gamma}$. This shows that $aq \in \operatorname{Hom}_A(P, A)_{\alpha+\beta} = P^*_{\alpha+\beta}$. On the other hand, if $q \in P^*_\alpha$ and $g \in \operatorname{Hom}(P, P)_\beta$, then $q.g \in P^*$, where $q.g(p) = qg(p)$. So if $p \in P_\gamma$ then $(q.g)(p) = qg(p) \in A_{\alpha+\beta+\gamma}$. So $q.g \in P^*_{\alpha+\beta}$ as well.

By Theorem 2.2.5 and Proposition 1.2.12, if P is a graded finitely generated projective and graded generator, then P is finitely generated projective and a generator. Since the grading is preserved under the tensor products, the restriction of the functors $-\otimes_A P^*$ and $-\otimes_B P$ of (2.4) induce an equivalence

$$-\otimes_A P^* : \text{Gr-}A \to \text{Gr-}B, \qquad \text{and} \qquad -\otimes_B P : \text{Gr-}B \to \text{Gr-}A. \tag{2.5}$$

Furthermore, these functors commute with suspensions. Thus we get a commutative diagram where the vertical maps are forgetful functor (see §1.2.7),

$$\begin{tikzcd} \text{Gr-}A \arrow[r, "-\otimes_A P^*"] \arrow[d, "U"'] & \text{Gr-}B \arrow[d, "U"] \\ \text{Mod-}A \arrow[r, "-\otimes_A P^*"] & \text{Mod-}B. \end{tikzcd} \tag{2.6}$$

We call $-\otimes_A P^*$ a graded equivalence functor (see Definition 2.3.2).

Example 2.3.1. Let A be a graded ring and e be a *full homogeneous idempotent* of A, i.e., e is a homogeneous element, $e^2 = e$ and $AeA = A$. Clearly e has degree zero. Consider $P = eA$. One can readily see that P is a right graded progenerator. Then $P^* = \operatorname{Hom}_A(eA, A) \cong_{\text{gr}} Ae$ and $B = \operatorname{End}_A(eA, eA) \cong_{\text{gr}} eAe$ (see §1.2.10). The maps ϕ and ψ described above as a part of Morita context, takes the form $\phi : Ae \otimes_{eAe} eA \to A$ and $\psi : eA \otimes_A Ae \to eAe$ which are graded isomorphisms. Thus we get an (graded) equivalence between Gr-A and Gr-eAe which lifts to an (graded) equivalence between Mod-A and Mod-eAe, as it is shown in the diagram below.

$$\begin{tikzcd} \text{Gr-}A \arrow[r, "-\otimes_A Ae"] \arrow[d, "U"'] & \text{Gr-}eAe \arrow[d, "U"] \\ \text{Mod-}A \arrow[r, "-\otimes_A Ae"] & \text{Mod-}eAe. \end{tikzcd} \tag{2.7}$$

Before stating the general graded Morita equivalence, we need to make some definitions. Recall from §1.1 that for $\alpha \in \Gamma$, the α-suspension functor $\mathcal{T}_\alpha : \text{Gr-}A \to \text{Gr-}A$, $M \mapsto M(\alpha)$ is an isomorphism with the property $\mathcal{T}_\alpha \mathcal{T}_\beta = \mathcal{T}_{\alpha+\beta}$, $\alpha, \beta \in \Gamma$.

Definition 2.3.2. Let A and B be Γ-graded rings.

1. A functor $\phi : \text{Gr-}A \to \text{Gr-}B$ is called a *graded functor* if $\phi \mathcal{T}_\alpha = \mathcal{T}_\alpha \phi$.

2. A graded functor $\phi : \text{Gr-}A \to \text{Gr-}B$ is called a *graded equivalence* if there is a graded functor $\psi : \text{Gr-}B \to \text{Gr-}A$ such that $\psi\phi \cong 1_{\text{Gr-}A}$ and $\phi\psi \cong 1_{\text{Gr-}B}$.

3. If there is a graded equivalence between Gr-A and Gr-B, we say A and B are *graded equivalent* or *graded Morita equivalence* and we write $\text{Gr-}A \approx_{\text{gr}} \text{Gr-}B$, or $\text{Gr}^\Gamma\text{-}A \approx_{\text{gr}} \text{Gr}^\Gamma\text{-}B$ to emphasis the categories are Γ-graded.

4. A functor $\phi' : \text{Mod-}A \to \text{Mod-}B$ is called a *graded functor* if there is a graded functor $\phi : \text{Gr-}A \to \text{Gr-}B$ such that the following diagram, where the vertical functors are forgetful functors (see §1.2.7), is commutative.

$$\begin{tikzcd} \text{Gr-}A \arrow[r, "\phi"] \arrow[d, "U"'] & \text{Gr-}B \arrow[d, "U"] \\ \text{Mod-}A \arrow[r, "\phi'"] & \text{Mod-}B. \end{tikzcd} \tag{2.8}$$

   The functor ϕ is called an *associated graded functor* of ϕ'.

5. A functor $\phi : \text{Mod-}A \to \text{Mod-}B$ is called a *graded equivalence* if it is graded and an equivalence.

Definition 2.3.2 of graded functors is formulated for the category of (graded) right modules. A similar definition can be written for the category of (graded) left modules. We will see that the notion of graded equivalence is a left-right symmetric (see Remark 2.3.8).

Example 2.3.3. The equivalence between Mod-A and Mod-eAe in Example 2.3.1 is a graded equivalence as it is demonstrated in Diagram 2.7.

If Q is an object in Gr-A, then we denote $U(Q) \in \text{Mod-}A$ also by Q, forgetting its graded structure (see §1.2.7). Also when working with graded matrix rings, say $\mathbb{M}_n(A)(\overline{\delta})$, we write simply Mod-$\mathbb{M}_n(A)$ when considering the category of (non-graded) modules over this matrix algebra. These should not cause any confusion in the text.

Example 2.3.4. Proposition 2.1.1 can be written for the the category of A-modules, which gives us a non-graded version of Morita equivalence. We have the following commutative diagram which shows that the functor $- \otimes_A A^n : \text{Mod-}A \to \text{Mod-}\mathbb{M}_n(A)$ is a graded equivalence.

$$\begin{tikzcd} \text{Gr-}A \arrow[r, "-\otimes_A A^n(\overline{\delta})"] \arrow[d, "U"'] & \text{Gr-}\mathbb{M}_n(A)(\overline{\delta}) \arrow[d, "U"] \\ \text{Mod-}A \arrow[r, "-\otimes_A A^n"] & \text{Mod-}\mathbb{M}_n(A). \end{tikzcd}$$

We are in a position to state the main Theorem of this chapter.

Theorem 2.3.5. *Let A and B be two Γ-graded rings. Let $\phi : \text{Gr-}A \longrightarrow \text{Gr-}B$ be a graded equivalence. Then there is a graded equivalence $\phi' : \text{Mod-}A \longrightarrow \text{Mod-}B$ with an associated graded functor isomorphic to ϕ. Indeed, there is a graded $A-B$-bimodule Q such that $\phi \cong - \otimes_A Q$ and consequently the following diagram commutes.*

$$\begin{tikzcd}
\text{Gr-}A \arrow[r, "-\otimes_A Q"] \arrow[d, "U"'] & \text{Gr-}B \arrow[d, "U"] \\
\text{Mod-}A \arrow[r, "-\otimes_A Q"] & \text{Mod-}B.
\end{tikzcd}$$

Proof. Let the graded functor $\psi : \text{Gr-}B \to \text{Gr-}A$ be an inverse of the functor ϕ with

$$f : \psi\phi \cong 1_{\text{Gr-}A}.$$

Since B is a graded finitely generated projective and a graded generator in Gr-B, it follows that $P = \psi(B)$ is a graded finitely generated projective and a graded generator in Gr-A. Thus by Theorem 2.2.5 and Proposition 1.2.12, P is a finitely generated projective and a generator in Mod-A as well. This shows that $\text{Hom}_A(P, -) : \text{Mod-}A \to \text{Mod-}B$ is a graded equivalence. We will show that $\phi \cong \text{Hom}_A(P, -)$ on the category of Gr-A.

By Theorem 1.2.5, $\text{Hom}_B(B, B) = \bigoplus_{\alpha\in\Gamma} \text{Hom}_B(B, B)_\alpha$, and by (1.9) we can write $\text{Hom}(B, B)_\alpha = \text{Hom}_{\text{Gr-}B}(B, B(\alpha))$. Applying ψ on each of these components, since ψ is a graded functor, we get

$$\text{Hom}_B(B, B) = \bigoplus_{\alpha\in\Gamma} \text{Hom}_B(B, B)_\alpha \xrightarrow{\psi} \bigoplus_{\alpha\in\Gamma} \text{Hom}_A(P, P)_\alpha = \text{Hom}_A(P, P). \tag{2.9}$$

One can immediately see that this gives a graded isomorphism of rings between $\text{Hom}_B(B, B)$ and $\text{Hom}_A(P, P)$.

For any $b \in B$ consider the right B-module homomorphism

$$\begin{aligned} \eta_b : B &\to B \\ x &\mapsto bx. \end{aligned}$$

Then the regular representation map $\eta : B \to \text{Hom}_B(B, B)$, $\eta(b) = \eta_b$, is a graded isomorphism of rings. Thus we have a graded isomorphism of rings

$$B \xrightarrow{\eta} \text{Hom}_B(B, B) \xrightarrow{\psi} \text{Hom}_A(P, P), \tag{2.10}$$

where P is a graded A-progenerator.

Now since for any graded A-module X, $\text{Hom}_A(P, X)$ is a graded right $\text{Hom}_A(P, P)$-module, the isomorphisms 2.10 induces a graded B-module structure on $\text{Hom}_A(P, X)$. Namely, for homogeneous elements $b \in B$ and $t \in \text{Hom}_A(P, X)$

$$t.b = t\psi(\eta_b) \tag{2.11}$$

which extends linearly to all elements.

We show that $\phi \cong \text{Hom}_A(P, -)$. Let X be a graded A-module. Then

$$\begin{aligned} \phi(X) \cong_{\text{gr}} \text{Hom}_B(B, \phi(X)) = \bigoplus_{\alpha\in\Gamma} \text{Hom}_B(B, \phi(X))_\alpha &\xrightarrow{\psi} \bigoplus_{\alpha\in\Gamma} \text{Hom}_A(P, \psi\phi(X))_\alpha \\ &\xrightarrow{\text{Hom}(1,f)} \bigoplus_{\alpha\in\Gamma} \text{Hom}_A(P, X)_\alpha = \text{Hom}_A(P, X). \end{aligned} \tag{2.12}$$

This shows that $\phi(X)$ is isomorphic to $\operatorname{Hom}_A(P,X)$ as a graded abelian groups. We need to show that this isomorphism, call it Θ, is a B-module isomorphism. Let $z \in \phi(X)$ and $b \in B$ be homogeneous elements. Since $\phi(X) \cong_{\mathrm{gr}} \operatorname{Hom}_B(B, \phi(X))$, for $z \in \phi(X)$, we denote the corresponding homomorphism in $\operatorname{Hom}_B(B,\phi(X))$ also by z. Then

$$\Theta(z.b) = \Theta(z\eta_b) = \operatorname{Hom}(1,f)\psi(z\eta_b) = f\psi(z)\psi(\eta_b).$$

But

$$\Theta(z).b = \big(\operatorname{Hom}(1,f)\psi(z)\big).b = f\psi(z)\psi(\eta_b) \qquad \text{(by (2.11))}.$$

Thus $\phi \cong \operatorname{Hom}_A(P,-)$ on $\text{Gr-}A$. Now considering $\phi' = \operatorname{Hom}_A(P,-) : \text{Mod-}A \to \text{Mod-}B$. This gives the first part of the Theorem.

Setting $P^* = \operatorname{Hom}_A(P,A)$, one can easily check that for any graded right A-module M, the map

$$\begin{aligned} M \otimes_A P^* &\longrightarrow \operatorname{Hom}_A(P,M),\\ m \otimes q &\longmapsto (m\otimes q)(p) = mq(p), \end{aligned}$$

where $m \in M$, $q \in P^*$ and $p \in P$, is a graded B-homomorphism (recall that by (2.10), $B \cong_{\mathrm{gr}} \operatorname{End}_A(P)$). Since P is a graded progenerator, by Theorem 2.2.5, it a progenerator, which in return gives that the above homomorphism is in fact an isomorphism (see [62, Remark 18.25]). Thus $\phi \cong \operatorname{Hom}_A(P,-) \cong -\otimes_A P^*$. This gives the second part of the Theorem. □

Theorem 2.3.6. *Let A and B be two Γ-graded rings. The following are equivalent:*

(1) $\text{Mod-}A$ *is graded equivalent to* $\text{Mod-}B$*;*

(2) $\text{Gr-}A$ *is graded equivalent to* $\text{Gr-}B$*;*

(3) $B \cong_{\mathrm{gr}} \operatorname{End}_A(P)$ *for a graded A-progenerator P;*

(4) $B \cong_{\mathrm{gr}} e\,\mathbb{M}_n(A)(\overline{\delta})e$ *for a full homogeneous idempotent $e \in \mathbb{M}_n(A)(\overline{\delta})$, where $\overline{\delta} = (\delta_1,\dots,\delta_n)$, $\delta_i \in \Gamma$.*

Proof. (1) $\Rightarrow$ (2) Let $\phi : \text{Mod-}A \to \text{Mod-}B$ be a graded equivalence. Using (2.8), it follows that $\phi(A) = P$ is a graded right B-module. Also a similar argument as in (2.10) shows that P is a graded left A-module. Since ϕ is an equivalence, from the (non-graded) Morita theory it follows that $\phi \cong -\otimes_A P$ with an inverse $-\otimes_B P^*$. Since the tensor product respects the grading, the same functor ϕ induces a graded equivalence between $\text{Gr-}A$ and $\text{Gr-}B$.

(2) $\Rightarrow$ (3) This is (2.10) in the proof of Theorem 2.3.5.

(3) $\Rightarrow$ (4) Since P is a graded finitely generated projective A-module, $P \oplus Q \cong_{\mathrm{gr}} A^n(-\overline{\delta})$, where $\overline{\delta} = (\delta_1,\dots,\delta_n)$ and $n \in \mathbb{N}$. Let $e \in \operatorname{End}_A(A^n(-\overline{\delta})) \cong_{\mathrm{gr}} \mathbb{M}_n(A)(\overline{\delta})$ be the graded homomorphism which sends Q to zero and acts as identity on P. Thus $e \in \operatorname{End}_A(A^n(-\overline{\delta}))_0 \cong \mathbb{M}_n(A)(\overline{\delta})_0$ and $P = eA^n(-\overline{\delta})$. Define the map

$$\theta : \operatorname{End}_A(P) \longrightarrow e\operatorname{End}_A(A^n(-\overline{\delta}))e$$

by $\theta(f)|_P = f$ and $\theta(f)|_Q = 0$. Since e is homogeneous of degree zero, it is straightforward to see that this is a graded isomorphism of rings (which preserve the identity). Thus

$$B \cong_{\mathrm{gr}} \operatorname{End}_A(P) \cong_{\mathrm{gr}} e\operatorname{End}_A(A^n(-\overline{\delta}))e \cong_{\mathrm{gr}} e\,\mathbb{M}_n(A)(\overline{\delta})e.$$

We are left to show that e is full. By [62, Exercise 2.8], $\mathbb{M}_n(A)e\,\mathbb{M}_n(A) = \mathbb{M}_n(\mathrm{Tr}(P))$, where $P = eA^n$. Since P is graded progenerator, it is a finitely generated projective A-module and a generator (see Theorem 2.2.5), thus $\mathrm{Tr}(P) = A$ and therefore e is a (homogeneous) full idempotent.

(4) $\Rightarrow$ (1) Example 2.3.1 shows that there is a graded equivalence between Mod-$\mathbb{M}_n(A)$ and Mod-$e\,\mathbb{M}_n(A)e$. On the other hand, Example 2.3.4 shows that there is a graded equivalence between Mod-$\mathbb{M}_n(A)$ and Mod-A. This finishes the proof. $\square$

Example 2.3.7. <u>Gr-$A \cong$ Gr-B DOES NOT IMPLY Gr-$A \cong_{\mathrm{gr}}$ Gr-B</u>

One can easily construct examples of two Γ-graded rings A and B such that the categories Gr-A and Gr-B are equivalent, but not graded equivalent, i.e., the equivalent functors do not commute with suspensions (see Definition 2.3.2). Let A be a Γ-graded ring and $\phi : \Gamma \to \mathrm{Aut}(A)$ be a group homomorphism. Consider the group ring $A[\Gamma]$ and the skew group ring $A \star_\phi \Gamma$ (see §1.1.3). These rings are strongly Γ-graded, and thus by Dade's Theorem 1.5.1, Gr-$A[\Gamma]$ and Gr-$A \star_\phi \Gamma$ are equivalent to Mod-A and thus Gr-$A[\Gamma] \cong$ Gr-$A \star_\phi \Gamma$. However one can easily show that these two graded rings are not necessarily graded equivalent.

Remark 2.3.8.

1. The graded Morita theory has a left-right symmetry property. Indeed, starting from the category of graded left modules, one can prove a similar statement as in Proposition 2.3.6, which in turn, part (4) is independent of the left-right assumption. This shows that Gr-A is graded equivalent to Gr-B if and only if A-Gr is graded equivalent to B-Gr

2. Proposition 2.3.6(3) shows that if all graded finitely generated projective A-modules are graded free, then Mod-A is graded equivalence to Mod-B if and only if $B \cong_{\mathrm{gr}} \mathbb{M}_n(A)(\overline{\delta})$ for some $n \in \mathbb{N}$ and $\overline{\delta} = (\delta_1, \dots, \delta_n)$, $\delta_i \in \Gamma$.

Remark 2.3.9. <u>Gr^{Ω}-$A \approx_{\mathrm{gr}} \mathrm{Gr}^{\Omega}$-$B$ IMPLIES Gr^{Γ}-$A \approx_{\mathrm{gr}} \mathrm{Gr}^{\Gamma}$-$B$</u>

Recall from Definition 2.3.2 that we write Gr^{Γ}-$A \approx_{\mathrm{gr}} \mathrm{Gr}^{\Gamma}$-$B$, if there is a graded equivalence between the categories of Γ-graded A-modules Gr^{Γ}-A and Γ-graded B-modules Gr^{Γ}-B.

Let A and B be Γ-graded rings and Ω a subgroup of Γ such that $\Gamma_A, \Gamma_B \subseteq \Omega \subseteq \Gamma$. Then A and B can be naturally considered as Ω-graded ring. If Gr^{Ω}-$A \approx_{\mathrm{gr}} \mathrm{Gr}^{\Omega}$-$B$, then by Theorem 2.3.6 there is a Ω-isomorphism $\phi : B \cong_{\mathrm{gr}} e\,\mathbb{M}_n(A)(\overline{\delta})e$ for a full homogeneous idempotent $e \in \mathbb{M}_n(A)(\overline{\delta})$, where $\overline{\delta} = (\delta_1, \dots, \delta_n)$, $\delta_i \in \Omega$. Since $\Omega \subseteq \Gamma$, another application of Theorem 2.3.6 shows that Gr^{Γ}-$A \approx_{\mathrm{gr}} \mathrm{Gr}^{\Gamma}$-$B$. One can also use Theorem 1.2.8 to obtain this statement.

Remark 2.3.10. <u>Gr^{Γ}-$A \approx_{\mathrm{gr}} \mathrm{Gr}^{\Gamma}$-$B$ IMPLIES $\mathrm{Gr}^{\Gamma/\Omega}$-$A \approx_{\mathrm{gr}} \mathrm{Gr}^{\Gamma/\Omega}$-$B$</u>

Let A and B be two Γ-graded rings. Theorem 2.3.6 shows the equivalence Gr-$A \approx_{\mathrm{gr}}$ Gr-B induces an equivalence Mod-$A \approx$ Mod-B. Haefner in [42] observed that Gr-$A \approx_{\mathrm{gr}}$ Gr-B induces other equivalences between different "layers" of grading. We briefly recount this result here.

Let A be a Γ-graded ring and let Ω be subgroup of Γ. Recall from Example 1.1.8 and §1.2.8 that A can be considered at Γ/Ω-graded ring. Recall also that the category of Γ/Ω-graded A-modules, denoted by $\mathrm{Gr}^{\Gamma/\Omega}$-$A$, consists of the Γ/Ω-graded A-modules as objects and A-module homomorphisms $\phi : M \to N$ which are grade-preserving in the sense that $\phi(M_{\Omega+\alpha}) \subseteq N_{\Omega+\alpha}$ for all $\Omega + \alpha \in \Gamma/\Omega$ as morphisms of the category. In two extreme cases $\Omega = 0$ and $\Omega = \Gamma$ we have $\mathrm{Gr}^{\Gamma/\Omega}$-$A =$ Gr-A and $\mathrm{Gr}^{\Gamma/\Omega}$-$A =$ Mod-A, respectively.

In [42] Haefner shows that, for any two Γ-graded equivalent rings A and B and for any subgroup Ω of Γ, there are equivalences between the categories $\mathrm{Gr}^{\Gamma/\Omega}$-$A$ and $\mathrm{Gr}^{\Gamma/\Omega}$-$B$. In fact, Haefner works with an arbitrary (non-abelian) group Γ and any subgroup Ω. In this case one needs to adjust the definitions as follows.

Let Γ/Ω denote a set of (right) cosets of Ω (we use the multiplication notation here). An Γ/Ω-graded right A-module M is defined as a right A-module M with an internal direct sum decomposition $M = \bigoplus_{\Omega\alpha\in\Gamma/\Omega} M_{\Omega\alpha}$, where each $M_{\Omega\alpha}$ is an additive subgroup of M such that $M_{\Omega\alpha}A_\beta \subseteq M_{\Omega\alpha\beta}$ for all $\Omega\alpha \in \Gamma/\Omega$ and $\beta \in \Gamma$. The decomposition is called an Γ/Ω-grading of M. With the abuse of notation we denote the category of Γ/Ω-graded right A-modules with $\mathrm{Gr}^{\Gamma/\Omega}$-$A$. Then in [42] it was shown that, Gr^{Γ}-$A \approx_{\mathrm{gr}} \mathrm{Gr}^{\Gamma}$-$B$ implies $\mathrm{Gr}^{\Gamma/\Omega}$-$A \approx_{\mathrm{gr}} \mathrm{Gr}^{\Gamma/\Omega}$-$B$.

Consider the truncation of A at Ω, i.e., $A_\Omega = \oplus_{\gamma\in\Omega}A_\gamma$ (see Example 1.1.8). As usual, let Mod- A_Ω denote the category of right A_Ω-modules. If A is strongly Γ-graded, one can show that the categories $\mathrm{Gr}^{\Gamma/\Omega}$-$A$ and Mod- A_Ω are equivalent via the functors truncation $(-)_\Omega : \mathrm{Gr}^{\Gamma/\Omega}\text{-}A \to$ Mod- A_Ω and induction $-\otimes_{A_\Omega} A :$ Mod- $A_\Omega \to \mathrm{Gr}^{\Gamma/\Omega}$-$A$ (see [42, Lemma 7.3]). Again, in the case that the subgroup Ω is trivial, this gives the equivalences of Gr-A and Mod- A_0 (see Theorem 1.5.1).

Remark 2.3.11. <u>The case of Gr-$A \approx$ Mod- R</u>

If a Γ-graded ring A is strongly graded, then by Theorem 1.5.1, Gr-A is equivalent to Mod- A_0. In [69], the conditions under which Gr-A is equivalent to Mod- R, for some ring R with identity, are given. It was shown that, among other things, Gr-$A \approx$ Mod- R, for some ring R with identity, if and only if Gr-$A \approx_{\mathrm{gr}}$ Gr-B, for a strongly Γ-graded ring B, if and only if there is a finite subset Ω of Γ such that for any $\tau \in \Gamma$,

$$A_0 = \sum_{\gamma\in\Omega} A_{\tau-\gamma}A_{\gamma-\tau}. \tag{2.13}$$

Furthermore, it was shown that if Gr-$A \approx$ Mod- R then A-Gr $\approx$ R-Mod and $A_0 \cong e\,\mathbb{M}_n(R)e$ for some $n \in \mathbb{N}$, and some idempotent $e \in \mathbb{M}_n(R)$.

If the graded group Γ is finite, then (2.13) shows that there is a ring R such that Gr-A is equivalent to Mod- R. In fact, Cohen and Montgomery in [27] construct a so called *smash product* $A\#\mathbb{Z}[\Gamma]^*$ and show that Gr-A is equivalent to Mod- $A\#\mathbb{Z}[\Gamma]^*$. In [74] it was shown that $A\#\mathbb{Z}[\Gamma]^*$ is isomorphic to the ring $\mathrm{End}_{\mathrm{Gr}\text{-}A}\big(\bigoplus_{\alpha\in\Gamma} A(\alpha)\big)$ (see Example 1.3.6).

Remark 2.3.12. <u>The case of Gr^{Γ}- $A \approx \mathrm{Gr}^{\Lambda}$- B</u>

One can consider several variations under which two categories are equivalent. For example, for a Γ-graded ring A and Λ-graded ring B, the situation when the categories Gr^{Γ}-A and Gr^{Λ}-B are equivalent is investigated in [80, 81] (See also Remark 1.1.26).

Furthermore, for two Γ-graded rings A and B, when Gr-A is equivalent to Gr-B (not necessarily respect the shifting) has been studied in [85, 98].

Chapter 3

Graded Grothendieck Groups

One of the main aim of this note is to study the graded Grothendieck group, $K_0^{\mathrm{gr}}(A)$, where A is a Γ-graded ring. In fact, $K_0^{\mathrm{gr}}(A)$ not only is an abelian group but it has also an extra $\mathbb{Z}[\Gamma]$-module structure. As we will see throughout this note, this extra structure carries a substantial information about the graded ring A. In §3.1 we construct, in detail, the graded Grothendieck groups using the concept of group completions. Here we briefly give an overview of different equivalent constructions.

For an abelian monoid V, we denote by V^+ the group completion of V. This gives a left adjoint functor to the forgetful functor from the category of abelian groups to abelian monoids. When the monoid V has a Γ-module structure, where Γ is a group, then V^+ inherits a natural Γ-module structure, or equivalently, $\mathbb{Z}[\Gamma]$-module structure (see §3.1).

There is also a more direct way to construct the V^+ which we recall here. Consider the set of symbols $\big\{[m] \mid m \in V\big\}$ and let V^+ be the free abelian group generated by this set modulo the relations $[m]+[n]-[m+n]$, $m,n \in V$. There is a natural (monoid) homomorphism $V \longrightarrow V^+$, $m \mapsto [m]$, which is universal. Using the universality, one can show that the group V^+ obtained here coincides with the one constructed above using the group completion.

Now let Γ be a group which acts on a monoid V. Then Γ acts in a natural way on the free abelian group generated by symbols $\big\{[m] \mid m \in V\big\}$. Furthermore the subgroup generated by relations $[m]+[n]-[m+n]$, $m,n \in V$ is closed under this action. Thus V^+ has a natural structure of Γ-module, or equivalently, V^+ is a $\mathbb{Z}[\Gamma]$-module.

For a ring A with identity and a finitely generated projective (right) A-module P, let $[P]$ denote the class of A-modules isomorphic to P. Then the set

$$\mathcal{V}(A) = \big\{[P] \mid P \text{ is a finitely generated projective A-module}\big\} \tag{3.1}$$

with addition $[P]+[Q] = [P \bigoplus Q]$ forms an abelian monoid. The *Grothendieck group* of A, denoted by $K_0(A)$, is by definition $\mathcal{V}(A)^+$.

The graded Grothendieck group of a graded ring is constructed similarly, by using graded finitely generated projective modules everywhere in the above process. Namely, for a Γ-graded ring A with identity and a graded finitely generated projective (right) A-module P, let $[P]$ denote the class of graded A-modules graded isomorphic to P. Then the monoid

$$\mathcal{V}^{\mathrm{gr}}(A) = \big\{[P] \mid P \text{ is graded finitely generated projective A-module}\big\} \tag{3.2}$$

has a Γ-module structure defined as follows: for $\gamma \in \Gamma$ and $[P] \in \mathcal{V}^{\mathrm{gr}}(A)$, $\gamma.[P] = [P(\gamma)]$. The group $\mathcal{V}^{\mathrm{gr}}(A)^+$ is called the *graded Grothendieck group* and is denoted by $K_0^{\mathrm{gr}}(A)$, which as the above discussion shows is a $\mathbb{Z}[\Gamma]$-module.

The above construction of the graded Grothendieck groups is carried over the category of graded right A-modules. Since for a graded finitely generated right A-module P, the dual module $P^* = \operatorname{Hom}_A(P, A)$ is a graded left A-module, and furthermore, if P is projective, then $P^{**} \cong_{\mathrm{gr}} P$, taking duals gives an equivalence between the category of graded finitely generated projective right A-modules and the category of graded finitely generated projective left A-modules, i.e.,

$$A\text{-}\mathrm{Pgr} \approx_{\mathrm{gr}} \mathrm{Pgr}\text{-}A.$$

This implies that constructing a graded Grothendieck group, using the graded left-modules, call it $K_0^{\mathrm{gr}}(A)_l$, is isomorphic $K_0^{\mathrm{gr}}(A)_r$, the group constructed using graded right A-modules. Furthermore, defining the action of Γ on generators of $K_0^{\mathrm{gr}}(A)_l$, by $\alpha[P] = [P(-\alpha)]$ and extend it to the whole group, and defining the action of Γ on $K_0^{\mathrm{gr}}(A)_r$, in the usual way, by $\alpha[P] = [P(\alpha)]$, then (1.12) shows that these two groups are $\mathbb{Z}[\Gamma]$-module isomorphic.

As emphasised above, one of the main differences of a graded Grothendieck group versus the non-graded group, is that the former has an extra module structure. For instance, in Example 3.6.3, we will see that for the $\mathbb{Z}$-graded ring $A = K[x^n, x^{-n}]$, where K is a field and $n \in \mathbb{N}$,

$$K_0^{\mathrm{gr}}(A) \cong \bigoplus_n \mathbb{Z},$$

which is a $\mathbb{Z}[x, x^{-1}]$-module, with the action of x on $(a_1, \dots, a_n) \in \bigoplus_n \mathbb{Z}$ is as follows:

$$x(a_1, \dots, a_n) = (a_n, a_1, \dots, a_{n-1}).$$

In this chapter we study the graded Grothendieck groups in detail and calculate them for certain types of graded rings, including the graded division rings (§3.6), graded local rings (§3.7) and path algebras (§3.8). In §5 we will use the graded Grothendieck groups to classify the so called graded ultramatricial algebras. We will show that certain information decoded in the graded Grothendieck group could be used to give a complete invariant for such algebras. In §6 we compare the graded versus non-graded Grothendieck groups.

For a comprehensive study of non-graded Grothendieck groups see [59, 66, 82].

3.1 The graded Grothendieck group K_0^{gr}

3.1.1 Group completions

Let V be a monoid and Γ be a group which acts on V. The *group completion* of V (this is also called the *Grothendieck group* of V) has a natural Γ-module structure. We recall the construction here. Define a relation $\sim$ on $V \times V$ as follows: $(x_1, y_1) \sim (x_2, y_2)$ if there is a $z \in V$ such that

$$x_1 + y_2 + z = y_1 + x_2 + z. \tag{3.3}$$

This is an equivalence relation and we denote the equivalence classes of $V \times V / \sim$ by $V^+ := \{[(x, y)] \mid (x, y) \in V \times V\}$. Define

$$[(x_1, y_1)] + [(x_2, y_2)] = [(x_1 + x_2, y_1 + y_2)],$$

$$\alpha[(x_1, y_1)] = [(\alpha x_1, \alpha y_1)].$$

One can easily check that these operations are well-defined, V^+ is an abelian group and further it is a Γ-module. The map

$$\begin{aligned} \phi : V &\longrightarrow V^+ \\ x &\longmapsto [(x,0)] \end{aligned} \tag{3.4}$$

is a Γ-module homomorphism and ϕ is *universal*, i.e., if there is a Γ-module G and a Γ-module homomorphism $\psi : V \to G$, then there exists a unique Γ-module homomorphism $f : V^+ \to G$ such that $f\phi = \psi$.

We record the following properties of V^+ whose proofs are easy and are left to the reader.

Lemma 3.1.1. *Let V be a monoid, Γ a group which acts on V, and let $\phi : V \to V^+$, $x \mapsto [(x,0)]$, be the universal homomorphism.*

(1) *For $x, y \in V$, if $\phi(x) = \phi(y)$ then there is a $z \in V$ such that $x + z = y + z$ in V.*

(2) *Each element of V^+ is of the form $\phi(x) - \phi(y)$ for some $x, y \in V$.*

(3) *V^+ is generated by V as a group.*

Example 3.1.2. Let G be a group and $\mathbb{N}$ be the monoid of positive integers. Then $\mathbb{N}[G]$ is a monoid equipped by the natural action of G. Its group completion is $\mathbb{Z}[G]$ which has a natural G-module structure. This will be used in Proposition 3.6.1 to calculate the graded Grothendieck group of graded fields.

Example 3.1.3. <u>A NONZERO MONOID WHOSE GROUP COMPLETION IS ZERO</u>

Let V be a monoid with the trivial module structure (i.e., Γ is trivial). Suppose that $V \backslash \{0\}$ is an abelian group. Then one can check that $V^+ \cong V \backslash \{0\}$.

Now let $V = \{0, v\}$, with $v + v = v$ and 0 as the trivial element. One can check that with this operation V is a monoid and $V \backslash \{0\}$ is an abelian group which is a trivial group.

Example 3.1.4. <u>THE TYPE OF A RING, IBN AND THE MONOID OF PROJECTIVE MODULES</u>

Let A be a ring (with trivial grading) and $\mathcal{V}(A)$ the monoid of the isomorphism classes of finitely generated projective A-modules. Then one can observe that A has IBN if and only if the submonoid generated by $[A]$ in $\mathcal{V}(A)$ is isomorphic to $\mathbb{N}$. Furthermore, a ring A has type (n,k) if and only if the the submonoid generated by $[A]$ is isomorphic to a free monoid generated by a symbol v subject to $nv = (n+k)v$ (see §1.7).

3.1.2 K_0^{gr}-group

Let A be a Γ-graded ring (with identity as usual) and let $\mathcal{V}^{\mathrm{gr}}(A)$ denote the monoid of graded isomorphism classes of graded finitely generated projective modules over A with the direct sum as the addition operation. For a graded finitely generated projective A-module P, we denote the graded isomorphism class of P by $[P]$ which is an element of $\mathcal{V}^{\mathrm{gr}}(A)$ (see (3.2)). Thus for $[P], [Q] \in \mathcal{V}^{\mathrm{gr}}(A)$, we have $[P] + [Q] = [P \oplus Q]$. Note that for $\alpha \in \Gamma$, the α-suspension functor $\mathcal{T}_\alpha : \text{Gr-}A \to \text{Gr-}A$, $M \mapsto M(\alpha)$ is an isomorphism with the property $\mathcal{T}_\alpha \mathcal{T}_\beta = \mathcal{T}_{\alpha+\beta}$, $\alpha, \beta \in \Gamma$. Furthermore, $\mathcal{T}_\alpha$ restricts to Pgr-A, the category of graded finitely generated projective A-modules. Thus the abelian group Γ acts on $\mathcal{V}^{\mathrm{gr}}(A)$ via

$$(\alpha, [P]) \mapsto [P(\alpha)]. \tag{3.5}$$

The *graded Grothendieck group*, $K_0^{\mathrm{gr}}(A)$, is defined as the group completion of the monoid $\mathcal{V}^{\mathrm{gr}}(A)$ which naturally inherits the Γ-module structure via (3.5). This makes $K_0^{\mathrm{gr}}(A)$ a $\mathbb{Z}[\Gamma]$-module. In particular if A is a $\mathbb{Z}$-graded then $K_0^{\mathrm{gr}}(A)$ is a $\mathbb{Z}[x,x^{-1}]$-module. This extra structure plays a crucial role in the applications of graded Grothendieck groups.

For a graded finitely generated projective A-module P, we denote the image of $[P] \in \mathcal{V}^{\mathrm{gr}}(A)$ under the natural homomorphism $\mathcal{V}^{\mathrm{gr}}(A) \to K_0^{\mathrm{gr}}(A)$ by $[P]$ again (see (3.4)). When the ring has graded stable rank 1, this map is injective (see Corollary 3.1.8).

Example 3.1.5. In the following "trivial" cases one can determine the graded Grothendieck group based on the Grothendieck group of the ring.

Trivial group grading: Let A be a ring and Γ be a trivial group. Then A is a Γ-graded ring in an obvious way, $\text{Gr-}A = \text{Mod-}\,A$ and $K_0^{\mathrm{gr}}(A) = K_0(A)$ as a $\mathbb{Z}[\Gamma]$-module. However here $\mathbb{Z}[\Gamma] \cong \mathbb{Z}$ and $K_0^{\mathrm{gr}}(A)$ does not have any extra module structure. This shows that the statements we prove in the graded setting will cover the results in the classical non-graded setting, by considering rings with trivial gradings from the outset.

Trivial grading: Let A be a ring and Γ be a group. Consider A with the trivial Γ-grading, i.e., A is concentrated in degree zero (see §1.1.1). Then one can observe that $\text{Gr-}A = \bigoplus_\Gamma \text{Mod-}\,A$. (Here if $\mathcal{C}$ is an additive category, $\bigoplus_I \mathcal{C}$, where I is a nonempty index set, is defined in the obvious manner, with objects $\bigoplus_{i\in I} M_i$, where M_i are objects of $\mathcal{C}$.). Consequently, $K_0^{\mathrm{gr}}(A) \cong \bigoplus_\Gamma K_0(A)$. Considering the shifting which induces a $\mathbb{Z}[\Gamma]$-module structure, we have $K_0^{\mathrm{gr}}(A) \cong K_0(A)[\Gamma]$ as $\mathbb{Z}[\Gamma]$-module.

Remark 3.1.6. <u>THE GROUP COMPLETION OF ALL PROJECTIVE MODULES IS TRIVIAL</u>

One reason to restricting ourselves to the finitely generated projective modules is that the group completion of the monoid of the isomorphism classes of (all) projective modules gives a trivial group. To see this, first observe that for a monoid V, V^+ is the trivial group if and only if for any $x, y \in V$, there is a $z \in V$ such that $x + z = y + z$. Now consider graded projective modules P and Q. Then for $M = \bigoplus_\infty (P \oplus Q)$, we have $P \oplus M \cong Q \oplus M$. Indeed,

$$
\begin{aligned}
Q \oplus M &\cong Q \oplus (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots \\
&\cong (Q \oplus P) \oplus (Q \oplus P) \oplus \cdots \\
&\cong (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots \\
&\cong P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \cdots \\
&\cong P \oplus M.
\end{aligned}
$$

Lemma 3.1.7. *Let A be a Γ-graded ring.*

(1) *Each element of $K_0^{\mathrm{gr}}(A)$ is of the form $[P] - [Q]$ for some graded finitely generated projective A-modules P and Q.*

(2) *Each element of $K_0^{\mathrm{gr}}(A)$ is of the form $[P] - [A^n(\overline{\alpha})]$ for some graded finitely generated projective A-module P and some $\overline{\alpha} = (\alpha_1, \dots, \alpha_n)$.*

(3) *Let P, Q be graded finitely generated projective A-modules. Then $[P] = [Q]$ in $K_0^{\mathrm{gr}}(A)$ if and only if $P \oplus A^n(\overline{\alpha}) \cong_{\mathrm{gr}} Q \oplus A^n(\overline{\alpha})$, for some $\overline{\alpha} = (\alpha_1, \dots, \alpha_n)$.*

Proof. (1) This follows immediately from Lemma 3.1.1(2) by considering $\mathcal{V}^{\mathrm{gr}}(A)$ as the monoid and the fact that $[P]$ also represent the image of $[P] \in \mathcal{V}^{\mathrm{gr}}(A)$ in $K_0^{\mathrm{gr}}(A)$.

(2) This follows from (1) and the fact that for a graded finitely generated projective A-module Q, there is a graded finitely generated projective module Q' such that $Q \oplus Q' \cong_{\mathrm{gr}} A^n(\overline{\alpha})$, for some $\overline{\alpha} = (\alpha_1, \dots, \alpha_n)$ (see Proposition 1.2.12 and (1.27)).

(3) Suppose $[P] = [Q]$ in $K_0^{\mathrm{gr}}(A)$. Then by Lemma 3.1.1(1) (for $V = \mathcal{V}^{\mathrm{gr}}(A)$) there is a graded finitely generated projective A-module T such that $P \oplus T \cong_{\mathrm{gr}} Q \oplus T$. Since T is graded finitely generated projective, there is an S such that $T \oplus S \cong_{\mathrm{gr}} A^n(\overline{\alpha})$, for some $\overline{\alpha} = (\alpha_1, \dots, \alpha_n)$ (see (1.27)). Thus

$$P \oplus A^n(\overline{\alpha}) \cong_{\mathrm{gr}} P \oplus T \oplus S \cong_{\mathrm{gr}} Q \oplus T \oplus S \cong_{\mathrm{gr}} Q \oplus A^n(\overline{\alpha}).$$

Since $K_0^{\mathrm{gr}}(A)$ is a group, the converse is immediate. □

For graded finitely generated projective A-modules P and Q, P is called *graded stably isomorphic* to Q, if $[P] = [Q]$ in $K_0^{\mathrm{gr}}(A)$ or equivalently by Lemma 3.1.7(3), if $P \oplus A^n(\overline{\alpha}) \cong_{\mathrm{gr}} Q \oplus A^n(\overline{\alpha})$, for some $\overline{\alpha} = (\alpha_1, \dots, \alpha_n)$.

Corollary 3.1.8. *Let A be a Γ-graded ring with the graded stable rank 1. Then the natural map $\mathcal{V}^{\mathrm{gr}}(A) \to K_0^{\mathrm{gr}}(A)$ is injective.*

Proof. Let P and Q be graded finitely generated projective A-modules such that $[P] = [Q]$ in $K_0^{\mathrm{gr}}(A)$. Then by Lemma 3.1.7(3), $P \oplus A^n(\overline{\alpha}) \cong_{\mathrm{gr}} Q \oplus A^n(\overline{\alpha})$, for some $\overline{\alpha} = (\alpha_1, \dots, \alpha_n)$. Now by Corollary 1.8.4, $P \cong_{\mathrm{gr}} Q$. Thus $[P] = [Q]$ in $\mathcal{V}^{\mathrm{gr}}(A)$. □

3.1.3 The graded Grothendieck group of strongly graded rings

Let A be a strongly Γ-graded ring. By Dade's Theorem 1.5.1 and Remark 1.5.4, the functor $(-)_0 : \text{Pgr-}A \to \text{Pr-}\,A_0$, $M \mapsto M_0$, is an additive functor with an inverse $- \otimes_{A_0} A : \text{Pr-}\,A_0 \to \text{Pgr-}A$ so that it induces an equivalence between the category of graded finitely generated projective A-modules and the category of finitely generated A_0-module. This implies that

$$\boxed{K_0^{\mathrm{gr}}(A) \cong K_0(A_0).} \tag{3.6}$$

(In fact, this implies that $K_i^{\mathrm{gr}}(A) \cong K_i(A_0)$, for all $i \geq 0$, where $K_i^{\mathrm{gr}}(A)$ and $K_i(A_0)$ are Quillen's K-groups.) Furthermore, since $A_\alpha \otimes_{A_0} A_\beta \cong A_{\alpha+\beta}$ as A_0-bimodule, the functor $\mathcal{T}_\alpha$ on gr-A induces a functor on the level of mod-A_0, $\mathcal{T}_\alpha : \text{mod-}A_0 \to \text{mod-}A_0$, $M \mapsto M \otimes_{A_0} A_\alpha$ such that $\mathcal{T}_\alpha \mathcal{T}_\beta \cong \mathcal{T}_{\alpha+\beta}$, $\alpha, \beta \in \Gamma$, so that the following diagram is commutative up to isomorphism.

$$\begin{tikzcd} \text{Pgr-}A \arrow[r, "\mathcal{T}_\alpha"] \arrow[d, "(-)_0"'] & \text{Pgr-}A \arrow[d, "(-)_0"] \\ \text{Pr-}\,A_0 \arrow[r, "\mathcal{T}_\alpha"] & \text{Pr-}\,A_0 \end{tikzcd} \tag{3.7}$$

Therefore $K_i(A_0)$ is also a $\mathbb{Z}[\Gamma]$-module and

$$K_i^{\mathrm{gr}}(A) \cong K_i(A_0), \tag{3.8}$$

as $\mathbb{Z}[\Gamma]$-modules.

Also note that if A is a graded commutative ring then the isomorphism (3.8) is a ring isomorphism.

Example 3.1.9. K_0^{gr} of crossed products

Let A be a Γ-graded crossed-product ring. Thus $A = A_0{}_\psi^\phi[\Gamma]$ (see §1.1.3). By Proposition 1.1.16(3), A is strongly graded, and so $\text{Gr-}A \approx \text{Mod-}A_0$ and $K_0^{\mathrm{gr}}(A) \cong K_0(A_0)$ (see (3.6)).

On the other hand, by Corollary 1.3.12(4), the restriction of the shift functor on $\text{Pgr-}A$, $\mathcal{T}_\alpha : \text{Pgr-}A \to \text{Pgr-}A$, is isomorphic to the trivial functor. This shows that the action of Γ on $K_0^{\mathrm{gr}}(A) \cong K_0(A_0)$ (and indeed on all K-groups $K_i^{\mathrm{gr}}(A)$) is trivial.

Example 3.1.10. K_0^Γ verses $K_0^{\Gamma/\Omega}$

Let A be a Γ-graded ring and Ω be a subgroup of Γ. Recall the construction of Γ/Ω-graded ring A from Example 1.1.8. The canonical forgetful functor $U : \text{Gr}^\Gamma\text{-}A \to \text{Gr}^{\Gamma/\Omega}\text{-}A$ is an exact functor (see §1.2.8). Proposition 1.2.12 guarantees that U restricts to the categories of graded finitely generated projective modules, i.e., $U : \text{Pgr}^\Gamma\text{-}A \to \text{Pgr}^{\Gamma/\Omega}\text{-}A$. This induces a group homomorphism

$$\theta : K_0^\Gamma(A) \longrightarrow K_0^{\Gamma/\Omega}(A),$$

such that for $\alpha \in \Gamma$ and $a \in K_0^\Gamma(A)$, we have $\theta(\alpha a) = (\Omega + \alpha)\theta(a)$. Here, to distinguish the graded groups, we denote by K_0^Γ and $K_0^{\Gamma/\Omega}$ the graded Grothendieck groups of A as Γ and Γ/Ω-graded rings, respectively (we will use this notation again in §6).

Now let A be a strongly Γ-graded ring. Then A is also a strongly Γ/Ω-graded (Example 1.1.18). Using (3.6), we have

$$\begin{aligned} K_0^\Gamma(A) &\cong K_0(A_0), \\ K_0^{\Gamma/\Omega}(A) &\cong K_0(A_\Omega), \end{aligned}$$

where $A_\Omega = \bigoplus_{\gamma\in\Omega} A_\gamma$.

Example 3.1.11. K_0^Γ verses K_0^Ω

Let A be a Γ-graded ring and Ω be a subgroup of Γ such that $\Gamma_A \subseteq \Omega$. By Theorem 1.2.8 the category $\text{Gr}^\Gamma\text{-}A$ is equivalent to $\bigoplus_{\Gamma/\Omega} \text{Gr}^\Omega\text{-}A_\Omega$. Since this equivalence preserves the projective modules, we have

$$K_0^\Gamma(A) \cong \bigoplus_{\Gamma/\Omega} K_0^\Omega(A).$$

In fact, the same argument shows that for any (higher) K-groups (see §6), if A is a Γ-graded ring such that for $\Omega := \Gamma_A$, A_Ω is strongly Ω-graded, then we have

$$\boxed{K_i^\Gamma(A) \cong \bigoplus_{\Gamma/\Omega} K_i^\Omega(A) \cong \bigoplus_{\Gamma/\Omega} K_i(A_0).} \tag{3.9}$$

Furthermore, if A_Ω is a crossed-product, we can determine the action of Γ on K_i^{gr} concretely. First, using Corollary 1.3.12, in (1.20) for any $\omega \in \Omega$, $M(w)_{\Omega+\alpha_i} \cong M_{\Omega+\alpha_i}$ as graded A_Ω-module. Thus the functor $\overline{\rho}_\beta$ in (1.20) reduces to permutations of categories. Representing $\bigoplus_{\Gamma/\Omega} K_i(A_0)$ by the additive group of group ring $K_i(A_0)[\Gamma/\Omega]$, by Theorem 1.2.8, the $\mathbb{Z}[\Gamma]$-module structure of $K_i^\Gamma(A)$, can be descried as $K_i(A_0)[\Gamma/\Omega]$ with the natural $\mathbb{Z}[\Gamma]$-module structure.

This can be used to calculate the (lower) graded K-theory of graded division algebras (see Proposition 3.6.1, Remark 3.6.2 and Example 3.10.3).

3.1.4 The reduced graded Grothendieck group $\widetilde{K_0^{\mathrm{gr}}}$

For a ring A, the element $[A] \in K_0(A)$ generates the "obvious" part of the Grothendieck group, i.e., all the elements presented by $\pm[A^n]$ for some $n \in \mathbb{N}$. The *reduced Grothendieck group*, denoted by $\widetilde{K_0}(A)$ is the quotient of $K_0(A)$ by this cyclic subgroup. There is a unique ring homomorphism $\mathbb{Z} \to A$, which induces the group homomorphism $K_0(\mathbb{Z}) \to K_0(A)$. Since the ring of integers $\mathbb{Z}$ is a PID and finitely generated projective modules over a PID ring are free of unique rank (see [82, Theorem 1.3.1]), we have

$$\boxed{\widetilde{K_0}(A) = \operatorname{coker}(K_0(\mathbb{Z}) \to K_0(A)).} \tag{3.10}$$

This group appears quite naturally. For example when A is a Dedekind domain, $\widetilde{K_0}(A)$ coincides with $C(A)$, the class group of A (see [82, Chap. 1,§4]).

We can thus write an exact sequence,

$$0 \longrightarrow \ker\theta \longrightarrow K_0(\mathbb{Z}) \xrightarrow{\theta} K_0(A) \longrightarrow \widetilde{K_0}(A) \longrightarrow 0.$$

For a commutative ring A, one can show that $\ker\theta = 0$ and the exact sequence is split, i.e.,

$$K_0(A) \cong \mathbb{Z} \bigoplus \widetilde{K_0}(A). \tag{3.11}$$

In fact, $\ker\theta$ determines if a ring has IBN (see §1.7). Namely, $\ker\theta = 0$ if and only if A has IBN. For, suppose $\ker\theta = 0$. If $A^n \cong A^m$ as A-modules, for $n, m \in \mathbb{N}$, then $(n-m)[A] = 0$. It follows $n = m$. Thus A has IBN. On the other hand suppose A has IBN. If $n \in \ker\theta$, then we can consider $n \geq 0$. Then $[A^n] = 0$ implies $A^{n+k} \cong A^k$, for some $k \in \mathbb{N}$. Thus $n = 0$ and so $\ker\theta = 0$.

Here we develop the graded version of the concept of reduced Grothendieck group. Let A be a Γ-graded ring. In this setting the obvious part of $K_0^{\mathrm{gr}}(A)$ is not only the subgroup generated by $[A]$ but by all the shiftings of $[A]$, i.e., the $\mathbb{Z}[\Gamma]$-module generated by $[A]$. The *reduced graded Grothendieck group* of the Γ-graded ring A, denoted by $\widetilde{K_0^{\mathrm{gr}}}(A)$, is the $\mathbb{Z}[\Gamma]$-module defined by the quotient of $K_0^{\mathrm{gr}}(A)$ by the submodule generated by $[A]$. Similarly to the non-graded case, considering $\mathbb{Z}$ as a Γ-graded ring concentrated in degree zero, the unique graded ring homomorphism $\mathbb{Z} \to A$, indices a $\mathbb{Z}[\Gamma]$-module homomorphism $K_0^{\mathrm{gr}}(\mathbb{Z}) \to K_0^{\mathrm{gr}}(A)$ (see Example 3.1.5(2)). Then

$$\widetilde{K_0^{\mathrm{gr}}}(A) = \operatorname{coker}(K_0^{\mathrm{gr}}(\mathbb{Z}) \to K_0^{\mathrm{gr}}(A)). \tag{3.12}$$

In order to obtain a graded version of the splitting formula (3.11), we need to take out a part of Γ which its action on $[A]$ via shifting would not change $[A]$ in $K_0^{\mathrm{gr}}(A)$.

First note that the $\mathbb{Z}[\Gamma]$-module homomorphism $K_0^{\mathrm{gr}}(\mathbb{Z}) \to K_0^{\mathrm{gr}}(A)$ can be written as

$$\begin{aligned} \phi : \mathbb{Z}[\Gamma] &\longrightarrow K_0^{\mathrm{gr}}(A), \\ \sum_\alpha n_\alpha \alpha &\longmapsto \sum_\alpha n_\alpha [A(\alpha)]. \end{aligned} \tag{3.13}$$

Furthermore, since by Corollary 1.3.11, $A(\alpha) \cong_{\mathrm{gr}} A(\beta)$ as a right A-module if and only if $\alpha - \beta \in \Gamma_A^*$, the above map induces

$$\begin{aligned} \psi : \mathbb{Z}[\Gamma/\Gamma_A^*] &\longrightarrow K_0^{\mathrm{gr}}(A), \\ \sum_\alpha n_\alpha (\Gamma_A^* + \alpha) &\longmapsto \sum_\alpha n_\alpha [A(\alpha)], \end{aligned}$$

so that the following diagram is naturally commutative.

$$\begin{tikzcd}
\mathbb{Z}[\Gamma] \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & K_0^{\mathrm{gr}}(A), \\
& \mathbb{Z}[\Gamma/\Gamma_A^*] \arrow[ur, "\psi"'] &
\end{tikzcd} \tag{3.14}$$

Since π is surjective, $\widetilde{K_0^{\mathrm{gr}}}(A)$ is also the cokernel of the map ψ.

In Example 3.6.8 we calculate the reduced Grothendieck group of the graded division algebras. Further in Example 3.8.6 we show that the reduced Grothendieck group of a strongly graded ring does not necessarily coincide with the Grothendieck group of its ring of the zero homogeneous part.

3.1.5 The graded Grothendieck group as a $\mathbb{Z}[\Gamma]$-algebra

Let A be a Γ-graded commutative ring. Then, as in the non-graded case, $K_0^{\mathrm{gr}}(A)$ forms a commutative ring, with multiplication defined on generators by tensor products, i.e., $[P].[Q] = [P \otimes_A Q]$ (see §1.2.6). Consider $\mathbb{Z}[\Gamma]$ as a group ring. To get the natural group ring operations, here we use the multiplicative notation for the group Γ. With this convention, since for any $\alpha, \beta \in \Gamma$, $A(\alpha) \otimes_A A(\beta) \cong_{\mathrm{gr}} A(\alpha\beta)$, the map

$$\begin{aligned}
\mathbb{Z}[\Gamma] &\longrightarrow K_0^{\mathrm{gr}}(A), \\
\sum_\alpha n_\alpha \alpha &\longmapsto \sum_\alpha n_\alpha [A(\alpha)]
\end{aligned}$$

induces a ring homomorphism. This makes $K_0^{\mathrm{gr}}(A)$ a $\mathbb{Z}[\Gamma]$-algebra. In fact the argument before Diagram 3.14 shows that $K_0^{\mathrm{gr}}(A)$ can be considered as $\mathbb{Z}[\Gamma/\Gamma_A^*]$-algebra.

In general, the category of graded finitely generated projective modules is an exact category with the usual notion of (split) short exact sequence. The Quillen K-groups of this category (see [79, §2] for the construction of K-groups of an exact category) is denoted by $K_i^{\mathrm{gr}}(A)$, $i \geq 0$. The group Γ acts on this category via $(\alpha, P) \mapsto P(\alpha)$. By functoriality of K-groups this equips $K_i^{\mathrm{gr}}(A)$ with the structure of a $\mathbb{Z}[\Gamma]$-module. When $i = 0$, Quillen's construction coincides with the above construction via the group completion.

3.2 K_0^{gr} from idempotents

One can describe the K_0-group of a ring A with identity in terms of idempotent matrices of A. This description is quite helpful as we can work with the conjugate classes of idempotent matrices instead of isomorphism classes of finitely generated projective modules. For example we will use this description to show that the graded K_0 is a continuous map (Theorem 3.2.4.) Also, this description allows us to define the monoid $\mathcal{V}$ for rings without identity in a natural way (see §3.4). We briefly recall the construction of K_0 from idempotents here.

Let A be a ring with identity. In the following, we can always enlarge matrices of different sizes over A by adding zeros in the lower right hand corner, so that they can be considered in a ring $\mathbb{M}_k(A)$ for a suitable $k \in \mathbb{N}$. This means that we work in the matrix $\mathbb{M}_\infty(A) = \varinjlim_n \mathbb{M}_n(A)$, where the connecting maps are the non-unital ring homomorphism

$$\begin{aligned} \mathbb{M}_n(A) &\longrightarrow \mathbb{M}_{n+1}(A) \\ p &\longmapsto \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{3.15}$$

Any idempotent matrix $p \in \mathbb{M}_n(A)$ (i.e., $p^2 = p$) gives rise to the finitely generated projective right A-module pA^n. On the other hand any finitely generated projective right module P gives rise to an idempotent matrix p such that $pA^n \cong P$. We say two idempotent matrices p and q are equivalent if (after suitably enlarging them) there are matrices x and y such that $xy = p$ and $yx = q$. One can show that p and q are equivalent if and only if they are conjugate if and only if the corresponding finitely generated projective modules are isomorphic. Therefore $K_0(A)$ can be defined as the group completion of the monoid of equivalence classes of idempotent matrices with addition defined by, $[p] + [q] = \left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \right]$. In fact, this is the definition that one adapts for $\mathcal{V}(A)$ when the ring A does not have identity (see for example [28, Chapter 1] or [68, p.296]).

A similar construction can be given in the setting of graded rings. This does not seem to be documented in literature and we provide the details here.

Let A be a Γ-graded ring with identity and let $\overline{\alpha} = (\alpha_1, \dots, \alpha_n)$, where $\alpha_i \in \Gamma$. Recall that $\mathbb{M}_n(A)(\overline{\alpha})$ is a graded ring (see §1.3). In the following if we need to enlarge a homogeneous matrix $p \in \mathbb{M}_n(A)(\overline{\alpha})$, by adding zeroes in the lower right hand corner (and call it p), then we add zeros in the right hand side of $\overline{\alpha} = (\alpha_1, \dots, \alpha_n)$ as well accordingly (and call it $\overline{\alpha}$ again) so that p is a homogeneous matrix in $\mathbb{M}_k(A)(\overline{\alpha})$, where $k \geq n$. Recall the definition of $\mathbb{M}_k(A)[\overline{\alpha}][\overline{\delta}]$ from §1.3.2 and note that if $x \in \mathbb{M}_k(A)[\overline{\alpha}][\overline{\delta}]$ and $y \in \mathbb{M}_k(A)[\overline{\delta}][\overline{\alpha}]$, then by Lemma 1.3.9 and (1.43), $xy \in \mathbb{M}_k(-\overline{\alpha})_0$ and $yx \in \mathbb{M}_k(-\overline{\delta})_0$.

Definition 3.2.1. Let A be a Γ-graded ring, $\overline{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $\overline{\delta} = (\delta_1, \dots, \delta_m)$, where $\alpha_i, \delta_j \in \Gamma$. Let $p \in \mathbb{M}_n(A)(\overline{\alpha})_0$ and $q \in \mathbb{M}_m(A)(\overline{\delta})_0$ be idempotent matrices. Then p and q are *graded equivalent*, denoted by $p \sim q$, if (after suitably enlarging them) there are $x \in \mathbb{M}_k(A)[-\overline{\alpha}][-\overline{\delta}]$ and $y \in \mathbb{M}_k(A)[-\overline{\delta}][-\overline{\alpha}]$ such that $xy = p$ and $yx = q$. Furthermore, we say p and q are *graded conjugate* if there is an invertible matrix $g \in \mathbb{M}_{2k}(A)[-\overline{\delta}][-\overline{\alpha}]$ with $g^{-1} \in \mathbb{M}_{2k}(A)[-\overline{\alpha}][-\overline{\delta}]$ such that $gpg^{-1} = q$.

The relation $\sim$ defined above is an equivalence relation. Indeed, if $p \in \mathbb{M}_n(A)(\overline{\alpha})_0$ is a homogeneous idempotent, then considering $x = p \in \mathbb{M}_n(A)[-\overline{\alpha}][-\overline{\alpha}]$ (see (1.43)) and $y = 1 \in \mathbb{M}_n(A)[-\overline{\alpha}][-\overline{\alpha}]$, Definition 3.2.1 shows that $p \sim p$. Clearly $\sim$ is reflexive. The following trick shows that $\sim$ is also transitive. Suppose $p \sim q$ and $q \sim r$. Then $p = xy$, $q = yx$ and $q = vw$ and $r = wv$. Thus $p = p^2 = xyxy = xqy = (xv)(wy)$ and $r = r^2 = wvwv = wqv = (wy)(xv)$. This shows that $p \sim r$.

Let $p \in \mathbb{M}_n(A)(\overline{\alpha})_0$ be a homogeneous idempotent. Then one can easily see that for any $\delta = (\delta_1, \dots, \delta_m)$, the idempotent $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{M}_{n+m}(A)(\overline{\alpha}, \overline{\beta})_0$ is graded equivalent to p. We call this element an *enlargement* of p.

Lemma 3.2.2. *Let $p \in \mathbb{M}_n(A)(\overline{\alpha})_0$ and $q \in \mathbb{M}_m(A)(\overline{\delta})_0$ be idempotent matrices. Then p and q are graded equivalent if and only if some enlargements of p and q are conjugate.*

Proof. Let p and q be graded equivalent idempotent matrices. By Definition 3.2.1, there are $x' \in \mathbb{M}_k(A)[-\overline{\alpha}][-\overline{\delta}]$ and $y' \in \mathbb{M}_k(A)[-\overline{\delta}][-\overline{\alpha}]$ such that $x'y' = p$ and $y'x' = q$. Let $x = px'q$ and

$y = qy'p$. Then $xy = px'qy'p = p(x'y')^2p = p$ and similarly $yx = q$. Furthermore $x = px = xq$ and $y = yp = qy$. Using Lemma 1.3.9, one can check that $x \in \mathbb{M}_k(A)[-\overline{\alpha}][-\overline{\delta}]$ and $y \in \mathbb{M}_k(A)[-\overline{\delta}][-\overline{\alpha}]$. We now use the standard argument, taking into account the shifting of the matrices. Consider the matrix

$$\begin{pmatrix} 1-p & x \\ y & 1-q \end{pmatrix} \in \mathbb{M}_{2k}(A)[-\overline{\alpha}, -\overline{\delta}][-\overline{\alpha}, -\overline{\delta}].$$

This matrix has order two and thus is its own inverse. Then

$$\begin{pmatrix} 1-p & x \\ y & 1-q \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1-p & x \\ y & 1-q \end{pmatrix} = \begin{pmatrix} 1-p & x \\ y & 1-q \end{pmatrix} \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}.$$

Furthermore,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}.$$

Now considering the enlargement $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{M}_{2k}(A)(\overline{\alpha}, \overline{\delta})_0$ of p and $\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{M}_{2k}(A)(\overline{\delta}, \overline{\alpha})_0$ of q, we have $g \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} g^{-1} = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$, where

$$g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1-p & x \\ y & 1-q \end{pmatrix} \in \mathbb{M}_{2k}(A)[-\overline{\delta}, -\overline{\alpha}][-\overline{\alpha}, -\overline{\delta}].$$

This shows that some enlargements of p and q are conjugates. Conversely, suppose some enlargements of p and q are conjugates. Thus there is a $g \in \mathbb{M}_{2k}(A)[-\overline{\delta}, -\overline{\mu}][-\overline{\alpha}, -\overline{\beta}]$ such that $g \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} g^{-1} = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$, where $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{M}_k(A)(\alpha, \beta)_0$ and $\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{M}_k(A)(\delta, \mu)_0$. Now setting $x = g \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$ and $y = g^{-1}$, gives that the enlargements of p and q and consequently p and q are graded equivalent. $\square$

The following lemma relates the graded finitely generated projective modules to homogeneous idempotent matrices which eventually leads to an equivalent definition of the graded Grothendieck group by idempotents.

Lemma 3.2.3. *Let A be a Γ-graded ring.*

(1) *Any graded finitely generated projective A-module P gives rise to a homogeneous idempotent matrix $p \in \mathbb{M}_n(A)(\overline{\alpha})_0$, for some $n \in \mathbb{N}$ and $\overline{\alpha} = (\alpha_1, \dots, \alpha_n)$, such that $P \cong_{\mathrm{gr}} pA^n(-\overline{\alpha})$.*

(2) *Any homogeneous idempotent matrix $p \in \mathbb{M}_n(A)(\overline{\alpha})_0$ gives rise to a graded finitely generated projective A-module $pA^n(-\overline{\alpha})$.*

(3) *Two homogeneous idempotent matrices are graded equivalent if and only if the corresponding graded finitely generated projective A-modules are graded isomorphic.*

Proof. (1) Let P be a graded finitely generated projective (right) A-module. Then there is a graded module Q such that $P \oplus Q \cong_{\mathrm{gr}} A^n(-\overline{\alpha})$ for some $n \in \mathbb{N}$ and $\overline{\alpha} = (\alpha_1, \dots, \alpha_n)$, where $\alpha_i \in \Gamma$ (see (1.27)). Identify $A^n(-\overline{\alpha})$ with $P \oplus Q$ and define the homomorphism $p \in \mathrm{End}_A(A^n(-\overline{\alpha}))$ which sends Q to zero and acts as identity on P. Clearly, p is an idempotent and graded homomorphism of degree 0. By (1.10) $p \in \operatorname{End}_A(A^n(-\overline{\alpha}))_0$. By (1.34), the homomorphism p can be represented by a matrix $p \in \mathbb{M}_n(A)(\overline{\alpha})_0$ acting from the left so, $P \cong_{\mathrm{gr}} pA^n(-\overline{\alpha})$.

(2) Let $p \in \mathbb{M}_n(A)(\overline{\alpha})_0$, $\overline{\alpha} = (\alpha_1, \dots, \alpha_n)$, where $\alpha_i \in \Gamma$. Since for any $\gamma \in \Gamma$, $pA^n(-\overline{\alpha})_\gamma \subseteq A^n(-\overline{\alpha})_\gamma$, we have

$$pA^n(-\overline{\alpha}) = \bigoplus_{\gamma \in \Gamma} pA^n(-\overline{\alpha})_\gamma.$$

This shows that $pA^n(-\overline{\alpha})$ is a graded finitely generated A-module. Furthermore, $1-p \in \mathbb{M}_n(A)(\overline{\alpha})_0$ and

$$A^n(-\overline{\alpha}) = pA^n(-\overline{\alpha}) \bigoplus (1-p)A^n(-\overline{\alpha}).$$

Thus $pA^n(-\overline{\alpha})$ is a graded finitely generated projective A-module.

(3) Let $p \in \mathbb{M}_n(A)(\overline{\alpha})_0$ and $q \in \mathbb{M}_m(A)(\overline{\delta})_0$ be graded equivalent idempotent matrices. The first part is similar to the proof of Lemma 3.2.2. By Definition 3.2.1, there are $x' \in \mathbb{M}_k(A)[-\overline{\alpha}][-\overline{\delta}]$ and $y' \in \mathbb{M}_k(A)[-\overline{\delta}][-\overline{\alpha}]$ such that $x'y' = p$ and $y'x' = q$. Let $x = px'q$ and $y = qy'p$. Then $xy = px'qy'p = p(x'y')^2p = p$ and similarly $yx = q$. Furthermore $x = px = xq$ and $y = yp = qy$. Now the left multiplication by x and y induce graded right A-homomorphisms $qA^k(-\overline{\delta}) \to pA^k(-\overline{\alpha})$ and $pA^k(-\overline{\alpha}) \to qA^k(-\overline{\delta})$, respectively, which are inverse of each other. Therefore $pA^k(-\overline{\alpha}) \cong_{\mathrm{gr}} qA^k(-\overline{\delta})$.

On the other hand if $f : pA^k(-\overline{\alpha}) \cong_{\mathrm{gr}} qA^k(-\overline{\delta})$, then extend f to $A^k(-\overline{\alpha})$ by sending $(1-p)A^k(-\overline{\alpha})$ to zero and thus define a map

$$\theta : A^k(-\overline{\alpha}) = pA^k(-\overline{\alpha}) \oplus (1-p)A^k(-\overline{\alpha}) \longrightarrow qA^k(-\overline{\delta}) \oplus (1-q)A^k(-\overline{\delta}) = A^k(-\overline{\delta}).$$

Similarly, extending f^{-1} to $A^k(-\overline{\delta})$, we get a map

$$\phi : A^k(-\overline{\delta}) = qA^k(-\overline{\delta}) \oplus (1-q)A^k(-\overline{\delta}) \longrightarrow pA^k(-\overline{\alpha}) \oplus (1-p)A^k(-\overline{\alpha}) = A^k(-\overline{\alpha})$$

such that $\theta\phi = p$ and $\phi\theta = q$. It follows $\theta \in \mathbb{M}_k(A)[-\overline{\alpha}][-\overline{\delta}]$ whereas $\phi \in \mathbb{M}_k(A)[-\overline{\delta}][-\overline{\alpha}]$ (see §1.3.2). This gives that p and q are graded equivalent. $\square$

For a homogeneous idempotent matrix p of degree zero, we denote the graded equivalence class of p, by $[p]$ (see Definition 3.2.1) and we define $[p]+[q] = \left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \right]$. This makes the set of equivalence classes of homogeneous idempotent matrices of degree zero a monoid. Lemma 3.2.3 shows that this monoid is isomorphic to $\mathcal{V}^{\mathrm{gr}}(A)$, via $[p] \mapsto [pA^n(-\overline{\alpha})]$, where $p \in \mathbb{M}_n(A)(\overline{\alpha})_0$ is a homogeneous matrix. Thus $K_0^{\mathrm{gr}}(A)$ can be defined as the group completion of this monoid. In fact, this is the definition we adapt for $\mathcal{V}^{\mathrm{gr}}$ when the graded ring A does not have identity (see §3.4).

3.2.1

Suppose $p \in \mathbb{M}_n(A)(\overline{\alpha})_0$ and $q \in \mathbb{M}_m(A)(\overline{\delta})_0$ are homogeneous idempotent matrices of degree zero such that $[p] = [q]$ in $K_0^{\mathrm{gr}}(A)$. As in Definition 3.2.1, we add zeros to the lower right hand corner of p and q so that p, q can be considered as matrices in $\mathbb{M}_k(A)$ for some $k \in \mathbb{N}$. Now $[p]$ represents the isomorphism class $[pA^k(-\overline{\alpha})]$ and $[q]$ represents $[qA^k(-\overline{\delta})]$ in $K_0^{\mathrm{gr}}(A)$ (see Lemma 3.2.3). Since $[p] = [q]$, $[pA^k(-\overline{\alpha})] = [qA^k(-\overline{\delta})]$ which by Lemma 3.1.7(3), implies $pA^k(-\overline{\alpha}) \oplus A^n(\overline{\beta}) \cong_{\mathrm{gr}} qA^k(-\overline{\delta}) \oplus A^n(\overline{\beta})$, for some $n \in \mathbb{N}$. Again by Lemma 3.2.3 this implies $p \oplus I_n = \begin{pmatrix} p & 0 \\ 0 & I_n \end{pmatrix}$ is equivalent to $q \oplus I_n = \begin{pmatrix} q & 0 \\ 0 & I_n \end{pmatrix}$, where I_n is the identity element of the ring $\mathbb{M}_n(A)$. (This can also be seen directly from Definition 3.2.1.) This observation will be used in Theorem 3.2.4 and later in Lemma 5.1.5.

3.2.2 Action of Γ on idempotents

We define the action of Γ on the idempotent matrices as follows: For $\gamma \in \Gamma$ and $p \in \mathbb{M}_n(A)(\overline{\alpha})_0$, γp is represented by the same matrix as p but considered in $\mathbb{M}_n(A)(\overline{\alpha}-\gamma)_0$ where $\overline{\alpha}-\gamma = (\alpha_1-\gamma, \cdots, \alpha_n-\gamma)$. Note that if $p \in \mathbb{M}_n(A)(\overline{\alpha})_0$ and $q \in \mathbb{M}_m(A)(\overline{\delta})_0$ are equivalent, then there are $x \in \mathbb{M}_k(A)[-\overline{\alpha}][-\overline{\delta}]$ and $y \in \mathbb{M}_k(A)[-\overline{\delta}][-\overline{\alpha}]$ such that $xy = p$ and $yx = q$ (Definition 3.2.1). Since $x \in \mathbb{M}_k(A)[\gamma-\overline{\alpha}][\gamma-\overline{\delta}]$ and $y \in \mathbb{M}_k(A)[\gamma-\overline{\delta}][\gamma-\overline{\alpha}]$, it follows that γp is equivalent to γq. Thus $K_0^{\mathrm{gr}}(A)$ becomes a $\mathbb{Z}[\Gamma]$-module with this definition.

Now a quick inspection of the proof of Lemma 3.2.3 shows that the action of Γ is compatible in both definitions of K_0^{gr}.

Let A and B be Γ-graded rings and $\phi : A \to B$ be a Γ-graded homomorphism. Using the graded homomorphism ϕ, one can consider B as a graded $A-B$-bimodule in a natural way (§1.2.5). Furthermore, if P is a graded right A-module, then $P \otimes_A B$ is a graded B-module (§1.2.6). Furthermore, if P is finitely generated projective, so is $P \otimes_A B$. Thus one can define a group homomorphism $\overline{\phi} : K_0^{\mathrm{gr}}(A) \to K_0^{\mathrm{gr}}(B)$, where $[P] \mapsto [P \otimes_A B]$ and extended to all $K_0^{\mathrm{gr}}(A)$. On the other hand, if $p \in \mathbb{M}_n(A)(\overline{\alpha})_0$ is an idempotent matrix over A then $\phi(p) \in \mathbb{M}_n(B)(\overline{\alpha})_0$ is an idempotent matrix over B obtained by applying ϕ to each entry of p. This also induces a homomorphism on the level of K_0^{gr} using the idempotent presentations. Since

$$pA^n(-\overline{\alpha}) \otimes_A B \cong_{\mathrm{gr}} \phi(p)B^n(-\overline{\alpha}),$$

we have the following commutative diagram

$$\begin{tikzcd}
{[p]} \arrow[r] \arrow[d] & {[\phi(p)]} \arrow[d] \\
{[pA^n(-\overline{\alpha})]} \arrow[r] & {[pA^n(-\overline{\alpha}) \otimes_A B].}
\end{tikzcd} \tag{3.16}$$

This shows that the homomorphisms induced on K_0^{gr} by ϕ are compatible, whether using the idempotent presentation or the module presentation for the graded Grothendieck groups.

3.2.3 K_0^{gr} is a continuous functor

Recall the construction of direct limit of graded rings from Example 1.1.10. We are in a position to determine their graded Grothendieck groups.

Theorem 3.2.4. *Let A_i, $i \in I$, be a direct system of Γ-graded rings and $A = \varinjlim A_i$ be a Γ-graded ring. Then $K_0^{\mathrm{gr}}(A) \cong \varinjlim K_0^{\mathrm{gr}}(A_i)$ as $\mathbb{Z}[\Gamma]$-modules.*

Proof. First note that $K_0^{\mathrm{gr}}(A_i)$, $i \in I$, is a direct system of abelian groups so $\varinjlim K_0^{\mathrm{gr}}(A_i)$ exists with $\mathbb{Z}[\Gamma]$-module homomorphisms $\phi_i : K_0^{\mathrm{gr}}(A_i) \to \varinjlim K_0^{\mathrm{gr}}(A_i)$. On the other hand, for any $i \in I$, there is a map $\psi_i : A_i \to A$ which induces $\overline{\psi}_i : K_0^{\mathrm{gr}}(A_i) \to K_0^{\mathrm{gr}}(A)$. Due to the universality of the direct limit, we have a $\mathbb{Z}[\Gamma]$-module homomorphism $\phi : \varinjlim K_0^{\mathrm{gr}}(A_i) \to K_0^{\mathrm{gr}}(A)$ such that for any $i \in I$, the diagram

$$\begin{tikzcd}
\varinjlim K_0^{\mathrm{gr}}(A_i) \arrow[r, "\phi"] & K_0^{\mathrm{gr}}(A) \\
K_0^{\mathrm{gr}}(A_i) \arrow[u, "\phi_i"] \arrow[ur, "\overline{\psi}_i"'] &
\end{tikzcd} \tag{3.17}$$

is commutative. We show that ϕ is in fact an isomorphism. We use the idempotent description of K_0^{gr} group to show this. Note that if p is an idempotent matrix over A_i for some $i \in I$, which gives the element $[p] \in K_0^{\mathrm{gr}}(A_i)$, then $\overline{\psi}_i([p]) = [\psi(p)]$, where $\psi(p)$ is an idempotent matrix over A obtained by applying ψ to each entry of p.

Let p be an idempotent matrix over $A = \varinjlim A_i$. Then p has a finite number of entires and each is coming from some A_i, $i \in I$. Since I is directed, there is a $j \in I$, such that p is the image of an idempotent matrix in A_j. Thus the class $[p]$ in $K_0^{\mathrm{gr}}(A)$ is the image of an element of $K_0^{\mathrm{gr}}(A_j)$. Since the Diagram 3.17 is commutative, there is an element in $\varinjlim K_0(A_i)$ which maps to [p] in $K_0^{\mathrm{gr}}(A)$. Since $K_0^{\mathrm{gr}}(A)$ is generated by elements $[p]$, this shows that ϕ is surjective. We are left to show that ϕ is injective. Suppose $x \in \varinjlim K_0^{\mathrm{gr}}(A_i)$ such that $\phi(x) = 0$. Since there is $j \in I$ such that x is the image of an element of $K_0^{\mathrm{gr}}(A_j)$, we have $[p]-[q] \in K_0^{\mathrm{gr}}(A_j)$ such that $\phi_j([p]-[q]) = x$, where p and q are idempotent matrices over A_j. Again, since the Diagram 3.17 is commutative, we have $\overline{\psi}_j([p]-[q]) = 0$. Thus $[\psi_j(p)] = \overline{\psi}_j([p]) = \overline{\psi}_j([q]) = [\psi_j(q)]$ in $K_0^{\mathrm{gr}}(A)$. This shows that $a = \begin{pmatrix} \psi_j(p) & 0 \\ 0 & I_n \end{pmatrix}$ is equivalent to $b = \begin{pmatrix} \psi_j(q) & 0 \\ 0 & I_n \end{pmatrix}$ in R (see 3.2.3). Thus there are matrices x and y over R such that $xy = a$ and $yx = b$. Since the entires of x and y are finite, one can find $k \geq j$ such that x and y are images of matrices from R_k. Thus $\begin{pmatrix} \psi_{jk}(p) & 0 \\ 0 & I_n \end{pmatrix}$ is equivalent to $\begin{pmatrix} \psi_{jk}(q) & 0 \\ 0 & I_n \end{pmatrix}$ in R_k. This shows that $\overline{\psi}_{jk}([p]) = \overline{\psi}_{jk}([q])$, i.e., the image of $[p]-[q]$ in $K_0^{\mathrm{gr}}(A_j)$ is zero. Thus x being the image of this element, is also zero. This finishes the proof. $\square$

3.2.4 Hattori-Stallings (Chern) trace map

Recall that, for a ring A, one can relate $K_0(A)$ to the Hochschild homology $HH_0(A) = A/[A, A]$, where $[A, A]$ is the subgroup generated by additive commutators $ab - ba$, $a, b \in A$. (Or more generally, if A is a k-algebra, where k is a commutative ring, then $HH_0(A)$ is a k-module.) The construction is as follows.

Let P be a finitely generated projective right A-module. Then as in the introduction of §3.2, there is an idempotent matrix $p \in \mathbb{M}_n(A)$ such that $P \cong pA^n$. Define

$$\begin{aligned} T : \mathcal{V}(A) &\longrightarrow A/[A, A], \\ [P] &\longmapsto [A, A] + \mathrm{Tr}(p), \end{aligned}$$

where Tr is the trace map of the matrices. Note that if P is isomorphic to Q then the idempotent matrices associated to them, call them p and q, are equivalent, i.e, $p = xy$ and $q = yx$. This shows that $\mathrm{Tr}(p) - \mathrm{Tr}(q) \in [A, A]$, so the map T is a well-define homomorphism of groups. Since K_0 is the universal group completion (§3.1.1), the map T induces a map on the level of K_0, which is called T again, i.e., $T : K_0(A) \to A/[A, A]$. This map is called *Hattori-Stallings trace map* or *Chern map*.

We carry out a similar construction in the graded setting. Let A be a Γ-graded ring and P be a finitely generated graded projective A-module. Then there is a homogeneous idempotent $p \in \mathbb{M}_n(A)(\overline{\alpha})_0$, where $\alpha = (\alpha_1, \dots, \alpha_n)$, such that $P \cong_{\mathrm{gr}} pA^n(-\overline{\alpha})$. Note that by (1.36), $\mathrm{Tr}(p) \in A_0$. Set $[A, A]_0 := A_0 \cap [A, A]$ and define

$$\begin{aligned} T : \mathcal{V}^{\mathrm{gr}}(A) &\longrightarrow A_0/[A, A]_0, \\ [P] &\longmapsto [A, A]_0 + \mathrm{Tr}(p). \end{aligned}$$

Similar to the non-graded case, Lemma 3.2.3 applies to show that T is a well-defined homomorphism. Note that the description of action of Γ on idempotents (§3.2.2) shows that $T([P]) =$

$T([P(\alpha)])$ for any $\alpha \in \Gamma$. Again, this map induces a group homomorphism $T : K_0^{\mathrm{gr}}(A) \to A_0/[A,A]_0$. Further, the forgetful functor $U : \text{Gr-}A \to \text{Mod-}A$ (§1.2.7) induces the right hand side of the following commutative diagram, whereas the left hand side is the natural map induces by inclusion $A_0 \subseteq A$,

$$\begin{tikzcd}
K_0^{\mathrm{gr}}(A) \arrow[r, "T"] \arrow[d, "U"'] & A_0/[A,A]_0 \arrow[d] \\
K_0(A) \arrow[r, "T"] & A/[A,A].
\end{tikzcd}$$

3.3 Relative K_0^{gr}-group

Let A be a ring and I be a two sided ideal of A. The canonical epimorphism $f : A \to A/I$ induces a homomorphism on the level of K-groups. For example on the level of Grothendieck groups, we have $\overline{f} : K_0(A) \to K_0(A/I)$, $\overline{f}([P]-[Q]) = [P/PI]-[Q/QI]$. In order to complete this into a long exact sequence, one needs to introduce the relative K-groups, $K_i(A,I)$, $i \geq 0$. This is done for lower K-groups by Bass and Milnor and the following long exact sequence has been established (see [82, Theorem 4.3.1]),

$$K_2(A) \to K_2(A/I) \to K_1(A,I) \to K_1(A) \to K_1(A/I) \to K_0(A,I) \to K_0(A) \to K_0(A/I).$$

In this section, we define the relative K_0^{gr} group and establish the sequence only on the level of graded Grothendieck groups. As we will see this requires a careful arrangements of the degrees of the matrices.

Let A be a Γ graded ring and I be a graded two sided ideal of A. Define the *graded double ring* of A along I

$$D^{\Gamma}(A,I) = \big\{(a,b) \in A \times A \mid a-b \in I\big\}.$$

One can check that $D^{\Gamma}(A,I)$ is a Γ-graded ring with

$$D^{\Gamma}(A,I)_{\gamma} = \big\{(a,b) \in A_{\gamma} \times A_{\gamma} \mid a-b \in I_{\gamma}\big\},$$

for any $\gamma \in \Gamma$.

Let $\pi_1, \pi_2 : D^{\Gamma}(A,I) \to A$, $i = 1,2$, be the projections of the first and second components of $D^{\Gamma}(A,I)$ to A, respectively. The maps π_i, $i = 1,2$, are Γ-graded ring homomorphisms and induce $\mathbb{Z}[\Gamma]$-module homomorphisms $\overline{\pi}_1, \overline{\pi}_2 : K_0^{\mathrm{gr}}(D^{\Gamma}(A,I)) \to K_0^{\mathrm{gr}}(A)$, $i = 1,2$. Define the *relative graded Grothendieck group* of the ring A with respect to I as

$$\boxed{K_0^{\mathrm{gr}}(A,I) := \ker\big(K_0^{\mathrm{gr}}(D^{\Gamma}(A,I)) \xrightarrow{\overline{\pi}_1} K_0^{\mathrm{gr}}(A)\big).} \tag{3.18}$$

The restriction of $\overline{\pi}_2$ to $K_0^{\mathrm{gr}}(A,I)$ gives a $\mathbb{Z}[\Gamma]$-module homomorphism $\overline{\pi}_2 : K_0^{\mathrm{gr}}(A,I) \to K_0^{\mathrm{gr}}(A)$. The following theorem relates these groups.

Theorem 3.3.1. *Let A be a Γ-graded ring and I be a graded two sided ideal of A. Then there is an exact sequence of $\mathbb{Z}[\Gamma]$-modules*

$$K_0^{\mathrm{gr}}(A,I) \xrightarrow{\overline{\pi}_2} K_0^{\mathrm{gr}}(A) \xrightarrow{\overline{f}} K_0^{\mathrm{gr}}(A/I).$$

Proof. We will use the presentation of K_0^{gr} by idempotents (§3.2) to prove the theorem. If p is a matrix over the ring A, we will denote by $\overline{p}$ the image of p under the canonical graded homomorphism $f : A \to A/I$. Let $[p]-[q] \in K_0^{\mathrm{gr}}(A, I)$. By the construction of $D^{\Gamma}(A, I)$, $p = (p_1, p_2)$, where p_1, p_2 are homogeneous idempotent matrices of A such that $p_1 - p_2$ is a matrix over I (in fact over I_0), namely $\overline{p}_1 = \overline{p}_2$. Similarly $q = (q_1, q_2)$, where q_1, q_2 are homogeneous idempotent matrices and $\overline{q}_1 = \overline{q}_2$. Furthermore, since $[(p_1, p_2)] - [(q_1, q_2)] \in K_0^{\mathrm{gr}}(A, I)$, by the definition of the relative K-group, $[p_1] - [q_1] = 0$. So

$$\overline{f}([p_1] - [q_1]) = [\overline{p}_1] - [\overline{q}_1] = 0 \tag{3.19}$$

Taking into account that $\overline{p}_1 = \overline{p}_2$ and $\overline{q}_1 = \overline{q}_2$ we have

$$\overline{f}\overline{\pi}_2([p] - [q]) = \overline{f}\overline{\pi}_2([(p_1, p_2)] - [(q_1, q_2)]) = [\overline{p}_2] - [\overline{q}_2] = [\overline{p}_1] - [\overline{q}_1] = 0.$$

This shows that $\operatorname{Im}(\overline{\pi}_2) \subseteq \ker \overline{f}$.

Next we show that $\ker \overline{f} \subseteq \operatorname{Im}(\overline{\pi}_2)$. Let $p \in \mathbb{M}_n(A)(\overline{\alpha})_0$ and $q \in \mathbb{M}_m(A)(\overline{\delta})_0$ be idempotent matrices with $\overline{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $\overline{\delta} = (\delta_1, \dots, \delta_m)$, where $\alpha_i, \delta_j \in \Gamma$. Suppose $x = [p] - [q] \in K_0^{\mathrm{gr}}(A)$ and $\overline{f}(x) = \overline{f}([p] - [q]) = [\overline{p}] - [\overline{q}] = 0$. Since $[\overline{p}] = [\overline{q}]$ in $K_0^{\mathrm{gr}}(A/I)$, there is an $l \in \mathbb{N}$, such that $\overline{p} \oplus 1_l$ is graded equivalent to $\overline{q} \oplus 1_l$ in A/I (see §3.2.1). We can replace p by $p \oplus 1_l$ and q by $q \oplus 1_l$ without changing x, and consequently we get that $\overline{p}$ is graded equivalent to $\overline{q}$. By Lemma 3.2.2 there is an invertible matrix $g \in \mathbb{M}_{2k}(A/I)[-\overline{\delta}][-\overline{\alpha}]$ such that

$$\overline{q} = g\overline{p}g^{-1}. \tag{3.20}$$

The following standard trick let us lift an invertible matrix over A/I to an invertible matrix over A. Consider the invertible matrix

$$\begin{pmatrix} g & 1 \\ 0 & g^{-1} \end{pmatrix} \in \mathbb{M}_{4k}(A/I)[-\overline{\delta}, -\overline{\alpha}][-\overline{\alpha}, -\overline{\delta}]. \tag{3.21}$$

This matrix is a product of the following matrices, each can be lifted from A/I to an invertible matrix over A with the same shifting.

$$\begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} \in \mathbb{M}_{4k}(A/I)[-\overline{\delta}, -\overline{\alpha}][-\overline{\delta}, -\overline{\alpha}], \qquad \begin{pmatrix} 1 & 0 \\ -g^{-1} & 1 \end{pmatrix} \in \mathbb{M}_{4k}(A/I)[-\overline{\delta}, -\overline{\alpha}][-\overline{\delta}, -\overline{\alpha}],$$
$$\begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} \in \mathbb{M}_{4k}(A/I)[-\overline{\delta}, -\overline{\alpha}][-\overline{\delta}, -\overline{\alpha}], \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathbb{M}_{4k}(A/I)[-\overline{\delta}, -\overline{\alpha}][-\overline{\alpha}, -\overline{\delta}],$$

and

$$\begin{pmatrix} g & 1 \\ 0 & g^{-1} \end{pmatrix} = \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -g^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let $h \in \mathbb{M}_{4k}(A)[-\overline{\delta}, -\overline{\alpha}][-\overline{\alpha}, -\overline{\delta}]$ be a matrix that lifts (3.21) with $h^{-1} \in \mathbb{M}_{4k}(A)[-\overline{\alpha}, -\overline{\delta}][-\overline{\delta}, -\overline{\alpha}]$.

Consider the enlargements of p and q as $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{M}_{4k}(A)(\overline{\alpha}, \overline{\delta})_0$ and $\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{M}_{4k}(A)(\overline{\delta}, \alpha)_0$. By Lemma 3.2.2, $h \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} h^{-1}$ and $\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ are graded equivalent to p and q, respectively, so replacing them does not change x. Replacing p and q with the new representatives, from (3.20), it follows $\overline{p} = \overline{q}$. This means $p - q$ is a matrix over I, so (p, q) is an idempotent matrix over $D^{\Gamma}(A, I)$. Now $[(p, p)] - [(p, q)] \in K_0^{\mathrm{gr}}(D^{\Gamma}(A, I))$ which maps to x under $\overline{\pi}_2$. This completes the proof. □

3.4 K_0^{gr} of non-unital rings

Let A be a ring which does not necessarily have identity. Let R be a ring with identity such that A is a two sided ideal of R. Then $\mathcal{V}(A)$ is defined as

$$\mathcal{V}(A) := \ker\big(\mathcal{V}(R) \longrightarrow \mathcal{V}(R/A)\big).$$

It is easy to see that

$$\mathcal{V}(A) = \big\{[P] \mid P \text{ is a finitely generated projective } R\text{-module and } PA = P\big\},$$

where $[P]$ is the class of R-modules isomorphic to P and addition is defined via direct sum as before (compare this with (3.1)). One can show that this definition is independent of the choice of the ring R by interpreting P as an idempotent matrix of $\mathbb{M}_k(A)$ for a suitable $k \in \mathbb{N}$.

In order to define $K_0(A)$ for a non-unital ring A, consider the *unitization ring* $\tilde{A} = \mathbb{Z} \times A$. The addition is component-wise and multiplication is defined as follows

$$(n, a)(m, b) = (nm, ma + nb + ab), \tag{3.22}$$

where $m, n \in \mathbb{Z}$ and $a, b \in A$. This is a ring with $(1, 0)$ as the identity element and A a two sided ideal of $\tilde{A}$ such that $\tilde{A}/A \cong \mathbb{Z}$. The canonical epimorphism $\tilde{A} \to \tilde{A}/A$ gives a natural homomorphism on the level of K_0 and then $K_0(A)$ is defined as

$$K_0(A) := \ker\big(K_0(\tilde{A}) \longrightarrow K_0(\tilde{A}/A)\big). \tag{3.23}$$

This construction extends the functor K_0 from the category of rings with identity, to the category of rings (not necessarily with identity) to the category of abelian groups.

Remark 3.4.1. Notice that, for a ring A without unit, we didn't define $K_0(A)$ as the group completion of $\mathcal{V}(A)$ (as defined in the unital case). This is because the group completion is not in general a left exact functor. For a non-unital ring A, let R be a ring with identity containing A as a two sided ideal. Then we have the following exact sequence of K-theory:

$$K_1(R) \longrightarrow K_1(R/A) \longrightarrow K_0(A) \longrightarrow K_0(R) \longrightarrow K_0(R/A),$$

(see §3.3 and [82, Theorem 2.5.4]). There is yet another construction, the *relative Grothendieck group of R and A*, $K_0(R, A)$, which one can show is isomorphic to $K_0(A)$ (see [9, §7] for comparisons between the groups $K_0(A)$ and $\mathcal{V}(A)^+$ in the non-graded setting).

A similar construction can be carried over to the graded setting as follows. Let A be a Γ-graded ring which does not necessarily have identity (Remark 1.1.15). Let R be a Γ-graded ring with identity such that A is a graded two-sided ideal of R. For example, consider $\tilde{A} = \mathbb{Z} \times A$ with multiplication given by (3.22). Furthermore, $\tilde{A} = \mathbb{Z} \times A$ is a Γ-graded with

$$\begin{aligned} \tilde{A}_0 &= \mathbb{Z} \times A_0, \\ \tilde{A}_\gamma &= 0 \times A_\gamma, \quad \text{for } \gamma \neq 0. \end{aligned} \tag{3.24}$$

Define

$$\mathcal{V}^{\mathrm{gr}}(A) = \big\{[P] \mid P \text{ is a graded finitely generated projective } R\text{-module and } PA = P\big\},$$

where $[P]$ is the class of graded R-modules, graded isomorphic to P and addition is defined via direct sum. Parallel to the non-graded setting we define

$$\mathcal{V}^{\mathrm{gr}}(A) := \ker\big(\mathcal{V}^{\mathrm{gr}}(R) \longrightarrow \mathcal{V}^{\mathrm{gr}}(R/A)\big). \tag{3.25}$$

As in the proof of Lemma 3.2.3(1), since $P \oplus Q \cong R^n(-\overline{\alpha})$, this gives an idempotent matrix p in $\mathbb{M}_n(R)(\overline{\alpha})_0$. However, since $PA = P$, we have $P \oplus QA = PA \oplus QA \cong A^n(-\overline{\alpha})$. This shows that $p \in \mathbb{M}_n(A)(\overline{\alpha})_0$. On the other hand, if $p \in \mathbb{M}_n(A)(\overline{\alpha})_0 \subseteq \mathbb{M}_n(R)(\overline{\alpha})_0$ is an idempotent, then $pR^n(-\overline{\alpha})$ is a graded finitely generated projective R-module such that

$$pR^n(-\overline{\alpha})A = pA^n(-\overline{\alpha}) = pR^n(-\overline{\alpha}).$$

So $[pR^n(-\overline{\alpha})] \in \mathcal{V}^{\mathrm{gr}}(A)$. Now a repeat of Lemma 3.2.3 shows that $\mathcal{V}^{\mathrm{gr}}(A)$ is isomorphic to the monoid of equivalence classes of graded idempotent matrices over A as in §3.2. This shows that the construction of $\mathcal{V}^{\mathrm{gr}}(A)$ is independent of the choice of the graded ring R. It also shows that if A has identity, the two constructions (using the graded projective A-modules versus the graded projective R-modules) coincide.

To define the graded Grothendieck group for non-unital rings, we use a similar approach as in (3.23): Let A be a ring (without identity) and let $\tilde{A} = \mathbb{Z} \times A$ with the multiplication defined in (3.22), so that A is a graded two sided ideal of $\tilde{A}$. The graded canonical epimorphism $\tilde{A} \to \tilde{A}/A$ gives a natural homomorphism on the level of K_0^{gr} and then $K_0^{\mathrm{gr}}(A)$ is defined as

$$K_0^{\mathrm{gr}}(A) := \ker\big(K_0^{\mathrm{gr}}(\tilde{A}) \longrightarrow K_0^{\mathrm{gr}}(\tilde{A}/A)\big). \tag{3.26}$$

Thus $K_0^{\mathrm{gr}}(A)$ is a $\mathbb{Z}[\Gamma]$-module. Since the graded homomorphism $\phi : \tilde{A} \to \tilde{A}/A \cong \mathbb{Z}, (n,a) \mapsto n$ splits, we obtain the split exact sequence

$$0 \longrightarrow K_0^{\mathrm{gr}}(A) \longrightarrow K_0^{\mathrm{gr}}(\mathbb{Z} \times A) \longrightarrow K_0^{\mathrm{gr}}(\mathbb{Z}) \longrightarrow 0.$$

Thus

$$K_0^{\mathrm{gr}}(\mathbb{Z} \times A) \cong K_0^{\mathrm{gr}}(A) \oplus \mathbb{Z}[x, x^{-1}].$$

Interpreting this in the language of the reduced K_0^{gr} (§3.1.4), we have

$$\widetilde{K_0^{\mathrm{gr}}}(\tilde{A}) = K_0^{\mathrm{gr}}(A).$$

This construction extends the functor K_0^{gr} from the category of graded rings with identity, to the category of graded rings (not necessarily with identity) to the category of $\mathbb{Z}[\Gamma]$-module.

In Example 3.6.9, we calculate the graded Grothendieck group of the non-unital ring of (countable) square matrices with finite number of nonzero entires using the idempotent representation.

Remark 3.4.2. If Γ-graded ring A has identity, then the ring $\tilde{A} = \mathbb{Z} \times A$ defined by the multiplication (3.22) is graded isomorphic to the cartesian product ring $\mathbb{Z} \times A$, where $\mathbb{Z}$ is a Γ-graded ring concentrated in degree zero (see Example 1.1.11). Indeed the map

$$\begin{aligned} \tilde{A} &\longrightarrow \mathbb{Z} \times A \\ (n,a) &\longmapsto (n, a + n1_A) \end{aligned}$$

is a graded ring isomorphism of unital rings. This shows that if A has an identity, the two definitions of K_0^{gr} for A coincides. Note also that the unitization ring $\tilde{A}$ never is a strongly graded ring.

Similar to the non-graded setting (see from example [82, Theorem 1.5.9]), one can prove that for a graded ideal A of the graded ring R, $K_0^{\mathrm{gr}}(R, A) \cong K_0^{\mathrm{gr}}(A)$ as a $\mathbb{Z}[\Gamma]$-modules. This shows that $K_0^{\mathrm{gr}}(R, A)$ depends only on the structure of the non-unital ring A.

3.4.1 Graded inner automorphims

Let A be a graded ring and $f : A \to A$ an inner-automorphism defined by $f(a) = rar^{-1}$, where r is a homogeneous and invertible element of A of degree δ. Clearly f is a graded automorphism. Then considering A as a graded left A-module via f, it is easy to observe, for any graded right A-module P, that there is a right graded A-module isomorphism

$$\begin{aligned} P(-\delta) &\longrightarrow P \otimes_f A, \\ p &\longmapsto p \otimes r, \end{aligned}$$

(with the inverse $p \otimes a \mapsto pr^{-1}a$). This induces an isomorphism between the functors $- \otimes_f A$: Pgr-$A \to$ Pgr-A and δ-suspension functor $\mathcal{T}_\delta$: Pgr-$A \to$ Pgr-A. Recall that a Quillen's K_i-group, $i \geq 0$, is a functor from the category of exact categories with exact functors to the category of abelian groups (see §6). Furthermore, isomorphic functors induce the same map on the K-groups [79, p.19]. Thus $K_i^{\mathrm{gr}}(f) = K_i^{\mathrm{gr}}(\mathcal{T}_\delta)$. Therefore if r is a homogeneous element of degree zero, i.e., $\delta = 0$, then $K_i^{\mathrm{gr}}(f) : K_i^{\mathrm{gr}}(A) \to K_i^{\mathrm{gr}}(A)$ is the identity map.

3.5 K_0^{gr} is a pre-ordered module

3.5.1 Γ-pre-ordered modules

An abelian group G is called a *pre-ordered abelian group* if there is a relation, denoted by $\geq$, on G which is reflexive and transitive and it respects the group structure. It follows that the set $G_+ := \{x \in G \mid x \geq 0\}$ forms a monoid. Conversely, any monoid C in G, induces a pre-ordering on G, by defining $x \geq y$ if $x - y \in C$. It follows that with this pre-ordering $G_+ = C$ which is called the cone of the ordering. (Note that G_+ should not be confused by G^+ used for the group completion in §3.1.)

Example 3.5.1. Let V be a monoid and V^+ its completion (see §3.1). There exist a natural homomorphism $\phi : V \to V^+$ (see (3.4)), which makes V^+ a pre-ordered abelian group with the image of V under this homomorphism as a cone of V^+.

Let G has a pre-ordering. An element $u \in G$ is called an order-unit if $u \geq 0$ and for any $x \in G$, there is $n \in \mathbb{N}$, such that $nu \geq x$.

For a ring R, by Example 3.5.1, the Grothendieck group $K_0(R)$ is a pre-ordered abelian group. Concretely, Consider the set of isomorphism classes of finitely generated projective R-modules in $K_0(R)$. This set forms a monoid and thus induces an ordering on $K_0(R)$. We check that with this pre-ordering $[R]$ is an order-unit. Let $u \in K_0(R)$. Then $u = [P] - [Q]$, where P, Q are finitely generated projective R-modules. But there is a finitely generated projective module P' such that $P \oplus P' \cong R^n$ as a right R-module, for some $n \in \mathbb{N}$. Then $n[R] \geq u$. Indeed,

$$n[R] - u = n[R] - [P] + [Q] = [R^n] - [P] + [Q] = [P \oplus P'] - [P] + [Q] = [P'] + [Q] \in K_0(R)_+.$$

Example 3.5.2. Suppose R is a ring such that $K_0(R) \neq 0$, but the order unit $[R] = 0$. It follows that all elements of $K_0(R)$ are either zero or negative. Furthermore, it is possible that there is an element $x \in K_0(R)$ such $x > 0$ and $0 > x$ simultaneously. One instance of such a ring is constructed in Example 3.8.3

The Grothendieck group of a ring as a pre-ordered group is studied extensively in [38, §15]. In particular it was established that for the so called ultramatricial algebras R, the abelian group $K_0(R)$ along with its positive cone and the order-unit $[R]$ is a complete invariant (see [38, Theorem 15.26] and the introduction to §5). This invariant is also called the *dimension group* in the literature as it coincides with an invariant called the dimension group by Elliot [34] to classify such algebras.

Since we will consider the graded Grothendieck groups, which have an extra $\mathbb{Z}[\Gamma]$-module structure, we need to adopt the above definitions on ordering to the graded setting. For this reason, here we define the category of Γ-pre-ordered modules.

Let Γ be a group and G be a (left) Γ-module. Let $\geq$ be a reflexive and transitive relation on G which respects the monoid and the module structures, i.e., for $\gamma \in \Gamma$ and $x, y, z \in G$, if $x \geq y$, then $x + z \geq y + z$ and $\gamma x \geq \gamma y$. We call G a *Γ-pre-ordered module*. We call G a *pre-ordered module* when Γ is clear from the context. The *cone* of G is defined as $\{x \in G \mid x \geq 0\}$ and denoted by G_+. The set G_+ is a Γ-submonoid of G, i.e., a submonoid which is closed under the action of Γ. In fact, G is a Γ-pre-ordered module if and only if there exists a Γ-submonoid of G. (Since G is a Γ-module, it can be considered as a $\mathbb{Z}[\Gamma]$-module.) An element $u \in G_+$ is called an *order-unit* if for any $x \in G$, there are $\alpha_1, \dots, \alpha_n \in \Gamma$, $n \in \mathbb{N}$, such that

$$\sum_{i=1}^{n} \alpha_i u \geq x. \tag{3.27}$$

As usual, in this setting, we only consider homomorphisms which preserve the pre-ordering, i.e., a Γ-homomorphism $f : G \to H$, such that $f(G_+) \subseteq H_+$. We denote by $\mathcal{P}_\Gamma$ the category of pointed Γ-pre-ordered modules with pointed ordered preserving homomorphisms, i.e., the objects are the pairs (G, u), where G is a Γ-pre-ordered module and u is an order-unit, and $f : (G, u) \to (H, v)$ is an ordered-preserving Γ-homomorphism such that $f(u) = v$. Note that when Γ is a trivial group, we are in the classical setting of pre-ordered abelian groups. When $\Gamma = \mathbb{Z}$ and so $\mathbb{Z}[\Gamma] = \mathbb{Z}[x, x^{-1}]$, we simply write $\mathcal{P}$ for $\mathcal{P}_\mathbb{Z}$.

Example 3.5.3. $K_0^{\mathrm{gr}}(A)$ AS A Γ-PRE-ORDERED MODULE

Let A be a Γ-graded ring. Then $K_0^{\mathrm{gr}}(A)$ is a Γ-pre-ordered module with the set of isomorphic classes of graded finitely generated projective right A-modules as the cone of ordering (i.e., the image of $\mathcal{V}^{\mathrm{gr}}(A)$ under the natural homomorphism $\mathcal{V}^{\mathrm{gr}}(A) \to K_0^{\mathrm{gr}}(A)$) and $[A]$ as an order-unit. Indeed, if $x \in K_0^{\mathrm{gr}}(A)$, then by Lemma 3.1.7(1) there are graded finitely generated projective modules P and P' such that $x = [P] - [P']$. But there is a graded module Q such that $P \oplus Q \cong A^n(\overline{\alpha})$, where $\overline{\alpha} = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in \Gamma$ (see (1.27)). Now

$$[A^n(\overline{\alpha})] - x = [P] + [Q] - [P] + [P'] = [Q] + [P'] = [Q \oplus P'] \in K_0^{\mathrm{gr}}(A)_+.$$

This shows that $\sum_{i=1}^{n} \alpha_i [A] = [A^n(\overline{\alpha})] \geq x$.

In §5 we will show that for the so called graded ultramatricial algebras R, the $\mathbb{Z}[\Gamma]$-module $K_0^{\mathrm{gr}}(R)$ along with its positive cone and the order-unit $[R]$ is a complete invariant (see Theorem 5.2.5). We call this invariant the *graded dimension module*.

Theorem 3.5.4. *Let A and B be Γ-graded rings. If A is graded Morita equivalent to B, then there is an order preserving $\mathbb{Z}[\Gamma]$-module isomorphism*

$$K_0^{\mathrm{gr}}(A) \cong K_0^{\mathrm{gr}}(B).$$

Proof. By Theorem 2.3.5, there is a graded $A-B$-bimodule Q such that the functor $-\otimes_A Q$: Gr-$A \to$ Gr-B is a graded equivalence. The restriction to Pgr-B induces a graded equivalence $-\otimes_A Q$: Pgr-$A \to$ Pgr-B. This in return induces an order preserving $\mathbb{Z}[\Gamma]$-module isomorphism $\phi : K_0^{\mathrm{gr}}(A) \to K_0^{\mathrm{gr}}(B)$ $\square$

If Γ-graded rings A and B are graded Morita equivalent, i.e., Gr-$A \approx$ Gr-B, then by Theorem 2.3.5, Mod-$A \approx$ Mod-B and thus not only by Theorem 3.5.4, $K_0^{\mathrm{gr}}(A) \cong K_0^{\mathrm{gr}}(B)$, but also $K_0(A) \cong K_0(B)$ as well.

In a more specific case, we have the following.

Lemma 3.5.5. *Let A be a Γ-graded ring and P be a A-graded progenerator. Let $B = \mathrm{End}_A(P)$. Then $[P]$ is an order-unit in $K_0^{\mathrm{gr}}(A)$ and there is an order preserving $\mathbb{Z}[\Gamma]$-module isomorphism*

$$\big(K_0^{\mathrm{gr}}(B), [B]\big) \cong \big(K_0^{\mathrm{gr}}(A), [P]\big).$$

Proof. Since P is a graded generator, by Theorem 2.2.2, there are $\alpha_i \in \Gamma$, $1 \leq i \leq n$, such that $[A] \leq \sum_i \alpha_i[P]$. Since $[A]$ is an order-unit, this immediately implies that $[P]$ is an order-unit. By (2.5), the functor $-\otimes_B P$: Gr-$B \to$ Gr-A is a graded equivalence. The restriction to Pgr-B induces a graded equivalence $-\otimes_B P$: Pgr-$B \to$ Pgr-A. This in return induces an order preserving $\mathbb{Z}[\Gamma]$-module isomorphism $\phi : K_0^{\mathrm{gr}}(B) \to K_0^{\mathrm{gr}}(A)$, $\phi([Q]-[L]) = [Q\otimes_B P] - [L \otimes_B P]$, where Q and L are graded finitely generated projective B-modules. In particular $\phi([B]) = [P]$. This completes the proof. $\square$

3.6 K_0^{gr} of graded division rings

In the case of graded division rings, one can compute the graded Grothendieck group completely, using the description of graded free modules.

Proposition 3.6.1. *Let A be a Γ-graded division ring with the support the subgroup Γ_A. Then the monoid of isomorphism classes of Γ-graded finitely generated projective A-modules is isomorphic to $\mathbb{N}[\Gamma/\Gamma_A]$ and $K_0^{\mathrm{gr}}(A) \cong \mathbb{Z}[\Gamma/\Gamma_A]$ as $\mathbb{Z}[\Gamma]$-modules. Furthermore, $[A^n(\delta_1,\dots,\delta_n)] \in K_0^{\mathrm{gr}}(A)$ corresponds to $\sum_{i=1}^n \underline{\delta}_i$ in $\mathbb{Z}[\Gamma/\Gamma_A]$, where $\underline{\delta}_i = \Gamma_A + \delta_i$. In particular if A is a (trivially graded) division ring, Γ a group and A considered as a Γ-graded division ring concentrated in degree zero, then $K_0^{\mathrm{gr}}(A) \cong \mathbb{Z}[\Gamma]$ and $[A^n(\delta_1,\dots,\delta_n)] \in K_0^{\mathrm{gr}}(A)$ corresponds to $\sum_{i=1}^n \delta_i$ in $\mathbb{Z}[\Gamma]$.*

Proof. By Proposition 1.3.10, $A(\delta_1) \cong_{\mathrm{gr}} A(\delta_2)$ as graded A-module if and only if $\delta_1 - \delta_2 \in \Gamma_A$. Thus any graded free module of rank 1 is graded isomorphic to some $A(\delta_i)$, where $\{\delta_i\}_{i\in I}$ is a complete set of coset representative of the subgroup Γ_A in Γ, i.e., $\{\Gamma_A + \delta_i, i \in I\}$ represents Γ/Γ_A. Since any graded finitely generated module M over A is graded free (see §1.1), it follows that

$$M \cong_{\mathrm{gr}} A(\delta_{i_1})^{r_1} \oplus \cdots \oplus A(\delta_{i_k})^{r_k}, \tag{3.28}$$

where $\delta_{i_1}, \dots \delta_{i_k}$ are distinct elements of the coset representative. Now suppose

$$M \cong_{\mathrm{gr}} A(\delta_{i'_1})^{s_1} \oplus \cdots \oplus A(\delta_{i'_{k'}})^{s_{k'}}. \tag{3.29}$$

Considering the A_0-module $M_{-\delta_{i_1}}$, from (3.28) we have $M_{-\delta_{i_1}} = A_0^{r_1}$. This implies one of $\delta_{i'_j}$, $1 \leq j \leq k'$, say, $\delta_{i'_1}$, has to be δ_{i_1} and so $r_1 = s_1$ as A_0 is a division ring. Repeating the same argument for each δ_{i_j}, $1 \leq j \leq k$, we see $k = k'$, $\delta_{i'_j} = \delta_{i_j}$ and $r_j = s_j$, for all $1 \leq j \leq k$

(possibly after suitable permutation). Thus any graded finitely generated projective A-module can be written uniquely as $M \cong_{\mathrm{gr}} A(\delta_{i_1})^{r_1} \oplus \cdots \oplus A(\delta_{i_k})^{r_k}$, where $\delta_{i_1}, \dots \delta_{i_k}$ are distinct elements of the coset representative. The (well-defined) map

$$\begin{aligned} \mathcal{V}^{\mathrm{gr}}(A) &\to \mathbb{N}[\Gamma/\Gamma_A] \\ [A(\delta_{i_1})^{r_1} \oplus \cdots \oplus A(\delta_{i_k})^{r_k}] &\mapsto r_1(\Gamma_A + \delta_{i_1}) + \cdots + r_k(\Gamma_A + \delta_{i_k}), \end{aligned} \tag{3.30}$$

gives a $\mathbb{N}[\Gamma]$-monoid isomorphism between the monoid of isomorphism classes of Γ-graded finitely generated projective A-modules $\mathcal{V}^{\mathrm{gr}}(A)$ and $\mathbb{N}[\Gamma/\Gamma_A]$. The rest of the proof follows easily. $\square$

Remark 3.6.2. One can use Equation 3.9 to calculate the graded K-theory of division algebras as well.

Example 3.6.3. Using Proposition 3.6.1, we calculate the graded K_0 of two types of graded fields and we determine the action of $\mathbb{Z}[x, x^{-1}]$ on these groups. These are graded fields obtained from Leavitt path algebras of acyclic and C_n-comet graphs, respectively (see Theorem 1.6.15 and 1.6.17).

1. Let K be a field. Consider $A = K$ as a $\mathbb{Z}$-graded field with the support $\Gamma_A = 0$, i.e., A is concentrated in degree 0. By Proposition 3.6.1, $K_0^{\mathrm{gr}}(A) \cong \mathbb{Z}[x, x^{-1}]$ as a $\mathbb{Z}[x, x^{-1}]$-module. With this presentation $[A(i)]$ corresponds to x^i in $\mathbb{Z}[x, x^{-1}]$.

2. Let $A = K[x^n, x^{-n}]$ be a $\mathbb{Z}$-graded field with $\Gamma_A = n\mathbb{Z}$. By Proposition 3.6.1,

$$K_0^{\mathrm{gr}}(A) \cong \mathbb{Z}\big[\mathbb{Z}/n\mathbb{Z}\big] \cong \bigoplus_n \mathbb{Z},$$

is a $\mathbb{Z}[x, x^{-1}]$-module. The action of x on $(a_1, \dots, a_n) \in \bigoplus_n \mathbb{Z}$ is

$$x(a_1, \dots, a_n) = (a_n, a_1, \dots, a_{n-1}).$$

With this presentation $[A]$ corresponds to $(1, 0, \dots, 0)$ in $\bigoplus_n \mathbb{Z}$. Furthermore, the map

$$U : K_0^{\mathrm{gr}}(A) \to K_0(A)$$

induced by the forgetful functor (§1.2.7), gives a group homomorphism

$$\begin{aligned} U : \bigoplus_n \mathbb{Z} &\longrightarrow \mathbb{Z}, \\ (a_i) &\longmapsto \sum_i a_i. \end{aligned}$$

In fact, the sequence

$$\bigoplus_n \mathbb{Z} \xrightarrow{f} \bigoplus_n \mathbb{Z} \xrightarrow{U} \mathbb{Z} \longrightarrow 0,$$

where $f(a_1, \dots, a_n) = (a_n - a_1, a_1 - a_2, \dots, a_{n-1} - a_n)$ is exact. In §6.3 we systematically related K_0^{gr} to K_0 for certain rings.

Example 3.6.4. Consider the Hamilton quaternion algebra $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$. By Example 1.1.21, $\mathbb{H}$ is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded division ring with $\Gamma_{\mathbb{H}} = \mathbb{Z}_2 \times \mathbb{Z}_2$. By Proposition 3.6.1, $K_0^{\mathrm{gr}}(\mathbb{H}) \cong \mathbb{Z}$. In fact since $\mathbb{H}$ is a strongly $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded division ring, one can deduce the result using the Dade's theorem (see §3.1.3), i.e., $K_0^{\mathrm{gr}}(\mathbb{H}) \cong K_0(\mathbb{H}_0) = K_0(\mathbb{R}) \cong \mathbb{Z}$.

Example 3.6.5. Let (D, v) be a valued division algebra, where $v : D^* \to \Gamma$ is the valuation homomorphism. By Example 1.4.7, there is a Γ-graded division algebra $\mathrm{gr}(D)$ associated to D, where $\Gamma_{\mathrm{gr}(D)} = \Gamma_D$. By Proposition 3.6.1,

$$K_0^{\mathrm{gr}}(\mathrm{gr}(D)) \cong \Gamma/\Gamma_D.$$

The following example generalises Example 3.6.4 of the Hamilton quaternion algebra $\mathbb{H}$ as a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded ring.

Remark 3.6.6. We saw in Example 1.1.21 that $\mathbb{H}$ can also be considered as a $\mathbb{Z}_2$-graded division ring. So $\mathbb{H}$ is also strongly $\mathbb{Z}_2$-graded, and $K_0^{\mathrm{gr}}(\mathbb{H}) \cong K_0(\mathbb{H}_0) = K_0(\mathbb{C}) \cong \mathbb{Z}$. Then $Z(\mathbb{H}) = \mathbb{R}$, which we can consider as a trivially $\mathbb{Z}_2$-graded field, so by Proposition 3.6.1, $K_0^{\mathrm{gr}}(\mathbb{R}) = \mathbb{Z} \oplus \mathbb{Z}$. We note that for both grade groups, $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$, we have $K_0^{\mathrm{gr}}(\mathbb{H}) \cong \mathbb{Z}$, but the $K_0^{\mathrm{gr}}(\mathbb{R})$ are different. So the graded K-theory of a graded ring depends not only on the ring, but also on its grade group.

Example 3.6.7. Let K be a field and let $R = K[x^2, x^{-2}]$. Then R is a $\mathbb{Z}$-graded field, with the support $2\mathbb{Z}$, where R can be written as $R = \bigoplus_{n \in \mathbb{Z}} R_n$, with $R_n = Kx^n$ if n is even and $R_n = 0$ if n is odd. Consider the shifted graded matrix ring $A = \mathbb{M}_3(R)(0, 1, 1)$, which has support $\mathbb{Z}$. Then we will show that A is a graded central simple algebra over R.

It is clear that the centre of A is R, and A is finite dimensional over R. Recall that a graded ideal is generated by homogeneous elements. If J is a non-zero graded ideal of A, then using the elementary matrices, we can show that $J = A$ (see [52, Ex. III.2.9]), so A is graded simple.

By Theorem 1.6.17, A is the Leavitt path algebra of the following graph. Thus by Theorem 1.6.12 A is a strongly $\mathbb{Z}$-graded ring.

$$\begin{tikzcd} E : \bullet \arrow[r] & \bullet \arrow[r, bend left] & \bullet \arrow[l, bend left] \end{tikzcd}$$

Here, we also show that A is a strongly $\mathbb{Z}$-graded ring by checking the conditions of Proposition 1.1.16. Since $\mathbb{Z}$ is finitely generated, it is sufficient to show that $I_3 \in A_1 A_{-1}$ and $I_3 \in A_{-1} A_1$. But

$$I_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ x^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & x^{-2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x^2 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & x^{-2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$I_3 = \begin{pmatrix} 0 & x^{-2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ x^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

As in the previous examples, using the Dade's theorem (see §3.1.3), we have $K_0^{\mathrm{gr}}(A) \cong K_0(A_0)$. Since $R_0 = K$, by (1.36) there is a ring isomorphism

$$A_0 = \begin{pmatrix} R_0 & R_1 & R_1 \\ R_{-1} & R_0 & R_0 \\ R_{-1} & R_0 & R_0 \end{pmatrix} = \begin{pmatrix} K & 0 & 0 \\ 0 & K & K \\ 0 & K & K \end{pmatrix} \cong K \times \mathbb{M}_2(K).$$

Then

$$K_0^{\mathrm{gr}}(A) \cong K_0(A_0) \cong K_0(K) \oplus K_0\big(\mathbb{M}_2(K)\big) \cong \mathbb{Z} \oplus \mathbb{Z},$$

since K_0 respects Cartesian products and Morita equivalence. Note that

$$K_0(A) = K_0\big(\mathbb{M}_3(R)(0,1,1)\big) \cong K_0(R) = K_0\big(K[x^2, x^{-2}]\big) \cong \mathbb{Z}.$$

The isomorphism $K_0(K[x^2, x^{-2}]) \cong \mathbb{Z}$ comes from the fundamental theorem of algebraic K-theory [82, Thm. 3.3.3] (see also [66, p. 484]), $K_0\big(K[x, x^{-1}]\big) \cong K_0(K) \cong \mathbb{Z}$, and that $K[x^2, x^{-2}] \cong K[x, x^{-1}]$ as rings. So the K-theory of A is isomorphic to one copy of $\mathbb{Z}$, which is not the same as the graded K-theory of A.

Example 3.6.8. <u>REDUCED K_0^{gr} OF GRADED CENTRAL SIMPLE ALGEBRAS</u>

Recall from §3.1.4, that for a Γ-graded ring A, $\widetilde{K_0^{\mathrm{gr}}}$ is the cokernel of the homomorphism

$$\begin{aligned} \phi : \mathbb{Z}[\Gamma] &\longrightarrow K_0^{\mathrm{gr}}(A), \\ \sum_\alpha n_\alpha \alpha &\longmapsto \sum_\alpha n_\alpha [A(\alpha)]. \end{aligned}$$

Let A be a Γ-graded division ring. We calculate $\widetilde{K_0^{\mathrm{gr}}}(\mathbb{M}_n(A))$, where $\mathbb{M}_n(A)$ is a Γ-graded ring (with no shifting). The $\mathbb{Z}[\Gamma]$-module homomorphism ϕ above takes the form

$$\begin{aligned} \mathbb{Z}[\Gamma] \longrightarrow K_0^{\mathrm{gr}}(\mathbb{M}_n(A)) &\xrightarrow{\cong} K_0^{\mathrm{gr}}(A) \xrightarrow{\cong} \mathbb{Z}[\Gamma/\Gamma_A], \\ \alpha \longmapsto [\mathbb{M}_n(A)(\alpha)] &\longmapsto \bigoplus_n [A(\alpha)] \longmapsto n(\Gamma_A + \alpha), \end{aligned}$$

where the second map is induced by the Morita theory (see Proposition 2.1.1) and the third map is induced by Proposition 3.6.1. The cokernel of the composition maps gives $\widetilde{K_0^{\mathrm{gr}}}(\mathbb{M}_n(A))$, which one can immediately calculate

$$\widetilde{K_0^{\mathrm{gr}}}(\mathbb{M}_n(A)) \cong \frac{\mathbb{Z}}{n\mathbb{Z}}\Big[\frac{\Gamma}{\Gamma_A}\Big].$$

In particular, for a graded division ring A, we have

$$\widetilde{K_0^{\mathrm{gr}}}(\mathbb{M}_n(A)) \cong 0.$$

Example 3.6.9. Let K be a field. Consider the following sequence of graded matrix rings

$$K \subseteq \mathbb{M}_2(K)(0,1) \subseteq \mathbb{M}_3(K)(0,1,2) \subseteq \mathbb{M}_4(K)(0,1,2,3) \subseteq \cdots \tag{3.31}$$

where the inclusion comes from the non-unital (graded) ring homomorphism of (3.15). Let R be the graded ring

$$R = \bigcup_{i=1}^{\infty} \mathbb{M}_i(K)(\overline{\alpha}_i),$$

where $\overline{\alpha}_i = (0, 1, \dots, i-1)$. Note that R is a non-unital ring. We calculate $K_0^{\mathrm{gr}}(R)$. Since K_0^{gr} respects the direct limit (Theorem 3.2.4), we have

$$K_0^{\mathrm{gr}}(R) = K_0^{\mathrm{gr}}(\varinjlim R_i) = \varinjlim K_0^{\mathrm{gr}}(R_i),$$

where $R_i = \mathbb{M}_i(K)(\overline{\alpha}_i)$. The non-unital homomorphism $\phi_i : R_i \to R_{i+1}$ (see (3.15)) induces the homomorphism $\overline{\phi}_i : K_0^{\mathrm{gr}}(R_i) \to K_0^{\mathrm{gr}}(R_{i+1})$. We will observe that each of these K-groups is $\mathbb{Z}[x, x^{-1}]$

and each map ϕ_i is identity. First, note that the ring R_i, $i \in \mathbb{N}$, is unital, so using the Morita theory (see Proposition 2.1.1) and Proposition 3.6.1, $K_0^{\mathrm{gr}}(R_i) = K_0^{\mathrm{gr}}(K) \cong \mathbb{Z}[x, x^{-1}]$. Switching to the idempotent representation of the graded Grothendieck group, observe that the idempotent matrix p_i having 1 on the upper left corner and zero everywhere else is in $R_{i0} = \mathbb{M}_i(K)(0, 1, \dots, i-1)_0$ and $p_i R_i \cong_{\mathrm{gr}} K \oplus K(1) \oplus \cdots \oplus K(i-1)$. Again Proposition 2.1.1 shows that $[p_i R_i] = [K \oplus K(1) \oplus \cdots \oplus K(i-1)]$ corresponds to $1 \in \mathbb{Z}[x, x^{-1}]$. Now $p_{i+1} := \phi_i(p_i)$ gives again a matrix with 1 on the upper left corner and zero everywhere else. So $p_{i+1} R_{i+1} \cong_{\mathrm{gr}} K \oplus K(1) \oplus \cdots \oplus K(i)$ and consequently $[p_{i+1} R_{i+1}] = 1$. So $\overline{\phi}_i(1) = 1$. Since $\overline{\phi}_i$ respects the shifting, this shows that $\overline{\phi}_i$ is the identity map.

A similar argument shows that for the graded ring

$$S = \bigcup_{i=1}^{\infty} \mathbb{M}_i(K)(\overline{\alpha}_i),$$

where $\overline{\alpha}_i = (0, 1, 1, \dots, 1)$, with one zero and 1 repeated $i-1$ times, we have an ordered $\mathbb{Z}[x, x^{-1}]$-module isomorphism

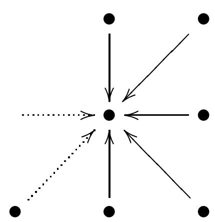
$$K_0^{\mathrm{gr}}(R) \cong K_0^{\mathrm{gr}}(S).$$

Using a similar argument as in the proof of Theorem 1.6.15, one can show that R is isomorphic to the Leavitt path algebra associated to the infinite graph

$$E: \quad \cdots\cdots \to \bullet \longrightarrow \bullet \longrightarrow \bullet$$

whereas S is the Leavitt path algebra of a graph F consisting of infinite vertices, one in the middle and the rest are connected to this vertex with an edge.

$F:$



Clearly the rings R and S are not graded isomorphic, as the support of R is $\mathbb{Z}$ whereas the support of S is $\{-1, 0, 1\}$.

3.7 K_0^{gr} of graded local rings

Recall that a ring is a *local ring* if the set of non-invertible elements form a two sided ideal. When A is a commutative ring, then A is local if and only if A has a unique maximal ideal.

A Γ-graded ring A is called a *graded local ring* if the two sided ideal M generate by non-invertible homogeneous elements is a proper ideal. One can easily observe that the graded ideal M is the unique graded maximal left, right, and graded two sided ideal of A. When A is a graded commutative ring, then A is graded local if and only if A has a unique graded maximal ideal.

If A is a graded local ring, then the graded ring A/M is a graded division ring. One can further show that A_0 is a local ring with the unique maximal ideal $A_0 \cap M$. In fact we have the following proposition.

Proposition 3.7.1. *Let A be a Γ-graded ring. Then A is a graded local ring if and only if A_0 is a local ring.*

Proof. Suppose A is a graded local ring. Then by definition, the two sided ideal M generated by non-invertible homogeneous elements is a proper ideal. Consider $m = A_0 \cap M$ which is a proper ideal of A_0. Suppose $x \in A_0 \backslash m$. Then x is a homogeneous element which is not in M. Thus x has to be invertible. This shows that A_0 is a local ring with the unique maximal ideal m.

Conversely, suppose A_0 is a local ring. We first show that any left or right invertible homogeneous element is a two sided invertible element. Let a be a left invertible homogeneous element. Then there is a homogeneous element b such that $ba = 1$. If ab is not right invertible, then $ab \in m$, where m is the unique maximal ideal of the local ring A_0. Thus $1 - ab \notin m$ which implies that $1 - ab$ is invertible. But $(1 - ab)a = a - aba = a - a = 0$, and since $1 - ab$ is invertible, we get $a = 0$ which is a contradiction to the fact that a has a left inverse. Thus a has a right inverse and so is invertible. A similar argument can be written for right invertible elements. Now let M be the ideal generated by all non-invertible homogeneous elements of A. We will show that M is proper, and thus A is a graded local ring. Suppose M is not proper. Thus $1 = \sum_i r_i a_i s_i$, where a_i are non-invertible homogeneous elements and r_i, s_i are homogeneous elements such that $\deg(r_i a_i s_i) = 0$. If $r_i a_i s_i$ is invertible for some i, using the fact that right and left invertibles are invertibles, it follows that a_i is invertible which is a contradiction. Thus $r_i a_i s_i$, for all i, are homogeneous elements of degree zero and not invertible. So they are all in m. This implies that $1 \in m$ which is a contradiction. Thus M is a proper ideal of A. $\square$

For more on graded local rings (graded by a cancellative monoid) see [64].

In Theorem 3.7.4, we will explicitly calculate the graded Grothendieck group of graded local rings. There are two ways to do this. One can prove that a graded finitely generated projective module over a graded local ring is a graded free with a unique rank, as in the case of graded division rings (see §1.4) and adopt the same approach to calculate the graded Grothendieck group (see §3.6). However, there is a more direct way to do so, which we develop here.

Recall the definition of graded Jacobson radical of a graded ring A, $J^{\mathrm{gr}}(A)$, from §1.1.5. We also need the graded version of the Nakayama lemma which holds in this setting as well. Namely, if $J \subseteq J^{\mathrm{gr}}(A)$ is a graded right ideal of A and P is a graded finitely generated right A-module, and Q is a graded submodule of P such that $P = Q + PJ$, then $P = Q$ (see [73, Corollary 2.9.2]).

We need the following two lemmas in order to calculate the graded Grothendieck group of a graded local ring.

Lemma 3.7.2. *Let A be a Γ-graded ring, $J \subseteq J^{\mathrm{gr}}(A)$ a homogeneous two sided ideal and P and Q be graded finitely generated projective A-modules. If $\overline{P} = P \otimes_A A/J = P/PJ$ and $\overline{Q} = Q \otimes_A A/J = Q/QJ$ are isomorphic as graded A/J-modules, then P and Q are isomorphic as graded A-module.*

Proof. Let $\phi : \overline{P} \to \overline{Q}$ be a graded A/J-module isomorphism. Clearly ϕ is also a A-module isomorphism. Consider the Diagram 3.32. Since π_2 is an epimorphism, and P is a graded finitely generated projective, there is a graded homomorphism $\psi : P \to Q$ which makes the diagram commutative.

$$\begin{tikzcd} P \arrow[r, "\pi_1"] \arrow[d, dotted, "\psi"'] & \overline{P} \arrow[r] \arrow[d, "\phi"] & 0 \\ Q \arrow[r, "\pi_2"] & \overline{Q} \arrow[r] & 0 \end{tikzcd} \tag{3.32}$$

We will show that ψ is a graded A-module isomorphism. Since $\pi_2\psi$ is an epimorphism, $Q = \psi(P) + \ker \pi_2 = \psi(P) + QJ$. The Nakayama lemma then implies that $Q = \psi(P)$, i.e., ψ is an epimorphism. Now since Q is graded projective, there is a graded homomorphism $i : Q \to P$ such that $\psi i = 1_Q$ and $P = i(Q) \oplus \ker \psi$. But

$$\ker \psi \subseteq \ker \pi_2 \psi = \ker \phi \pi_1 = \ker \pi_1 = PJ.$$

Thus $P = i(Q) + PJ$. Again the Nakayama lemma implies $P = i(Q)$. Thus $\ker \psi = 0$, and so $\psi : P \to Q$ is an isomorphism. $\square$

Lemma 3.7.3. *Let A and B be Γ-graded rings and $\phi : A \to B$ be a graded epimorphism such that $\ker \phi \subseteq J^{\mathrm{gr}}(A)$. Then the induced homomorphism $\overline{\phi} : K_0^{\mathrm{gr}}(A) \to K_0^{\mathrm{gr}}(B)$ is a $\mathbb{Z}[\Gamma]$-module monomorphism.*

Proof. Let $J = \ker(\phi)$. Without the loss of generality we can assume $B = A/J$. Let $x \in \ker \overline{\phi}$. Write $x = [P] - [Q]$ for two graded finitely generated projective A-modules P and Q. So $\phi(x) = [\overline{P}] - [\overline{Q}] = 0$, where $\overline{P} = P \otimes_A A/J = P/PJ$ and $\overline{Q} = Q \otimes_A A/J = Q/QJ$ are graded A/J-modules. Since $\phi(x) = 0$, by Lemma 3.1.7(3), $\overline{P} \bigoplus B^n(\overline{\alpha}) \cong_{\mathrm{gr}} \overline{Q} \bigoplus B^n(\overline{\alpha})$, where $\overline{\alpha} = (\alpha_1, \dots, \alpha_n)$. So

$$\overline{P \bigoplus A^n(\overline{\alpha})} \cong_{\mathrm{gr}} \overline{Q \bigoplus A^n(\overline{\alpha})}.$$

By Lemma 3.7.2, $P \bigoplus A^n(\overline{\alpha}) \cong_{\mathrm{gr}} Q \bigoplus A^n(\overline{\alpha})$. Therefore $x = [P] - [Q] = 0$ in $K_0^{\mathrm{gr}}(A)$. This completes the proof. $\square$

For a Γ-graded ring A, recall that Γ_A is the support of A and $\Gamma_A^* = \{\alpha \in \Gamma \mid A_\alpha^* \neq \emptyset\}$ is a subgroup of Γ_A.

Proposition 3.7.4. *Let A be a Γ-graded local ring. Then there is a $\mathbb{Z}[\Gamma]$-module isomorphism*

$$K_0^{\mathrm{gr}}(A) \cong \mathbb{Z}[\Gamma/\Gamma_A^*].$$

Proof. Let $M = J^{\mathrm{gr}}(A)$ be the unique graded maximal ideal of A. Then by Lemma 3.7.3 the homomorphism

$$\begin{aligned} \phi : K_0^{\mathrm{gr}}(A) &\longrightarrow K_0^{\mathrm{gr}}(A/M) \\ [P] &\longmapsto [P \otimes_A A/M], \end{aligned} \tag{3.33}$$

is a monomorphism. Since A/M is a graded division ring, by Proposition 3.6.1, $K_0^{\mathrm{gr}}(A/M) \cong \mathbb{Z}[\Gamma/\Gamma_{A/M}]$. Observe that $\Gamma_{A/M} = \Gamma_A^*$. On the other hand, for the graded A-module $A(\alpha)$, where $\alpha \in \Gamma$, we have

$$A(\alpha) \otimes_A A/M \cong_{\mathrm{gr}} (A/M)(\alpha), \tag{3.34}$$

as the graded A/M-module. Since $K_0^{\mathrm{gr}}(A/M) \cong \mathbb{Z}[\Gamma/\Gamma_A^*]$ is generated by $[(A/M)(\alpha_i)]$, where $\{\alpha_i\}_{i \in I}$ is a complete set of coset representative of Γ/Γ_A^* (see Proposition 3.6.1), from (3.33) and (3.34) we get $\phi([A(\alpha_i)] = [(A/M)(\alpha_i)]$. So ϕ is an epimorphism as well. This finishes the proof. $\square$

3.8 K_0^{gr} of Leavitt path algebras

For a graph E, its associated path algebra $\mathcal{P}(E)$ is a positively graded ring. In §6.1, we will use Quillen's theorem on graded K-theory of such rings to calculate the graded Grothendieck group of paths algebras (see Theorem 6.1.2).

In this section we calculate the K_0 and K_0^{gr} of Leavitt path algebras. For one thing, they provide very nice examples. We first calculate the non-graded K_0 of the Leavitt path algebras in §3.8.1 and in §3.8.2, we determine their K_0^{gr}-groups.

3.8.1

For a Leavitt path algebra $\mathcal{L}_K(E)$, the monoid $\mathcal{V}(\mathcal{L}_K(E))$ is studied in [6]. In particular using [6, Theorem 3.5], one can calculate the Grothendieck group of a Leavitt path algebra from the adjacency matrix of a graph (see [3, p.1998]). We present the calculation of the Grothendieck group of a Leavitt path algebra here.

Let F be a free abelian monoid generated by a countable set X. The nonzero elements of F can be written as $\sum_{t=1}^{n} x_t$, where $x_t \in X$. Let r_i, s_i be elements of F, where $i \in I \subseteq \mathbb{N}$. We define an equivalence relation on F denoted by $\langle r_i = s_i \mid i \in I\rangle$ as follows: Define a binary relation $\rightarrow$ on $F \backslash \{0\}$, $r_i + \sum_{t=1}^{n} x_t \rightarrow s_i + \sum_{t=1}^{n} x_t$, $i \in I$ and generate the equivalence relation on F using this binary relation. Namely, $a \sim a$ for any $a \in F$ and for $a, b \in F\backslash\{0\}$, $a \sim b$ if there is a sequence $a = a_0, a_1, \dots, a_n = b$ such that for each $t = 0, \dots, n-1$ either $a_t \rightarrow a_{t+1}$ or $a_{t+1} \rightarrow a_t$. We denote the quotient monoid by $F/\langle r_i = s_i \mid i \in I\rangle$. Completing the monoid (see §3.1.1), one can see that there is a canonical group isomorphism

$$\Big(\frac{F}{\langle r_i = s_i \mid i \in I\rangle}\Big)^+ \cong \frac{F^+}{\langle r_i - s_i \mid i \in I\rangle}. \tag{3.35}$$

Let E be a graph (as usual we consider only graphs with no sinks) and A_E be the *adjacency matrix* $(n_{ij}) \in \mathbb{Z}^{E^0 \oplus E^0}$, where n_{ij} is the number of edges from v_i to v_j. Clearly the adjacency matrix depends on the ordering we put on E^0. We usually fix an ordering on E^0.

Multiplying the matrix $A_E^t - I$ from the left defines a homomorphism

$$\mathbb{Z}^{E^0} \longrightarrow \mathbb{Z}^{E^0},$$

where $\mathbb{Z}^{E^0}$ is the direct sum of copies of $\mathbb{Z}$ indexed by E^0. The next theorem shows that the cokernel of this map gives the Grothendieck group of Leavitt path algebras.

Theorem 3.8.1. *Let E be finite graph with no sinks and $\mathcal{L}(E)$ be the Leavitt path algebra associated to E. Then*

$$K_0(\mathcal{L}(E)) \cong \operatorname{coker}\big(A_E^t - I : \mathbb{Z}^{E^0} \longrightarrow \mathbb{Z}^{E^0}\big). \tag{3.36}$$

Proof. Let M_E be the abelian monoid generated by $\{v \mid v \in E^0\}$ subject to the relations

$$v = \sum_{\{\alpha \in E^1 \mid s(\alpha) = v\}} r(\alpha), \tag{3.37}$$

for every $v \in E^0$. The relations (3.37) can be then written as $A_E^t \overline{v}_i = I\overline{v}_i$, where $v_i \in E^0$ and $\overline{v}_i$ is the $(0, \dots, 1, 0, \dots)$ with 1 in the i-th component. Therefore,

$$M_E \cong \frac{F}{\langle A_E^t \overline{v}_i = I\overline{v}_i, v_i \in E^0\rangle},$$

where F is the free abelian monoid generated by the vertices of E. By [6, Theorem 3.5] there is a natural monoid isomorphism

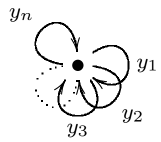
$$\mathcal{V}(\mathcal{L}_K(E)) \cong M_E.$$

So using (3.35) we have,

$$K_0(\mathcal{L}(E)) \cong \mathcal{V}(\mathcal{L}_K(E))^+ \cong M_E^+ \cong \frac{F^+}{\langle (A_E^t - I)\overline{v}_i, v_i \in E^0\rangle}. \tag{3.38}$$

Now F^+ is $\mathbb{Z}^{E^0}$ and it is easy to see that the denominator in (3.38) is the image of $A_E^t - I : \mathbb{Z}^{E^0} \longrightarrow \mathbb{Z}^{E^0}$. $\square$

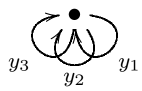
Example 3.8.2. Let E be the following graph.



Then the Leavitt path algebra associated to E, $\mathcal{L}(E)$, is the algebra constructed by Leavitt in (1.44). By Theorem 3.8.1,

$$K_0^{\mathrm{gr}}(\mathcal{L}(E)) \cong \mathbb{Z}/(n-1)\mathbb{Z}.$$

Example 3.8.3. Here is an example of a ring R such that $K_0(R) \neq 0$ but $[R] = 0$. Let A be the Leavitt path algebra associated to the graph



Thus as a right A-module, $A^3 \cong A$ (see Example 1.3.13). By Example 3.8.2, $K_0(A) = \mathbb{Z}/2\mathbb{Z}$ and $2[A] = 0$. The ring $R = \mathbb{M}_2(A)$ is Morita equivalent to A (using the assignment $P_{\mathbb{M}_2(A)} \mapsto P \otimes_{\mathbb{M}_2(A)} A^2$, see Proposition 2.1.1). Thus $K_0(R) \cong K_0(A) \cong \mathbb{Z}/2\mathbb{Z}$. Under this assignment, $[R] = [\mathbb{M}_2(A)]$ is sent to $2[A]$ which is zero, thus $[R] = 0$.

3.8.2

Recall that the Leavitt path algebras have a natural $\mathbb{Z}$-graded structure (see §1.6.3). In this section we calculate the graded Grothendieck group of Leavitt path algebras. The graded Grothendieck group as a possible invariant for these algebras was first considered in [47]. In the case of finite graphs with no sinks, there is a good description of the action of $\mathbb{Z}$ on the graded Grothendieck group which we recall here.

Let E be a finite graph with no sinks. Set $\mathcal{A} = \mathcal{L}(E)$ which is a strongly $\mathbb{Z}$-graded ring by Theorem 1.6.11. For any $u \in E^0$ and $i \in \mathbb{Z}$, $u\mathcal{A}(i)$ is a right graded finitely generated projective $\mathcal{A}$-module and any graded finitely generated projective $\mathcal{A}$-module is generated by these modules up to isomorphism, i.e.,

$$\boxed{\mathcal{V}^{\mathrm{gr}}(\mathcal{A}) = \Big\langle \big[u\mathcal{A}(i)\big] \mid u \in E^0, i \in \mathbb{Z} \Big\rangle.}$$

By §3.1, $K_0^{\mathrm{gr}}(\mathcal{A})$ is the group completion of $\mathcal{V}^{\mathrm{gr}}(\mathcal{A})$. The action of $\mathbb{N}[x, x^{-1}]$ on $\mathcal{V}^{\mathrm{gr}}(\mathcal{A})$ and thus the action of $\mathbb{Z}[x, x^{-1}]$ on $K_0^{\mathrm{gr}}(\mathcal{A})$ is defined on generators by $x^j[u\mathcal{A}(i)] = [u\mathcal{A}(i+j)]$, where $i, j \in \mathbb{Z}$. We first observe that for $i \geq 0$,

$$x[u\mathcal{A}(i)] = [u\mathcal{A}(i+1)] = \sum_{\{\alpha \in E^1 \mid s(\alpha) = u\}} [r(\alpha)\mathcal{A}(i)]. \tag{3.39}$$

First notice that for $i \geq 0$, $\mathcal{A}_{i+1} = \sum_{\alpha \in E^1} \alpha \mathcal{A}_i$. It follows

$$u\mathcal{A}_{i+1} = \bigoplus_{\{\alpha \in E^1 \mid s(\alpha) = u\}} \alpha \mathcal{A}_i$$

as $\mathcal{A}_0$-modules. Using the fact that $\mathcal{A}_n \otimes_{\mathcal{A}_0} \mathcal{A} \cong \mathcal{A}(n)$, $n \in \mathbb{Z}$, and the fact that $\alpha\mathcal{A}_i \cong r(\alpha)\mathcal{A}_i$ as $\mathcal{A}_0$-module, we get

$$u\mathcal{A}(i+1) \cong \bigoplus_{\{\alpha \in E^1 | s(\alpha)=u\}} r(\alpha)\mathcal{A}(i)$$

as graded $\mathcal{A}$-modules. This gives (3.39).

Recall that for a Γ-graded ring A, $K_0^{\mathrm{gr}}(A)$ is a pre-ordered abelian group with the set of isomorphic classes of graded finitely generated projective right A-modules as the cone of ordering, denoted by $K_0^{\mathrm{gr}}(A)_+$ (i.e., the image of $\mathcal{V}^{\mathrm{gr}}(A)$ under the natural homomorphism $\mathcal{V}^{\mathrm{gr}}(A) \to K_0^{\mathrm{gr}}(A)$). Furthermore, $[A]$ is an order-unit. We call the triple, $(K_0^{\mathrm{gr}}(A), K_0^{\mathrm{gr}}(A)_+, [A])$ the *graded dimension group* (see [38, §15] for some background on dimension groups).

In [47] it was conjectured that the graded dimension group is a complete invariant for Leavitt path algebras. Namely, for graphs E and F, $\mathcal{L}(E) \cong_{\mathrm{gr}} \mathcal{L}(F)$ if and only if there is an order preserving $\mathbb{Z}[x, x^{-1}]$-module isomorphism

$$\phi : K_0^{\mathrm{gr}}(\mathcal{L}(E)) \to K_0^{\mathrm{gr}}(\mathcal{L}(F)) \tag{3.40}$$

such that $\phi([\mathcal{L}(E)] = \mathcal{L}(F)$.

3.8.3

For the Leavitt path algebra $\mathcal{L}(E)$, the structure of the ring of homogeneous elements of degree zero, $\mathcal{L}(E)_0$, is known. Since $K_0^{\mathrm{gr}}(\mathcal{L}(E) \cong K_0(\mathcal{L}(E)_0)$, we recall the description of $\mathcal{L}(E)_0$ in the setting of finite graphs with no sinks (see the proof of Theorem 5.3 in [6]). Let A_E be the adjacency matrix of E. Let $L_{0,n}$ be the linear span of all elements of the form pq^* with $r(p) = r(q)$ and $|p| = |q| \leq n$. Then

$$\mathcal{L}(E)_0 = \bigcup_{n=0}^{\infty} L_{0,n}, \tag{3.41}$$

where the transition inclusion $L_{0,n} \to L_{0,n+1}$ is to identify pq^* with $r(p) = v$ by

$$\sum_{\{\alpha | s(\alpha)=v\}} p\alpha(q\alpha)^*.$$

Note that since E does not have sinks, for any $v \in E_0$ the set $\{\alpha | s(\alpha) = v\}$ is not empty.

For a fixed $v \in E^0$, let $L_{0,n}^v$ be the linear span of all elements of the form pq^* with $|p| = |q| = n$ and $r(p) = r(q) = v$. Arrange the paths of length n with the range v in a fixed order $p_1^v, p_2^v, \ldots, p_{k_n^v}^v$, and observe that the correspondence of $p_i^v p_j^{v*}$ to the matrix unit e_{ij} gives rise to a ring isomorphism $L_{0,n}^v \cong \mathbb{M}_{k_n^v}(K)$. Furthermore, $L_{0,n}^v$, $v \in E^0$ form a direct sum. This implies that

$$L_{0,n} \cong \bigoplus_{v \in E^0} \mathbb{M}_{k_n^v}(K),$$

where k_n^v, $v \in E^0$, is the number of paths of length n with the range v. The inclusion map $L_{0,n} \to L_{0,n+1}$ is

$$A_E^t : \bigoplus_{v \in E^0} \mathbb{M}_{k_n^v}(K) \longrightarrow \bigoplus_{v \in E^0} \mathbb{M}_{k_{n+1}^v}(K). \tag{3.42}$$

This means $(A_1,\dots,A_l)\in\bigoplus_{v\in E^0}\mathbb{M}_{k_n^v}(K)$ is sent to

$$(\sum_{j=1}^{l} n_{j1}A_j,\dots,\sum_{j=1}^{l} n_{jl}A_j)\in\bigoplus_{v\in E^0}\mathbb{M}_{k_{n+1}^v}(K),$$

where n_{ji} is the number of edges connecting v_j to v_i and

$$\sum_{j=1}^{l} k_jA_j=\begin{pmatrix} A_1 & & & \\ & A_1 & & \\ & & \ddots & \\ & & & A_l \end{pmatrix}$$

in which each matrix is repeated k_j times down the leading diagonal and if $k_j=0$, then A_j is omitted. This shows that $\mathcal{L}(E)_0$ is an *ultramatricial algebra*, i.e., it is isomorphic to the union of an increasing chain of a finite product of matrix algebras over a field K (see §5).

Writing $\mathcal{L}(E)_0=\varinjlim_n L_{0,n}$, since the Grothendieck group K_0 respects the direct limit, we have $K_0(\mathcal{L}(E)_0)\cong\varinjlim_n K_0(L_{0,n})$. Since K_0 of (Artinian) simple algebras are $\mathbb{Z}$, the ring homomorphism $L_{0,n}\to L_{0,n+1}$ induces the group homomorphism

$$\mathbb{Z}^{E^0}\xrightarrow{A_E^t}\mathbb{Z}^{E^0},$$

where $A_E^t:\mathbb{Z}^{E^0}\to\mathbb{Z}^{E^0}$ is multiplication from left which is induced by the homomorphism (3.42).

For a finite graph E with no sinks, with n vertices and the adjacency matrix A, by Theorem 1.6.11, $K_0^{\mathrm{gr}}(\mathcal{L}(E))\cong K_0(\mathcal{L}(E)_0)$. Thus $K_0^{\mathrm{gr}}(\mathcal{L}(E))$ is the direct limit of the ordered direct system

$$\mathbb{Z}^n\xrightarrow{A^t}\mathbb{Z}^n\xrightarrow{A^t}\mathbb{Z}^n\xrightarrow{A^t}\cdots,\tag{3.43}$$

where the ordering in $\mathbb{Z}^n$ is defined point-wise.

In general, the direct limit of the system, $\varinjlim_A\mathbb{Z}^n$, where $A\in\mathbb{M}_n(\mathbb{Z})$, is an ordered group and can be described as follows. Consider the pair (a,k), where $a\in\mathbb{Z}^n$ and $k\in\mathbb{N}$, and define the equivalence relation $(a,k)\sim(b,k')$ if $A^{k''-k}a=A^{k''-k'}b$ for some $k''\in\mathbb{N}$. Let $[a,k]$ denote the equivalence class of (a,k). Clearly $[A^na,n+k]=[a,k]$. Then it is not difficult to show that the direct limit $\varinjlim_A\mathbb{Z}^n$ is the abelian group consists of equivalent classes $[a,k]$, $a\in\mathbb{Z}^n$, $k\in\mathbb{N}$, with addition defined by

$$[a,k]+[b,k']=[A^{k'}a+A^kb,k+k'].\tag{3.44}$$

The positive cone of this ordered group is the set of elements $[a,k]$, where $a\in{\mathbb{Z}^+}^n$, $k\in\mathbb{N}$. Furthermore, there is automorphism $\delta_A:\varinjlim_A\mathbb{Z}^n\to\varinjlim_A\mathbb{Z}^n$ defined by $\delta_A([a,k])=[Aa,k]$.

There is another presentation for $\varinjlim_A\mathbb{Z}^n$ which is sometimes easier to work with. Consider the set

$$\Delta_A=\big\{v\in A^n\mathbb{Q}^n\mid A^kv\in\mathbb{Z}^n,\ \text{for some } k\in\mathbb{N}\big\}.\tag{3.45}$$

The set Δ_A forms an ordered abelian group with the usual addition of vectors and the positive cone

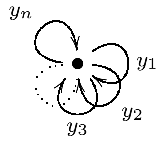
$$\Delta_A^+=\big\{v\in A^n\mathbb{Q}^n\mid A^kv\in{\mathbb{Z}^+}^n,\ \text{for some } k\in\mathbb{N}\big\}.\tag{3.46}$$

Furthermore, there is automorphism $\delta_A:\Delta_A\to\Delta_A$ defined by $\delta_A(v)=Av$. The map

$$\begin{aligned}\phi:\Delta_A&\to\varinjlim_A\mathbb{Z}^n\\ v&\mapsto[A^kv,k],\end{aligned}\tag{3.47}$$

where $k \in \mathbb{N}$ such that $A^k v \in \mathbb{Z}^n$, is an isomorphism which respects the action of A and the ordering, i.e., $\phi(\Delta_A^+) = (\varinjlim_A \mathbb{Z}^n)^+$ and $\phi(\delta_A(v)) = \delta_A \phi(v)$.

Example 3.8.4. Let E be the following graph.

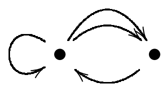


The non-graded K_0 of $\mathcal{L}(E)$ was computed in Example 3.8.2. The graph E has no sinks, and so by (3.43),

$$K_0^{\mathrm{gr}}(\mathcal{L}(E)) \cong \varinjlim \mathbb{Z},$$

of the inductive system $\mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{n} \cdots$. This gives that $K_0^{\mathrm{gr}}(\mathcal{L}(E)) \cong \mathbb{Z}[1/n]$.

Example 3.8.5. For the graph

$E :$ 

with the adjacency $A_E = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$, the ring of homogeneous element of degree zero, $\mathcal{L}(E)_0$, is the direct limit of the system

$$\begin{aligned} &K \oplus K \xrightarrow{A_E^t} \mathbb{M}_2(K) \oplus \mathbb{M}_2(K) \xrightarrow{A_E^t} \mathbb{M}_4(K) \oplus \mathbb{M}_4(K) \xrightarrow{A_E^t} \cdots \\ &(a,b) \mapsto \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \oplus \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \end{aligned}$$

So $K_0^{\mathrm{gr}}(\mathcal{L}(E))$ is the direct limit of the direct system

$$\mathbb{Z}^2 \xrightarrow{A_E^t} \mathbb{Z}^2 \xrightarrow{A_E^t} \mathbb{Z}^2 \xrightarrow{A_E^t} \cdots,$$

Since $\det(A_E^t) = -2$, one can easily calculate that

$$K_0^{\mathrm{gr}}(\mathcal{L}(E)) \cong \mathbb{Z}[1/2] \bigoplus \mathbb{Z}[1/2].$$

Furthermore $[\mathcal{L}(E)] \in K_0^{\mathrm{gr}}(\mathcal{L}(E))$ is represented by $(1,1) \in \mathbb{Z}[1/2] \bigoplus \mathbb{Z}[1/2]$. Adopting (3.45) for the description of $K_0^{\mathrm{gr}}(\mathcal{L}(E))$, since the action of x on $K_0^{\mathrm{gr}}(\mathcal{L}(E))$ represented by action of A_E^t from the left, we have $x(a,b) = (a+b, 2a)$. Furthermore, considering (3.46) for the positive cone, ${A_E^t}^k(a,b)$ is eventually positive, if $v(a,b) > 0$, where $v = (2,1)$ is the Perron eigenvector of A_E (see [53, Lemma 7.3.8]). It follows that

$$K_0^{\mathrm{gr}}(\mathcal{L}(E))^+ = \Delta_{A_E^t}^+ = \big\{(a,b) \in \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/2] \mid 2a + b > 0\big\} \cup \{(0,0)\}.$$

Example 3.8.6. <u>Reduced K_0^{gr} of strongly graded rings</u>

When A is a strongly graded ring, the graded K-groups coincide with K-groups of the zero homogeneous part (see (3.8)). However this example shows that this is not the case for the reduced graded Grothendieck groups.

Let A be the Leavitt's algebra generated by $2n$ symbols (which is associated to a graph with one vertex and n-loops) (see 1.3.13). By Theorem 1.6.11, this is a strongly graded ring. The homomorphism 3.13 takes the form

$$\begin{aligned} \phi : \mathbb{Z}[x, x^{-1}] &\longrightarrow K_0^{\mathrm{gr}}(A) \cong \mathbb{Z}[1/n], \\ \sum_i n_i x^i &\longmapsto \sum_i n_i [A(i)]. \end{aligned}$$

This shows that ϕ is surjective and thus $\widetilde{K_0^{\mathrm{gr}}}(A)$ is trivial. On the other hand, by Example 3.8.4, $K_0(A_0) \cong K_0^{\mathrm{gr}}(A) \cong \mathbb{Z}[1/n]$. But

$$\begin{aligned} \phi_0 : \mathbb{Z} &\longrightarrow K_0(A_0) \cong \mathbb{Z}[1/n], \\ n &\longmapsto n[A_0]. \end{aligned}$$

This shows that $\widetilde{K_0}(A_0)$ is a nontrivial torsion group $\mathbb{Z}[1/n]/\mathbb{Z}$. Thus $\widetilde{K_0^{\mathrm{gr}}}(A) \ncong \widetilde{K_0}(A_0)$.

Remark 3.8.7. K_0^{gr} OF WEYL ALGEBRAS

Let $A = K(x, y)/\langle xy - yx - 1\rangle$ be the Weyl algebra, where K is an algebraically closed field of characteristic 0. By Example 1.6.4, this is a $\mathbb{Z}$-graded ring. The graded Grothendieck group of this ring is calculated in [86]. It is shown that $K_0^{\mathrm{gr}}(A) \cong \bigoplus_{\mathbb{Z}} \mathbb{Z}$ (i.e., a direct sum of a countably many $\mathbb{Z}$), $\widetilde{K_0^{\mathrm{gr}}}(A) = 0$ and $K_0(A) = 0$.

3.9 Symbolic dynamics

One of the central objects in the theory of symbolic dynamics is a *shift of finite type* (i.e., a topological Markov chain). Every finite directed graph E with no sinks and sources gives rise to a shift of finite type X_E by considering the set of bi-infinite paths and the natural shift of the paths to the left. This is called an *edge shift*. Conversely any shift of finite type is conjugate to an edge shift (for a comprehensive introduction to symbolic dynamics see [53]). Several invariant have been proposed in order to classify shifts of finite type, among them Krieger's dimension group. In this section we see that Krieger's invariant can be expressible as the graded Grothendieck group of a Leavitt path algebra.

We briefly recall the objects of our interest. Let $\mathcal{A}$ be a finite alphabet (i.e., a finite set). A *full shift space* is defined as

$$\mathcal{A}^{\mathbb{Z}} := \big\{(a_i)_{i\in\mathbb{Z}} \mid a_i \in \mathcal{A}\big\},$$

and a *shift map* $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is defined as

$$\sigma\big((a_i)_{i\in\mathbb{Z}}\big) = (a_{i+1})_{i\in\mathbb{Z}}.$$

Furthermore, a *subshift* $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is a closed σ-invariant subspace of $\mathcal{A}^{\mathbb{Z}}$.

Given a finite graph E (see §1.6.3 for terminologies related to graphs), a *subshift of finite type associated to E* is defined as

$$X_E := \big\{(e_i)_{i\in\mathbb{Z}} \in (E^1)^{\mathbb{Z}} \mid r(e_i) = s(e_{i+1})\big\}.$$

We say X_E is essential if the graph E has no sinks and sources. Furthermore, X_E is called *irreducible* if the adjacency matrix A_E is irreducible. For a square nonnegative integer matrix A, we denote by X_A the subshift of finite type associated to the graph with the adjacency matrix A. Finally, two shifts of finite type X_A and X_B are called *conjugate* (or *topologically conjugate of subshifts*) and denoted by $X_A \cong X_B$, if there exists a homeomorphism $h : X_A \to X_B$ such that $\sigma_B\, h = h\, \sigma_A$.

The notion of the shift equivalence for matrices was introduced by Williams [96] (see also [53, §7]) in an attempt to provide a computable machinery for determining the conjugacy between two shifts of finite type. Two square nonnegative integer matrices A and B are called *elementary shift equivalent*, and denoted by $A \sim_{ES} B$, if there are nonnegative matrices R and S such that $A = RS$ and $B = SR$. The equivalence relation $\sim_S$ on square nonnegative integer matrices generated by elementary shift equivalence is called *strong shift equivalence*.

Example 3.9.1. Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$. We show that A is strongly shift equivalent to B.

Let $R_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $S_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $A = R_1S_1$. Set $E_1 := S_1R_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. Let $R_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $S_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. Then $E_1 = R_2S_2$. Set $E_2 := S_2R_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Finally, let $R_3 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $S_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. Then $E_2 = R_3S_3$ and $B = S_3R_3$. This shows

$$A \sim_{ES} E_1 \sim_{ES} E_2 \sim_{ES} B.$$

Thus $A \sim_S B$.

There is a weaker notion, called shift equivalence defined as follows. The nonnegative integer square matrices A and B are called *shift equivalent* if there are nonnegative matrices R and S such that $A^l = RS$ and $B^l = SR$, for some $l \in \mathbb{N}$, and $AR = RB$ and $SA = BS$. Clearly the strongly shift equivalence implies the shift equivalence, but the converse does not hold [53].

Theorem 3.9.2 (Williams [96, 53])**.** *Let A and B be two square nonnegative integer matrices and let E and F be two essential graphs.*

(1) *X_A is conjugate to X_B if and only if A is strongly shift equivalent to B.*

(2) *X_E is conjugate to X_F if and only if E can be obtained from F by a sequence of in/out-splitting and their converses.*

Krieger in [57] defined an invariant for classifying the irreducible shifts of finite type up to shift equivalence. Later Wagoner systematically used this invariant to relate it with higher K-groups. Surprisingly. Krieger's dimension group and Wagoner's dimension module in symbolic dynamics turn out to be expressible as the graded Grothendieck groups of Leavitt path algebras. Here, we briefly describe this relation.

In general, a nonnegative integral $n \times n$ matrix A gives rise to a stationary system. This in turn gives a direct system of order free abelian groups with A acting as an order preserving group homomorphism as follows

$$\mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^n \xrightarrow{A} \cdots,$$

where the ordering in $\mathbb{Z}^n$ is defined point-wise (i.e., the positive cone is $\mathbb{N}^n$). The direct limit of this system, $\Delta_A := \varinjlim_A \mathbb{Z}^n$, (i.e, the K_0 of the stationary system,) along with its positive cone, Δ^+, and the automorphism which induced by A on the direct limit, $\delta_A : \Delta_A \to \Delta_A$, is the invariant considered by Krieger, now known as *Krieger's dimension group*. Following [53], we denote this triple by $(\Delta_A, \Delta_A^+, \delta_A)$.

The following theorem was proved by Krieger ([57, Theorem 4.2], and [53, Theorem 7.5.8], see also [53, §7.5] for a detailed algebraic treatment).

Theorem 3.9.3. *Let A and B be two square nonnegative integer matrices. Then A and B are shift equivalent if and only if*

$$(\Delta_A, \Delta_A^+, \delta_A) \cong (\Delta_B, \Delta_B^+, \delta_B).$$

Wagoner noted that the induced structure on Δ_A by the automorphism δ_A makes Δ_A a $\mathbb{Z}[x, x^{-1}]$-module which was systematically used in [92, 93] (see also [23, §3]).

Recall that the graded Grothendieck group of a $\mathbb{Z}$-graded ring has a natural $\mathbb{Z}[x, x^{-1}]$-module structure and the following observation (Theorem 3.9.4) shows that the graded Grothendieck group of the Leavitt path algebra associated to a matrix A coincides with the Krieger dimension group of the shift of finite type associated to A^t, i.e., the graded dimension group of a Leavitt path algebra coincides with Krieger's dimension group,

$$\big(K_0^{\mathrm{gr}}(\mathcal{L}(E)), (K_0^{\mathrm{gr}}(\mathcal{L}(E))^+\big) \cong (\Delta_{A^t}, \Delta_{A^t}^+).$$

This will provide a link between the theory of Leavitt path algebras and symbolic dynamics.

Theorem 3.9.4. *Let E be a finite graph with no sinks with the adjacency matrix A. Then there is an isomorphism $\phi : K_0^{\mathrm{gr}}(\mathcal{L}(E)) \longrightarrow \Delta_{A^t}$ such that $\phi(x\alpha) = \delta_{A^t}\phi(\alpha)$, $\alpha \in \mathcal{L}(E)$, $x \in \mathbb{Z}[x, x^{-1}]$ and $\phi(K_0^{\mathrm{gr}}(\mathcal{L}(E))^+) = \Delta_{A^t}^+$.*

Proof. Since by Theorem 1.6.11, $\mathcal{L}(E)$ strongly graded, there is an ordered isomorphism $K_0^{\mathrm{gr}}(\mathcal{L}(E)) \to K_0(\mathcal{L}(E)_0)$. Thus the ordered group $K_0^{\mathrm{gr}}(\mathcal{L}(E))$ coincides with the ordered group Δ_{A^t} (see (3.43)). We only need to check that their module structures are compatible. It is enough to show that the action of x on K_0^{gr} coincides with the action of A^t on $K_0(\mathcal{L}(E)_0)$, i.e., $\phi(x\alpha) = \delta_{A^t}\phi(\alpha)$.

Set $\mathcal{A} = \mathcal{L}(E)$. Since graded finitely generated projective modules are generated by $u\mathcal{A}(i)$, where $u \in E^0$ and $i \in \mathbb{Z}$, it suffices to show that $\phi(x[u\mathcal{A}]) = \delta_{A^t}\phi([u\mathcal{A}])$. Since the image of $u\mathcal{A}$ in $K_0(\mathcal{A}_0)$ is $[u\mathcal{A}_0]$, and $\mathcal{A}_0 = \bigcup_{n=0}^{\infty} L_{0,n}$, (see (3.41)) using the presentation of K_0 given in (3.44), we have

$$\phi([u\mathcal{A}]) = [u\mathcal{A}_0] = [uL_{0,0}, 1] = [u, 1].$$

Thus

$$\delta_{A^t}\phi([u\mathcal{A}]) = \delta_{A^t}([u, 1]) = [A^t u, 1] = \sum_{\{\alpha \in E^1 \mid s(\alpha) = u\}} [r(\alpha), 1].$$

On the other hand,

$$\phi(x[u\mathcal{A}]) = \phi([u\mathcal{A}(1)]) = \phi\Big(\sum_{\{\alpha \in E^1 \mid s(\alpha) = u\}} [r(\alpha)\mathcal{A}]\Big) =$$
$$\sum_{\{\alpha \in E^1 \mid s(\alpha) = u\}} [r(\alpha)\mathcal{A}_0] = \sum_{\{\alpha \in E^1 \mid s(\alpha) = u\}} [r(\alpha)L_{0,0}, 1] = \sum_{\{\alpha \in E^1 \mid s(\alpha) = u\}} [r(\alpha), 1]. \quad (3.48)$$

Thus $\phi(x[u\mathcal{A}]) = \delta_{A^t}\phi([u\mathcal{A}])$. This finishes the proof. $\square$

It is easy to see that two matrices A and B are shift equivalent if and only if A^t and B^t are shift equivalent. Combining this with Theorem 3.9.4 and the fact that Krieger's dimension group is a complete invariant for shift equivalent we have the following corollary.

Corollary 3.9.5. *Let E and F be finite graphs with no sinks and A_E and A_F be their adjacency matrices, respectively. Then A_E is shift equivalent to A_F if and only if there is an order preserving $\mathbb{Z}[x,x^{-1}]$-module isomorphism $K_0^{\mathrm{gr}}(\mathcal{L}(E)) \cong K_0^{\mathrm{gr}}(\mathcal{L}(F))$.*

3.10 K_1^{gr}-theory

For a Γ-graded ring A, the category of finitely generated Γ-graded projective right A-modules, $\mathrm{Pgr}^{\Gamma}\text{-}A$, is an exact category. Thus using Quillen's Q-construction one defines

$$K_i^{\mathrm{gr}}(A) := K_i(\mathrm{Pgr}^{\Gamma}\text{-}A), \quad i \geq 0.$$

Furthermore, the shift functors induce auto-equivalences (in fact, automorphisms) $\mathcal{T}_\alpha : \mathrm{Pgr}\text{-}A \to \mathrm{Pgr}\text{-}A$. These in return give a group homomorphism $\Gamma \to \mathrm{Aut}(K_i^{\mathrm{gr}}(A))$ or equivalently, a $\mathbb{Z}[\Gamma]$-module on $K_i^{\mathrm{gr}}(A)$. This general construction will be used in §6.

In this section we concretely construct the graded K_1-group, using a graded version of Bass' construction of K_1-group. For a concise introduction of the groups K_0 and K_1, see [59].

Definition 3.10.1. K_0 AND K_1 OF AN EXACT CATEGORY

Let $\mathcal{P}$ be an *exact category*, i.e., a full additive subcategory of an abelian category $\mathcal{A}$ such that, if

$$0 \longrightarrow P_1 \longrightarrow P \longrightarrow P_2 \longrightarrow 0,$$

is an exact sequence in $\mathcal{A}$ and $P_1, P_2 \in \mathcal{P}$, then $P \in \mathcal{P}$ (i.e., $\mathcal{P}$ is closed under extension). Furthermore, we assume $\mathcal{P}$ has a small skeleton, i.e., $\mathcal{P}$ has a full subcategory $\mathcal{P}_0$ which is small and $\mathcal{P}_0 \hookrightarrow \mathcal{P}$ is an equivalence.

The groups $K_0(\mathcal{P})$ and $K_1(\mathcal{P})$ are defined as follows.

1. $K_0(\mathcal{P})$ is the free abelian group generated by objects of $\mathcal{P}_0$, subject to the relation $[P] = [P_1] + [P_2]$ if there is an exact sequence

   $$0 \longrightarrow P_1 \longrightarrow P \longrightarrow P_2 \longrightarrow 0,$$

   in $\mathcal{P}$.

2. $K_1(\mathcal{P})$ is the free abelian group generated by pairs (P, f), where P is an object of $\mathcal{P}_0$ and $f \in \mathrm{Aut}(P)$, subject to the relations

   $$[P, f] + [P, g] = [P, fg],$$

   and

   $$[P, f] = [P_1, g] + [P_2, h],$$

   if there is a commutative diagram in $\mathcal{P}_0$

   $$\begin{tikzcd} 0 \arrow[r] & P_1 \arrow[r, "i"] \arrow[d, "g"] & P \arrow[r, "\pi"] \arrow[d, "f"] & P_2 \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & P_1 \arrow[r, "i"] & P \arrow[r, "\pi"] & P_2 \arrow[r] & 0. \end{tikzcd}$$

Note that from the relations of K_1 it follows

$$[P, fg] = [P, gf],$$
$$[P, fg] = [P \oplus P, f \oplus g].$$

If Γ is a group and $\mathcal{T}_\alpha : \mathcal{P} \to \mathcal{P}$, $\alpha \in \Gamma$, are auto-equivalences such that $\mathcal{T}_\beta \mathcal{T}_\alpha \cong \mathcal{T}_{\alpha+\beta}$, then $K_0(\mathcal{P})$ and $K_1(\mathcal{P})$ have a Γ-module structures.

One can easily see that $K_0^{\mathrm{gr}}(A) = K_0(\mathrm{Pgr}^\Gamma\text{-}A)$. Following Bass, one defines $K_1^{\mathrm{gr}}(A) = K_1(\mathrm{Pgr}^\Gamma\text{-}A)$.

Since the category Gr-A is an abelian category and Pgr-A is an exact category, the main theorems of K-theory are valid for the grade Grothendieck group, such as Dévissage, Resolution theorem and the localisation exact sequences of K-theory (see §6.2.1 and [82, Chapter 3] and [79, 94]).

Example 3.10.2. Let F be a field and $F[x_1, \dots, x_r]$ be the polynomial ring with r variables, which is considered as a $\mathbb{Z}$-grade ring with support $\mathbb{N}$ (i.e., $\deg(x_i) = 1, 1 \leq i \leq r$). Then we will prove in §6 (see Proposition 6.1.1) that

$$K_1^{\mathrm{gr}}(F[x_1, \dots, x_r]) = F^* \otimes_{\mathbb{Z}} \mathbb{Z}[x, x^{-1}].$$

Example 3.10.3. <u>K_1^{gr} OF GRADED DIVISION ALGEBRAS</u>

Let A be a Γ-graded division ring. Then A_0 is a division ring and $\Omega := \Gamma_A$ is a group (§1.1.4). By (3.9), for $i = 1$, and the description of K_1-group of division ring due to Dieudonné ([33, § 20]), we have

$$K_1^\Gamma(A) \cong \bigoplus_{\Gamma/\Omega} K_1^\Omega(A) \cong \bigoplus_{\Gamma/\Omega} K_1(A_0) = \bigoplus_{\Gamma/\Omega} A_0^*/[A_0^*, A_0^*], \tag{3.49}$$

where A_0^* is the group of invertible elements of division ring A_0 and $[A_0^*, A_0^*]$ is the multiplicative commutator subgroup. Representing $\bigoplus_{\Gamma/\Omega} K_1(A_0)$ as the additive group of the group ring $K_1(A_0)[\Gamma/\Omega]$, (see Example 3.1.11) the action of Γ can be described as follows: for $\beta \in \Gamma$,

$$\beta\Big(\bigoplus_{\Omega+\alpha \in \Gamma/\Omega} K_1(A_0)(\Omega + \alpha) \Big) = \bigoplus_{\Omega+\alpha \in \Gamma/\Omega} K_1(A_0)(\Omega + \alpha + \beta).$$

As a concrete example, let $A = K[x^n, x^{-n}]$ be a $\mathbb{Z}$-graded field with $\Gamma_A = n\mathbb{Z}$. Then by (3.49),

$$K_1^{\mathrm{gr}}(A) \cong \bigoplus_n K^*,$$

is a $\mathbb{Z}[x, x^{-1}]$-module. The action of x on $(a_1, \dots, a_n) \in \bigoplus_n K^*$ is

$$x(a_1, \dots, a_n) = (a_n, a_1, \dots, a_{n-1}).$$

Compare this with the computation of $K_0^{\mathrm{gr}}(A)$ in Example 3.6.3(2).

Remark 3.10.4. <u>THE MATRIX DESCRIPTION OF K_1^{gr}-GROUP</u>

Let A be a strongly Γ-graded ring. Then the map

$$\begin{aligned} K_1^{\mathrm{gr}}(A) &\longrightarrow K_1(A_0), \\ [P, f] &\longmapsto [P_0, f_0] \end{aligned} \tag{3.50}$$

is an isomorphism of groups (see §1.5). Here for a graded isomorphism $f : P \to P$, we denote by f_α, the restriction of f to P_α, where $\alpha \in \Gamma$. Note that $K_1^{\mathrm{gr}}(A)$ is a $\mathbb{Z}[\Gamma]$-graded module, where the action of Γ on the generators is defined by $\alpha[P, f] = [P(\alpha), f]$.

Since $K_1(A_0)$ has a matrix description ([82, Theorem 3.1.7]), from (3.50) we get a matrix representation

$$K_1^{\mathrm{gr}}(A) \cong K_1(A_0) \cong \mathrm{GL}(A_0)/E(A_0).$$

We don't know whether for an arbitrary graded ring, one can give a matrix description for $K_1^{\mathrm{gr}}(A)$. For some work in this direction see [99].

Chapter 4

Graded Picard Groups

Let A be a commutative ring. If M is a finitely generated projective A-module of constant rank 1, then there is an A-module N such that $M \otimes_A N \cong A$. In fact this is an equivalent condition. The module M above is called an *invertible module*. The isomorphism classes of invertible modules with the tensor product form a group, denoted by $\mathrm{Pic}(A)$ and called the *Picard group* of A. Since A is commutative, $K_0(A)$ is a ring with the multiplication defined by the tensor product and one can prove that there is an exact sequence

$$1 \longrightarrow \mathrm{Pic}(A) \xrightarrow{\phi} K_0(A)^*, \tag{4.1}$$

where $K_0(A)^*$ is the group of invertible elements of $K_0(A)$ and $\phi([A]) = [A]$.

When A is a graded commutative ring, a parallel construction, using the graded modules, gives the graded Picard group $\mathrm{Pic}^{\mathrm{gr}}(A)$. As one expects when A is strongly graded commutative ring then, using Dade's theorem 1.5.1 one immediately gets

$$\mathrm{Pic}^{\mathrm{gr}}(A) \cong \mathrm{Pic}(A_0). \tag{4.2}$$

However when A is a noncommutative ring, the situation is substantially more involved. One needs to define the invertible bimodules in order to define the Picard group. Furthermore, since for bimodules M and N, $M \otimes_A N$ is not necessarily isomorphic to $N \otimes_A M$ as bimodules, the Picard group is not abelian. Even when A is strongly graded, the identities such as (4.2) does not hold in the noncommutative setting (see [41]).

4.1 $\mathrm{Pic}^{\mathrm{gr}}$ of a graded commutative ring

Let A be a commutative Γ-graded ring. A graded A-module P is called a *graded invertible* module if there is a graded module Q such that $P \otimes_A Q \cong_{\mathrm{gr}} A$ as graded A-modules. It is clear that if P is a graded invertible module, so is $P(\alpha)$ for any $\alpha \in \Gamma$. The *graded Picard group*, $\mathrm{Pic}^{\mathrm{gr}}(A)$, is defined as the set of graded isomorphism classes of graded invertible A-modules with tensor product as multiplication. This is a well-defined binary operation and makes $\mathrm{Pic}^{\mathrm{gr}}(A)$ an abelian group with the isomorphism class of A as identity element. Since a graded invertible module is an invertible module, we have a group homomorphism $\mathrm{Pic}^{\mathrm{gr}}(A) \to \mathrm{Pic}(A)$, $[P] \mapsto [P]$, where $[P]$ represents the isomorphism class of P in both groups.

Since for any $\alpha, \beta \in \Gamma$, $A(\alpha) \otimes_A A(\beta) \cong_{\mathrm{gr}} A(\alpha+\beta)$ (see §1.2.6), the map

$$\begin{aligned} \phi : \Gamma &\to \operatorname{Pic}^{\mathrm{gr}}(A), \\ \alpha &\mapsto [A(\alpha)] \end{aligned} \tag{4.3}$$

is a group homomorphism. This map will be used in the next lemma to calculate the graded Picard group of graded fields.

For a Γ-graded ring A, recall from §1.1.2 that Γ_A is the support of A and $\Gamma_A^* = \{\alpha \in \Gamma \mid A_\alpha^* \neq \emptyset\}$. Furthermore, for a graded field A, Γ_A is a subgroup of Γ.

Proposition 4.1.1. *Let A be a Γ-graded field with the support Γ_A. Then* $\operatorname{Pic}^{\mathrm{gr}}(A) \cong \Gamma/\Gamma_A$.

Proof. Consider the map $\phi : \Gamma \to \operatorname{Pic}^{\mathrm{gr}}(A)$ from (4.3). Since any graded invertible module is graded projective and graded projective modules over graded fields are graded free (Proposition 1.4.1), it follows that the graded invertible modules have to be of the form $A(\alpha)$ for some $\alpha \in \Gamma$. This shows that ϕ is an epimorphism. Now if $\phi(\alpha) = [A(\alpha)] = [A]$, then $A(\alpha) \cong_{\mathrm{gr}} A$, which by Corollary 1.3.11, it follows that $\alpha \in \Gamma_A$. Conversely, if $\alpha \in \Gamma_A$, then there is an element of degree α, which has to be invertible, as A is a graded field. Thus $A(\alpha) \cong_{\mathrm{gr}} A$, again by Corollary 1.3.11. This shows that the kernel of ϕ is Γ_A. This completes the proof. ☐

The graded Grothendieck group of a graded local ring was determined in Proposition 3.7.4. Here we determine its graded Picard group.

Proposition 4.1.2. *Let A be a commutative Γ-graded local ring with the support Γ_A. Then* $\operatorname{Pic}^{\mathrm{gr}}(A) \cong \Gamma/\Gamma_A^*$.

Proof. Let M be the unique graded maximal ideal of A. By Lemma 3.7.2, if for graded projective A-modules P and Q, $\overline{P} = P/PM$ is isomorphic to $\overline{Q} = Q/QM$ as graded A/M-modules, then P is isomorphic to Q as graded A-modules. This immediately implies that the natural map

$$\begin{aligned} \phi : \operatorname{Pic}^{\mathrm{gr}}(A) &\longrightarrow \operatorname{Pic}^{\mathrm{gr}}(A/M), \\ [P] &\longmapsto [\overline{P}] \end{aligned}$$

is a monomorphism. But by the proof of Proposition 4.1.1, any graded invertible A/M-module is of the form $(A/M)(\alpha)$ for some $\alpha \in \Gamma$. Since $\phi([A(\alpha)]) = [(A/M)(\alpha)]$, ϕ is an isomorphism. Since $\Gamma_{A/M} = \Gamma_A^*$, by Proposition 4.1.1, $\operatorname{Pic}^{\mathrm{gr}}(A/M) = \Gamma/\Gamma_A^*$. This completes the proof. ☐

In the next theorem (Theorem 4.1.5) we will be using the calculus of exterior algebras in the graded setting. Recall that if M is an A-module, the n-th *exterior power* of M is the quotient of the tensor product of n copies of M over A, denoted by $\bigotimes^n M$ (or $T_n(M)$ as in Example 1.1.3), by the submodule generated by $m_1 \otimes \cdots \otimes m_n$, where $m_i = m_j$ for some $1 \leq i \neq j \leq n$. The n-th exterior power of M is denoted by $\bigwedge^n M$. We set $\bigwedge^0 M = A$ and clearly $\bigwedge^1 M = M$.

If A is a commutative Γ-graded ring and M is a graded A-module, then $\bigotimes^n M$ is a graded A-module (§1.2.6) and the submodule generated by $m_1 \otimes \cdots \otimes m_n$, where $m_i = m_j$ for some $1 \leq i \neq j \leq n$ coincides with the submodule generated by $m_1 \otimes \cdots \otimes m_n$, where all m_i are homogeneous and $m_i = m_j$ for some $1 \leq i \neq j \leq n$, and $m_1 \otimes \cdots \otimes m_n + m_1' \otimes \cdots \otimes m_n'$, where all m_i and m_i' are homogeneous, $m_i = m_i'$ for all $1 \leq i \leq n$ except for two indices i, j, where $i \neq j$ and $m_i = m_j'$ and $m_j = m_i'$. Thus this submodule is a graded submodule of $\bigotimes^n M$ and therefore

$\bigwedge^n M$ is a graded A-module as well. We will use the following isomorphism which is valid in the non-graded setting and it is easy to see it respects the grading as well.

$$\bigwedge^n (M \oplus N) \cong_{\mathrm{gr}} \bigoplus_{r=0}^{n} \Big(\bigwedge^r M \otimes_A \bigwedge^{n-r} N \Big). \tag{4.4}$$

Lemma 4.1.3. *Let A be a commutative Γ-graded ring. Then*

$$\bigwedge^n A^m(\alpha_1, \dots, \alpha_m) \cong_{\mathrm{gr}} \bigoplus_{1 \leq i_1 < i_2 < \cdots < i_n \leq m} A(\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_n}). \tag{4.5}$$

Proof. We prove the lemma by induction on m. For $m = 1$ and $n = 1$ we clearly have $\bigwedge^1 A(\alpha_1) \cong_{\mathrm{gr}} A(\alpha_1)$. For $n \geq 2$, since $\bigwedge^2 A = 0$ it follows that $\bigwedge^n A(\alpha) = 0$. This shows that (4.5) is valid for $m = 1$. Now using the induction, by (4.4), since $\bigwedge^n A(\alpha) = 0$ for any $\alpha \in \Gamma$ and $n \geq 2$, we have

$$\begin{aligned}
\bigwedge^n A^m(\alpha_1, \dots, \alpha_m) &\cong_{\mathrm{gr}} \bigwedge^n \Big(A(\alpha_1) \oplus A^{m-1}(\alpha_2, \dots, \alpha_m) \Big) \\
&\cong_{\mathrm{gr}} A \otimes_A \bigwedge^n A^{m-1}(\alpha_2, \dots, \alpha_m) \bigoplus A(\alpha_1) \otimes_A \bigwedge^{n-1} A^{m-1}(\alpha_2, \dots, \alpha_m) \\
&\cong_{\mathrm{gr}} \bigoplus_{2 \leq i_1 < i_2 < \cdots < i_n \leq m} A(\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_n}) \bigoplus \\
&\quad A(\alpha_1) \otimes_A \bigoplus_{2 \leq i_1 < i_2 < \cdots < i_{n-1} \leq m} A(\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_n}) \\
&\cong_{\mathrm{gr}} \bigoplus_{2 \leq i_1 < i_2 < \cdots < i_n \leq m} A(\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_n}) \bigoplus \\
&\quad \bigoplus_{2 \leq i_1 < i_2 < \cdots < i_{n-1} \leq m} A(\alpha_1 + \alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_n}) \\
&\cong_{\mathrm{gr}} \bigoplus_{1 \leq i_1 < i_2 < \cdots < i_n \leq m} A(\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_n}).
\end{aligned}$$

□

The following corollary is immediate and will be used in Theorem 4.1.5.

Corollary 4.1.4. *Let A be a commutative Γ-graded ring. Then*

$$\bigwedge^n A^n(\alpha_1, \dots, \alpha_n) \cong_{\mathrm{gr}} A(\alpha_1 + \alpha_2 + \cdots + \alpha_n).$$

Recall from §3.1.5 that if A is a Γ-graded commutative ring, then $K_0^{\mathrm{gr}}(A)$ is a commutative ring. Denote by $K_0^{\mathrm{gr}}(A)^*$ the group of invertible elements of this ring. The following theorem establishes the graded version of the exact sequence (4.1).

Theorem 4.1.5. *Let A be a commutative Γ-graded ring. Then there is an exact sequence,*

$$1 \longrightarrow \mathrm{Pic}^{\mathrm{gr}}(A) \xrightarrow{\phi} K_0^{\mathrm{gr}}(A)^*,$$

where $\phi([P]) = [P]$.

Proof. It is clear that $\phi : \mathrm{Pic}^{\mathrm{gr}}(A) \to K_0^{\mathrm{gr}}(A)^*$ is a well-defined group homomorphism. We only need to show that ϕ is injective. Suppose $\phi([P]) = \phi([Q])$. Thus $[P] = [Q]$ in $K_0^{\mathrm{gr}}(A)$. By Lemma 3.1.7, $P \oplus A^n(\overline{\alpha}) \cong_{\mathrm{gr}} Q \oplus A^n(\overline{\alpha})$, where $\overline{\alpha} = (\alpha_1, \dots, \alpha_n)$. Since P and Q are graded invertible, they are in particular invertible, and so are of constant rank 1. Thus $\bigwedge^n(P) = 0$, for $n \geq 2$. Now by (4.4)

$$\begin{aligned}
\bigwedge^{n+1}(P \oplus A^n(\overline{\alpha})) &\cong_{\mathrm{gr}} \bigoplus_{i+j=n+1} \bigwedge^{i} A^n(\overline{\alpha}) \otimes \bigwedge^{j} P \\
&\cong_{\mathrm{gr}} \bigwedge^{n} A^n(\overline{\alpha}) \otimes \bigwedge^{1} P \\
&\cong_{\mathrm{gr}} A(\alpha_1 + \cdots + \alpha_n) \otimes P \cong_{\mathrm{gr}} P(\alpha_1 + \cdots + \alpha_n),
\end{aligned}$$

thanks to Corollary 4.1.4. Similarly, $\bigwedge^{n+1}\big(Q \oplus A^n(\overline{\alpha})\big) \cong_{\mathrm{gr}} Q(\alpha_1 + \cdots + \alpha_n)$. Thus

$$P(\alpha_1 + \cdots + \alpha_n) \cong_{\mathrm{gr}} Q(\alpha_1 + \cdots + \alpha_n)$$

which implies $P \cong_{\mathrm{gr}} Q$. So ϕ is an injective map. □

4.2 $\mathrm{Pic}^{\mathrm{gr}}$ of a graded noncommutative ring

When A is a noncommutative ring, the definition of the (graded) Picard group is more involved (see [11, Chapter 2], [15], [29, §55], and [36] for non-graded Picard groups of noncommutative rings). Note that if P are Q are $A-A$-bimodules then $P \otimes_A Q$ is not necessarily isomorphic to $Q \otimes_A P$ as A-bimodules. This is an indication that the Picard group, in this setting, is not necessarily an abelian group.

Let A and B be Γ-graded rings and P be a graded $A-B$-bimodule. Then P is called a *graded invertible* $A-B$-bimodule, if there is a graded $B-A$-bimodule Q such that $P \otimes_B Q \cong_{\mathrm{gr}} A$ as $A-A$-bimodules and $Q \otimes_A P \cong_{\mathrm{gr}} B$ as $B-B$-bimodules and the following diagrams are commutative.

$$\begin{tikzcd}
P \otimes_B Q \otimes_A P \arrow[r] \arrow[d] & A \otimes_A P \arrow[d] \\
P \otimes_B B \arrow[r] & P
\end{tikzcd}
\qquad
\begin{tikzcd}
Q \otimes_A P \otimes_B Q \arrow[r] \arrow[d] & B \otimes_B Q \arrow[d] \\
Q \otimes_A A \arrow[r] & Q.
\end{tikzcd}
\tag{4.6}$$

As in the commutative case (§4.1), for a noncommutative graded ring A, the *graded Picard group*, $\mathrm{Pic}^{\mathrm{gr}}(A)$, is defined as the set of graded isomorphism classes of graded invertible $A-A$-bimodules with tensor product as multiplications. The graded isomorphism class of the graded bimodule P is denoted by $[P]$. Since P is invertible, it has an inverse $[Q]$ in $\mathrm{Pic}^{\mathrm{gr}}(A)$ and the Diagram 4.6 guarantees that $([P][Q])[P] = [P]([Q][P]) = [P]$.

Theorem 4.2.1. *Let A and B be Γ-graded rings. If A is graded Morita equivalent to B, then*

$$\mathrm{Pic}^{\mathrm{gr}}(A) \cong \mathrm{Pic}^{\mathrm{gr}}(B).$$

Proof. Let $\text{Gr-}A \approx_{\mathrm{gr}} \text{Gr-}B$. Then there is a graded equivalence $\phi : \text{Gr-}A \to \text{Gr-}B$ with an inverse $\psi : \text{Gr-}B \to \text{Gr-}A$. By Theorem 2.3.5 (and its proof), $\psi(B) = P$ is a graded $B-A$-bi-module and

$\phi \cong - \otimes_A P^*$ and $\psi \cong - \otimes_B P$. Now one can easily check that the map

$$\begin{aligned} \operatorname{Pic}^{\mathrm{gr}}(A) &\longrightarrow \operatorname{Pic}^{\mathrm{gr}}(B), \\ [M] &\longmapsto [P \otimes_A M \otimes_A P^*] \end{aligned}$$

is an isomorphism of groups. $\square$

Let A be a Γ-graded ring. Consider the group $\operatorname{Aut}^{\mathrm{gr}}(A)$ of all the graded automorphisms and its subgroup $\operatorname{Inn}^{\mathrm{gr}}_{A_0}(A)$ which consists of graded inner automorphisms induced by the homogeneous elements of degree zero as follows.

$$\operatorname{Inn}^{\mathrm{gr}}_{A_0}(A) = \{f : A \to A \mid f(x) = uxu^{-1}, u \in A_0^*\}.$$

The graded A-bimodule structure on A induced by $f, g \in \operatorname{Aut}^{\mathrm{gr}}(A)$ will be denoted by ${}_fA_g$. Namely, A acts on ${}_fA_g$ as follows, $a.x.b = f(a)xg(b)$. One can prove the following graded A-bimodule isomorphisms.

$$\begin{aligned} {}_fA_g &\cong_{\mathrm{gr}} {}_{hf}A_{hg} \\ {}_fA_1 \otimes_A {}_gA_1 &\cong_{\mathrm{gr}} {}_{fg}A_1 \\ {}_fA_1 \otimes_A {}_1A_f &\cong_{\mathrm{gr}} {}_1A_f \otimes_A {}_fA_1 \cong_{\mathrm{gr}} A. \end{aligned} \tag{4.7}$$

The following theorem provides two exact sequences between the groups Γ, $\operatorname{Inn}^{\mathrm{gr}}_{A_0}(A)$ and $\operatorname{Aut}^{\mathrm{gr}}(A)$ and the graded Picard group. The first sequence belongs to the graded setting, whereas the second sequence is a graded version of a similar result in the non-graded setting (see [29, Theorem 55.11]).

Recall that $Z(A)$ stands for the centre of the ring A which is a graded subring of A (when the grade group Γ is abelian, see Example 1.1.25). Furthermore, it is easy to see that the map

$$\begin{aligned} \phi : \Gamma &\longrightarrow \operatorname{Pic}^{\mathrm{gr}}(A), \\ \alpha &\longmapsto [A(\alpha)] \end{aligned} \tag{4.8}$$

(which was considered in the case of commutative graded rings in (4.3)) is well-defined and is a homomorphism.

Theorem 4.2.2. *Let A be a Γ-graded ring. Then the following sequences are exact.*

(1) *The sequence* $1 \longrightarrow \Gamma^*_{Z(A)} \longrightarrow \Gamma \xrightarrow{\phi} \operatorname{Pic}^{\mathrm{gr}}(A)$, *where* $\phi(\alpha) = [A(\alpha)]$.

(2) *The sequence* $1 \longrightarrow \operatorname{Inn}^{\mathrm{gr}}_{A_0}(A) \longrightarrow \operatorname{Aut}^{\mathrm{gr}}(A) \xrightarrow{\phi} \operatorname{Pic}^{\mathrm{gr}}(A)$, *where* $\phi(f) = [{}_fA_1]$.

Proof. (1) Consider the group homomorphism ϕ defined in (4.8). If $u \in Z(A)^*$ is a homogeneous element of degree α, then the map $\psi : A \to A(\alpha)$, $a \mapsto ua$, is a graded A-bimodule isomorphism. This shows that $\Gamma^*_{Z(A)} \subseteq \ker(\phi)$. On the other hand if $\phi(\alpha) = [A(\alpha)] = 1_{\operatorname{Pic}^{\mathrm{gr}}(A)} = [A]$, then there is a graded A-bimodule isomorphism $\psi : A \to A(\alpha)$. From this it follows that there is an invertible homogeneous element $u \in Z(A)$ of degree α such that $\psi(x) = ux$ (see also Corollary 1.3.11 and Proposition 1.3.10). This completes the proof.

(2) The fact that the map ϕ is well-defined and is a homomorphism follows from (4.7). We only need to show that $\ker \phi$ coincides with $\operatorname{Inn}^{\mathrm{gr}}_{A_0}(A)$. Let $f \in \operatorname{Inn}^{\mathrm{gr}}_{A_0}(A)$, so that $f(x) = uxu^{-1}$, for any

$x \in A$, where $u \in A_0^*$. Then the A-graded bimodules ${}_fA_1$ and A are isomorphic. Indeed for the map $\theta : {}_fA_1 \to A$, $x \mapsto u^{-1}x$, we have

$$\theta(a.x.b) = \theta(uau^{-1}xb) = au^{-1}xb = a.\theta(x).b,$$

which gives a graded A-bimodule isomorphisms. This shows that $\mathrm{Inn}^{\mathrm{gr}}_{A_0}(A) \subseteq \ker\phi$. Conversely, suppose $\phi(f) = [{}_fA_1] = [A]$. Thus there is a graded A-bimodule isomorphism $\theta : {}_fA_1 \to A$ such that

$$\theta(a.x.b) = \theta(f(a)xb) = a\theta(x)b. \tag{4.9}$$

Since θ is bijective, there is a $u \in A$ such that $\theta(u) = 1$. Also, since θ is graded and $1 \in A_0$ it follows that $u \in A_0$. But $A = \theta(A) = \theta(A1) = Au$ and similarly $A = uA$. This implies $u \in A_0^*$. Plugging $x = a = 1$ in (4.9), we have $\theta(b) = ub$, for any $b \in A$. Using this identity, by plugging $x = b = 1$ in (4.9), we obtain $uf(a) = au$. Since u is invertible, we get $f(a) = u^{-1}au$, so $f \in \mathrm{Inn}^{\mathrm{gr}}_{A_0}(A)$ and we are done. □

For a graded noncommutative ring A the problem of determining the Picard group $\mathrm{Pic}(A)$ is difficult. There are several cases in the literature where $\mathrm{Picent}(A) := \mathrm{Pic}_R(A)$ is determined, where R is a centre of A and $\mathrm{Pic}_R(A)$ is a subgroup of $\mathrm{Pic}(A)$ consisting of all isomorphism classes of invertible bimodules P of A which are centralised by R, (i.e., $rp = pr$, for all $r \in R$ and $p \in P$) i.e., P is a $A \otimes_R A^{\mathrm{op}}$-left module (see [36], and [29, §55]). For a graded division algebra A (i.e., a graded division ring which is finite dimensional over its centre) we will be able to determine $\mathrm{Pic}^{\mathrm{gr}}(A)$. In fact since graded division algebras are graded Azumaya algebras, we first determine $\mathrm{Pic}^{\mathrm{gr}}_R(A)$, where A is a graded Azumaya algebra over its centre R. The approach follows the idea in Bass [15, §3, Corollary 4.5], where it is shown that for an Azumaya algebra A over R, the Picard group $\mathrm{Pic}_R(A)$ coincides with $\mathrm{Pic}(R)$.

Recall that a graded algebra A over a commutative graded ring R is called a *graded Azumaya algebra* if A is a graded faithfully projective R-module and the natural map $\phi : A \otimes_R A^{\mathrm{op}} \to \mathrm{End}_R(A)$, $\phi(a \otimes b)(x) = axb$, where $a, x \in A$ and $b \in A^{\mathrm{op}}$, is a graded R-isomorphism. This implies that A is an Azumaya algebra over R. Conversely, if A is a graded algebra over graded ring R and A is an Azumaya algebra over R, then A is faithfully projective as an R-module and the natural map $\phi : A \otimes_R A^{\mathrm{op}} \to \mathrm{End}_R(A)$ is naturally graded. Thus A is also a graded Azumaya algebra.

A graded division algebra is a graded central simple algebra. We first define these type of rings and show that they are graded Azumaya algebras.

A graded algebra A over a graded commutative ring R is said to be a *graded central simple algebra* over R if A is graded simple ring, i.e., A does not have any proper graded two-sided ideals, $Z(A) \cong_{\mathrm{gr}} R$ and A is finite dimensional as an R-module. Note that since A is graded simple, the centre of A (identified with R), is a graded field. Thus A is graded free as a graded module over its centre by Proposition 1.4.1, so the dimension of A over R is uniquely defined.

Proposition 4.2.3. *Let A be a Γ-graded central simple algebra over a graded field R. Then A is a graded Azumaya algebra over R.*

Proof. Since A is graded free of finite dimension over R, it follows that A is faithfully projective over R. Consider the natural graded R-algebra homomorphism $\psi : A \otimes_R A^{\mathrm{op}} \to \mathrm{End}_R(A)$ defined by $\psi(a \otimes b)(x) = axb$ where $a, x \in A$, $b \in A^{\mathrm{op}}$. Since graded ideals of A^{op} coincide with graded ideals of A, A^{op} is also graded simple. Thus $A \otimes A^{\mathrm{op}}$ is also graded simple (see [51, Proposition 1.1], so ψ is injective. Hence the map is surjective by dimension count, using Theorem 1.4.3. This shows that A is an Azumaya algebra over R, as required. □

Let A be a graded algebra over a graded commutative ring R. For a graded A-bimodule P centralised by R, define
$$P^A = \{p \in P \mid ap = pa, \text{ for all } a \in A\}.$$
Denote by $\text{Gr-}A_R\text{-Gr}$ the category of graded A-bimodules centralised by R. The functors

$$\begin{aligned} \text{Gr-}A_R\text{-Gr} &\longrightarrow \text{Gr-}R, \\ P &\longmapsto P^A \end{aligned} \qquad\qquad \begin{aligned} \text{Gr-}R &\longrightarrow \text{Gr-}A_R\text{-Gr}, \\ N &\longmapsto N \otimes_R A \end{aligned}$$

are inverse equivalence of categories which graded projective modules correspond to graded projective modules and furthermore, invertible modules correspond to invertible modules (see [24, Proposition III.4.1]). This immediately implies that
$$\operatorname{Pic}^{\mathrm{gr}}_R(A) \cong \operatorname{Pic}^{\mathrm{gr}}(R). \tag{4.10}$$

Lemma 4.2.4. *Let A be a Γ-graded division algebra with centre R. Then $\operatorname{Pic}^{\mathrm{gr}}_R(A) \cong \Gamma/\Gamma_R$.*

Proof. By Proposition 4.2.3, A is a graded Azumaya algebra over R. By (4.10), $\operatorname{Pic}^{\mathrm{gr}}_R(A) \cong \operatorname{Pic}^{\mathrm{gr}}(R)$. Since R is a graded field, by Proposition 4.1.1, $\operatorname{Pic}^{\mathrm{gr}}(R) = \Gamma/\Gamma_R$. □

Example 4.2.5. Recall from Example 1.1.21 that the Hamilton quaternion algebra $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ is a graded division algebra with two different gradings, i.e., $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$, respectively. Since in both gradings, the centre $\mathbb{R}$ is concentrated in degree 0, by Lemma 4.2.4, $\operatorname{Pic}^{\mathrm{gr}}_{\mathbb{R}}(\mathbb{H}) = \mathbb{Z}_2$ or $\operatorname{Pic}^{\mathrm{gr}}_{\mathbb{R}}(\mathbb{H}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, depending on the grading. This also shows that contrary to the graded Grothendieck group, $\operatorname{Pic}^{\mathrm{gr}}_{\mathbb{R}}(\mathbb{H}) \not\cong \operatorname{Pic}(\mathbb{H}_0)$, despite the fact that in both gradings, $\mathbb{H}$ is a strongly graded ring.

Example 4.2.6. Let (D, v) be a tame valued division algebra over a henselian field F, where $v : D^* \to \Gamma$ is the valuation homomorphism. By Example 1.4.7, there is a Γ-graded division algebra $\operatorname{gr}(D)$ associated to D with the centre $\operatorname{gr}(F)$, where $\Gamma_{\operatorname{gr}(D)} = \Gamma_D$ and $\Gamma_{\operatorname{gr}(F)} = \Gamma_F$. Since by Lemma 4.2.3, graded division algebras are graded Azumaya, by Lemma 4.2.4,
$$\operatorname{Pic}^{\mathrm{gr}}_{\operatorname{gr}(F)}\big(\operatorname{gr}(D)\big) = \Gamma/\Gamma_F.$$

Let A be a strongly graded Γ-graded ring. Then for any $\alpha \in \Gamma$, A_α is a finitely generated projective invertible A_0-bimodule (see Theorem 1.5.11) and the map $\psi : \Gamma \to \operatorname{Pic}(A_0)$, $\alpha \mapsto [A_\alpha]$ is a group homomorphism. We then have a natural commutative diagram

$$\begin{tikzcd} \Gamma \arrow[r, "\phi"] \arrow[dr, "\psi"'] & \operatorname{Pic}^{\mathrm{gr}}(A) \arrow[d, "(-)_0"] \\ & \operatorname{Pic}(A_0) \end{tikzcd} \tag{4.11}$$

Remark 4.2.7. <u>Relating $\operatorname{Pic}^{\mathrm{gr}}(A)$ to $\operatorname{Pic}(A_0)$ for a strongly graded ring A</u>

Example 4.2.5 shows that for a strongly graded ring A, $\operatorname{Pic}^{\mathrm{gr}}(A)$ is not necessarily isomorphic to $\operatorname{Pic}(A_0)$. However one can relate these two groups with an exact sequence as follows.
$$1 \longrightarrow H^1(\Gamma, Z(A_0)^*) \longrightarrow \operatorname{Pic}^{\mathrm{gr}}(A) \xrightarrow{\Psi} \operatorname{Pic}(A_0)^\Gamma \longrightarrow H^2(\Gamma, Z(A_0)^*).$$
Here Γ acts on $\operatorname{Pic}(A_0)$ by $\gamma[P] = [A_\gamma \otimes_{A_0} P \otimes_{A_0} A_{-\gamma}]$, so the notation $\operatorname{Pic}(A_0)^\Gamma$ refers to the group of Γ-invariant elements of $\operatorname{Pic}(A_0)$. Furthermore, $Z(A_0)^*$ denotes the units of the centre of A_0 and H^1, H^2 denote the first and second cohomology groups. The map Ψ is defined by $\Psi([P]) = [P_0]$ (see [67] for details. Also see [16]).

We include a result from [75] which describes if A_0 has IBN, then A has gr-IBN, based on a condition on $\operatorname{Pic}(A_0)$. In general, one can produce an example of strongly graded ring A such that A_0 has IBN whereas A is a non-IBN ring.

For a ring R, define $\operatorname{Pic}_n(R) = \big\{[X] \in \operatorname{Pic}(R) \mid X^n \cong R^n \text{ as right } R\text{-module}\big\}$. This is a subgroup of $\operatorname{Pic}(R)$ and $\operatorname{Pic}_n(R)$ and $\operatorname{Pic}_m(R)$ are subgroups of $\operatorname{Pic}_{nm}(R)$. Thus $\operatorname{Pic}_\infty(R) = \bigcup_{n\geq 1}\operatorname{Pic}_n(R)$ is a subgroup of $\operatorname{Pic}(R)$.

Proposition 4.2.8. *Let A be a strongly Γ-graded ring such that $\{[A_\alpha] \mid \alpha \in \Gamma\} \subseteq \operatorname{Pic}_\infty(A_0)$. If A_0 has IBN then A has gr-IBN. Furthermore, if Γ is finite, then A has IBN if and only if A_0 has IBN.*

Proof. Let $A^n(\overline{\alpha}) \cong_{\text{gr}} A^m(\overline{\beta})$ as graded right A-modules, where $\overline{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $\overline{\beta} = (\beta_1, \dots, \beta_m)$. Using Dade's theorem 1.5.1, we have

$$A_{\alpha_1} \oplus \cdots \oplus A_{\alpha_n} \cong A_{\beta_1} \oplus \cdots \oplus A_{\beta_m}, \tag{4.12}$$

as right A_0-modules. Since each $[A_{\alpha_i}]$ and $[A_{\beta_j}]$ is in some $\operatorname{Pic}_{n_i}(A_0)$ and $\operatorname{Pic}_{n_j}(A_0)$, respectively, there is a large enough t such that $[A_{\alpha_i}], [A_{\beta_j}] \in \operatorname{Pic}_t(A_0)$, for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Thus $A_{\alpha_i} \cong A_0^t$ and $A_{\beta_j} \cong A_0^t$, for all i and j. Replacing this into (4.12) we get $A_0^{nt} \cong A_0^{mt}$. Since A_0 has IBN, it follows $n = m$.

Now suppose Γ is finite. If A_0 has IBN, then we will prove that A has IBN. From what we proved above, it follows that A has gr-IBN. If $A^n \cong A^m$, then $U(A^n) \cong_{\text{gr}} U(A^m)$, where U is the forgetful functor (see §1.2.7). Since $U(A^n) = U(A)^n$ and $U(A) = \bigoplus_{\gamma\in\Gamma} A(\gamma)$, using the fact that A has gr-IBN, it follows immediately that $n = m$. Conversely, suppose A has IBN. If $A_0^n \cong A_0^m$, then tensoring with A over A_0 we have $A^n \cong_{\text{gr}} A_0^n \otimes_{A_0} A \cong A_0^m \otimes_{A_0} A \cong_{\text{gr}} A^m$. So $n = m$ and we are done. $\square$

Chapter 5

Graded Ultramatricial Algebras, Classification via K_0^{gr}

Let F be a field. An F-algebra is called an *ultramatricial* algebra, if it is isomorphic to the union of an increasing chain of a finite product of matrix algebras over F. When F is the field of complex numbers, these algebras are also called *locally semisimple* algebras (or LS-algebras for short), as they are isomorphic to a union of a chain of semisimple $\mathbb{C}$-algebras. An important example of such rings is the group ring $\mathbb{C}[S_\infty]$, where S_∞ is the infinite symmetric group (see Example 5.2.2). These rings appeared in the setting of C^*-algebras and then von Neumann regular algebras in the work of Grimm, Brattelli, Elliott, Goodearl, Handelman and many others after them.

Despite their simple constructions, the study of ultramatricial algebras are far from over. As it is noted in [90] "The current state of the theory of LS-algebras and its applications should be considered as the initial one; one has discovered the first fundamental facts and noted a general circle of questions. To estimate it in perspective, one must consider the enormous number of diverse and profound examples of such algebras. In addition one can observe the connections with a large number of areas of mathematics."

One of the sparkling examples of the Grothendieck group as a complete invariant is in the setting of ultramatricial algebras (and AF C^*-algebras). It is by now a classical result that the K_0-group along with its positive cone and the position of identity is a complete invariant for such algebras ([38, Theorem 15.26]). To be precise, let R and S be (unital) ultramatricial algebras. Then $R \cong S$ if and only if there is an order isomorphism $(K_0(R), [R]) \cong (K_0(S), [S])$.

The theory has also been worked out for the non-unital ultramatricial algebras (see [39, Chapter XII]). Two valuable surveys on ultramatricial algebras and their relations with other branches of mathematics are [90, 97]. The lecture notes by Effros [35] is also an excellent detailed account of this theory.

In this section we initiate the graded version of this theory. We define the graded ultramatricial algebras. We then show that the graded Grothendieck group, equipped with its module structure and its ordering is a complete invariant for such algebras (Theorem 5.2.5). Again, when the graded group considered to be trivial, we retrieve the Brattelli-Elliott Theorem (Corollary 5.2.7).

5.1 Graded matricial algebras

We begin with the graded version of matricial algebras.

Definition 5.1.1. Let A be a Γ-graded field. A Γ-*graded matricial* A-algebra is a graded A-algebra of the form

$$\mathbb{M}_{n_1}(A)(\overline{\delta}_1) \times \cdots \times \mathbb{M}_{n_l}(A)(\overline{\delta}_l),$$

where $\overline{\delta}_i = (\delta_1^{(i)}, \ldots, \delta_{n_i}^{(i)})$, $\delta_j^{(i)} \in \Gamma$, $1 \leq j \leq n_i$ and $1 \leq i \leq l$.

If Γ is a trivial group, then Definition 5.1.1 reduces to the definition of matricial algebras ([38, §15]). Note that if R is a graded matricial A-algebra, then R_0 is a matricial A_0-algebra (see §1.4.1).

In general, when two graded finitely generated projective modules represent the same element in the graded Grothendieck group, then they are not necessarily graded isomorphic but rather graded stably isomorphic (see Lemma 3.1.7(3)). However for graded matricial algebras one can prove that the graded stably isomorphic modules are in fact graded isomorphic. Later in Lemma 5.1.5 we show this is also the case in the larger category of graded ultramatricial algebras.

Lemma 5.1.2. *Let A be a Γ-graded field and R be a Γ-graded matricial A-algebra. Let P and Q be graded finitely generated projective R-modules. Then $[P] = [Q]$ in $K_0^{\mathrm{gr}}(R)$, if and only if $P \cong_{\mathrm{gr}} Q$.*

Proof. Since the functor K_0^{gr} respects the direct sum, it suffices to prove the statement for a graded matricial algebra of the form $R = \mathbb{M}_n(A)(\overline{\delta})$. Let P and Q be graded finitely generated projective R-modules such that $[P] = [Q]$ in $K_0^{\mathrm{gr}}(R)$. By Proposition 2.1.2, $R = \mathbb{M}_n(A)(\overline{\delta})$ is graded Morita equivalent to A. So there are equivalent functors ψ and ϕ such that $\psi\phi \cong 1$ and $\phi\psi \cong 1$, which also induce an isomorphism $K_0^{\mathrm{gr}}(\psi) : K_0^{\mathrm{gr}}(R) \to K_0^{\mathrm{gr}}(A)$ such that $[P] \mapsto [\psi(P)]$. Now since $[P] = [Q]$, it follows $[\psi(P)] = [\psi(Q)]$ in $K_0^{\mathrm{gr}}(A)$. But since A is a graded field, by the proof of Proposition 3.6.1, any graded finitely generated projective A-module can be written uniquely as a direct sum of appropriate shifted A. Writing $\psi(P)$ and $\psi(Q)$ in this form, the homomorphism (3.30) shows that $\psi(P) \cong_{\mathrm{gr}} \psi(Q)$. Now applying the functor ϕ to this we obtain $P \cong_{\mathrm{gr}} Q$. ☐

Let A be a Γ-graded field (with the support Γ_A) and $\mathcal{C}$ be a category consisting of Γ-graded matricial A-algebras as objects and A-graded algebra homomorphisms as morphisms. We consider the quotient category $\mathcal{C}^{\mathrm{out}}$ obtained from $\mathcal{C}$ by identifying homomorphisms which differ up to a degree zero graded inner-automorphim. That is, the graded homomorphisms $\phi, \psi \in \mathrm{Hom}_{\mathcal{C}}(R, S)$ represent the same morphism in $\mathcal{C}^{\mathrm{out}}$ if there is an inner-automorpshim $\theta : S \to S$, defined by $\theta(s) = xsx^{-1}$, where $\deg(x) = 0$, such that $\phi = \theta\psi$. The following theorem shows that K_0^{gr} "classifies" the category of $\mathcal{C}^{\mathrm{out}}$. This is a graded analog of a similar result for matricial algebras (see [38, Lemma 15.23]).

Theorem 5.1.3. *Let A be a Γ-graded field and $\mathcal{C}^{\mathrm{out}}$ be the category consisting of Γ-graded matricial A-algebras as objects and A-graded algebra homomorphisms modulo graded inner-automorphisms as morphisms. Then $K_0^{\mathrm{gr}} : \mathcal{C}^{\mathrm{out}} \to \mathcal{P}$ is a fully faithful functor. Namely,*

(1) *(well-defined and faithful) For any graded matricial A-algebras R and S and $\phi, \psi \in \mathrm{Hom}_{\mathcal{C}}(R, S)$, we have $\phi(r) = x\psi(r)x^{-1}$, $r \in R$, for some invertible homogeneous element x of degree 0 in S, if and only if $K_0^{\mathrm{gr}}(\phi) = K_0^{\mathrm{gr}}(\psi)$.*

(2) *(full) For any graded matricial A-algebras R and S and the morphism*

$$f : (K_0^{\mathrm{gr}}(R), [R]) \to (K_0^{\mathrm{gr}}(S), [S])$$

in $\mathcal{P}$, there is $\phi \in \mathrm{Hom}_{\mathcal{C}}(R, S)$ such that $K_0^{\mathrm{gr}}(\phi) = f$.

Proof. (1) (well-defined.) Let $\phi, \psi \in \mathrm{Hom}_{\mathcal{C}}(R,S)$ such that $\phi = \theta\psi$, where $\theta(s) = xsx^{-1}$ for some invertible homogeneous element x of S of degree 0. Then $K_0^{\mathrm{gr}}(\phi) = K_0^{\mathrm{gr}}(\theta\psi) = K_0^{\mathrm{gr}}(\theta)K_0^{\mathrm{gr}}(\psi) = K_0^{\mathrm{gr}}(\psi)$ since $K_0^{\mathrm{gr}}(\theta)$ is the identity map (see §3.4.1).

(faithful.) The rest of the proof is similar to the non-graded version with an extra attension given to the grading (cf. [38, p.218]). We give it here for the completeness. Let $K_0^{\mathrm{gr}}(\phi) = K_0^{\mathrm{gr}}(\psi)$. Let $R = \mathbb{M}_{n_1}(A)(\overline{\delta}_1) \times \cdots \times \mathbb{M}_{n_l}(A)(\overline{\delta}_l)$ and set $g_{jk}^{(i)} = \phi(e_{jk}^{(i)})$ and $h_{jk}^{(i)} = \psi(e_{jk}^{(i)})$ for $1 \leq i \leq l$ and $1 \leq j,k \leq n_i$, where $e_{jk}^{(i)}$ are the standard basis for $\mathbb{M}_{n_i}(A)$. Since ϕ and ψ are graded homomorphism, $\deg(e_{jk}^{(i)}) = \deg(g_{jk}^{(i)}) = \deg(h_{jk}^{(i)}) = \delta_j^{(i)} - \delta_k^{(i)}$.

Since $e_{jj}^{(i)}$ are pairwise graded orthogonal idempotents (of degree zero) in R and $\sum_{i=1}^{l}\sum_{j=1}^{n_i} e_{jj}^{(i)} = 1$, the same is also the case for $g_{jj}^{(i)}$ and $h_{jj}^{(i)}$. Then

$$[g_{11}^{(i)}S] = K_0^{\mathrm{gr}}(\phi)([e_{11}^{(i)}R]) = K_0^{\mathrm{gr}}(\psi)([e_{11}^{(i)}R]) = [h_{11}^{(i)}S].$$

By Lemma 5.1.2, $g_{11}^{(i)}S \cong_{\mathrm{gr}} h_{11}^{(i)}S$. Thus there are homogeneous elements of degree zero x_i and y_i such that $x_iy_i = g_{11}^{(i)}$ and $y_ix_i = h_{11}^{(i)}$ (see §1.2.10). Let $x = \sum_{i=1}^{l}\sum_{j=1}^{n_i} g_{j1}^{(i)}x_ih_{1j}^{(i)}$ and $y = \sum_{i=1}^{l}\sum_{j=1}^{n_i} h_{j1}^{(i)}y_ig_{1j}^{(i)}$. Note that x and y are homogeneous elements of degree zero. One can easily check that $xy = yx = 1$. Now for $1 \leq i \leq l$ and $1 \leq j,k \leq n_i$ we have

$$\begin{aligned} xh_{jk}^{(i)} &= \sum_{s=1}^{l}\sum_{t=1}^{n_s} g_{t1}^{(s)}x_sh_{1t}^{(s)}h_{jk}^{(i)} \\ &= g_{j1}^{(i)}x_ih_{1j}^{(i)}h_{jk}^{(i)} = g_{jk}^{(i)}g_{k1}^{(i)}x_ih_{1k}^{(i)} \\ &= \sum_{s=1}^{l}\sum_{t=1}^{n_s} g_{jk}^{(i)}g_{t1}^{(s)}x_sh_{1t}^{(i)} = g_{jk}^{(i)}x \end{aligned}$$

Let $\theta : S \to S$ be the graded inner-automorphism $\theta(s) = xsy$. Then

$$\theta\psi(e_{jk}^{(i)})) = xh_{jk}^{(i)}y = g_{jk}^{(i)} = \phi(e_{jk}^{(i)}).$$

Since $e_{jk}^{(i)}$, $1 \leq i \leq l$ and $1 \leq j,k \leq n_i$, form a homogeneous A-basis for R, $\theta\psi = \phi$.

(2) Let $R = \mathbb{M}_{n_1}(A)(\overline{\delta}_1) \times \cdots \times \mathbb{M}_{n_l}(A)(\overline{\delta}_l)$. Consider $R_i = \mathbb{M}_{n_i}(A)(\overline{\delta}_i)$, $1 \leq i \leq l$. Each R_i is a graded finitely generated projective R-module, so $f([R_i])$ is in the cone of $K_0^{\mathrm{gr}}(S)$, i.e., there is a graded finitely generated projective S-module P_i such that $f([R_i]) = [P_i]$. Then

$$[S] = f([R]) = f([R_1] + \cdots + [R_l]) = [P_1] + \cdots + [P_l] = [P_1 \oplus \cdots \oplus P_l].$$

Since S is a graded matricial algebra, by Lemma 5.1.2, $P_1 \oplus \cdots \oplus P_l \cong_{\mathrm{gr}} S$ as a right S-module. So there are homogeneous orthogonal idempotents $g_1, \dots, g_l \in S$ such that $g_1 + \cdots + g_l = 1$ and $g_iS \cong_{\mathrm{gr}} P_i$ (see §1.2.10). Note that each of $R_i = \mathbb{M}_{n_i}(A)(\overline{\delta}_i)$ are graded simple algebras. Set $\overline{\delta}_i = (\delta_1^{(i)}, \dots, \delta_n^{(i)})$ (here $n = n_i$). Let $e_{jk}^{(i)}$,$1 \leq j,k \leq n$, be matrix units of R_i and consider the graded finitely generated projective (right) R_i-module, $V = e_{11}^{(i)}R_i = A(\delta_1^{(i)} - \delta_1^{(i)}) \oplus A(\delta_2^{(i)} - \delta_1^{(i)}) \oplus \cdots \oplus A(\delta_n^{(i)} - \delta_1^{(i)})$. Then (1.33) shows

$$R_i \cong_{\mathrm{gr}} V(\delta_1^{(i)} - \delta_1^{(i)}) \oplus V(\delta_1^{(i)} - \delta_2^{(i)}) \oplus \cdots \oplus V(\delta_1^{(i)} - \delta_n^{(i)}),$$

as graded R-module. Thus

$$[P_i] = [g_iS] = f([R_i]) = f([V(\delta_1^{(i)} - \delta_1^{(i)})]) + f([V(\delta_1^{(i)} - \delta_2^{(i)})]) + \cdots + f([V(\delta_1^{(i)} - \delta_n^{(i)})]). \tag{5.1}$$

There is a graded finitely generated projective S-module Q such that $f([V]) = f([V(\delta_1^{(i)} - \delta_1^{(i)})]) = [Q]$. Since f is a $\mathbb{Z}[\Gamma]$-module homomorphism, for $1 \leq k \leq n$,

$$f([V(\delta_1^{(i)} - \delta_k^{(i)})]) = f((\delta_1^{(i)} - \delta_k^{(i)})[V]) = (\delta_1^{(i)} - \delta_k^{(i)})f([V]) = (\delta_1^{(i)} - \delta_k^{(i)})[Q] = [Q(\delta_1^{(i)} - \delta_k^{(i)})].$$

From (5.1) and Lemma 5.1.2 now follows

$$g_iS \cong_{\text{gr}} Q(\delta_1^{(i)} - \delta_1^{(i)}) \oplus Q(\delta_1^{(i)} - \delta_2^{(i)}) \oplus \cdots \oplus Q(\delta_1^{(i)} - \delta_n^{(i)}). \tag{5.2}$$

Let $g_{jk}^{(i)} \in \operatorname{End}(g_iS) \cong_{\text{gr}} g_iSg_i$ maps the j-th summand of the right hand side of (5.2) to its k-th summand and everything else to zero. Observe that $\deg(g_{jk}^{(i)}) = \delta_j^{(i)} - \delta_k^{(i)}$ and $g_{jk}^{(i)}$, $1 \leq j,k \leq n$, form the matrix units. Furthermore, $g_{11}^{(i)} + \cdots + g_{nn}^{(i)} = g_i$ and $g_{11}^{(i)}S = Q(\delta_1^{(i)} - \delta_1^{(i)}) = Q$. Thus $[g_{11}^{(i)}S] = [Q] = f([V]) = f([e_{11}^{(i)}R_i])$.

Now for $1 \leq i \leq l$, define the A-algebra homomorphism $R_i \to g_iSg_i$, $e_{jk}^{(i)} \mapsto g_{jk}^{(i)}$. This is a graded homomorphism, and induces a graded homorphism $\phi : R \to S$ such that $\phi(e_{jk}^{(i)}) = g_{jk}^{(i)}$. Clearly

$$K_0^{\text{gr}}(\phi)([e_{11}^{(i)}R_i]) = [\phi(e_{11}^{(i)})S] = [g_{11}^{(i)}S] = f([e_{11}^{(i)}R_i]),$$

for $1 \leq i \leq l$. Now $K_0^{\text{gr}}(R)$ is generated by $[e_{11}^{(i)}R_i]$, $1 \leq i \leq l$, as $\mathbb{Z}[\Gamma]$-module. This implies that $K_0^{\text{gr}}(\phi) = f$. $\square$

Remark 5.1.4. In Theorem 5.1.3, both parts (1) and (2) are valid when the ring S is a graded ultramatricial algebra. In fact, in the proofs of (1) and (2), the only property of S which is used is that if for two graded finitely generated projective S-modules P and Q, we have $[P] = [Q]$ in $K_0^{\text{gr}}(S)$, then $P \cong_{\text{gr}} Q$. Lemma 5.1.5 shows that this is the case for ultramatricial algebras.

Lemma 5.1.5. *Let A be a Γ-graded field and R be a Γ-graded ultramatricial A-algebra. Let P and Q be graded finitely generated projective R-modules. Then $[P] = [Q]$ in $K_0^{\text{gr}}(R)$, if and only if $P \cong_{\text{gr}} Q$.*

Proof. We will use the description of K_0^{gr} based on homogeneous idempotents to prove this Lemma (see §3.2). Let p and q be homogeneous idempotents matrices over R corresponding to the graded finitely generated R-modules P and Q, respectively (see Lemma 3.2.3(2)). Suppose $[P] = [Q]$ in $K_0^{\text{gr}}(R)$. We will show that p and q are graded equivalent in R, which by Lemma 3.2.3(3), implies that $P \cong_{\text{gr}} Q$. Since by Definition 5.2.1, $R = \bigcup_{i \in I} R_i$, there is $j \in I$ such that p and q are homogeneous idempotent matrices over R_j. But since $[p] = [q]$ in $K_0^{\text{gr}}(R)$, there is an $n \in \mathbb{N}$, such that $p \oplus 1_n$ is graded equivalent to $q \oplus 1_n$ in R (see §3.2.1). So there is $k \in I$, $k \geq j$, such that $p \oplus 1_n$ and $q \oplus 1_n$ are graded equivalent in R_k. Thus $[p \oplus 1_{R_k}] = [q \oplus 1_{R_k}]$ in $K_0^{\text{gr}}(R_k)$. This implies $[p] = [q]$ in $K_0^{\text{gr}}(R_k)$. Since R_k is a graded matricial algebra, by Lemma 5.1.5, p is graded equivalent to q in R_k. So p is graded equivalent to q in R and consequently $P \cong_{\text{gr}} Q$ as R-module. The converse is immediate. $\square$

5.2 Graded ultramatricial algebras, classification via K_0^{gr}

The direct limit of a direct system of graded rings is a ring with a graded structure (see Example 1.1.10). In this section, we study a particular case of such graded rings, namely the direct limit of graded matricial algebras. Recall from Definition 5.1.1 that a Γ-graded matricial algebra over the graded field A is of the form

$$\mathbb{M}_{n_1}(A)(\overline{\delta}_1) \times \cdots \times \mathbb{M}_{n_l}(A)(\overline{\delta}_l),$$

for some shifts $\overline{\delta}_i$, $1 \leq i \leq l$.

Definition 5.2.1. Let A be a Γ-graded field. Then the ring R is called a Γ-*graded ultramatricial* A-algebra if $R = \bigcup_{i=1}^{\infty} R_i$, where $R_1 \subseteq R_2 \subseteq \dots$ is a sequence of graded matricial A-subalgebras. Here the inclusion respects the grading, i.e., $R_{i\alpha} \subseteq R_{i+1\alpha}$. Furthermore, under the inclusion $R_i \subseteq R_{i+1}$, we have $1_{R_i} = 1_{R_{i+1}}$.

Clearly R is also a Γ-graded A-algebra with $R_\alpha = \bigcup_{i=1}^{\infty} R_{i\alpha}$. If Γ is a trivial group, then Definition 5.2.1 reduces to the definition of ultramatricial algebras ([38, §15]). Note that if R is a graded ultramatricial A-algebra, then R_0 is a ultramatricial A_0-algebra (see §1.4.1).

Example 5.2.2. Let $\mathbb{C}$ be a field of complex numbers. Furthermore let $\Gamma_1 \subseteq \Gamma_2 \subseteq \dots$ be a chain of finite abelian groups. Consider the *locally finite* group $\Gamma := \bigcup_{i=1}^{\infty} \Gamma_i$. Consider $\mathbb{C}$ as Γ-graded field concentrated in degree 0. Then the group ring $\mathbb{C}[\Gamma]$ is a Γ-graded ultramatricial $\mathbb{C}$-algebra. The construction is as follows. For each i, the group ring $\mathbb{C}[\Gamma_i]$ is considered as a Γ-graded ring with the support Γ_i (with the standard grading of group rings as in Example 1.1.7). Then the inclusion $\mathbb{C}[\Gamma_i] \subseteq \mathbb{C}[\Gamma_{i+1}]$ gives a sequence of graded $\mathbb{C}$-algebras. By Maschke's theorem, each of $\mathbb{C}[\Gamma_i]$ is a matricial algebra with the grading coming from the group ring. This makes $\mathbb{C}[\Gamma] = \bigcup_i \mathbb{C}[\Gamma_i]$ a graded ultramatricial $\mathbb{C}$-algebra.

Example 5.2.3. Let A be a ring. We identify $\mathbb{M}_n(A)$ as a subring of $\mathbb{M}_{2n}(A)$ under the monomorphism $X \in \mathbb{M}_n(A) \mapsto \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}$. With this identification, we have a sequence

$$\mathbb{M}_2(A) \subseteq \mathbb{M}_4(A) \subseteq \dots.$$

Now let A be a Γ-graded field, $\alpha_1, \alpha_2 \in \Gamma$ and consider the sequence of graded subalgebras

$$\mathbb{M}_2(A)(\alpha_1,\alpha_2) \subseteq \mathbb{M}_4(A)(\alpha_1,\alpha_2,\alpha_1,\alpha_2) \subseteq \dots.$$

Then

$$R = \bigcup_{i=1}^{\infty} \mathbb{M}_{2^i}(A)\big((\alpha_1,\alpha_2)^{2^{i-1}}\big),$$

where $(\alpha_1,\alpha_2)^k$ stands for k copies of (α_1,α_2), is a graded ultramatricial algebra.

Example 5.2.4. Let $A = K[x^3, x^{-3}]$ be a $\mathbb{Z}$-graded field, where K is a field. Consider the following two sequences of graded matricial algebras

$$\mathbb{M}_2(A)(0,1) \subseteq \mathbb{M}_4(A)(0,1,0,1) \subseteq \mathbb{M}_8(A)(0,1,0,1,0,1,0,1) \subseteq \dots \tag{5.3}$$

$$\mathbb{M}_2(A)(0,1) \subseteq \mathbb{M}_4(A)(0,1,1,2) \subseteq \mathbb{M}_8(A)(0,1,1,2,0,1,1,2) \subseteq \dots \tag{5.4}$$

Let R and S be graded ultramatricial algebras constructed as in Example 5.2.3, from the union of matricial algebras of sequences 5.3 and 5.4, respectively. We calculate $K_0^{\mathrm{gr}}(R)$ and $K_0^{\mathrm{gr}}(S)$. Since K_0^{gr} respects the direct limit (Theorem 3.2.4), we have

$$K_0^{\mathrm{gr}}(R) = K_0^{\mathrm{gr}}(\varinjlim R_i) = \varinjlim K_0^{\mathrm{gr}}(R_i),$$

where R_i corresponds to the i-th algebra in the sequence 5.3. Since all the matricial algebras R_i are strongly graded, we get $\varinjlim K_0^{\mathrm{gr}}(R_i) = \varinjlim K_0(R_{i_0})$.

Recall that (see Proposition 1.4.1) if

$$T \cong_{\mathrm{gr}} \mathbb{M}_m\big(K[x^n, x^{-n}]\big)(p_1, \dots, p_m),$$

then letting d_l, $0 \le l \le n-1$, be the number of i such that p_i represents $\bar{l}$ in $\mathbb{Z}/n\mathbb{Z}$, we have

$$T \cong_{\mathrm{gr}} \mathbb{M}_m\big(K[x^n, x^{-n}]\big)(0, \dots, 0, 1, \dots, 1, \dots, n-1, \dots, n-1),$$

where $0 \le l \le n-1$ occurring d_l times. It is now easy to see

$$T_0 \cong \mathbb{M}_{d_0}(K) \times \cdots \times \mathbb{M}_{d_{n-1}}(K).$$

Using this the zero homogeneous ring of the sequence 5.3 becomes

$$\begin{aligned}
&K \oplus K \subseteq \mathbb{M}_2(K) \oplus \mathbb{M}_2(K) \subseteq \cdots \\
&(x, y) \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \oplus \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} \mapsto \cdots
\end{aligned}$$

Thus

$$K_0^{\mathrm{gr}}(R) \cong \varinjlim K_0(R_{i_0}) \cong \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/2].$$

Similarly, the zero homogeneous ring of the sequence 5.4 becomes

$$\begin{aligned}
&K \oplus K \subseteq K \oplus \mathbb{M}_2(K) \oplus K \subseteq \mathbb{M}_2(K) \oplus \mathbb{M}_4(K) \oplus \mathbb{M}_2(K) \\
&(x, y) \mapsto (x, \begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}, y) \\
&\qquad\qquad (x, y, z) \mapsto (\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}).
\end{aligned}$$

Thus

$$K_0^{\mathrm{gr}}(S) \cong \varinjlim K_0(S_{i_0}) \cong \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/2].$$

We are now in a position to classify the graded ultramatricial algebras via K_0^{gr}-group. The following theorem shows that the *dimension module* is a complete invariant for the category of graded ultramatricial algebras.

Theorem 5.2.5. *Let R and S be Γ-graded ultramatricial algebras over a graded field A. Then $R \cong_{\mathrm{gr}} S$ as graded A-algebras if and only if there is an order preserving $\mathbb{Z}[\Gamma]$-module isomorphism*

$$\big(K_0^{\mathrm{gr}}(R), [R]\big) \cong \big(K_0^{\mathrm{gr}}(S), [S]\big).$$

Proof. One direction is clear. Let $R = \bigcup_{i=1}^{\infty} R_i$ and $S = \bigcup_{i=1}^{\infty} S_i$, where $R_1 \subseteq R_2 \subseteq \dots$ and $S_1 \subseteq S_2 \subseteq \dots$ are sequences of graded matricial A-algebras. Let $\phi_i : R_i \to R$ and $\psi_i : S_i \to S$, $i \in \mathbb{N}$, be inclusion maps. In order to show that the isomorphism

$$f : \big(K_0^{\mathrm{gr}}(R), [R]\big) \longrightarrow \big(K_0^{\mathrm{gr}}(S), [S]\big)$$

between the graded Grothendieck groups gives the isomorphism between the rings R and S, we will find a sequence $n_1 < n_2 < \cdots$ of positive numbers and graded A-module injections $\rho_k : R_{n_k} \to S$ such that ρ_{k+1} is an extension of ρ_k and $\bigcup_{k=1}^{\infty} R_{n_k} = R$. To achieve this we repeatedly use Theorem 5.1.3 (and Remark 5.1.4) (a "local" version of this theorem) and the fact that since R_n, $n \in \mathbb{N}$, is a finite dimensional A-algebra, for any A-graded homomorphism $\rho : R_n \to S$, we have $\rho(R_n) \subseteq S_i$, for some positive number i. Throughout the proof, for simplicity, we write $\overline{\theta}$ for the $\mathbb{Z}[\Gamma]$-homomorphism $K_0^{\mathrm{gr}}(\theta)$ induced by a graded A-algebra homomorphism $\theta : R \to S$.

We first prove two auxiliary facts.

I. If $\sigma : S_k \to R_n$ is a graded A-algebra such that

$$\overline{\phi}_n \overline{\sigma} = \overline{f}^{-1} \overline{\psi}_k, \tag{5.5}$$

(see Diagram 5.6) then there exist an integer $j > k$ and a graded A-algebra homomorphism $\rho : R_n \to S_j$ such that $\psi_j \rho \sigma = \psi_k$ and $\overline{\psi}_j \overline{\rho} = \overline{f}\, \overline{\phi}_n$.

$$\begin{tikzcd}
S_k \arrow[rr, "\sigma"] \arrow[dd, "\psi_k"'] & & R_n \arrow[dl, dotted, "\rho"] \arrow[dd, "\phi_n"] \\
 & S_j \arrow[dl, dotted, "\psi_j"] & \\
S & & R.
\end{tikzcd}
\qquad
\begin{tikzcd}
K_0^{\mathrm{gr}}(S_k) \arrow[rr, "\overline{\sigma}"] \arrow[dd, "\overline{\psi}_k"'] & & K_0^{\mathrm{gr}}(R_n) \arrow[dl, dotted, "\overline{\rho}"] \arrow[dd, "\overline{\phi}_n"] \\
 & K_0^{\mathrm{gr}}(S_j) \arrow[dl, dotted, "\overline{\psi}_j"] & \\
K_0^{\mathrm{gr}}(S) \arrow[rr, "\overline{f}^{-1}"] & & K_0^{\mathrm{gr}}(R).
\end{tikzcd} \tag{5.6}$$

Proof of I. Consider $\overline{f}\, \overline{\phi}_n : K_0^{\mathrm{gr}}(R_n) \to K_0^{\mathrm{gr}}(S)$. By Theorem 5.1.3 (and Remark 5.1.4), there is an A-graded homomorphism $\rho' : R_n \to S$ such that $\overline{\rho'} = \overline{f}\, \overline{\phi}_n$. Since R_n (a matricial A-algebra) is a finite dimensional A-algebra, $\rho'(R_n) \subseteq S_i$ for some i. Thus ρ' gives a graded homomorphism $\rho'' : R_n \to S_i$ such that $\psi_i \rho'' = \rho'$ (recall that ψ_i is just an inclusion). Furthermore

$$\overline{\psi}_i \overline{\rho''} = \overline{\rho'} = \overline{f}\, \overline{\phi}_n. \tag{5.7}$$

Then, using (5.5), $\overline{\psi}_i \overline{\rho''} \overline{\sigma} = \overline{f}\, \overline{\phi}_n \overline{\sigma} = \overline{\psi}_k$. Theorem 5.1.3 implies that there is a graded inner automorphism θ of S such that

$$\theta \psi_i \rho'' \sigma = \psi_k. \tag{5.8}$$

The restriction of θ on S_i gives $\theta|_{S_i} : S_i \to S$. Since S_i is finite dimensional A-algebra, it follows that there is j such that $\theta(S_i) \subseteq S_j$. So θ gives a graded homomorphism $\theta' : S_i \to S_j$ such that $\psi_j \theta' = \theta \psi_i$. Set $\rho = \theta' \rho'' : R_n \to S_i$. We get, using (5.8)

$$\psi_j \rho \sigma = \psi_j \theta' \rho'' \sigma = \theta \psi_i \rho'' \sigma = \psi_k.$$

This gives the first part of I. Using Theorem 5.1.3 and (5.7) we have

$$\overline{\psi}_j \overline{\rho} = \overline{\psi}_j \overline{\theta}' \overline{\rho}'' = \overline{\theta}\, \overline{\psi}_i \overline{\rho}'' = \overline{\psi}_i \overline{\rho}'' = \overline{f}\, \overline{\phi}_n.$$

This completes the proof of I.

The second auxiliary fact we need is the to replace R_i's and S_i's in I as follows. The proof is similar to I.

II. If $\rho : R_n \to S_k$ is a graded A-algebra such that

$$\overline{\psi}_k \overline{\rho} = \bar{f}\, \overline{\phi}_n.$$

Then there exist an integer $m > n$ and a graded A-algebra homomorphism $\sigma : S_k \to R_m$ such that $\phi_m \sigma \rho = \phi_n$ and $\overline{\phi}_m \overline{\sigma} = \bar{f}^{-1} \overline{\psi}_k$.

Now we are in a position to construct positive numbers $n_1 < n_2 < \cdots$ and graded A-algebra homomorphisms $\rho_k : R_{n_k} \to S$ such that

1. $S_k \subseteq \rho_k(R_{n_k})$ and $\overline{\rho}_k = \bar{f}\, \overline{\phi}_{n_k}$, for all $k \in \mathbb{N}$.
2. ρ_{k+1} is an extension of ρ_k and for all $k \in \mathbb{N}$, i.e, the following diagram commutes.

$$\begin{tikzcd}
R_{n_k} \arrow[r, "\rho_k"] \arrow[d, hook] & S \\
R_{n_{k+1}} \arrow[ur, "\rho_{k+1}"'] &
\end{tikzcd}$$

Furthermore, ρ_k is injective for all $k \in \mathbb{N}$.

Consider the morphism $\bar{f}^{-1} \overline{\psi}_1 : K_0^{\mathrm{gr}}(S_1) \to K_0^{\mathrm{gr}}(R)$. By Theorem 5.1.3, there is a graded A-algebra homomorphism $\sigma' : S_1 \to R$ such that $\overline{\sigma}' = \bar{f}^{-1} \overline{\psi}_1$. Since S_1 is a finite dimensional A-algebra, $\sigma'(S_1) \subseteq R_{n_1}$ for some positive number n_1. So σ' gives a graded A-algebra homomorphism $\sigma : S_1 \to R_{n_1}$ such that $\phi_{n_1} \sigma = \sigma'$ and $\overline{\phi}_{n_1} \overline{\sigma} = \bar{f}^{-1} \overline{\psi}_1$. Thus σ satisfies the conditions of part I. Therefore there is $j > 1$ and a graded A-algebra homomorphism $\rho : R_{n_1} \to S_j$ (see Diagram 5.9) such that $\psi_j \rho \sigma = \psi_1$ and $\overline{\psi}_j \overline{\rho} = \bar{f}\, \overline{\phi}_{n_1}$.

$$\begin{tikzcd}
S_1 \arrow[rr, "\sigma"] \arrow[dd, "\psi_1"'] & & R_{n_1} \arrow[dl, dotted, "\rho"] \\
 & S_j \arrow[dl, dotted, "\psi_j"] & \\
S & &
\end{tikzcd} \tag{5.9}$$

So $\rho_1 = \psi_j \rho : R_{n_1} \to S$ is a graded A-homomorphism such that $\rho_1 \sigma = \psi_1$ and $\overline{\rho}_1 = \bar{f}\, \overline{\phi}_{n_1}$. But

$$S_1 = \psi_1(S_1) = \rho_1 \sigma(S_1) \subseteq \rho_1(R_{n_1}).$$

This proves (1). We now proceed by induction. Suppose there are $\{n_1, \dots, n_k\}$ are $\{\rho_1, \dots, \rho_k\}$, for some positive integer k such that (1) and (2) above are satisfied. Since R_{n_k} is a finite dimensional A-algebra, there is $i \geq k+1$ such that $\rho_k(R_{n_k}) \subseteq S_i$. So ρ_k gives a graded A-homomorphism $\rho' : R_{n_k} \to S_i$ such that $\psi_i \rho' = \rho_k$ and $\overline{\psi}_i \overline{\rho'} = \bar{f}\, \overline{\phi}_{n_k}$. By II, there is $n_{k+1} > n_k$ and $\sigma : S_i \to R_{n_{k+1}}$ such that $\phi_{n_{k+1}} \sigma \rho' = \phi_{n_k}$ and $\overline{\phi}_{n_{k+1}} \overline{\sigma} = \bar{f}^{-1} \overline{\phi}_i$. Since $\phi_{n_{k+1}} \sigma \rho' = \phi_{n_k}$, and ϕ_{n_k} (being an inclusion) is injective, ρ' is injective and so $\rho_k = \psi_i \rho'$ is injective. Since $\sigma : S_i \to R_{n_{k+1}}$ satisfies conditions of Part I, there is $j > i$ such that $\rho : R_{n_{k+1}} \to S_j$ such that $\psi_j \rho \sigma = \psi_i$ and $\overline{\psi}_j \overline{\rho} = \bar{f}\, \overline{\phi}_{n_{k+1}}$. Thus for

$\rho_{k+1} = \psi_j \rho : R_{n_{k+1}} \to S$ we have $\overline{\rho}_{k+1} = \overline{\psi}_j \overline{\rho} = \bar{f}\,\overline{\phi}_{n_{k+1}}$. Since $\rho_{k+1}\sigma = \psi_j \rho\sigma = \psi_i$ and $i \geq k+1$, we have

$$S_{k+1} = \psi_i(S_{k+1}) = \rho_{k+1}\sigma(S_{k+1}) \subseteq \rho_{k+1}\sigma(S_i) \subseteq \rho_{k+1}(R_{n_{k+1}}).$$

Finally, since $\phi_{n_{k+1}}\sigma\rho' = \phi_{n_k}$ and $\rho_{k+1}\sigma\rho' = \psi_i\rho' = \rho_k$, it follows that ρ_{k+1} is an extension of ρ_k. Then by induction (1) and (2) follows.

We have $n_1 < n_2 < \cdots$ and $k \leq n_k$. So $\bigcup R_{n_k} = R$. So ρ_k induces an injection $\rho : R \to S$ such that $\rho\phi_{n_k} = \rho_k$, for any k. But

$$S_k \subseteq \rho_k(R_{n_k}) = \rho(R_{n_k}) \subseteq \rho(R).$$

It follows that ρ is an epimorphism as well. This finishes the proof. $\square$

Theorem 5.2.6. *Let R and S be Γ-graded ultramatricial algebras over a graded field A. Then R and S are graded Morita equivalent if and only if there is an order preserving $\mathbb{Z}[\Gamma]$-module isomorphism $K_0^{\mathrm{gr}}(R) \cong K_0^{\mathrm{gr}}(S)$.*

Proof. Let R be graded Morita equivalent to S. Then by Theorem 2.3.6, there is a R-graded progenerator P, such that $S \cong_{\mathrm{gr}} \mathrm{End}_R(P)$. Now using Lemma 3.5.5, we have

$$\big(K_0^{\mathrm{gr}}(S), [S]\big) \cong \big(K_0^{\mathrm{gr}}(R), [P]\big).$$

In particular $K_0^{\mathrm{gr}}(R) \cong K_0^{\mathrm{gr}}(S)$ as ordered $\mathbb{Z}[\Gamma]$-modules. (Note that the ultramatricial assumptions on rings are not used in this direction.)

For the converse, suppose $K_0^{\mathrm{gr}}(S) \cong K_0^{\mathrm{gr}}(R)$ as ordered $\mathbb{Z}[\Gamma]$-modules. Denote the image of $[S] \in K_0^{\mathrm{gr}}(S)$ under this isomorphism by $[P]$. Then

$$\big(K_0^{\mathrm{gr}}(S), [S]\big) \cong \big(K_0^{\mathrm{gr}}(R), [P]\big).$$

Since $[P]$ is an order-unit (3.27), there is $\overline{\alpha} = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in \Gamma$, such that

$$\sum_{i=1}^{n} \alpha_i[P] = [P^n(\overline{\alpha})] \geq [R].$$

This means there is a graded finitely generated projective A-module, Q such that $[P^n(\overline{\alpha})] = [R \oplus Q]$. Since R is a graded ultramatricial algebra, by Lemma 5.1.5, $P^n(\overline{\alpha}) \cong_{\mathrm{gr}} R \oplus Q$. Using Theorem 2.2.2, it follows that P is a graded progenerator. Let $T = \mathrm{End}_R(P)$. Using Lemma 3.5.5, we get an order preserving $\mathbb{Z}[\Gamma]$-module isomorphism

$$\big(K_0^{\mathrm{gr}}(T), [T]\big) \cong \big(K_0^{\mathrm{gr}}(R), [P]\big) \cong \big(K_0^{\mathrm{gr}}(S), [S]\big).$$

Observe that T is also a graded ultramatricial algebras. By the main Theorem 5.2.5, $\mathrm{End}_R(P) = T \cong_{\mathrm{gr}} S$. By Theorem 2.3.6, this implies that R and S are graded Morita equivalent. $\square$

Considering a trivial group as the graded group, we retrieve the classical Brattelli-Elliott Theorem (see the introduction of this chapter).

Corollary 5.2.7 (The Brattelli-Elliott Theorem)**.** *Let R and S be ultramatricial algebras over a field F. Then R and S are Morita equivalent if and only if there is an order preserving isomorphism $K_0(R) \cong K_0(S)$.*

Furthermore, $R \cong S$ as F-algebras if and only if there is an order preserving isomorphism

$$\big(K_0(R), [R]\big) \cong \big(K_0(S), [S]\big).$$

Proof. The corollary follows by considering Γ to be a trivial group in Theorems 5.2.5 and 5.2.6. $\square$

Chapter 6

Graded Versus Non-graded (Higher) K-Theory

Recall that for a Γ-graded ring A, the category of finitely generated Γ-graded projective right A-modules is denoted by $\mathrm{Pgr}^{\Gamma}\text{-}A$. This is an exact category with the usual notion of (split) short exact sequence. Thus one can apply Quillen's Q-construction [79] to obtain K-groups

$$K_i(\mathrm{Pgr}^{\Gamma}\text{-}A),$$

for $i \geq 0$, which we denote by $K_i^{\mathrm{gr}}(A)$. If there are more than one grading involve (which is the case in this chapter) we denote this group by $K_i^{\Gamma}(A)$. The group Γ acts on the category $\mathrm{Pgr}^{\Gamma}\text{-}A$ from the right via $(P,\alpha) \mapsto P(\alpha)$. By functoriality of K-groups this equips $K_i^{\mathrm{gr}}(A)$ with the structure of a right $\mathbb{Z}[\Gamma]$-module.

The relation between graded K-groups and non-graded K-groups is not always apparent. For example consider the $\mathbb{Z}$-graded matrix ring $A = \mathbb{M}_5(K)(0,1,2,2,3)$, where K is a field. Using graded Morita theory one can show that $K_0^{\mathbb{Z}}(A) \cong \mathbb{Z}[x,x^{-1}]$ (see Example 3.6.3), whereas $K_0(A) \cong \mathbb{Z}$ and $K_0(A_0) \cong \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$ (see Example 1.3.5).

In this chapter we describe a theorem due to van den Bergh [19], relating the graded K-theory to non-graded K-theory. In order to do so, we need a generalisation of a result of Quillen on relating K_i^{gr}-groups of a positively graded rings to their non-graded K_i-groups (§6.1). We also need to recall a proof of the fundamental theorem of K-theory, i.e., for a regular Noetherian ring A, $K_i(A[x]) \cong K_i(A)$ (§6.2). We will see that in the proof of this theorem, graded K-theory appears in a crucial way. Putting all these together we can relate the graded K-theory of regular Noetherian $\mathbb{Z}$-graded rings to their non-graded K-theory (§6.3).

As a conseqeunce, we will see that if A is a $\mathbb{Z}$-graded right regular Noetherian ring, then one has an exact sequence (Corollary 6.3.2),

$$K_0^{\mathrm{gr}}(A) \xrightarrow{[P]\mapsto[P(1)]-[P]} K_0^{\mathrm{gr}}(A) \xrightarrow{U} K_0(A) \longrightarrow 0,$$

where U is induced by the forgetful functor (§1.2.7). This shows that

$$K_0^{\mathrm{gr}}(A)/\langle [P]-[P(1)]\rangle \cong K_0(A),$$

where P is a graded finitely generated projective A-module. Further the action of $\mathbb{Z}[x,x^{-1}]$ on the quotient group is trivial. This means, as soon as we discard the shifting in K_0^{gr}, this group reduces to the usual K_0-group.

However this is not in general the case. For example consider the group ring $\mathbb{Z}[G]$, where G is an abelian group. Since $\mathbb{Z}[G]$ has IBN, the canonical homomorphism $\theta : K_0(\mathbb{Z}) \to K_0(\mathbb{Z}[G])$ is injective (see §3.1.4). The augmentation map $\mathbb{Z}[G] \to \mathbb{Z}, g \mapsto 1$, induces $\vartheta : K_0(\mathbb{Z}[G]) \to K_0(\mathbb{Z})$, such that $\vartheta\theta = 1$. Thus

$$K_0(\mathbb{Z}[G]) \cong \mathbb{Z} \bigoplus \widetilde{K_0}(\mathbb{Z}[G]).$$

Now since $\mathbb{Z}[G]$ is a group ring (so a crossed product), $K_0^{\mathrm{gr}}(\mathbb{Z}[G]) \cong K_0(\mathbb{Z}) \cong \mathbb{Z}$ and the action of $\mathbb{Z}[x, x^{-1}]$ on the $K_0^{\mathrm{gr}}(\mathbb{Z}[G])$ is trivial (see Example 3.1.9). So $K_0^{\mathrm{gr}}(\mathbb{Z}[G])/\langle [P] - [P(1)] \rangle \cong \mathbb{Z}$. This shows that $K_0^{\mathrm{gr}}(\mathbb{Z}[G])$ does not shed any light on the group $K_0(\mathbb{Z}[G])$. The Grothendieck group of group rings is far from easy to compute (see for example [14] and [94, Example 2.4]).

In this chapter we compare the graded K-theory to its non-graded counterpart. We start with a generalisation of Quillen's theorem.

6.1 K_*^{gr} of positively graded rings

For a $\mathbb{Z}$-graded ring A with the support in $\mathbb{N}$, in his seminal paper [79], Quillen calculated the graded K-theory of A in terms of K-theory of its zero homogeneous component.

Theorem 6.1.1. [79, §3, PROPOSITION] *Let A be a $\mathbb{Z}$-graded ring with the support in $\mathbb{N}$. Then for $i \geq 0$, there is a $\mathbb{Z}[x, x^{-1}]$-module isomorphism,*

$$K_i^{\mathrm{gr}}(A) \cong K_i(A_0) \otimes_{\mathbb{Z}} \mathbb{Z}[x, x^{-1}]. \tag{6.1}$$

Furthermore, the element $K_i(A_0) \otimes x^r$ is the image of $\overline{f}_r$ where f_r is the exact functor

$$\begin{aligned} f_r : \text{Pr-}A_0 &\longrightarrow \text{Pgr-}A, \\ P &\longmapsto (P \otimes_{A_0} A)(-r). \end{aligned}$$

In Theorem 6.1.3, we prove a generalised version of this proposition. Proposition 6.1.1 was used in an essential way to prove the fundamental theorem of K-theory (see §6.2.2). In particular, Proposition 6.1.1 gives a powerful tool to calculate the graded Grothendieck group of positively graded rings. We demonstrate this by calculating K_0^{gr} of path algebras.

Theorem 6.1.2. *Let E be a finite graph and $\mathcal{P}_K(E)$ be the path algebra with coefficients in the field K. Then*

$$K_0^{\mathrm{gr}}(\mathcal{P}_K(E)) \cong \bigoplus_{|E^0|} \mathbb{Z}[x, x^{-1}].$$

Proof. Recall from (1.6.3), that $\mathcal{P}_K(E)$ is a graded ring with the support $\mathbb{N}$. Furthermore $\mathcal{P}_K(E)_0 = \bigoplus_{|E^0|} K$. Then by (6.1)

$$K_0^{\mathrm{gr}}(\mathcal{P}_K(E)) \cong K_0(\bigoplus_{|E^0|} K) \otimes_{\mathbb{Z}} \mathbb{Z}[x, x^{-1}] \cong \bigoplus_{|E^0|} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}[x, x^{-1}] \cong \bigoplus_{|E^0|} \mathbb{Z}[x, x^{-1}]. \qquad \square$$

The proof of Proposition 6.1.2 shows that, in the case of $i = 0$, $[A] \in K_i^{\mathrm{gr}}(A)$ is sent to $[A_0] \otimes 1 \in K_i(A_0) \otimes_{\mathbb{Z}} \mathbb{Z}[x, x^{-1}]$ under the isomorphism (6.1). Thus for the graphs

$$E : \quad \begin{tikzcd} \bullet \arrow[r] & \bullet \end{tikzcd} \qquad\qquad F : \quad \begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \arrow[l, bend left] \end{tikzcd}$$

We have

$$\left(K_0^{\mathrm{gr}}(\mathcal{P}_K(E), [\mathcal{P}_K(E)]\right) \cong \left(\mathbb{Z}[x,x^{-1}] \oplus \mathbb{Z}[x,x^{-1}], (1,1)\right) \cong \left(K_0^{\mathrm{gr}}(\mathcal{P}_K(E), [\mathcal{P}_K(E)]\right).$$

This in particular shows that the ordered group K_0^{gr} in itself would not classify the path algebras.

Contrary to other fundamental theorems in the subject, such as fundamental theorem of K-theory (i.e., $K_i(R[x,x^{-1}]) = K_i(R) \times K_{i-1}(R)$, for R a regular ring), one cannot use an easy induction on (6.1) to write a similar statement for "multi-variables" rings. For example, it appears that there is no obvious inductive approach to generalise (6.1) to $\mathbb{Z}^m \times G$-graded rings. However, by generalising Quillen's argument to take gradings into account on both sides of the isomorphism, such a procedure becomes feasible. The details have been worked out in [49] and we present it here.

We will prove the following statement.

Theorem 6.1.3. *Let G be a group, and let A be a $\mathbb{Z} \times G$-graded ring with support in $\mathbb{N} \times G$. Then there is a $\mathbb{Z}[\mathbb{Z} \times G]$-module isomorphism*

$$K_i^G(A_{(0,-)}) \otimes_{\mathbb{Z}[G]} \mathbb{Z}[\mathbb{Z} \times G] \cong K_i^{\mathbb{Z}\times G}(A),$$

where $A_{(0,-)} = \bigoplus_{g\in G} A_{(0,g)}$.

By a straightforward induction this now implies:

Corollary 6.1.4. *For a $\mathbb{Z}^m \times G$-graded ring A with support in $\mathbb{N}^m \times G$ there is a $\mathbb{Z}[x_1^{\pm 1}, \cdots, x_m^{\pm 1}]$-module isomorphism*

$$K_i^G(A_{(0,-)}) \otimes_{\mathbb{Z}} \mathbb{Z}[x_1^{\pm 1}, \cdots, x_m^{\pm 1}] \cong K_i^{\mathbb{Z}^m \times G}(A)\ .$$

For a trivial group G this is a direct generalisation of Quillen's theorem to $\mathbb{Z}^m$-graded rings.

As in Quillen's calculation the proof of the Theorem 6.1.3 is based on a version of Swan's Theorem, modified to the present situation: it provides a correspondence between isomorphism classes of $\mathbb{Z} \times G$-graded finitely generated projective A-modules and of G-graded finitely generated projective $A_{(0,-)}$-modules.

Proposition 6.1.5. *Let Γ be a (possibly non-abelian) group. Let A be a Γ-graded ring, A_0 a graded subring of A and $\pi\colon A \to A_0$ a graded ring homomorphism such that $\pi|_{A_0} = 1$. (In other words, A_0 is a retract of A in the category of Γ-graded rings.) We denote the kernel of π by A_+.*

Suppose that for any graded finitely generated right A-module M the condition $MA_+ = M$ implies $M = 0$. Then the natural functor

$$S = - \otimes_{A_0} A\colon \mathrm{Pgr}^{\Gamma}\text{-}A_0 \to \mathrm{Pgr}^{\Gamma}\text{-}A$$

induces a bijective correspondence between the isomorphism classes of graded finitely generated projective A_0-modules and of graded finitely generated projective A-modules. An inverse of the bijection is given by the functor

$$T = - \otimes_A A_0\colon \mathrm{Pgr}^{\Gamma}\text{-}A \to \mathrm{Pgr}^{\Gamma}\text{-}A_0\ .$$

There is a natural isomorphism $T \circ S \cong \mathrm{id}$, and for each $P \in \mathrm{Pgr}^{\Gamma}\text{-}A$ a non-canonical isomorphism $S \circ T(P) \cong P$. The latter is given by

$$T(P) \otimes_{A_0} A \to P\ , \quad x \otimes a \mapsto g(x) \cdot a, \tag{6.2}$$

where g is an A_0-linear section of the epimorphism $P \to T(P)$.

Proof. For any graded finitely generated projective A_0-module Q we have a natural isomorphism $TS(Q) \cong Q$ given by

$$\nu_Q \colon TS(Q) = Q \otimes_{A_0} A \otimes_A A_0 \to Q \ , \quad q \otimes a \otimes a_0 \mapsto q\pi(a)a_0 \ . \tag{6.3}$$

We will show that for a graded projective A-module P there is a non-canonical graded isomorphism $ST(P) \cong_{\mathrm{gr}} P$. The lemma then follows.

Consider the natural graded A-module epimorphism

$$f \colon P \to T(P) = P \otimes_A A_0 \ , \quad p \mapsto p \otimes 1 \ .$$

Here $T(P)$ is considered as an A-module via the map π. Since $T(P)$ is a graded projective A_0-module, the map f has a graded A_0-linear section $g \colon T(P) \to P$. This section determines an A-linear graded map

$$\psi \colon ST(P) = P \otimes_A A_0 \otimes_{A_0} A \to P \ , \quad p \otimes a_0 \otimes a \mapsto g(p \otimes a_0) \cdot a \ ,$$

and we will show that ψ is an isomorphism. First note that the map

$$T(f) \colon T(P) \to TT(P) \ , \quad p \otimes a_0 \mapsto f(p) \otimes a_0 = p \otimes 1 \otimes a_0$$

is an isomorphism (consider $T(P)$ as an A-module via π here). In fact the inverse is given by the isomorphism $TT(P) = P \otimes_A A_0 \otimes_A A_0 \to P \otimes_A A_0$ which maps $p \otimes a_0 \otimes b_0$ to $p \otimes (a_0 b_0)$. Tracing the definitions now shows that both composites

$$\begin{tikzcd} TST(P) \arrow[r, shift left, "T(\psi)"] \arrow[r, shift right, "\nu_{T(P)}"'] & T(P) \arrow[r, "T(f)", "\cong"'] & TT(P) \end{tikzcd}$$

map $p \otimes a_0 \otimes a \otimes b_0 \in P \otimes_A A_0 \otimes_{A_0} A \otimes_A A_0 = TST(P)$ to the element

$$f\big(g(p \otimes a_0) \cdot a\big) \otimes b_0 = p \otimes (a_0 \pi(a) b_0) \otimes 1 \in TT(P).$$

This implies that $T(\psi) = \nu_{T(P)}$, which is an isomorphism.

The exact sequence

$$0 \longrightarrow \ker \psi \longrightarrow ST(P) \xrightarrow{\psi} P \longrightarrow \operatorname{coker} \psi \longrightarrow 0 \tag{6.4}$$

gives rise, upon application of the right exact functor T, to an exact sequence

$$TST(P) \xrightarrow{T(\psi)} T(P) \longrightarrow T(\operatorname{coker} \psi) \longrightarrow 0 \ .$$

Since $T(\psi)$ is an isomorphism we have $T(\operatorname{coker} \psi) = \operatorname{coker} T(\psi) = 0$. Since $\operatorname{coker} \psi$ is finitely generated by (6.4) this implies $\operatorname{coker} \psi = 0$ (note that $T(M) = MA_+$ for every finitely generated module M). In other words, ψ is surjective and (6.4) becomes the short exact sequence $0 \to \ker \psi \to ST(P) \xrightarrow{\psi} P \to 0$. This latter sequence splits since P is projective; this immediately implies that $\ker \psi$ is finitely generated, and since $T(\psi)$ is injective we also have $T(\ker \psi) = \ker T(\psi) = 0$. The hypotheses guarantee $\ker \psi = 0$ now so that ψ is injective as well as surjective, and thus is an isomorphism as claimed. □

Lemma 6.1.6. *Let G and Γ be groups, and let A be a G-graded ring. Then, considering A as a $\Gamma \times G$-graded ring in a trivial way where necessary, the functorial assignment $(M, \gamma) \mapsto M(\gamma, 0)$ induces a $\mathbb{Z}[\Gamma \times G]$-module isomorphism*

$$K_i^{\Gamma \times G}(A) \cong K_i^G(A) \otimes_{\mathbb{Z}[G]} \mathbb{Z}[\Gamma \times G] \; .$$

Proof. Let $P = \bigoplus_{(\gamma,g) \in \Gamma \times G} P_{(\gamma,g)}$ be a $\Gamma \times G$-graded finitely generated projective A-module. Since the support of A is $G = 1 \times G$, there is a unique decomposition $P = \bigoplus_{\gamma \in \Gamma} P_\gamma$, where the $P_\gamma = \bigoplus_{g \in G} P_{(\gamma,g)}$ are finitely generated G-graded projective A-modules. This gives a natural isomorphism of categories (see Corollary 1.2.10)

$$\Psi \colon \mathrm{Pgr}^{\Gamma\times\mathrm{G}}\text{-}A \xrightarrow{\cong} \bigoplus_{\gamma \in \Gamma} \mathrm{Pgr}^{\mathrm{G}}\text{-}A \; .$$

The natural right action of $\Gamma \times G$ on these categories is described as follows: for a given module $P \in \mathrm{Pgr}^{\Gamma\times\mathrm{G}}\text{-}A$ as above and elements $(\gamma, g) \in \Gamma \times G$ we have

$$P(\gamma, g)_{(\delta,h)} = P_{(\gamma+\delta, g+h)} \text{ and } \Psi(P)_\delta = P_{\gamma+\delta}(g)$$

so that $\Psi\big(P(\gamma, g)\big) = \Psi(P)(\gamma, g)$. Since K-groups respect direct sums we thus have a chain of $\mathbb{Z}[\Gamma \times G]$-linear isomorphisms

$$\begin{aligned} K_i^{\Gamma\times G}(A) = K_i(\mathrm{Pgr}^{\Gamma\times\mathrm{G}}\text{-}A) &\cong K_i\big(\bigoplus_{\gamma\in\Gamma} \mathrm{Pgr}^{\mathrm{G}}\text{-}A\big) \\ &= \bigoplus_{\gamma\in\Gamma} K_i^G(A) \cong K_i^G(A) \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma] = K_i^G(A) \otimes_{\mathbb{Z}[G]} \mathbb{Z}[\Gamma \times G] \; . \end{aligned}$$ □

We are in a position to prove the Theorem 6.1.3.

Proof of Theorem 6.1.3. Let A be a $\mathbb{Z} \times G$-graded ring with support in $\mathbb{N} \times G$. That is, A comes equipped with a decomposition

$$A = \bigoplus_{\omega \in \mathbb{N}} A_{(\omega,-)} \quad \text{where} \quad A_{(\omega,-)} = \bigoplus_{g \in G} A_{(\omega,g)} \; .$$

The ring A has a $\mathbb{Z} \times G$-graded subring $A_{(0,-)}$ (with trivial grading in $\mathbb{Z}$-direction). The projection map $A \to A_{(0,-)}$ is a $\mathbb{Z} \times G$-graded ring homomorphism; its kernel is denoted A_+. Explicitly, A_+ is the two-sided ideal

$$A_+ = \bigoplus_{\omega > 0} A_{(\omega,-)} \; .$$

We identify the quotient ring A/A_+ with the subring $A_{(0,-)}$ via the projection.

If P is a graded finitely generated projective A-module, then $P \otimes_A A_{(0,-)}$ is a finitely generated $\mathbb{Z} \times G$-graded projective $A_{(0,-)}$-module. Similarly, if Q is a graded finitely generated projective $A_{(0,-)}$-module then $Q \otimes_{A_{(0,-)}} A$ is a $\mathbb{Z} \times G$- graded finitely generated projective A-module. We can thus define functors

$$\begin{aligned} T = - \otimes_A A_{(0,-)} \colon \mathrm{Pgr}^{\mathbb{Z}\times\mathrm{G}}\text{-}A &\longrightarrow \mathrm{Pgr}^{\mathbb{Z}\times\mathrm{G}}\text{-}A_{(0,-)} \\ \text{and} \quad S = - \otimes_{A_{(0,-)}} A \colon \mathrm{Pgr}^{\mathbb{Z}\times\mathrm{G}}\text{-}A_{(0,-)} &\longrightarrow \mathrm{Pgr}^{\mathbb{Z}\times\mathrm{G}}\text{-}A \; . \end{aligned}$$

Since $T(P) = P/PA_+$ we see that *the support of $T(P)$ is contained in the support of P.*

Observe now that *if M is a finitely generated $\mathbb{Z} \times G$-graded A-module and $MA_+ = M$ then $M = 0$*; for if $M \neq 0$ there is a minimal $\omega \in \mathbb{Z}$ such that $M_{(\omega,-)} \neq 0$, but $(MA_+)_{(\omega,-)} = 0$. It follows from Proposition 6.1.5 that for each graded finitely generated projective A-module P there is a non-canonical isomorphism $P \cong T(P) \otimes_{A_{(0,-)}} A$ as in (6.2) which respects the $\mathbb{Z} \times G$-grading. Explicitly, for a given $(\omega, g) \in \mathbb{Z} \times G$ we have an isomorphism of abelian groups

$$P_{(\omega,g)} \cong \bigoplus_{(\kappa,h)} T(P)_{(\kappa,h)} \otimes A_{(-\kappa+\omega,-h+g)} \ ; \tag{6.5}$$

the tensor product $T(P)_{(\kappa,h)} \otimes A_{(-\kappa+\omega,-h+g)}$ denotes, by convention, the abelian subgroup of $T(P) \otimes_{A_{(0,-)}} A$ generated by primitive tensors of the form $x \otimes y$ with homogeneous elements $x \in T(P)$ of degree (κ, h) and $y \in A$ of degree $(-\kappa + \omega, -h + g)$.

For a $\mathbb{Z} \times G$-graded A-module P write $P = \bigoplus_{\omega \in \mathbb{Z}} P_{(\omega,-)}$, where $P_{(\omega,-)} = \bigoplus_{g \in G} P_{(\omega,g)}$. For $\lambda \in \mathbb{Z}$ let $F^\lambda P$ denote the A-submodule of P generated by the elements of $\bigcup_{\omega \leq \lambda} P_{(\omega,-)}$; this is $\mathbb{Z} \times G$-graded again. As an explicit example, we have

$$F^\lambda A(\omega, g) = \begin{cases} A(\omega, g) & \text{if } \lambda \geq -\omega \ , \\ 0 & \text{else.} \end{cases}$$

Suppose that P is a graded finitely generated projective A-module. Since the support of A is contained in $\mathbb{N} \times G$ there exists $n \in \mathbb{Z}$ such that $F^{-n} P = 0$ and $F^n P = P$. Write $\mathrm{Pgr}_\mathrm{n}^{\mathbb{Z}\times\mathrm{G}}\text{-}A$ for the full subcategory of $\mathrm{Pgr}^{\mathbb{Z}\times\mathrm{G}}\text{-}A$ spanned by those modules P which satisfy $F^{-n} P = 0$ and $F^n P = P$. Then $\mathrm{Pgr}^{\mathbb{Z}\times\mathrm{G}}\text{-}A$ is the filtered union of the $\mathrm{Pgr}_\mathrm{n}^{\mathbb{Z}\times\mathrm{G}}\text{-}A$.

Let $P \in \mathrm{Pgr}_\mathrm{n}^{\mathbb{Z}\times\mathrm{G}}\text{-}A$; we want to identify $F^\lambda P$. By definition, the A-module $F^\lambda P$ is generated by the elements of $P_{(\omega,g)}$ for $\omega \leq \lambda$, with $P_{(\omega,g)}$ having been identified in (6.5). We remark that the direct summands in (6.5) indexed by $\kappa > \omega$ are trivial as A has support in $\mathbb{N} \times G$. On the other hand, for $\omega \geq \kappa$ a given primitive tensor $x \otimes y \in P_{(\omega,g)}$ with $x \in T(P)_{(\kappa,h)}$ and $y \in A_{(-\kappa+\omega,-h+g)}$ can always be re-written, using the right A-module structure of $T(P) \otimes_{A_{(0,-)}} A$, as

$$x \otimes y = (x \otimes 1) \cdot y \qquad \text{where } x \otimes 1 \in T(P)_{(\kappa,h)} \otimes A_{(0,0)} \subseteq P_{(\kappa,h)} \ .$$

That is, the A-module $F^\lambda P$ is generated by those summands of (6.5) with $\kappa = \omega \leq \lambda$. We claim now that $F^\lambda P$ is isomorphic to

$$M^{(\lambda)} = \bigoplus_{\kappa \leq \lambda} T(P)_{(\kappa,-)} \otimes_{A_{(0,-)}} A(-\kappa, 0) \ , \tag{6.6}$$

considering $T(P)_{(\kappa,-)}$ as a $\mathbb{Z} \times G$-graded $A_{(0,-)}$-module with support in $\{0\} \times G$. The homogeneous components of $M^{(\lambda)}$ are given by

$$M^{(\lambda)}_{(\omega,g)} = \bigoplus_{\kappa \leq \lambda} \bigoplus_{h \in G} T(P)_{(\kappa,h)} \otimes A(-\kappa, 0)_{(\omega,-h+g)} \ .$$

Now elements of the form

$$x \otimes 1 \in T(P)_{(\kappa,h)} \otimes A(-\kappa, 0)_{(\kappa,-h+h)} \subseteq M^{(\lambda)}_{(\kappa,h)}$$

clearly form a set of A-module generators for $M^{(\lambda)}$ so that, by the argument given above, $F^\lambda P$ and $M^{(\lambda)}$ have the same generators in the same degrees. The claim follows. The module $F^\lambda P$ is finitely generated (*viz.*, by those generators of P that have $\mathbb{Z}$-degree at most λ). Since $T(P)$ is a finitely generated projective $A_{(0,-)}$-module so is its summand $T(P)_{(\lambda_k,-)}$; consequently, $P \mapsto F^kP$ is an endofunctor of $\mathrm{Pgr}^{\mathbb{Z}\times\mathrm{G}}_{\mathrm{n}}\text{-}A$. It is exact as can be deduced from the (non-canonical) isomorphism in (6.6), using exactness of tensor products.

From the isomorphism $F^kP \cong M^{(\lambda_k)}$, cf. (6.6), we obtain an isomorphism

$$F^{k+1}P/F^kP \cong T(P)_{(\lambda_k,-)} \otimes_{A_{(0,-)}} A(-\lambda_k, 0) \ ; \tag{6.7}$$

in particular, $F^{k+1}P/F^kP \in \mathrm{Pgr}^{\mathbb{Z}\times\mathrm{G}}_{\mathrm{n}}\text{-}A$.

The isomorphism (6.7) depends on the isomorphism (6.6), and thus ultimately on (6.2). The latter depends on a choice of a section g of $P \to T(P)$. Given another section g_0 the difference $g - g_0$ has image in $\ker(P \to T(P)) = PA_+$. Since A_+ consists of elements of positive $\mathbb{Z}$-degree only, this implies that the isomorphism $F^{k+1}P \cong M^{(\lambda_{k+1})}$ does not depend on g up to elements in F^kP; in other words, the quotient $F^{k+1}P/F^kP$ is independent of the choice of g. Thus the isomorphism (6.7) is, in fact, a natural isomorphism of functors.

We are now in a position to perform the K-theoretical calculations. First define the exact functor

$$\begin{aligned} \Theta_q : \mathrm{Pgr}^{\mathbb{Z}\times\mathrm{G}}_{\mathrm{q}}\text{-}A_{(0,-)} &\longrightarrow \mathrm{Pgr}^{\mathbb{Z}\times\mathrm{G}}_{\mathrm{q}}\text{-}A \\ P = \bigoplus_\omega P_{(\omega,-)} &\mapsto \bigoplus_\omega P_{(\omega,-)} \otimes_{A_{(0,-)}} A(-\omega,0) \ ; \end{aligned}$$

here $\mathrm{Pgr}^{\mathbb{Z}\times\mathrm{G}}_{\mathrm{q}}\text{-}A_{(0,-)}$ denotes the full subcategory of $\mathrm{Pgr}^{\mathbb{Z}\times\mathrm{G}}\text{-}A_{(0,-)}$ spanned by modules with support in $[-q,q] \times G$, and $P_{(\omega,-)}$ on the right is considered as a $\mathbb{Z} \times G$-graded $A_{(0,-)}$-module with support in $\{0\} \times G$.

Next define the exact functor

$$\begin{aligned} \Psi_q : \mathrm{Pgr}^{\mathbb{Z}\times\mathrm{G}}_{\mathrm{q}}\text{-}A &\longrightarrow \mathrm{Pgr}^{\mathbb{Z}\times\mathrm{G}}_{\mathrm{q}}\text{-}A_{(0,-)} \\ P &\mapsto \bigoplus_{\omega\in\mathbb{Z}} T(P)_{(\omega,-)} \ ; \end{aligned}$$

here $T(P)_{(\omega,-)}$ is considered as an $A_{(0,-)}$-module with support in $\{\omega\} \times G$.

Now $\Psi_q \circ \Theta_q \cong \mathrm{id}$; indeed, the composition sends the summand $P_{(\omega,-)}$ of P to the κ-indexed direct sum of

$$T\big(P_{(\omega,-)} \otimes_{A_{(0,-)}} A(-\omega,0)\big)_{(\kappa,-)} \cong \begin{cases} P_{(\omega,-)} & \text{if } \kappa = \omega \ , \\ 0 & \text{else.} \end{cases}$$

In particular, $\Psi_q \circ \Theta_q$ induces the identity on K-groups. As for the other composition, we have

$$\Theta_q \circ \Psi_q(P) = \bigoplus_\omega T(P)_{(\omega,-)} \otimes_{A_{(0,-)}} A(-\omega,0) \underset{(6.7)}{=} \bigoplus_{j=-q}^{q-1} F^{j+1}P/F^jP \ .$$

Since $F^q = \mathrm{id}$, additivity for characteristic filtrations [79, p. 107, Corollary 2] implies that $\Theta_q \circ \Psi_q$ induces the identity on K-groups.

For any $P \in \mathrm{Pgr}^{\mathbb{Z}\times\mathrm{G}}\text{-}A_{(0,-)}$ we have

$$\big(P \otimes_{A_{(0,-)}} A(-\omega,0)\big)(0,g) = P(0,g) \otimes_{A_{(0,-)}} A(-\omega,0) \ ,$$

by direct calculation. Hence the functor Θ_q induces a $\mathbb{Z}[G]$-linear isomorphism on K-groups. Since K-groups are compatible with direct limits, letting $q \to \infty$ yields a $\mathbb{Z}[G]$-linear isomorphism $K_i^{\mathbb{Z}\times G}(A_{(0,-)}) \cong K_i^{\mathbb{Z}\times G}(A)$ and thus, by Lemma 6.1.6, a $\mathbb{Z}[\mathbb{Z}\times G]$-module isomorphism

$$K_i^G(A_{(0,-)}) \otimes_{\mathbb{Z}[G]} \mathbb{Z}[\mathbb{Z}\times G] \cong K_i^{\mathbb{Z}\times G}(A) \ . \qquad \square$$

6.2 The fundamental theorem of K-theory

6.2.1 Quillen's K-theory of exact categories

Recall that an exact category $\mathcal{P}$ is a full additive subcategory of an abelian category $\mathcal{A}$ which is closed under extension (see Definition 3.10.1). An *exact functor* $\mathcal{F} : \mathcal{P} \to \mathcal{P}'$, between exact categories $\mathcal{P}$ and $\mathcal{P}'$, is an additive functor which preserves the short exact sequences.

From an exact category $\mathcal{P}$, Quillen constructed abelian groups $K_n(\mathcal{P})$, $n \geq 0$ (see [79]). In fact, for a fixed n, K_n is a functor from the category of exact categories, with exact functors as morphisms to the category of abelian groups. If $f : \mathcal{A} \to \mathcal{B}$ is an exact functor, then we denote the corresponding group homomorphism by $\overline{f} : K_n(\mathcal{A}) \longrightarrow K_n(\mathcal{B})$.

When $\mathcal{P} = \text{Pr-}A$, the category of finitely generated projective A-module, then $K_0(\mathcal{P}) = K_0(A)$. Quillen proved that basic theorems established for K_0-group, such as Dévissage, resolution and localisation theorems ([88, 94]), can be established for these higher K-groups. We briefly mention the main theorems of K-theory which will be used in §6.2.2. The following statements were established by Quillen in [79] (see also [87, 94]).

1. **The Grothendieck group.** For an exact category $\mathcal{P}$, the abelian group $K_0(\mathcal{P})$ coincides with the construction given in Definition 3.10.1(1). In particular, Quillen's construction gives the Grothendieck group $K_0(A) = K_0(\text{Pr-}A)$ and graded Grothendieck group $K_0^{\text{gr}}(A) = K_0(\text{Pgr-}A)$, for non-graded and graded rings, respectively.

2. **Dévissage.** Let $\mathcal{A}$ be an abelian category and $\mathcal{B}$ be a non-empty full subcategory closed under subobjects, quotient objects and finite products in $\mathcal{A}$. Thus $\mathcal{B}$ is an abelian category and the inclusion $\mathcal{B} \hookrightarrow \mathcal{A}$ is exact. If any element A in $\mathcal{A}$, has a finite filtration
$$0 = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = A,$$
with A_i/A_{i-1} in $\mathcal{B}$, $1 \leq i \leq n$, then the inclusion $\mathcal{B} \hookrightarrow \mathcal{A}$ induces isomorphisms $K_n(\mathcal{B}) \cong K_n(\mathcal{A})$, $n \geq 0$.

3. **Resolution.** Let $\mathcal{M}$ be an exact category and $\mathcal{P}$ be a full subcategory closed under extensions in $\mathcal{M}$. Suppose that if $0 \to M \to P \to P' \to 0$ is exact in $\mathcal{M}$ with P and P' in $\mathcal{P}$, then M is in $\mathcal{P}$. Furthermore, for every M in $\mathcal{M}$, there is a finite $\mathcal{P}$ resolution of M,
$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$
Then the inclusion $\mathcal{P} \hookrightarrow \mathcal{M}$ induces isomorphisms $K_n(\mathcal{P}) \cong K_n(\mathcal{M})$, $n \geq 0$.

4. **Localisation.** Let $\mathcal{S}$ be a *Serre subcategory* of abelian category $\mathcal{A}$ (i.e., $\mathcal{S}$ is an abelian subcategory of $\mathcal{A}$, closed under subobjects, quotient objects and extensions in $\mathcal{A}$), and let $\mathcal{A}/\mathcal{S}$ be the quotient abelian category. Then there is a natural long exact sequence
$$\cdots \longrightarrow K_{n+1}(\mathcal{A}/\mathcal{S}) \xrightarrow{\delta} K_n(\mathcal{S}) \longrightarrow K_n(\mathcal{A}) \longrightarrow K_n(\mathcal{A}/\mathcal{S}) \longrightarrow \cdots .$$

5. **Exact Sequence of Functors.** Let $\mathcal{P}$ and $\mathcal{P}'$ be exact categories and

$$0 \longrightarrow f' \longrightarrow f \longrightarrow f'' \longrightarrow 0,$$

be an exact sequence of exact functors from $\mathcal{P}$ to $\mathcal{P}'$. Then

$$\overline{f} = \overline{f}' + \overline{f}'' : K_n(\mathcal{P}) \longrightarrow K_n(\mathcal{P}').$$

Remark 6.2.1. Base change and transfer functors

Let A be a ring with identity. If A is a right Noetherian then the category of finitely generated right A-modules, mod-A, form an abelian category. Thus one can define the K-groups of this category, $G_n(A) := K_n(\text{mod-}A)$, $n \in \mathbb{N}$. If further A is regular, then by resolution theorem (§6.2.1), $G_n(A) = K_n(A)$.

Let $R \to S$ be a ring homomorphism. This makes S an R-module. If S is a finitely generated R-module, then any finitely generated S-module can be considered as a finitely generated R-module. This induces an exact functor, called a *transfer functor*, mod-$S \to$ mod-R. Since K-theory is a functor from the category of exact categories with exact functors as morphisms, we get a transfer map $G_n(S) \to G_n(R)$, $n \in \mathbb{N}$. A similar argument shows that, if $R \to S$ is a graded homomorphism, then we have a transfer map $G_n^{\mathrm{gr}}(S) \to G_n^{\mathrm{gr}}(R)$, $n \in \mathbb{N}$.

If under the ring homomorphism $R \to S$, S is a flat R-module, then the *base change functor* $- \otimes_R S : \text{Mod-}R \to \text{Mod-}S$ is exact, which restricts to $- \otimes_R S : \text{mod-}R \to \text{mod-}S$. This in turn induces the group homomorphism $G_n(R) \to G_n(S)$, $n \in \mathbb{N}$. Note that any ring homomorphism $R \to S$, induces an exact functor $- \otimes_R S : \text{Pr-}R \to \text{Pr-}S$, which induces the homomorphism $K_n(R) \to K_n(S)$. However, to prove the fundamental theorem of K-theory (see (6.12)), we need to use the Dévissage and localisation theorems, which are only valid for abelian categories. This forces us to work in the abelian category gr-A (assuming A is right Noetherian), rather than the exact category Pr-A.

Example 6.2.2. A long localisation exact sequence for graded rings

Let A be a right regular Noetherian Γ-graded ring, i.e., A is graded right regular and graded right Noetherian. We assume the ring is graded regular so that we could work with K-theory instead of G-theory. We recall the concept of graded regular rings. For a graded right module M, the minimum length of all graded projective resolutions of M is defined as the *graded projective dimension* of M and denoted by $\operatorname{pdim}^{\mathrm{gr}}(M)$. If M does not admit a finite graded projective resolution, then we set $\operatorname{pdim}^{\mathrm{gr}}(M) = \infty$. The *graded right global dimension* of the graded ring A is the supremum of the graded projective dimension of all the graded right modules over A and denoted by $\operatorname{gldim}^{\mathrm{gr}}(A)$. We say A is a *graded right regular* if the graded right global dimension is finite. As soon as Γ is a trivial group, the above definitions become the standard definitions of projective dimension, denoted by, pdim, and the global dimension, denoted by, gldim, in ring theory. In the case of $\mathbb{Z}$-graded ring, we have the following relation between the dimensions (see [72, Theorem II.8.2])

$$\operatorname{gldim}^{\mathrm{gr}}(A) \leq \operatorname{gldim}(A) \leq 1 + \operatorname{gldim}^{\mathrm{gr}}(A). \tag{6.8}$$

This shows, if A is graded regular, then A is regular. This will be used in §6.3.

Let S be a central multiplicative closed subsets of A, consisting of homogeneous elements. Then $S^{-1}A$ is a Γ-graded ring (see Example 1.1.12). Let $\text{gr}_S\text{-}A$ be a category of S-torsion graded finitely generated right A-modules, i.e., a graded finitely generated A-module M such that $Ms = 0$ for some $s \in S$. Clearly $\text{gr}_S\text{-}A$ is a Serre subcategory of the abelian category gr-A and

$$\text{gr-}A/\text{gr}_S\text{-}A \approx \text{gr-}(S^{-1}A). \tag{6.9}$$

Now let $S = \{s^k \mid k \in \mathbb{N}\}$, where s is a homogeneous element in the centre of A. Then any module M in $\text{gr}_S\text{-} A$ has a filtration

$$0 = Ms^k \subseteq \cdots \subseteq Ms \subseteq M,$$

and clearly the consecutive quotients in the filtration are Γ-graded A/sA-modules. Considering $\text{gr-} A/sA$ as a subcategory of $\text{gr}_S\text{-} A$ via the transfer map $A \to A/sA$ (see Remark 6.2.1), and using Dévissage theorem, we get, for any $n \in \mathbb{N}$,

$$K_n(\text{gr}_S\text{-} A) \cong K_n^{\text{gr}}(A/sA).$$

Now using the localisation theorem for (6.9), we obtain a long localisation exact sequence

$$\cdots \longrightarrow K_{n+1}^{\text{gr}}(A_s) \longrightarrow K_n^{\text{gr}}(A/sA) \longrightarrow K_n^{\text{gr}}(A) \longrightarrow K_n^{\text{gr}}(A_s) \longrightarrow \cdots, \tag{6.10}$$

where $A_s := S^{-1}A$.

In particular, consider the Γ-graded rings $A[y]$ and $A[y, y^{-1}]$, where $\deg(y) = \alpha$. For $A[y]$ and $S = \{y^n \mid n \in \mathbb{N}\}$, the sequence 6.10 reduces to

$$\cdots \longrightarrow K_{n+1}^{\text{gr}}(A[y, y^{-1}]) \longrightarrow K_n^{\text{gr}}(A) \longrightarrow K_n^{\text{gr}}(A[y]) \longrightarrow K_n^{\text{gr}}(A[y, y^{-1}]) \longrightarrow \cdots. \tag{6.11}$$

This long exact sequence will be used in §6.3 (with $\deg(y) = 1$) to relate K_n^{gr}-groups to K_n-groups.

6.2.2 The fundamental theorem

Let A be a right Noetherian and regular ring. Consider the polynomial ring $A[x]$. The fundamental theorem of K-theory gives that, for any $n \in \mathbb{N}$,

$$K_n(A[x]) \cong K_n(A). \tag{6.12}$$

In this section we prove that if A is a positively graded $\mathbb{Z}$-graded ring, then $K_n(A) \cong K_n(A_0)$, which in particular gives (6.12). We will follow Gersten's treatment in [37]. The proof shows how the graded K-theory is effectively used to prove a theorem on (non-graded) K-theory.

Let A be a positively $\mathbb{Z}$-graded ring, i.e., $A = \bigoplus_{i \in \mathbb{N}} A_i$. Define a $\mathbb{Z}$-grading on the polynomial ring $A[x]$ as follows (see Example 1.1.7):

$$A[x]_n = \bigoplus_{i+j=n} A_i x^j. \tag{6.13}$$

Clearly this is also a positively graded ring and $A[x]_0 = A_0$.

Throughout the proof of the fundamental theorem we consider two evaluation maps on $A[x]$ as follows.

1. Consider the evaluation homomorphism at 0, i.e.,

$$\begin{aligned} e_0 : A[x] &\longrightarrow A, \\ f(x) &\longmapsto f(0). \end{aligned} \tag{6.14}$$

This is a graded homomorphism (with the grading defined in (6.13)), and A becomes a graded finitely generated $A[x]$-module. This induces an exact transfer functor (see Remark 6.2.1)

$$\begin{aligned} i : \text{gr-}A &\longrightarrow \text{gr-}A[x], \\ M &\longmapsto M \end{aligned} \tag{6.15}$$

and in turn a homomorphism

$$\bar{i} : G_n^{\mathrm{gr}}(A) \to G_n^{\mathrm{gr}}(A[x]).$$

Let M be a graded A-module. Under the above homomorphism, M is also a grade $A[x]$-module (see (6.15)). On the other hand, the canonical graded homomorphism $A \to A[x]$, gives the graded $A[x]$-module $M \otimes_A A[x] \cong_{\mathrm{gr}} M[x]$. Then we have the following short exact sequence of graded $A[x]$-modules,

$$0 \longrightarrow M[x](-1) \xrightarrow{x} M[x] \xrightarrow{e_0} M \longrightarrow 0. \tag{6.16}$$

Note that this short exact sequence is not an split sequence.

2. Consider the evaluation homomorphism at 1, i.e.,

$$\begin{aligned} e_1 : A[x] &\longrightarrow A, \\ f(x) &\longmapsto f(1). \end{aligned} \tag{6.17}$$

Note that this homomorphism is not graded. Clearly $\ker(e_1) = (1-x)$ (i.e., the ideal generated by $1 - x$). Thus $A[x]/(1 - x) \cong A$. If M is a (graded) $A[x]$-module, then

$$M \otimes_{A[x]} A \cong M \otimes_{A[x]} A[x]/(1 - x) \cong M/M(1 - x),$$

is an A-module. Thus the homomorphism e_1 induces the functor

$$\begin{aligned} \mathcal{F} := - \otimes_{A[x]} A : \text{gr-}A[x] &\longrightarrow \text{mod-}A \\ M &\longmapsto M/M(1 - x). \end{aligned} \tag{6.18}$$

We need the following Lemma (see [72, Lemma II.8.1]).

Lemma 6.2.3. *Consider the functor $\mathcal{F}$ defined in (6.18). Then we have the following.*

(1) *$\mathcal{F}$ is an exact functor.*

(2) *$\mathcal{F}(M) = 0$ if and only if $Mx^n = 0$ for some $n \in \mathbb{N}$.*

Proof. (1) Since the tensor product is a right exact functor, we are left to show that if $0 \to M' \to M$ is exact, then by (6.18),

$$0 \longrightarrow M'/M'(1 - x) \longrightarrow M/M(1 - x),$$

is exact, i.e., $M' \cap M(1 - x) = M'(1 - x)$.

Let $m' \in M' \cap M(1 - x)$. Then

$$m' = m(1 - x), \tag{6.19}$$

for some $m \in M$. Since M' and M are graded modules, $m' = \sum m'_i$ and $m = \sum m_i$, with $\deg(m'_i) = \deg(m_i) = i$. Comparing the degrees in the Equation 6.19, we have $m_i - m_{i-1}x = m'_i \in M'_i$. Let j be the smallest i in the support of m such that $m_j \notin M'_j$. Then $m_{j-1}x \in M'_j$ which implies that $m_j = m'_j - m_{j-1}x \in M'_j$, which is a contradiction. Thus $m \in M'$, and so $m' \in M'(1 - x)$.

(2) The proof is elementary and left to the reader. ∎

By Lemma 6.2.3(2), $\ker(\mathcal{F})$ is an (abelian) full subcategory of $\text{gr-}A[x]$, with objects the graded $A[x]$-modules M such that $Mx^n = 0$, for some $n \in \mathbb{N}$. Define $\mathcal{C}_1$ the (abelian) full subcategory of $\text{gr-}A[x]$ with objects the graded $A[x]$-modules M such that $Mx = 0$. Thus $\mathcal{C}_1 \hookrightarrow \ker(\mathcal{F})$ is an exact functor. Furthermore, for any object M in $\ker(\mathcal{F})$, we have a finite filtration of graded $A[x]$-modules

$$0 = Mx^n \subseteq Mx^{n-1} \subseteq \cdots \subseteq Mx \subseteq M,$$

such that $Mx^i/Mx^{i+1} \in \mathcal{C}_1$. Thus by Dévissage theorem (§6.2.1), we have

$$K_n(\mathcal{C}_1) \cong K_n(\ker(\mathcal{F})). \tag{6.20}$$

Next recall the exact functor $i : \text{gr-}A \longrightarrow \text{gr-}A[x]$ from (6.15). It is easy to see that this functor induces an equivalence between the categories

$$i : \text{gr-}A \longrightarrow \mathcal{C}_1, \tag{6.21}$$

and thus on the level of K-groups. Combing this with (6.20) we get

$$K_n^{\text{gr}}(A) = K_n(\mathcal{C}_1) \cong K_n(\ker(\mathcal{F})). \tag{6.22}$$

Since by Lemma 6.2.3, $\mathcal{F}$ is exact, $\ker(\mathcal{F})$ is closed under subobjects, quotients and extensions, i.e, it is a Serre subcategory of $\text{gr-}A[x]$. We next prove that

$$\text{gr-}A[x]/\ker(\mathcal{F}) \cong \text{mod-}A. \tag{6.23}$$

Any finitely generated module over a Noetherian ring is finitely presented. We need the following general lemma to invoke the localisation.

Lemma 6.2.4. *Let M be a finitely presented A-module. Then there is a finitely presented graded $A[x]$-module N such that $\mathcal{F}(N) \cong M$, where $\mathcal{F}$ is the functor defined in (6.18).*

Proof. Since $\mathcal{F} = - \otimes_{A[x]} A$, for any free A-module M of rank k we have $\mathcal{F}(\bigoplus_k A[x]) \cong \bigoplus_k A \cong M$. Let M be a finitely presented A-module. Then there is an exact sequence

$$\bigoplus_m A \xrightarrow{f} \bigoplus_n A \longrightarrow M \longrightarrow 0.$$

If we show that there are graded free $A[x]$-modules F_1 and F_2 and a graded homomorphism $g : F_1 \to F_2$ such that the following diagram is commutative,

$$\begin{tikzcd}
\mathcal{F}(F_1) \arrow[r, "\mathcal{F}(g)"] \arrow[d, "\cong"'] & \mathcal{F}(F_2) \arrow[d, "\cong"] \\
\bigoplus_m A \arrow[r, "f"] & \bigoplus_n A,
\end{tikzcd}$$

then since $\mathcal{F}$ is right exact, one can complete the diagram

$$\begin{tikzcd}
\mathcal{F}(F_1) \arrow[r, "\mathcal{F}(g)"] \arrow[d, "\cong"'] & \mathcal{F}(F_2) \arrow[r] \arrow[d, "\cong"] & \mathcal{F}(\text{coker}(g)) \arrow[r] \arrow[d, dotted, "\cong"] & 0 \\
\bigoplus_m A \arrow[r, "f"] & \bigoplus_n A \arrow[r] & M \arrow[r] & 0.
\end{tikzcd}$$

This shows $\mathcal{F}(\mathrm{coker}(g)) \cong M$ and we will be done.

Thus, suppose $f : \bigoplus_m A \to \bigoplus_n A$ is an A-module homomorphism. With the misuse of notation, denote the standard basis for both A-modules A^m and A^n, by e_i. Then

$$f(e_i) = \sum_{j=1}^{n} a_{ij} e_j,$$

where $1 \leq i \leq m$ and $a_{ij} \in A$. Decomposing a_{ij} into its homogeneous components, $a_{ij} = \sum_k a_{ij_k}$, since the number of a_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$, are finite, there is $N \in \mathbb{N}$ such that $a_{ij_N} = 0$ for all i, j. Now consider the graded free $A[x]$-modules $\bigoplus_m A[x](-N)$ and $\bigoplus_n A[x]$, with the standard bases (which are of degrees N and 0 in $\bigoplus_m A[x](-N)$ and $\bigoplus_n A[x]$, respectively). Define

$$\begin{aligned} g : \bigoplus_m A[x](-N) &\longrightarrow \bigoplus_n A[x], \\ e_i &\longmapsto \sum_j \big(\sum_k x^{N-k} a_{ij_k}\big) e_j. \end{aligned}$$

This is a graded $A[x]$-module homomorphism. Further, $\mathcal{F}(g)$ amount to evaluation of g at $x = 1$ which coincides with the map f. This finishes the proof. $\square$

Remark 6.2.5. The proof of Lemma 6.2.4, in particular, shows that if $f \in \mathbb{M}_n(A)$, then there is a $g \in \mathbb{M}_n(A[x])(\overline{\delta})$ such that evaluation of g at 1, gives f.

Lemma 6.2.4 along with standard results of localisation theory (see [88, Theorem 5.11]) implies that

$$\text{gr-}\,A[x]/\ker(\mathcal{F}) \cong \text{mod-}\,A.$$

Now we are ready to apply the localisation theorem (§6.2.1) to the sequence

$$\ker(\mathcal{F}) \hookrightarrow \text{gr-}\,A[x] \longrightarrow \text{gr-}\,A[x]/\ker(\mathcal{F}) \cong \text{mod-}\,A,$$

to obtain a long exact sequence

$$\cdots \longrightarrow K_{n+1}(A) \longrightarrow K_n(\ker(\mathcal{F})) \longrightarrow K_n^{\mathrm{gr}}(A[x]) \longrightarrow K_n(A) \longrightarrow \cdots .$$

Now using (6.22) to replace $K_n(\ker(\mathcal{F}))$, we get

$$\cdots \longrightarrow K_{n+1}(A) \longrightarrow K_n^{\mathrm{gr}}(A) \xrightarrow{\overline{i}} K_n^{\mathrm{gr}}(A[x]) \longrightarrow K_n(A) \longrightarrow \cdots . \tag{6.24}$$

Next, we show that the homomorphism $\overline{i}$ is injective. Note that the homomorphism $\overline{i}$ on the level of K-groups induced by the composition of exact functors (see (6.15) and (6.21)),

$$\text{Pgr-}A \hookrightarrow \text{gr-}\,A \longrightarrow \mathcal{C}_1 \hookrightarrow \ker(\mathcal{F}) \hookrightarrow \text{gr-}\,A[x],$$

which send a finitely generated graded projective A-module M, to M considered as graded $A[x]$-module. Define three exact functors $\phi_i : \text{Pgr-}A \to \text{gr-}\,A[x]$, $1 \leq i \leq 3$, as follows,

$$\begin{aligned} \phi_1(M) &= M[x](-1) \\ \phi_2(M) &= M[x] \\ \phi_3(M) &= M. \end{aligned} \tag{6.25}$$

Now the exact sequence 6.16 and the exact sequence of the functors Theorem (§6.2.1) immediately give,

$$\overline{\phi}_3 = \overline{\phi}_2 - \overline{\phi}_1. \tag{6.26}$$

Note that $\overline{\phi}_3 = \overline{i}$.

Now invoking Quillen's Theorem 6.1.1, from exact sequence 6.24, we get

$$\cdots \longrightarrow K_{n+1}(A) \longrightarrow \mathbb{Z}[x, x^{-1}] \otimes K_n(A_0) \xrightarrow{\overline{i}} \mathbb{Z}[x, x^{-1}] \otimes K_n(A_0) \longrightarrow K_n(A) \longrightarrow \cdots. \tag{6.27}$$

Further, the maps $\overline{\phi}_2$ and $\overline{\phi}_1$ becomes $1 \otimes 1$ and $t \otimes 1$, respectively. Thus from (6.26) we get

$$\overline{i} = (1 - t) \otimes 1 : \mathbb{Z}[x, x^{-1}] \otimes_{\mathbb{Z}} K_n(A_0) \longrightarrow \mathbb{Z}[x, x^{-1}] \otimes_{\mathbb{Z}} K_n(A_0).$$

This shows that $\overline{i}$ is an injective map. Thus the exact sequence 6.27 reduces to

$$0 \longrightarrow \mathbb{Z}[x, x^{-1}] \otimes_{\mathbb{Z}} K_n(A_0) \xrightarrow{\overline{i}} \mathbb{Z}[x, x^{-1}] \otimes_{\mathbb{Z}} K_n(A_0) \longrightarrow K_n(A) \longrightarrow 0. \tag{6.28}$$

Again, from the description of $\overline{i}$ it follows,

$$K_n(A) \cong K_n(A_0).$$

6.3 Relating K_*^{gr} to K_*

In [19], van den Bergh, following the methods to prove the fundamental theorem of K-theory (§6.2.2), established a long exact sequence relating graded K-theory of a ring to non-graded K-theory in the case of right regular Noetherian $\mathbb{Z}$-graded rings. In this section we will present van den Bergh's observation. As a consequence, we will see that in the case of graded Grothendieck group, the shifting of modules is all the difference between the K_0^{gr} and K_0-groups (Corollary 6.3.2).

Let A be a right regular Noetherian $\mathbb{Z}$-graded ring. Thus G^{gr}-theory and G-theory become K^{gr}-theory and K-theory (see Example 6.2.2). Consider the $\mathbb{Z}$-graded ring $A[y]$, where $\deg(y) = 1$ and $\mathbb{Z} \times \mathbb{Z}$-graded ring $B = A[t]$, where the homogeneous components defined by $B_{(n,m)} = A_m t^n$. Furthermore, consider the $\mathbb{Z} \times \mathbb{Z}$-graded rings $B[z]$ and $B[z, z^{-1}]$, where $\deg(z) = (1, -1)$. It is easy to check that

(1) The support of B is $\mathbb{N} \times \mathbb{Z}$ and $B_{(0,-)} = A$ as $\mathbb{Z}$-graded rings.

(2) The support of $B[z]$ is $\mathbb{N} \times \mathbb{Z}$ and $B[z]_{(0,-)} = A$ as $\mathbb{Z}$-graded rings.

(3) $B[z, z^{-1}]$ is a $\mathbb{Z} \times \mathbb{Z}$-strongly graded ring and $\mathrm{Gr}^{\mathbb{Z}\times\mathbb{Z}}\text{-}B[z, z^{-1}] \approx \mathrm{Gr}^{\mathbb{Z}}\text{-}A[y]$.

Proof. We first show that $B[z, z^{-1}]$ is a $\mathbb{Z} \times \mathbb{Z}$-strongly graded ring, by proving for any $(n, m) \in \mathbb{Z} \times \mathbb{Z}$,

$$1 \in B[z, z^{-1}]_{(n,m)} B[z, z^{-1}]_{(-n,-m)}.$$

But note that if $m \geq 0$, then $B_{(m+n,0)} z^{-m} \in B[z, z^{-1}]_{(n,m)}$ and $B_{(-m-n,0)} z^m \in B[z, z^{-1}]_{(-n,-m)}$. Further, $B_{(m+n,0)} = A_0 t^{m+n}$ and $B_{(-m-n,0)} = A_0 t^{-m-n}$. This shows that

$$1 \in A_0 \subseteq B[z, z^{-1}]_{(n,m)} B[z, z^{-1}]_{(-n,-m)}.$$

Now by Example 1.5.8,

$$\mathrm{Gr}^{\mathbb{Z}\times\mathbb{Z}}\text{-}B[z, z^{-1}] \approx \mathrm{Gr}^{\mathbb{Z}}\text{-}B[z, z^{-1}]_{(0,-)}.$$

But it is easy to see that $B[z, z^{-1}]_{(0,-)} \cong_{\mathrm{gr}} A[y]$ as $\mathbb{Z}$-graded rings. ∎

By Example 6.2.2, we have a long exact sequence

$$\cdots \longrightarrow K_{n+1}^{\mathrm{gr}}(B[z,z^{-1}]) \longrightarrow K_n^{\mathrm{gr}}(B) \xrightarrow{\bar{i}} K_n^{\mathrm{gr}}(B[z]) \longrightarrow K_n^{\mathrm{gr}}(B[z,z^{-1}]) \longrightarrow \cdots. \tag{6.29}$$

By Corollary 6.1.4 and (1) and (2) above,

$$K_n^{\mathrm{gr}}(B) = K_n^{\mathbb{Z}\times\mathbb{Z}}(B) \cong \mathbb{Z}[x,x^{-1}] \otimes K_n^{\mathbb{Z}}(B_{(0,-)}) \cong \mathbb{Z}[x,x^{-1}] \otimes K_n^{\mathrm{gr}}(A)$$

and

$$K_n^{\mathrm{gr}}(B[z]) = K_n^{\mathbb{Z}\times\mathbb{Z}}(B[z]) \cong \mathbb{Z}[x,x^{-1}] \otimes K_n^{\mathbb{Z}}(B[z]_{(0,-)}) \cong \mathbb{Z}[x,x^{-1}] \otimes K_n^{\mathrm{gr}}(A).$$

Furthermore, from (3) we have

$$K_n^{\mathrm{gr}}(B[z,z^{-1}]) = K_n^{\mathbb{Z}\times\mathbb{Z}}(B[z,z^{-1}]) \cong K_n^{\mathbb{Z}}(A[y]) = K_n^{\mathrm{gr}}(A[y]).$$

So, from the sequence 6.29, we get

$$\cdots \longrightarrow K_{n+1}^{\mathrm{gr}}(A[y]) \longrightarrow \mathbb{Z}[x,x^{-1}] \otimes K_n^{\mathrm{gr}}(A) \xrightarrow{\bar{i}} \mathbb{Z}[x,x^{-1}] \otimes K_n^{\mathrm{gr}}(A) \longrightarrow K_n^{\mathrm{gr}}(A[y]) \longrightarrow \cdots. \tag{6.30}$$

The rest is similar to the method used to prove the fundamental theorem in §6.2.2. First note that we have a short exact sequence of $\mathbb{Z} \times \mathbb{Z}$-graded $S[z]$-modules (compare this with (6.16))

$$0 \longrightarrow M[z](-1,1) \xrightarrow{z} M[z] \xrightarrow{e_0} M \longrightarrow 0. \tag{6.31}$$

Here M is a graded S-module which becomes a graded $S[y]$-module under the evaluation map $e_0 : S[y] \to S, y \mapsto 0$. Again, similar to (6.25), defining

$$\begin{aligned} \phi_1(M) &= M[z](-1,1) \\ \phi_2(M) &= M[z] \\ \phi_3(M) &= M, \end{aligned}$$

the exact sequence 6.31 and the exact sequence of the functors Theorem (§6.2.1) immediately give,

$$\overline{\phi}_3 = \overline{\phi}_2 - \overline{\phi}_1, \tag{6.32}$$

and $\overline{\phi}_3 = \bar{i}$. One can then observe that $\bar{i}$ is injective, so the exact sequence 6.30 reduces to

$$0 \longrightarrow \mathbb{Z}[x,x^{-1}] \otimes K_n^{\mathrm{gr}}(A) \xrightarrow{\bar{i}} \mathbb{Z}[x,x^{-1}] \otimes K_n^{\mathrm{gr}}(A) \longrightarrow K_n^{\mathrm{gr}}(A[y]) \longrightarrow 0.$$

Finally, from this exact sequence it follows that

$$K_n^{\mathrm{gr}}(A[y]) \cong K_n^{\mathrm{gr}}(A). \tag{6.33}$$

We are in a position to relate graded K-theory to non-graded K-theory.

Theorem 6.3.1. *Let A be a right regular Noetherian $\mathbb{Z}$-graded ring. Then there is a long exact sequence*

$$\cdots \longrightarrow K_{n+1}(A) \longrightarrow K_n^{\mathrm{gr}}(A) \xrightarrow{\bar{i}} K_n^{\mathrm{gr}}(A) \xrightarrow{U} K_n(A) \longrightarrow \cdots. \tag{6.34}$$

Here $\bar{i} = \bar{s} - 1$, where $s : \mathrm{Gr}\text{-}A \to \mathrm{Gr}\text{-}A, M \mapsto M(1)$ and U is the forgetful functor.

Proof. By Example 6.2.2, we have a long exact sequence

$$\cdots \longrightarrow K^{\mathrm{gr}}_{n+1}(A[y,y^{-1}]) \longrightarrow K^{\mathrm{gr}}_n(A) \xrightarrow{i} K^{\mathrm{gr}}_n(A[y]) \longrightarrow K^{\mathrm{gr}}_n(A[y,y^{-1}]) \longrightarrow \cdots.$$

By (6.33), $K^{\mathrm{gr}}_n(A[y]) \cong K^{\mathrm{gr}}_n(A)$. Since $A[y,y^{-1}]$ is strongly graded, by Dade's Theorem 1.5.1, $K^{\mathrm{gr}}_n(A[y,y^{-1}]) \cong K_n(A)$. Replacing these into above long exact sequence, the theorem follows. □

The following is an immediate corollary of the Theorem 6.3.1. It shows that for the graded regular Noetherian rings, the shifting of modules is all the difference between the graded Grothendieck group and the usual Grothendieck group.

Corollary 6.3.2. *Let A be a right regular Noetherian $\mathbb{Z}$-graded ring. Then*

$$K^{\mathrm{gr}}_0(A)/\langle [P(1)] - [P] \rangle \cong K_0(A),$$

where P is a graded projective A-module.

Proof. The long exact sequence 6.3.1, for $n = 0$, reduces to

$$K^{\mathrm{gr}}_0(A) \xrightarrow{[P]\mapsto[P(1)]-[P]} K^{\mathrm{gr}}_0(A) \xrightarrow{U} K_0(A) \longrightarrow 0.$$

The corollary is now immediate. □

Remark 6.3.3. Both the fundamental theorem (§6.2.2) and Theorem 6.3.1 can be written for the case of graded coherent regular rings, as it was demonstrated by Gersten in [37] for the non-graded case.

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