NON–HYPERBOLIC SPLITTING OF QUADRATIC PAIRS

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Abstract. We show that a non–hyperbolic quadratic pair on a central simple algebra Brauer equivalent to a quaternion algebra stays non–hyperbolic over some splitting field of the quaternion algebra. This extends a result previously only known for fields of characteristic different from two. Our presentation is free from restrictions on the characteristic of the base field.

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1. Introduction

It was proven independently in [16, (3.3)] and [6] that a non-hyperbolic orthogonal involution on a central simple algebra of Schur index two over a field of characteristic different from two cannot become hyperbolic after scalar extension to a generic splitting field of the algebra. This was generalised in [12] by dropping the assumption on the Schur index. In [16] a stronger version, addressing isotropy in place of hyperbolicity, was proven, which is still open in the case of Schur index larger than two.

In characteristic two, orthogonal involutions are never hyperbolic. This motivates replacing hyperbolicity by metabolicity, as both condition are equivalent in characteristic different from two. However, the splitting behaviour of orthogonal involutions in characteristic two is quite particular. In [8] it is shown that a non-metabolic orthogonal algebra with involution over a field of characteristic two and of Schur index two may become metabolic over every splitting field of the algebra.

Orthogonal involutions correspond after splitting to symmetric bilinear forms, up to similarity. In characteristic different from two, symmetric bilinear forms and quadratic forms are equivalent concepts, but this is not the case in characteristic two. This motivates the search for objects that correspond to quadratic forms in an analogous way to how involutions correspond to symmetric bilinear forms. This led to the introduction of quadratic pairs in [14, Section 5].

In this article we present a characteristic free version of the aforementioned result, with quadratic pairs in place of orthogonal involutions. This covers the original result as the two concepts are equivalent in characteristic different from two. We approach the problem by working directly with the involutions and associated maps involved in quadratic pairs, rather than using that they are associated with hermitian or generalised quadratic forms. The presentation is free from any assumption on the characteristic of the base field. It also provides a different proof of the statement in characteristic different from two.

In [16] and [6], the statements and proofs are given in terms of hermitian forms and exact sequences of the corresponding Witt groups, rather than in terms of involutions. The result for orthogonal involutions can be obtained by taking the adjoint involution to the hermitian forms. It is possible to adapt this approach directly by using the notion of generalised quadratic forms over an algebra with involution from [2]. In our approach we emphasise the role of involutions as independently from hermitian form theory as much as possible.

A major motivation for this work stems from the crucial role of the result within the proof of the Pfister Factor Conjecture in [4] and of the corresponding characteristic free version for quadratic pairs in [9].

2. Quadratic forms

In this section we recall the basic terminology and results we use from quadratic form theory. We refer to [10, Chapters 1 and 2] as a general reference on symmetric bilinear and quadratic forms.

For two objects α and β in a certain category, we write $\alpha \simeq \beta$ to indicate that they are *isomorphic*, i.e. that there exists an isomorphism between them. This applies in particular to algebras with involution or with quadratic pair, but also to quadratic, bilinear and hermitian forms, where the corresponding isomorphisms are usually called *isometries*.

Throughout, let F be a field. We denote the characteristic of F by $\mathrm{char}(F)$ and the multiplicative group of F by $F^\times$.

An F–bilinear map $b : V \times V \longrightarrow F$ on a finite dimensional F–vector space V is called *degenerate* if there exists $x \in V \setminus \{0\}$ such that $b(x, y) = 0$ for all $y \in V$ and *nondegenerate* otherwise.

A *bilinear form over* F is a pair (V, b) where V is a finite dimensional F–vector space and b is a nondegenerate F–bilinear map $b : V \times V \to F$. Let $\varphi = (V, b)$ be a bilinear form over F. We say that φ is *symmetric* if $b(x, y) = b(y, x)$ for all $x, y \in V$, and *alternating* if $b(x, x) = 0$ for all $x \in V$. We call $\dim_F(V)$ the *dimension of* φ and denote it by $\dim(\varphi)$. For $c \in F^\times$ we denote by $c\varphi$ the bilinear form (V, cb), where $(cb)(x, y) = c(b(x, y))$ for $x, y \in V$. Through the map $0 \times 0 \longrightarrow 0$ we obtain a bilinear form of dimension zero, which we call the *zero form*.

Let $\varphi = (V, b)$ and $\psi = (W, b')$ be two symmetric or alternating bilinear forms over F. By an *isometry of bilinear forms* $f : \varphi \longrightarrow \psi$ we mean an isomorphism of F–vector spaces $f : V \longrightarrow W$ such that $b(x, y) = b'(f(x), f(y))$ for all $x, y \in V$. If such an isometry exists, we say φ and ψ are *isometric* and write $\varphi \simeq \psi$. We say that φ and ψ are *similar* if there exists $c \in F^\times$ such that $\varphi \simeq c\psi$. The *orthogonal sum of* φ *and* ψ is defined to be the pair $(V \times W, b'')$ where the map $b'' : (V \times W) \times (V \times W) \to F$ is given by $b''((v_1, w_1), (v_2, w_2)) = b(v_1, v_2) + b'(w_1, w_2)$ for all $v_1, v_2 \in V$ and $w_1, w_2 \in W$, and we write $\varphi \perp \psi = (V \times W, b'')$. The *tensor product of* φ *and* ψ is defined to be the pair $(V \otimes W, b'')$ where the F–bilinear map $b'' : (V \otimes W) \times (V \otimes W) \to F$ is given by $b''(v_1 \otimes w_1, v_2 \otimes w_2) = b(v_1, v_2) \cdot b'(w_1, w_2)$ for all $v_1, v_2 \in V$ and $w_1, w_2 \in W$, and we write $\varphi \otimes \psi = (V \otimes W, b'')$.

Let $\varphi = (V, b)$ be a bilinear form over F. We say φ is *isotropic* if there exists an $x \in V \setminus \{0\}$ such that $b(x, x) = 0$. Otherwise we say that φ is *anisotropic*. We call a subspace $W \subset V$ *totally isotropic* (with respect to b) if $b|_{W \times W} = 0$. We call φ *metabolic* if it has a totally isotropic subspace W with $\dim_F(W) = \frac{1}{2}\dim(\varphi)$. Note that an alternating form is always metabolic.

For $a_1,\dots,a_n \in F^\times$ we denote by $\langle a_1,\dots,a_n\rangle$ the symmetric bilinear space (F^n,b) where

$$b: F^n\times F^n \to F,\qquad (x,y)\mapsto \sum_{i=1}^n a_ix_iy_i.$$

We call such a form *diagonalised*. A symmetric bilinear space that is isometric to a diagonalised form is called *diagonalisable*.

For $a\in F^\times$ we denote $\langle 1,-a\rangle$ by $\langle\langle a\rangle\rangle$. Let $m\in\mathbb{N}$. For $a_1,\dots,a_m\in F^\times$ we denote by $\langle\langle a_1,\dots,a_m\rangle\rangle$ the symmetric bilinear form $\langle\langle a_1\rangle\rangle\otimes\ldots\otimes\langle\langle a_m\rangle\rangle$. We call any bilinear form isometric to $\langle\langle a_1,\dots,a_m\rangle\rangle$ for some $a_1,\dots,a_m\in F$ a *bilinear m–fold Pfister form*. We consider $\langle 1\rangle$ as the 0-fold bilinear Pfister form.

By a *quadratic form over F* we mean a pair (V,q) of a finite dimensional F-vector space V and a map $q:V\to F$ such that $q(\lambda x)=\lambda^2 q(x)$ for all $x\in V$ and $\lambda\in F$, and such that $b_q: V\times V\to F, (x,y)\longmapsto q(x+y)-q(x)-q(y)$ is F–bilinear and nondegenerate. (Hence, q is assumed to be nonsingular in the sense of [10, p. 42].) Then (V,b_q) is a symmetric bilinear form over F, called the *polar form of* (V,q).

Consider a quadratic form $\rho=(V,q)$ over F. We call $\dim_F(V)$ the *dimension of* ρ and denote it by $\dim(\varphi)$. We say ρ is *isotropic* if $q(x)=0$ for some $x\in V\setminus\{0\}$ and *anisotropic* otherwise. By *a totally isotropic subspace of ρ* we mean an F–subspace W of V such that $q|_W=0$. If there exists a totally isotropic subspace W of ρ such that $\dim_F(W)=\frac{1}{2}\dim(\rho)$, we say that ρ is *hyperbolic*. We say that ρ *represents an element* $a\in F^\times$ if there exists an $x\in V\setminus\{0\}$ such that $q(x)=a$. For $c\in F^\times$ let $c\rho$ denote the quadratic form (V,cq), where $(cq)(x)=c(q(x))$ for $x\in V$. The map $0\longrightarrow 0$ yields a quadratic form of dimension zero, which we call the *zero form*.

Let $\rho_1=(V,q)$ and $\rho_2=(W,q')$ be two quadratic forms over F. By an *isometry of quadratic forms* $\phi:\rho_1\longrightarrow\rho_2$ we mean an isomorphism of F–vector spaces $f:V\longrightarrow W$ such that $q=q'\circ f$. If such an isometry exists, we say ρ_1 and ρ_2 are *isometric* and write $\rho_1\simeq\rho_2$. We say that ρ_1 and ρ_2 are *similar* if there exists a $c\in F^\times$ such that $\rho_1\simeq c\rho_2$. The *orthogonal sum of the quadratic forms ρ_1 and ρ_2* is defined to be pair $(V\times W,q'')$ where the map $q'':(V\times W)\to F$ is given by $q''((v,w))=q'(v)+q(w)$ for all $v\in V$ and $w\in W$, and we write $\rho_1\perp\rho_2=(V\times W,q'')$.

Let $\varphi=(V,b)$ be a symmetric or alternating bilinear form over F and $\rho=(W,q)$ be a quadratic form over F. There is a natural F–linear map $b\otimes q: V\otimes_F W\to F$ determined by the rule that $(b\otimes q)(w\otimes v)=b(v,v)\cdot q(w)$ for all $w\in W, v\in V$, and $(V\otimes_F W, b\otimes q)$ is a quadratic form over F, called the *tensor product of φ and ρ* and denoted $\varphi\otimes\rho$.

Consider $a\in F$ with $4a\neq -1$. We denote by $\langle\langle a]]$ the 2-dimensional quadratic form $(F\times F,q)$ with

$$q: F\times F\to F, (x,y)\mapsto x^2+xy+ay^2.$$

Such a form is called a 1*–fold quadratic Pfister form.* Let m be a positive integer. By an *m–fold (quadratic) Pfister form over F* we mean a quadratic form over F that is isometric to the tensor product of a 1–fold quadratic Pfister form over F and an $(m-1)$-fold bilinear Pfister form over F. For $a_1,\dots,a_m\in F$ we denote by $\langle\langle a_1,\dots,a_m]]$ the m–fold Pfister form $\langle\langle a_1,\dots,a_{m-1}\rangle\rangle\otimes\langle\langle a_m]]$.

In the case where $\mathrm{char}(F)\neq 2$, the following is well known, see e.g. [10, (6.25)].

2.1. **Proposition.** *Let ρ be an anisotropic quadratic Pfister form over F and φ a symmetric bilinear form over F. Then there exist symmetric bilinear forms φ_1 and*

φ_2 over F with $\varphi \otimes \rho \simeq (\varphi_1 \otimes \rho) \perp (\varphi_2 \otimes \rho)$ such that $\varphi_1 \otimes \rho$ is anisotropic and $\varphi_2 \otimes \rho$ is hyperbolic.

Proof. If $\varphi \otimes \rho$ is anisotropic or hyperbolic, then the statement holds trivially, as we can choose each of the forms φ_1 and φ_2 to be φ or the zero form, in the appropriate order. Assume now that $\varphi \otimes \rho$ is isotropic, but not hyperbolic. Then φ is non-alternating and hence diagonalisable, by [10, (1.17)]. Let $a_1, \ldots, a_n \in F^\times$ be such that $\varphi \simeq \langle a_1, \ldots, a_n \rangle$. As $\varphi \otimes \rho$ is isotropic, ρ represents elements $b_1', \ldots, b_n' \in F$ not all equal to zero and such that $\sum_{i=1}^n a_i b_i' = 0$. For $i = 1, \ldots, n$, replacing among the elements $b_1', \ldots, b_n'$ any 0 by 1, we obtain nonzero elements $b_1, \ldots, b_n \in F^\times$ represented by ρ such that $\langle a_1 b_1, \ldots, a_n b_n \rangle$ is isotropic. By [10, (9.9)] for $i = 1, \ldots, n$ we have $b_i \rho \simeq \rho$. Hence

$$\varphi \otimes \rho \simeq a_1 \rho \perp \ldots \perp a_n \rho \simeq a_1 b_1 \rho \perp \ldots \perp a_n b_n \rho \simeq \langle a_1 b_1, \ldots, a_n b_n \rangle \otimes \rho.$$

As $\langle a_1 b_1, \ldots, a_n b_n \rangle$ is isotropic, there exists a symmetric bilinear form φ' and an element $c \in F^\times$ such that $\langle a_1 b_1, \ldots, a_n b_n \rangle \simeq \varphi' \perp \langle c, -c \rangle$. Hence

$$\varphi \otimes \rho \simeq \varphi' \otimes \rho \perp \langle c, -c \rangle \otimes \rho\,.$$

As $\langle c, -c \rangle \otimes \rho$ is hyperbolic and $\dim(\varphi') = \dim(\varphi) - 2$, an induction argument shows that φ is of the desired form. □

Recall that, if $\mathrm{char}(F) \neq 2$, then the discriminant of a quadratic form ρ over F is defined as the class in $F^\times / F^{\times 2}$ given by $(-1)^m d$ with $m = \binom{\dim(\rho)}{2}$ and where d is the determinant of the Gram matrix (with respect to an arbitrary basis) of the associated polar form of ρ (see [10, (13.5)]).

Suppose that $\mathrm{char}(F) = 2$. For $a \in F$ we write $\wp(a) = a^2 + a$. Then $\wp(F) = \{\wp(a) \mid a \in F\}$ is an additive subgroup of F. For every quadratic form ρ over F there exist $n \in \mathbb{N}$, $a_1, \ldots, a_n \in F$ and $b_1, \ldots b_n \in F^\times$ such that $\rho \simeq b_1 \langle\langle a_1]] \perp \ldots \perp b_n \langle\langle a_n]]$ (see [10, (7.32)]). Then the *discriminant of* ρ is given as the class of $a_1 + \ldots + a_n$ in $F/\wp(F)$ (see [10, (13.5)]).

In either case, the discriminant of a quadratic form ρ over F, denoted by $\Delta(\rho)$, is an element of some group given by F together with one of the field operations, either multiplication or addition, depending on the characteristic.

2.2. **Proposition.** *Any hyperbolic quadratic form over F has trivial discriminant. The converse holds for quadratic forms of dimension 2.*

Proof. This follows from [10, (13.3), (13.4) and (13.5)]. □

2.3. **Corollary.** *Let ρ be a non-hyperbolic two-dimensional quadratic form and φ a symmetric bilinear form over F such that $\varphi \otimes \rho$ is hyperbolic. Then $\dim(\varphi)$ is even.*

Proof. By (2.2) the form $\varphi \otimes \rho$ has trivial discriminant, whereas the form ρ has non-trivial discriminant. If $\mathrm{char}(F) \neq 2$, then $\Delta(\varphi \otimes \rho) = \Delta(\rho)^{\dim(\varphi)}$ in $F^\times / F^{\times 2}$. If $\mathrm{char}(F) = 2$, then $\Delta(\varphi \otimes \rho) = \dim(\varphi) \cdot \Delta(\rho)$ in $F/\wp F$. Hence, in either case we conclude that $\dim(\varphi)$ is even. □

Let $\rho = (V, q)$ be a quadratic form over F and let K/F be a field extension. Then we write $\rho_K = (V \otimes_F K, q_K)$ where the quadratic map $q_K : V \otimes_F K \longrightarrow K$ is determined by having $q_K(v \otimes k) = k^2 q(v)$ for all $v \in V$ and $k \in K$.

By a *rational function field over F* we mean a field extension K/F such that there exist $n \in \mathbb{N}$ and elements $t_1, \ldots, t_n \in K$ that are algebraically independent over F

such that $K = F(t_1, \dots, t_n)$; in this case K/F is finitely generated of transcendence degree n.

2.4. **Proposition.** *Let ρ be a quadratic form over F and K/F a rational function field. Then ρ_K is isotropic (resp. hyperbolic) if and only if ρ is isotropic (resp. hyperbolic).*

Proof. This follows from [10, (7.15) and (8.5)]. □

Let ρ be a quadratic form over F. If $\dim(\rho) \geqslant 3$ or if ρ is a anisotropic 2-dimensional form, then we call the function field of the projective quadric over F given by ρ the *function field of* ρ and denote it by $F(\rho)$. In the remaining cases we set $F(\rho) = F$. This agrees with the definition in [10, Section 22]. By [10, (22.9)] then $F(\rho)/F$ is a rational function field if and only if ρ is isotropic over F.

3. Algebras with involution

We refer to [17] as a general reference on finite-dimensional algebras over fields, and for central simple algebras in particular, and to [14] for involutions.

Let A be an (associative) F–algebra. We denote the centre of A by $Z(A)$. For a field extension K/F, the K–algebra $A \otimes_F K$ is denoted by A_K. For $a \in A^\times$ we denote by $\mathrm{Int}(a) : A \longrightarrow A$ the *inner automorphism* given by $c \mapsto aca^{-1}$. An element $e \in A$ is called an *idempotent* if $e^2 = e$. An *F–involution* on A is an F–linear map $\sigma : A \to A$ such that $\sigma(xy) = \sigma(y)\sigma(x)$ for all $x, y \in A$ and $\sigma^2 = \mathrm{id}_A$.

Assume now that A is finite-dimensional and simple (i.e. it has no nontrivial two sided ideals). Then $Z(A)$ is a field, and by Wedderburn's Theorem (see [14, (1.1)]) we have that $A \simeq \mathrm{End}_D(V)$ for an F–division algebra D and a right D–vector space V, and furthermore $\dim_{Z(A)}(A)$ is a square number, whose positive square root is called the *degree of* A and is denoted $\deg(A)$. The degree of D is called the *index of* A and is denoted $\mathrm{ind}(A)$. We call A *split* if $\mathrm{ind}(A) = 1$. We call a field extension K/F a *splitting field of* A if A_K is split. If $Z(A) = F$, then we call the F–algebra A *central simple.* Two central simple F–algebras A and B are called *Brauer equivalent* if A and B are isomorphic to endomorphism algebras of two right vector spaces over the same F–division algebra.

If A is a central simple F-algebra let $\mathrm{Trd}_A : A \longrightarrow F$ denote the reduced trace map and $\mathrm{Nrd}_A : A \longrightarrow F$ the reduced norm map (see [14, (1.6)] for the definitions).

3.1. **Lemma.** *If the F–algebra A is central simple, then for all $x \in A$ and $a \in A^\times$ we have*
$$\mathrm{Trd}_A(\mathrm{Int}(a)(x)) = \mathrm{Trd}_A(x)\,.$$

Proof. By [14, (1.8)] we have that $\mathrm{Trd}_A(bc) = \mathrm{Trd}_A(cb)$ for all $b, c \in A$. Hence $\mathrm{Trd}_A(a(xa^{-1})) = \mathrm{Trd}_A((xa^{-1})a) = \mathrm{Trd}_A(x)\,.$ □

An *F–algebra with involution* is a pair (A, σ) of a finite-dimensional F–algebra A and an F–involution σ on A such that one has $F = \{x \in Z(A) \mid \sigma(x) = x\}$, and such that either A is simple or A is a product of two simple F–algebras that are mapped to one another by σ. In this situation, there are two possibilities: either $Z(A) = F$, so that A is a central simple F–algebra, or $Z(A)/F$ is a quadratic étale extension with σ restricting to the nontrivial F–automorphism of $Z(A)$. To distinguish these two situations, we speak of algebras with involution of the *first* and *second kind*: we say that the F–algebra with involution (A, σ) is of the *first kind* if $Z(A) = F$ and of the *second kind* otherwise. For more information on involutions of the second kind, also called *unitary involutions*, we refer to [14, Section 2.B].

Let (A, σ) be an F–algebra with involution. If $Z(A)$ is a field, then A is a central simple $Z(A)$–algebra, and we say that (A, σ) is *split* if A is split as $Z(A)$–algebra. If $Z(A) \simeq F \times F$, then $(A, \sigma) \simeq (B \times B^{\mathsf{op}}, \varepsilon)$ where B is a central simple F–algebra, B^{op} is its opposite algebra, and ε is the map exchanging the components of elements of $B \times B^{\mathsf{op}}$; in this situation we say that (A, σ) is *split* if B is split as an F–algebra. Given a field extension K/F, we abbreviate $\sigma_K = \sigma \otimes \mathrm{id}_K$ and obtain a K–algebra with involution $(A, \sigma)_K = (A_K, \sigma_K)$, and we call K a *splitting field* of (A, σ) if $(A, \sigma)_K$ is split.

Let (A, σ) and (B, τ) be F–algebras with involution. Letting $(\sigma \otimes \tau)(a \otimes b) = \sigma(a) \otimes \tau(b)$ for $a \in A$ and $b \in B$ determines an F–involution $\sigma \otimes \tau$ on the F–algebra $A \otimes_F B$. We denote the pair $(A \otimes_F B, \sigma \otimes \tau)$ by $(A, \sigma) \otimes (B, \tau)$. By a *homomorphism of algebras with involution* $\Phi : (A, \sigma) \to (B, \tau)$ we mean an F–algebra homomorphism $\Phi : A \to B$ satisfying $\Phi \circ \sigma = \tau \circ \Phi$. An injective homomorphism is called an *embedding*, a bijective homomorphism is called an *isomorphism*.

To every (nondegenerate) symmetric or alternating bilinear form $\varphi = (V, b)$ over F we can associate a split F–algebra with involution of the first kind in the following way. There is a unique F–involution σ on $\mathrm{End}_F(V)$ such that

$$b(x, f(y)) = b(\sigma(f)(x), y) \quad \text{for all } x, y \in V \text{ and } f \in \mathrm{End}_F(V);$$

this involution σ is called the *adjoint involution of* b and denoted by ad_b. We use the notation $\mathrm{Ad}(\varphi) = (\mathrm{End}_F(V), \mathrm{ad}_b)$, and we call this the *$F$–algebra with involution adjoint to* φ. Conversely, to any split F–algebra with involution of the first kind (A, σ), we can find a bilinear form φ over F such that $(A, \sigma) \simeq \mathrm{Ad}(\varphi)$ (see [14, (2.1)]). Let φ and ψ be symmetric bilinear forms over F. Then it is easy to show that $\mathrm{Ad}(\varphi \otimes \psi) \simeq \mathrm{Ad}(\varphi) \otimes \mathrm{Ad}(\psi)$. Further, if φ and ψ are similar then $\mathrm{Ad}(\varphi) = \mathrm{Ad}(\psi)$.

Let (A, σ) be an F–algebra with involution. We say (A, σ) is *isotropic* if there exists an element a in $A \backslash \{0\}$ such that $\sigma(a)a = 0$. If no such element exists, we call (A, σ) *anisotropic*. We call an idempotent $e \in A$ *hyperbolic* (resp. *metabolic*) *with respect to* σ if $\sigma(e) = 1 - e$ (resp. $\sigma(e)e = 0$ and $\dim_F eA = \frac{1}{2}\dim_F A$). Any hyperbolic idempotent is in particular metabolic (see [7, (4.9)]). We say that (A, σ) is *hyperbolic* (resp. *metabolic*) if A contains a hyperbolic (resp. metabolic) idempotent with respect to σ.

3.2. **Proposition.** *Let φ be a symmetric or alternating bilinear form over F. Then φ is metabolic (resp. isotropic) if and only if $\mathrm{Ad}(\varphi)$ is metabolic (resp. isotropic).*

Proof. See [7, (4.8)] for the statement on metabolicity; we show the statement on isotropy. Let $\varphi = (V, b)$. Suppose that φ is isotropic. Let $x \in V \backslash \{0\}$ be such that $b(x, x) = 0$. We may choose $f \in \mathrm{End}_F(V) \backslash \{0\}$ such that $f(V) = Fx$. Then $b((\mathrm{ad}_b(f) \circ f)(y), z) = b(f(y), f(z)) = 0$ for all $y, z \in V$. As φ is nondegenerate, it follows that $\mathrm{ad}_b(f) \circ f = 0$. Therefore $\mathrm{Ad}(\varphi)$ is isotropic. Conversely, suppose that there exists $f \in \mathrm{End}_F(V) \backslash \{0\}$ with $\mathrm{ad}_b(f) \circ f = 0$. We choose $x \in V \backslash \{0\}$ with $f(x) \neq 0$ and obtain that $b(f(x), f(x)) = b((\mathrm{ad}_b(f) \circ f)(x), x) = 0$. Hence φ is isotropic. $\square$

Let (A, σ) be an F–algebra with involution. The F–subspaces of A of *symmetric*, *skew-symmetric*, and *alternating elements*, respectively, are defined as follows:

$$\begin{aligned}\mathrm{Sym}(A,\sigma) &= \{a \in A \mid \sigma(a) = a\},\\ \mathrm{Skew}(A,\sigma) &= \{a \in A \mid \sigma(a) = -a\},\\ \mathrm{Alt}(A,\sigma) &= \{a - \sigma(a) \mid a \in A\}.\end{aligned}$$

Any split F–algebra with involution of the first kind is adjoint to a (nondegenerate) symmetric or alternating bilinear form over F. One divides algebras with involutions of the first kind into two *types*: An F–algebra with involution of the first kind is *symplectic* if it becomes adjoint to an alternating bilinear form over some splitting field, and *orthogonal* otherwise. In characteristic different from two, these types are distinguished by the dimensions of the spaces of symmetric and of alternating elements, whereas in characteristic two these dimensions do not depend on the type (see [14, (2.6)]).

4. Semi-traces and quadratic pairs

In this section we revisit quadratic pairs and assemble a couple of partially known results in a consistent setup. Quadratic pairs on a central simple algebra consist of an involution and a certain map defined on the elements that are symmetric under this involution. We first take a closer look at these maps.

Let (A, σ) be an F–algebra with involution of the first kind. We call an F–linear map $f : \mathrm{Sym}(A, \sigma) \to F$ a *semi–trace on* (A, σ) if it satisfies $f(x+\sigma(x)) = \mathrm{Trd}_A(x)$ for all $x \in A$.

4.1. **Lemma.** *Let $\ell \in A$ with $\ell + \sigma(\ell) = 1$. Then for all $x \in A$ we have*

$$\mathrm{Trd}_A(x) = \mathrm{Trd}_A(\ell(x + \sigma(x)))\,.$$

Proof. We use that $\mathrm{Trd}_A : A \longrightarrow F$ is F-linear and satisfies $\mathrm{Trd}_A(ab) = \mathrm{Trd}_A(ba)$ for all $a, b \in A$, and further that by [14, (2.3)] we have $\mathrm{Trd}_A(a) = \mathrm{Trd}_A(\sigma(a))$ for any $a \in A$. This yields that

$$\mathrm{Trd}_A(\ell(x+\sigma(x))) = \mathrm{Trd}_A(\ell x)+\mathrm{Trd}_A(\ell\sigma(x)) = \mathrm{Trd}_A(\ell x)+\mathrm{Trd}_A(\sigma(\ell)x) = \mathrm{Trd}_A(x)$$

for all $x \in A$. □

By [14, (2.6)], if $\mathrm{char}(F) = 2$ then an element $\ell \in A$ with $\ell+\sigma(\ell) = 1$ exists if and only if (A, σ) is symplectic. In any case, given an element $\ell \in A$ with $\ell + \sigma(\ell) = 1$, we write

$$\mathrm{Trd}_A^{\sigma,\ell} : \mathrm{Sym}(A, \sigma) \longrightarrow F,\ x \mapsto \mathrm{Trd}_A(\ell x)\,.$$

4.2. **Proposition.** *For any $\ell \in A$ with $\ell+\sigma(\ell) = 1$, the map $\mathrm{Trd}_A^{\sigma,\ell}$ is a semi–trace on (A, σ). Conversely, every semi–trace on (A, σ) is of this form. For $\ell, \ell' \in A$ with $\ell + \sigma(\ell) = \ell' + \sigma(\ell') = 1$, we have $\mathrm{Trd}_A^{\sigma,\ell} = \mathrm{Trd}_A^{\sigma,\ell'}$ if and only if $\ell' - \ell \in \mathrm{Alt}(A, \sigma)$.*

Proof. The first statement is obvious from (4.1). The remaining parts of the statement are proven in [14, (5.7)]; although there the case where $\mathrm{char}(F) \neq 2$ and (A, σ) is symplectic is excluded, the same proof applies. □

4.3. **Proposition.** *Let (A, σ) be an F–algebra with involution of the first kind. If $\mathrm{char}(F) \neq 2$, then the unique semi–trace on (A, σ) is given by $\frac{1}{2} \cdot \mathrm{Trd}_A|_{\mathrm{Sym}(A,\sigma)}$. If $\mathrm{char}(F) = 2$ and (A, σ) is orthogonal, then there exists no semi–trace on (A, σ).*

Proof. Note that the existence of a semi–trace on (A,σ) implies that Trd_A vanishes on $\mathrm{Skew}(A,\sigma)$. Hence, if $\mathrm{char}(F)=2$ and if there exists a semi–trace on (A,σ), then Trd_A vanishes on $\mathrm{Sym}(A,\sigma)$, and therefore (A,σ) is symplectic, by [14, (2.6)].

Assume now that $\mathrm{char}(F)\neq 2$. We may apply (4.2) with $\ell=\frac{1}{2}$ to obtain a semi–trace on (A,σ). Given a semi-trace f on (A,σ), for all $a\in\mathrm{Sym}(A,\sigma)$ we have $a=\frac{1}{2}(a+\sigma(a))$ and thus $f(a)=\frac{1}{2}f(a+\sigma(a))=\frac{1}{2}\cdot\mathrm{Trd}_A(a)$. □

An *F–algebra with quadratic pair* is a triple (A,σ,f) where (A,σ) is an F–algebra with involution, assumed to be orthogonal if $\mathrm{char}(F)\neq 2$ and symplectic if $\mathrm{char}(F)=2$, and where f is a semi–trace on (A,σ). Given two F–algebras with quadratic pair (A,σ,f) and (B,τ,g), a *homomorphism of F–algebras with quadratic pair* $\Phi:(A,\sigma,f)\to(B,\tau,g)$ is a homomorphism of the underlying F–algebras with involution satisfying $f=g\circ\Phi$; if Φ is bijective then it is an *isomorphism.*

We describe following [14, Sect. 5] how a (nonsingular) quadratic form gives rise to an adjoint algebra with quadratic pair. Let $\rho=(V,q)$ be a quadratic form over F with polar form (V,b_q). By declaring

$$(v_1\otimes w_1)*(v_2\otimes w_2)=b_q(w_1,v_2)\cdot(v_1\otimes w_2)\quad\text{for}\quad v_1,v_2,w_1,w_2\in V$$

a product $*$ is defined on $V\otimes_F V$, which makes $V\otimes_F V$ into an F–algebra. By declaring $\sigma(v\otimes w)=w\otimes v$ for $v,w\in V$ we obtain an F–involution σ on $V\otimes_F V$. Then by [14, (5.1)], the F–linear map $\Phi:V\otimes_F V\to\mathrm{End}_F(V)$ determined by

$$\Phi(u\otimes v)(w)=b_q(v,w)u\quad\text{for }\ u,v,w\in V$$

yields an isomorphism of F–algebras with involution $\mathrm{Ad}(V,b_q)\longrightarrow(V\otimes_F V,\sigma)$. According to [14, (5.11)] there is a unique semi–trace $f_q:\mathrm{Sym}(\mathrm{Ad}(V,b_q))\to F$ such that

$$f_q(\Phi(v\otimes v))=q(v)\quad\text{for }\ v\in V,$$

which yields an F–algebra with quadratic pair

$$\mathrm{Ad}(\rho)=(\mathrm{End}_F(V),\mathrm{ad}_{b_q},f_q)\,,$$

called the *adjoint quadratic pair to ρ.* We say that a quadratic pair (A,σ,f) is *adjoint to a quadratic form ρ* if $(A,\sigma,f)\simeq\mathrm{Ad}(\rho)$. By [14, (5.11)], to any split F–algebra with quadratic pair (A,σ,f), there exists a (nonsingular) quadratic form ρ over F such that $(A,\sigma,f)\simeq\mathrm{Ad}(\rho)$. If ρ_1 and ρ_2 are similar quadratic forms over F we have $\mathrm{Ad}(\rho_1)\simeq\mathrm{Ad}(\rho_2)$.

We recall the construction of the tensor product of an F–algebra with involution of the first kind, assumed orthogonal if $\mathrm{char}(F)\neq 2$, with an F–algebra with quadratic pair. This has a close relation to the tensor product of a symmetric bilinear form with a quadratic form.

4.4. **Proposition.** *Let (A,σ) and (C,τ) be F–algebras with involution of the first kind and let f be a semi-trace on (A,σ). Then there is a unique semi-trace f_* on $\mathrm{Sym}((C,\tau)\otimes(A,\sigma))$ such that $f_*(c\otimes a)=\mathrm{Trd}_C(c)\cdot f(a)$ for all $a\in\mathrm{Sym}(A,\sigma)$ and $c\in\mathrm{Sym}(C,\tau)$.*

Proof. The statement follows from (4.3) if $\mathrm{char}(F)\neq 2$. See [14, (5.18)] for the proof in the case where $\mathrm{char}(F)=2$. □

Let (C,τ) be an F–algebra with involution of the first kind and such that τ is orthogonal if $\mathrm{char}(F)\neq 2$. Then by [14, (2.23)], $(C,\tau)\otimes(A,\sigma)$ is orthogonal if $\mathrm{char}(F)\neq 2$ and symplectic if $\mathrm{char}(F)=2$. Therefore the semi-trace f_* on

$(C,\tau)\otimes(A,\sigma)$ characterised in (4.4) makes $(C\otimes_F A, \tau\otimes\sigma, f_*)$ into an F–algebra with quadratic pair, which we also denote by $(C,\tau)\otimes(A,\sigma,f)$.

4.5. **Proposition.** *Let φ be a symmetric bilinear form and ρ a quadratic form over F. Then* $\mathrm{Ad}(\varphi\otimes\rho)\simeq\mathrm{Ad}(\varphi)\otimes\mathrm{Ad}(\rho)$.

Proof. See [14, (5.19)] □

We now describe how the notions of isotropy and hyperbolicity carry over from quadratic forms to algebras with quadratic pairs.

Let (A,σ,f) be an F–algebra with quadratic pair. We call (A,σ,f) *isotropic* if there exists an element $s\in\mathrm{Sym}(A,\sigma)\setminus\{0\}$ such that $s^2=0$ and $f(s)=0$, and *anisotropic* otherwise. We call an idempotent $e\in A$ *hyperbolic with respect to σ and f* if $\sigma(e)=1-e$ and $f(eA\cap\mathrm{Sym}(A,\sigma))=\{0\}$. We say that the F–algebra with quadratic pair (A,σ,f) is *hyperbolic* if A contains a hyperbolic idempotent with respect to σ and f; note that this implies that the F-algebra with involution (A,σ) is hyperbolic.

4.6. **Proposition.** *A quadratic form ρ over F is hyperbolic if and only if the adjoint F–algebra with quadratic pair* $\mathrm{Ad}(\rho)$ *is hyperbolic.*

Proof. See [14, (6.13)]. □

4.7. **Lemma.** *Let (A,σ,f) be an F–algebra with quadratic pair. Then (A,σ,f) is hyperbolic if and only if $f=\mathrm{Trd}_A^{\sigma,e}$ for some idempotent $e\in A$.*

Proof. See [14, (6.14)]. □

4.8. **Proposition.** *Let (A,σ,f) be an F–algebra with quadratic pair and (C,τ) an F–algebra with involution. Assume that (C,τ) is orthogonal if* $\mathrm{char}(F)\neq 2$*. If (C,τ) is metabolic or (A,σ,f) is hyperbolic, then $(C,\tau)\otimes(A,\sigma,f)$ is hyperbolic.*

Proof. If (C,τ) is metabolic, then $(C,\tau)\otimes(A,\sigma,f)$ is hyperbolic by [5, (A.5)]. Assume that (A,σ,f) is hyperbolic. Hence, by (4.7) there exists an idempotent $e\in A$ such that $f=\mathrm{Trd}_A^{\sigma,e}$. Let f_* be the semi-trace of the F–algebra with quadratic pair $(C,\tau)\otimes(A,\sigma,f)$. Since $\mathrm{Trd}_{C\otimes_F A}=\mathrm{Trd}_C\otimes\mathrm{Trd}_A$, it follows from (4.4) that

$$f_*(x)=(\mathrm{Trd}_C\otimes\mathrm{Trd}_A^{\sigma,e})(x)=\mathrm{Trd}_{C\otimes_F A}^{\tau\otimes\sigma,1\otimes e}(x)$$

for all $x\in\mathrm{Sym}(C,\tau)\otimes\mathrm{Sym}(A,\sigma)$. Using again (4.4) we conclude that

$$f_*=\mathrm{Trd}_{C\otimes_F A}^{\tau\otimes\sigma,1\otimes e},$$

and as $1\otimes e$ is idempotent, (4.7) yields that $(C,\tau)\otimes(A,\sigma,f)$ is hyperbolic. □

4.9. **Proposition.** *Let φ_1 and φ_2 be symmetric bilinear forms over F. If (A,σ) is an F–algebra with involution such that $\mathrm{Ad}(\varphi_i)\otimes(A,\sigma)$ is hyperbolic for $i=1,2$, then $\mathrm{Ad}(\varphi_1\perp\varphi_2)\otimes(A,\sigma)$ is hyperbolic. If (A,σ,f) is an F–algebra with quadratic pair such that $\mathrm{Ad}(\varphi_i)\otimes(A,\sigma,f)$ is hyperbolic for $i=1,2$, then $\mathrm{Ad}(\varphi_1\perp\varphi_2)\otimes(A,\sigma,f)$ is hyperbolic.*

Proof. Let (A,σ) be an F–algebra with involution. Let $\varphi=\varphi_1\perp\varphi_2$ and let e_1 and e_2 be the projections from φ onto φ_1 and φ_2, respectively. Set $(B,\tau)=\mathrm{Ad}(\varphi)$. Hence, e_1 and e_2 are symmetric idempotents in (B,τ) with $e_1+e_2=1_B$ and such $\mathrm{Ad}(\varphi_i)\simeq(e_iBe_i,\tau_i)$ where $\tau_i=\tau|_{e_iBe_i}$, for $i=1,2$.

Assume now that $(e_iBe_i,\tau_i)\otimes(A,\sigma)$ is hyperbolic for $i=1,2$. Hence, for $i=1,2$ there exists a hyperbolic idempotent a_i in $(e_iBe_i\otimes_F A,\tau_i\otimes\sigma)$. We consider B as an F–subalgebra of $B\otimes A$ and set $a=a_1+a_2$ in $B\otimes A$. Then

$$(\sigma\otimes\tau)(f)=e_1\otimes 1_A-a_1+e_2\otimes 1_A-a_2=1_{B\otimes A}-a\,,$$

showing that $(B,\tau)\otimes(A,\sigma)$ is hyperbolic.

Assume now that we are given a semi-trace f such that (A,σ,f) is an F–algebra with quadratic pair. Let

$$(C,\gamma,g)=(B,\tau)\otimes(A,\sigma,f)\,.$$

Assume further that $\mathrm{Ad}(\varphi_i)\otimes(A,\sigma,f)$ is hyperbolic for $i=1,2$. Then in particular, $\mathrm{Ad}(\varphi_i)\otimes(A,\sigma)$ is hyperbolic. Hence, we can choose $e_1,e_2,a_1,a_2\in B\otimes A$ as above such that $a=a_1+a_2$ is an idempotent with $(\tau\otimes\sigma)(a)=1-a$, and such that we further have

$$(\mathrm{Trd}_B\otimes f)\,(aC\cap\mathrm{Sym}(\mathrm{Ad}(\varphi_i)\otimes(A,\sigma)))=\{0\}.$$

We want to show that $g(eC\cap\mathrm{Sym}(C,\gamma))=\{0\}$. Since $e_1+e_2=1_B$, for all $x\in B$ we have $x=e_1xe_1+e_1xe_2+e_2xe_1+e_2xe_2$. Then, as $e_1e_2=e_2e_1=0$, we have

$$\mathrm{Trd}_B(e_1xe_2)=\mathrm{Trd}_B(e_2e_1x)=0=\mathrm{Trd}_B(e_1e_2x)=\mathrm{Trd}_B(e_2xe_1).$$

Therefore, as Trd_B is an F–linear map, we have

$$\mathrm{Trd}_B(x)=\mathrm{Trd}_B(e_1xe_1)+\mathrm{Trd}_B(e_2xe_2)\ \text{ for all } x\in B.$$

It follows that

$$g|_{\mathrm{Sym}(C,\gamma)}=\mathrm{Trd}_B\otimes f|_{\mathrm{Sym}(\mathrm{Ad}(\varphi_1)\otimes(A,\sigma))}+\mathrm{Trd}_B\otimes f|_{\mathrm{Sym}(\mathrm{Ad}(\varphi_2)\otimes(A,\sigma))}.$$

The result then follows. □

4.10. **Remark.** Note that statement and proof of (4.9) remain valid if 'symmetric bilinear forms' are replaced by 'λ–hermitian forms' over an F–algebra with involution of the first kind (see Section 6 for the definition).

Given a field extension K/F, we denote by $(A,\sigma,f)_K$ the K–algebra with quadratic pair (A_K,σ_K,f_K) where $f_K=f\otimes\mathrm{id}_K:\mathrm{Sym}(A,\sigma)_K\to K$. Note that for a field extension K/F and a quadratic form ρ over F we have $\mathrm{Ad}((\rho_K)\simeq(\mathrm{Ad}(\rho))_K$.

5. Quaternion algebras

For $a\in F$ with $4a\neq -1$ we denote by F_a the étale quadratic extension $F[T]/(T^2-T-a)$ of F. If T^2-T-a is reducible over F then $F_a\simeq F\times F$, otherwise F_a is a field. The linear substitution $T\mapsto 1-T$ induces the (unique) nontrivial F-automorphism τ of F_a, and the pair (F_a,τ) becomes a unitary F–algebra with involution. We use the notation $[a)_F$ for (F_a,τ).

With the norm map $\nu:F_a\to F, x\mapsto\tau(x)x$ we obtain a 2-dimensional quadratic space (F_a,ν) over F, which is isometric to $\langle\langle a]]$. Note that, by [1, Chap. IX, Lemma 8], every separable quadratic field extension of F is F-isomorphic to F_a for some $a\in F$ with $4a\neq -1$.

An *F–quaternion algebra* is a central simple F-algebra of degree 2. Any F–quaternion algebra has a basis $(1,u,v,w)$ such that

$$u^2=u+a, v^2=b\ \text{ and }\ w=uv=v-vu\,, \tag{5.1}$$

for some $a \in F$ with $4a \neq -1$ and $b \in F^\times$ (see [1, Chap. IX, Thm. 26]); such a basis is called a *quaternion basis*. Note that the F–subalgebra $F[u] = F \oplus uF$ is isomorphic to F_a, and $\mathrm{Int}(v)$ restricts to the non-trivial F–automorphism on $F[u]$. Conversely, given $a \in F$ with $4a \neq -1$ and $b \in F^\times$, the above relations determine an F–quaternion algebra, which we denote by $[a, b)_F$. By the above, up to isomorphism any F–quaternion algebra is of this form.

Let Q be an F–quaternion algebra. By [14, (2.21)], the map $\gamma : Q \to Q$, $x \mapsto \mathrm{Trd}_Q(x) - x$ is the unique symplectic involution on Q; it is called the *canonical involution of* Q. With an F–quaternion basis $(1, u, v, w)$ of Q as above, γ is determined by the conditions that $\gamma(u) = 1 - u$ and $\gamma(v) = -v$. By [14, (2.21)] every orthogonal involution σ on Q is of the form $\sigma = \mathrm{Int}(s) \circ \gamma$ for an invertible element $s \in \mathrm{Skew}(Q, \gamma) \setminus F$.

Considering Q as an F-vector space, the pair (Q, Nrd_Q) is a 4–dimensional quadratic form over F. If $Q \simeq [a, b)_F$ for some $a \in F$ with $4a \neq -1$ and $b \in F^\times$, then $(Q, \mathrm{Nrd}_Q) \simeq \langle\langle b, a]]$.

5.2. **Proposition.** *Let Q be an F–quaternion algebra. Then Q is split if and only if (Q, Nrd_Q) is hyperbolic.*

Proof. See [10, (12.5)]. □

We denote by $F(Q)$ the function field of the quadratic form (Q, Nrd_Q).

5.3. **Proposition.** *Let Q be an F–quaternion algebra and let ρ be an anisotropic even dimensional quadratic form over F. Then $\rho_{F(Q)}$ is hyperbolic if and only if $\rho \simeq \varphi \otimes (Q, \mathrm{Nrd}_Q)$ for some alternating or symmetric bilinear form φ over F.*

Proof. If the 2–fold Pfister form (Q, Nrd_Q) is an anisotropic, then the result follows from [10, (23.6)]. Otherwise (Q, Nrd_Q) is hyperbolic by (5.2), so that $F(Q)/F$ is a rational function field over F, and then the result follows from (2.4). □

Let $a \in F$ with $4a \neq -1$ and $b \in F^\times$. We consider the quaternion algebra $Q = [a, b)_F$, constructed from a basis $(1, u, v, w)$ through the relations in (5.1). Let γ denote the canonical involution on Q. Then $\tau = \mathrm{Int}(v) \circ \gamma$ is an orthogonal involution on Q, by [14, (2.21)]; it is characterised by the rules

$$\tau(u) = u \text{ and } \tau(v) = -v\,. \tag{5.4}$$

We denote by $[a \,|\!\cdot b)_F$ the F–algebra with involution (Q, τ).

5.5. **Proposition.** *If $a \in F^\times$ then* $\mathrm{Ad}(\langle\langle a\rangle\rangle) \simeq [0 \,|\!\cdot a)_F$.

Proof. Let $\langle\langle a\rangle\rangle = (V, b)$ and let $x, y \in V$ be such that $b(x, y) = 0$, $b(x, x) = 1$ and $b(y, y) = -a$. Let $f, g \in \mathrm{End}_F(V)$ be given by $f(x) = x$, $f(y) = 0$, $g(x) = y$ and $g(y) = ax$. Then $fg + gf = g$, $f^2 = f$ and $g^2 = a \cdot \mathrm{id}_F$. Hence $(1, f, g, fg)$ is a quaternion basis of $[0, a)_F$. For $w, z \in V$ we have $b(f(w), z) = b(w, f(z))$ and $b(g(w), z) = -b(w, g(z))$. Hence, $\mathrm{ad}_b(f) = f$ and $\mathrm{ad}_b(g) = -g$. Hence, the statement follows using (5.4). □

Let σ be the F–involution on Q determined by

$$\sigma(u) = 1 - u \text{ and } \sigma(v) = v\,.$$

As $4a \neq -1$ and hence $(2\ell - 1)^2 = 4a + 1 \in F^\times$, we have that $2\ell - 1 \in Q^\times$, and we obtain that $\sigma = \mathrm{Int}(2u - 1) \circ \gamma$; in particular, $\sigma = \gamma$ in the case where $\mathrm{char}(F) = 2$. We denote by $[a \,\cdot\!|\, b)_F$ the F–algebra with involution (Q, σ). We obtain a canonical homomorphism of F–algebras with involution $[a)_F \longrightarrow [a \;\cdot\!| \; b)_F$, and it is injective. Furthermore, as $\sigma(u) = 1 - u$, by (4.2) the map

$$f : \mathrm{Sym}(Q, \sigma) \longrightarrow F, x \mapsto \mathrm{Trd}_Q(ux)$$

is a semi-trace on (Q, σ). This gives rise to an F–algebra with quadratic pair (Q, σ, f), which we denote by $[a \,\|\, b)_F$.

5.6. **Proposition.** *Any F–quaternion algebra with quadratic pair is isomorphic to $[a \,\|\, b)_F$ for some $a \in F$ with $4a \neq -1$ and $b \in F^\times$.*

Proof. Let (Q, σ, f) be an F–algebra with quadratic pair. Let γ be the canonical involution on Q. Assume first that $\mathrm{char}(F) \neq 2$. Then σ is orthogonal, so by [14, (2.21)] we have $\sigma = \mathrm{Int}(s) \circ \gamma$ for some $s \in \mathrm{Skew}(Q, \gamma) \cap Q^\times$. Then $s^2 \in F^\times$, and for $u = s + \frac{1}{2}$ we obtain that $u^2 - u = s^2 - \frac{1}{4} \in F$, $4(u^2 - u) \neq -1$, and $\mathrm{Int}(2u - 1) \circ \gamma = \mathrm{Int}(s) \circ \gamma = \sigma$. Moreover, (4.2) and (4.3) yield that $f = \frac{1}{2}\mathrm{Trd}_Q|_{\mathrm{Sym}(Q,\sigma)} = \mathrm{Trd}_Q^{\sigma,u}$. Assume now that $\mathrm{char}(F) = 2$. Then σ is symplectic and thus $\sigma = \gamma$. By (4.2) there exists $u \in Q$ such that $1 = u + \gamma(u)$ and $f = \mathrm{Trd}_Q^{\sigma,u}$. Moreover, $u^2 - u = \gamma(u)u \in F$.

In either case we have found an element $u \in Q \setminus F$ with $u^2 - u \in F$ and $4(u^2 - u) \neq -1$ and such that $\sigma = \mathrm{Int}(2u - 1) \circ \gamma$ and $f = \mathrm{Trd}_Q^{\sigma,u}$. Then $F(u)$ is a separable quadratic extension of F contained in Q, and by the Skolem-Noether Theorem (see [17, (12.6)]) the non-trivial F–automorphism of $F(u)$ extends to an inner automorphism of Q. Hence, there exists $v \in Q^\times$ such that $vuv^{-1} = 1 - u$. It follows that $(1, u, v, uv)$ is a quaternion basis of Q and that with $u^2 - u = a$ and $v^2 = b$ we obtain that $(Q, \sigma, f) \simeq [a \,\|\, b)$. $\square$

5.7. **Proposition.** *Let $a, c \in F$ such that $4a \neq -1 \neq 4c$ and $b, d \in F^\times$. Then*

$$[c \,|\!\cdot\, d)_F \otimes [a \,\|\, b)_F \simeq [c \,|\!\cdot\, bd)_F \otimes [a + c + 4ac \,\|\, b)_F.$$

Proof. Let $(B, \sigma, f) = [c \,|\!\cdot\, d)_F \otimes [a \,\|\, b)_F$, $(Q_1, \sigma_1) = [c \,|\!\cdot\, d)_F$ and $(Q_2, \sigma_2) = [a \;\cdot\!|\; b)_F$. We fix $i, j \in Q_1$ such that $i^2 = i + c$, $j^2 = d$, $ij = j - ji$ as well as $u, v \in Q_2$ such that $u^2 = u + a$, $v^2 = b$ and $uv = v - vu$. In B we have that $\sigma(i \otimes 1) = i \otimes 1$, $\sigma(v \otimes 1) = -v \otimes 1$, $\sigma(1 \otimes u) = 1 \otimes 1 - 1 \otimes u$ and $\sigma(1 \otimes v) = 1 \otimes v$.

Let $i' = i \otimes 1$, $j' = j \otimes v$, $v' = 1 \otimes v$ and $u' = (1 - 2i) \otimes u + i \otimes 1$. Then one easily checks that

$$Q_1' = F \oplus Fi' \oplus Fj' \oplus Fi'j' \quad \text{and} \quad Q_2' = F \oplus Fu' \oplus Fv' \oplus Fu'v'$$

are σ–invariant F–subalgebras of B that commute element wise with one another. We set $\sigma_1' = \sigma|_{Q_1'}$ and $\sigma_2' = \sigma|_{Q_2'}$. We have

$$(Q_1', \sigma_1') \simeq [c \,|\!\cdot\, bd)_F \quad \text{and} \quad (Q_2', \sigma_2') \simeq [a{+}c{+}4ac \;\cdot\!|\; b)_F\,.$$

Hence, $(B, \sigma) \simeq (Q_1', \sigma_1') \otimes (Q_2', \sigma_2')$ and it remains to express the semi–trace f in these terms.

We have

$$(Q_2', \sigma_2', \mathrm{Trd}_{Q_2'}^{\sigma_2', u'}) \simeq [a + c + 4ac \,\|\, b)_F\,.$$

Since $i \otimes u - \sigma(i \otimes u) = i \otimes u - i \otimes (1 - u) = 2i \otimes u - i \otimes 1$ we have that

$$1 \otimes u - u' = 2i \otimes u - i \otimes 1 \in \mathrm{Alt}(B, \sigma),$$

whence $\mathrm{Trd}_B^{\sigma, 1\otimes u} = \mathrm{Trd}_B^{\sigma, u'}$ by (4.2). Let $g : Q_2 \longrightarrow F$ be the map given by $x \mapsto \mathrm{Trd}_{Q_2}(ux)$. Since $(B, \sigma) = (Q_1, \sigma_1) \otimes (Q_2, \sigma_2)$ and further $\mathrm{Trd}_B = \mathrm{Trd}_{Q_1} \otimes \mathrm{Trd}_{Q_2}$, we obtain that

$$f = \left(\mathrm{Trd}_{Q_1} \otimes g\right)\big|_{\mathrm{Sym}(B,\sigma)} = \mathrm{Trd}_B^u = \mathrm{Trd}_B^{\sigma,u'}.$$

Hence

$$(B,\sigma,f) = (Q_1' \otimes Q_2', \sigma_1' \otimes \sigma_2', \mathrm{Trd}_B^{\sigma,u'}) \simeq [c \mid\cdot bd)_F \otimes [a+c \,\|\, b)_F.$$

$\square$

5.8. **Corollary.** *For $b \in F^\times$ and $a \in F$ with $4a \neq -1$ the F–algebra with quadratic pair $\mathrm{Ad}(\langle\langle b\rangle\rangle) \otimes [a \,\|\, b)_F$ is hyperbolic.*

Proof. From (5.5) and (5.7) we get

$$\begin{aligned}
\mathrm{Ad}(\langle\langle b\rangle\rangle) \otimes [a \,\|\, b)_F &\simeq [0 \mid\cdot b)_F \otimes [a \,\|\, b)_F \\
&\simeq [0 \mid\cdot b^2)_F \otimes [a+0 \,\|\, b)_F \\
&\simeq [0 \mid\cdot 1)_F \otimes [a \,\|\, b)_F \\
&\simeq \mathrm{Ad}(\langle\langle 1\rangle\rangle) \otimes [a \,\|\, b)_F
\end{aligned}$$

As $\langle\langle 1\rangle\rangle$ is metabolic, $\mathrm{Ad}(\langle\langle 1\rangle\rangle)$ is metabolic by (3.2), and hence the conclusion follows by (4.8). $\square$

6. Unitary involutions and hermitian forms

Let (D,θ) be an F–division algebra with involution. In the sequel, let $\lambda \in Z(D)$ be such that $\lambda\theta(\lambda) = 1$. An bi–additive map $h : V\times V \longrightarrow D$ on a finite dimensional D–vector space V is called *degenerate* if there exists $x \in V\setminus\{0\}$ such that $h(x,y) = 0$ for all $y \in V$ and *nondegenerate* otherwise. A *λ–hermitian form* over (D,θ) is a pair (V,h) where V is a finite-dimensional right D-vector space and h is a nondegenerate bi-additive map $h : V \times V \to D$ such that

$$h(x, yd) = h(x,y)d \quad \text{and} \quad h(y,x) = \lambda\theta(h(x,y))$$

for all $x,y \in V$ and $d \in D$. Note that a symmetric bilinear form over F is the same as a 1–hermitian form over (F, id_F).

Let $\eta = (V,h)$ and $\eta' = (V',h')$ be λ–hermitian forms over (D,θ). By an *isometry of λ–hermitian forms $\phi : \eta \longrightarrow \eta'$* over (D,θ) we mean an isomorphism of D–vector spaces $\phi : V \longrightarrow V'$ such that $h(x,y) = h'(\phi(x),\phi(y))$ for all $x,y \in V$. If such an isometry exists, we write $\eta \simeq \eta'$. For $c \in F^\times$ we denote by $c\eta$ the λ–hermitian form (V, ch) on (D,θ), where $(ch)(x,y) = c(h(x,y))$ for all $x,y \in V$. We say that η is *similar to* η' if $c\eta \simeq \eta'$ for some $c \in F^\times$.

Let $\varphi = (V,b)$ be a symmetric bilinear form over F. Then $(V \otimes_F D, h)$ where $h : (V \otimes_F D) \times (V \otimes_F D) \to D$ is the F–bilinear map given by $h(x \otimes d_1, y \otimes d_2) = \theta(d_1)b(x,y)d_2$ for $x,y \in V$ and $d_1,d_2 \in D$ is a 1–hermitian form over (D,θ). We call $(V \otimes_F D, h)$ the *1–hermitian form over (D,θ) extended from φ*. Note that for a 1–hermitian form $(V \otimes_F D, h)$ over (D,θ) extended from a symmetric bilinear form (V,b) over F we have that h is degenerate if and only if b is degenerate.

There is well-known correspondence between (nondegenerate) λ-hermitian forms on V and F–involutions on $\mathrm{End}_D(V)$, generalising the correspondence between bilinear forms and involutions on a split algebra. We state it only in the unitary case.

6.1. **Proposition.** *Let (D,θ) be an F–division algebra with unitary involution, V a finite-dimensional right D–vector space and let $A = \mathrm{End}_D(V)$. For every λ-hermitian form (V,h), there is a unique F–involution σ on A such that $\sigma|_D = \theta$ and*

$$h(f(x),y) = h(x,\sigma(f)(y)) \text{ for all } x,y \in V \text{ and } f \in A.$$

This gives a one-to-one correspondence between λ–hermitian forms on (D,θ) up to a factor in $F^\times$ invariant under θ and unitary involutions on A.

Proof. See e.g. [14, (4.1) and (4.2)]. □

In the situation of (6.1), we call the F–algebra with involution (A,σ) the *adjoint involution to* (V,h).

6.2. **Proposition.** *Let (A,σ) be a split F–algebra with unitary involution. Let $K = Z(A)$ and $\tau = \sigma|_K$. Then $(A,\sigma) \simeq \mathrm{Ad}(\varphi) \otimes (K,\tau)$ for a non-alternating symmetric bilinear form φ over F.*

Proof. If K is not a field, then (K,τ) and (A,σ) are both hyperbolic, and we have $(A,\sigma) \simeq \mathrm{Ad}(\varphi) \otimes (K,\tau)$ for any symmetric bilinear form φ over F with $\dim(\varphi) = \deg(A)$. Suppose now that K is a field. We may then identify A with $\mathrm{End}_K(V)$ for a finite-dimensional right K–vector space V. Then by [14, (4.2)], the involution σ is adjoint to some λ–hermitian form (V,h) over (K,τ). Note that $h(v,v) \in \mathrm{Sym}(K,\tau) = F$ for all $v \in V$. By [13, Chap. I, (6.2.4)] there exists an orthogonal basis $(v_1,\ldots,v_r)$ of (V,h). Consider the F–vector space $U = Fv_1 \oplus \ldots \oplus Fv_r$. Then $\varphi = (U, h|_{U\times U})$ is a non-alternating symmetric bilinear form over F, and the natural isomorphism of K–spaces $U \otimes_F K \to V$ induces an isomorphism of F–algebras with involution $\mathrm{Ad}(\varphi) \otimes (K,\tau) \to (A,\sigma)$. □

6.3. **Proposition.** *Let A be a central simple F–algebra, let K be a quadratic separable extension of F contained in A, and let C be the centraliser of K in A. Any F-involution on A that restricts to the non-trivial F–automorphism of K is uniquely determined by its type and its restriction to C. Further, if* $\mathrm{char}(F) = 2$ *then any such involution is symplectic.*

Proof. Let τ denote the non-trivial F–automorphism of K. We fix $x \in K$ with $\tau(x) + x = 1$. Consider F–involutions σ and σ' on A with $\sigma|_K = \sigma'|_K = \tau$ and $\sigma|_C = \sigma'|_C$. Note that $1 \in \mathrm{Alt}(A,\sigma)$, so if $\mathrm{char}(F) = 2$ then (A,σ) is symplectic by [14, (2.6)]. Then $\sigma' \circ \sigma$ is an F–automorphism of A, so $\sigma' \circ \sigma = \mathrm{Int}(b)$ for some $b \in A^\times$ by the Skolem-Noether Theorem (see [17, (12.6)]). Since $\sigma|_C = \sigma'|_C$, we obtain that $\mathrm{Int}(b)|_C = \mathrm{id}_C$, whereby $b \in K$ by the Double-Centralizer-Theorem (see [17, (12.7)]). As $\mathrm{id}_A = \sigma'^2 = (\mathrm{Int}(b) \circ \sigma)^2 = \mathrm{Int}(b\sigma(b^{-1}))$, we obtain that $b\sigma(b^{-1}) \in Z(A) = F$, and therefore $\sigma(b) = \pm b$. If $\sigma(b) = b$ then $b \in F^\times$ and $\sigma' = \sigma$. Otherwise σ and σ' are of different type by [14, (2.7)]. □

6.4. **Proposition.** *Let B be a central simple F–algebra and $a \in F$ with $4a \neq -1$. Let σ_1 and σ_2 be F–involutions on B, both orthogonal in the case where* $\mathrm{char}(F) \neq 2$. *If $(B,\sigma_1) \otimes [a)_F \simeq (B,\sigma_2) \otimes [a)_F$ then $(B,\sigma_1) \otimes [a \,\|\, b)_F \simeq (B,\sigma_2) \otimes [a \,\|\, b)_F$.*

Proof. If F_a is not a field, then $[a)_F$ and $[a \,\|\, b)_F$ are hyperbolic and the statement is trivial. So we assume that F_a is a field.

Let $(Q,\gamma,f) = [a \,\|\, b)_F$. Fix $u \in Q$ with $u^2 + u = a$ and $\gamma(u) + u = 1$. Let $A = B \otimes_F Q$, $K = F(u)$, and let C be the centraliser of K in A. For $i = 1,2$ we let $\hat\sigma_i = \sigma_i \otimes \gamma$ and thus have that $(B,\sigma_i) \otimes [a)_F \simeq (C, \hat\sigma_i|_C)$.

Hence, assuming that $(B,\sigma_1)\otimes[a)_F \simeq (B,\sigma_2)\otimes[a)_F$ we have

$$(C,\hat{\sigma}_1|_C) \simeq (C,\hat{\sigma}_2|_C).$$

As C is a central simple K-algebra and $\hat{\sigma}_1|_K = \hat{\sigma}_2|_K$, by [14, (2.18)] we have that $\hat{\sigma}_2|_C = \mathrm{Int}(c)\circ\hat{\sigma}_1|_C$ for some $c\in C^\times$ with $\hat{\sigma}_1(c)=c$. By [14, (2.7)], the F-involutions $\hat{\sigma}_2$ and $\mathrm{Int}(c)\circ\hat{\sigma}_1$ on A have the same type. Hence, we obtain using (6.3) that $\hat{\sigma}_2 = \mathrm{Int}(c)\circ\hat{\sigma}_1$. Therefore we have an isomorphism of F–algebras with involution $\mathrm{Int}(c) : (A,\hat{\sigma}_2) \longrightarrow (A,\hat{\sigma}_1)$.

For $i=1,2$, we have that $(B,\sigma_i)\otimes[a\,\|\,b)_F \simeq (A,\hat{\sigma}_i,f_i)$ for the semi-trace $f_i : \mathrm{Sym}(A,\hat{\sigma}_i) \longrightarrow F, x\mapsto \mathrm{Trd}_A((1\otimes u)\cdot x)$. Since c is in the centraliser of K in A, it commutes with $1\otimes u$. It follows using (3.1) that

$$\mathrm{Trd}_A((1\otimes u)\cdot(\mathrm{Int}(c))(x)) = \mathrm{Trd}_A(\mathrm{Int}(c)((1\otimes u)\cdot x)) = \mathrm{Trd}_A((1\otimes u)\cdot x) \text{ for all } x\in A\,.$$

We conclude that $f_2 = f_1\circ\mathrm{Int}(c)$. Hence, we obtain an isomorphism of F–algebras with quadratic pairs $\mathrm{Int}(c) : (A,\hat{\sigma}_2,f_2) \longrightarrow (A,\hat{\sigma}_1,f_1)$, as desired. □

6.5. **Proposition.** *Let $a\in F$ with $4a\neq -1$. Let φ_1 and φ_2 be symmetric bilinear forms over F. We have $\mathrm{Ad}(\varphi_1)\otimes[a)_F \simeq \mathrm{Ad}(\varphi_2)\otimes[a)_F$ if and only if $\varphi_1\otimes\langle\langle a]]$ and $\varphi_2\otimes\langle\langle a]]$ are similar. Moreover, in this case $\mathrm{Ad}(\varphi_1)\otimes[a\,\|\,b)_F \simeq \mathrm{Ad}(\varphi_2)\otimes[a\,\|\,b)_F$ for any $b\in F^\times$.*

Proof. For $i=1,2$, the F–algebra with involution $\mathrm{Ad}(\varphi_i)\otimes[a)_F$ is adjoint to the hermitian form over $[a)_F$ over (D,θ) extended from φ_i. Hence by (6.1), we have $\mathrm{Ad}(\varphi_1)\otimes[a)_F \simeq \mathrm{Ad}(\varphi_2)\otimes[a)_F$ if and only if the 1–hermitian forms over $[a)_F$ extended from φ_1 and φ_2 are similar, which by [18, (10.1.1)] is if and only if $\varphi_1\otimes\langle\langle a]]$ and $\varphi_2\otimes\langle\langle a]]$ are similar. The second part of the statement follows directly from (6.4). □

6.6. **Corollary.** *Let $a\in F$ be such that $4a\neq -1$. Let φ be a symmetric bilinear form over F. Then $\mathrm{Ad}(\varphi)\otimes[a)_F$ is hyperbolic if and only if $\varphi\otimes\langle\langle a]]$ is hyperbolic, and in this case $\mathrm{Ad}(\varphi)\otimes[a\,\|\,b)_F$ is hyperbolic for any $b\in F^\times$.*

Proof. If F_a is not a field, then $\langle\langle a]]$ and $[a\,\|\,b)_F$ are hyperbolic and the statement is trivial. Suppose now that F_a is a field. Then $\langle\langle a]]$ is anisotropic. Assume that $\varphi\otimes\langle\langle a]]$ is hyperbolic. Then φ is even-dimensional by (2.3), so there exists a metabolic symmetric bilinear form φ' over F with $\dim(\varphi') = \dim(\varphi)$. Since all hyperbolic quadratic forms of the same dimension are isometric it follows that $\varphi\otimes\langle\langle a]] \simeq \varphi'\otimes\langle\langle a]]$. Hence $\mathrm{Ad}(\varphi)\otimes[a\,\|\,b)_F \simeq \mathrm{Ad}(\varphi')\otimes[a\,\|\,b)_F$ by (6.5), and the result follows from (3.2) and (4.8). □

7. Hyperbolicity over a separable quadratic extension

In this section we expand on a characterisation given in [5, (1.16)] of those algebras with quadratic pair that become hyperbolic over a given quadratic separable extension.

7.1. **Proposition.** *Let (A,σ,f) be an F–algebra with quadratic pair and K/F a quadratic separable extension with nontrivial F–automorphism τ. The K-algebra with quadratic pair $(A,\sigma,f)_K$ is hyperbolic if and only if one of the following holds:*

(1) *The F-algebra with quadratic pair (A,σ,f) is adjoint to a quadratic form of odd Witt index and whose anisotropic part is isometric to $\nu\otimes\varphi$ where ν is the norm form of K/F and φ is a symmetric bilinear form over F.*

(2) *There exists an embedding of F–algebras with involution* $\varepsilon : (K, \tau) \longrightarrow (A, \sigma)$ *such that* $\mathrm{Trd}_A(\varepsilon(c)x) = (c + \tau(c))f(x)$ *for all* $c \in K$ *and* $x \in \mathrm{Sym}(A, \sigma)$.

The conditions (1) *and* (2) *are mutually exclusive.*

Proof. This is a reformulation of [5, (1.16)] and the remark that follows it. □

7.2. **Proposition.** *Let* (A, σ) *be an* F*–algebra with involution of the first kind and let* K *be a* σ*–stable quadratic étale extension of* F *inside* A *such that* $\sigma|_K$ *is the nontrivial* F*–automorphism of* K*. Let*

$$C = \{x \in A \mid xy = yx \text{ for all } y \in K\}\,.$$

Assume that the K*–algebra with unitary involution* $(C, \sigma|_C)$ *is split. Then* K *is contained in a* σ*–stable quaternion algebra* Q *inside* A *that is Brauer equivalent to* A *and such that* $(Q, \sigma|_Q)$ *is of the same type as* (A, σ)*. Moreover, then*

$$(A, \sigma) \simeq \mathrm{Ad}(\varphi) \otimes (Q, \sigma|_Q)$$

for a non-alternating symmetric form φ *over* F*.*

Proof. Note that, if $\mathrm{char}(F) = 2$, the fact that $\sigma|_K$ is the non-trivial automorphism of K/F implies that $1 \in \mathrm{Alt}(A, \sigma)$ and thus that (A, σ) is symplectic, by [14, (2.6)]. By (6.2) we have $(C, \sigma|_C) \simeq \mathrm{Ad}(\varphi) \otimes (K, \sigma|_K)$ for a non-alternating symmetric bilinear form φ over F. Hence C contains a σ–invariant central simple F–subalgebra B such that $(B, \sigma|_B) \simeq \mathrm{Ad}(\varphi)$, $C \simeq B \otimes_F K$ and $\dim_F(A) = 2\dim_F(C) = 4\dim_F(B)$. Then

$$Q = \{x \in A \mid xy = yx \text{ for all } y \in B\}$$

is a σ–stable F-subalgebra of A containing K. By the Double Centralizer Theorem (see [17, (12.7)]), Q is a central simple F–algebra and $A = B \otimes_F Q$. Hence, $\dim_F(Q) = 4$, whereby Q is an F–quaternion algebra, and we obtain that $(A, \sigma) \simeq \mathrm{Ad}(\varphi) \otimes (Q, \sigma|_Q)$. As $\mathrm{Ad}(\varphi)$ is orthogonal, it follows from [14, (2.23)] that (A, σ) and $(Q, \sigma|_Q)$ have the same type. □

7.3. **Proposition.** *Let* $a \in F$ *with* $4a \neq -1$ *and* $b \in F^\times$ *and let* (A, σ, f) *be an* F*–algebra with quadratic pair such that* A *is Brauer equivalent to* $[a, b)_F$ *and* $(A, \sigma, f)_{F_a}$ *is hyperbolic. Then*

$$(A, \sigma, f) \simeq \mathrm{Ad}(\varphi) \otimes [a \,\|\, b)_F$$

for a non-alternating symmetric bilinear form φ *over* F*.*

Proof. Let τ be the non-trivial F–automorphism on F_a. As $(A, \sigma, f)_{F_a}$ is hyperbolic, (7.1) says that there exists an embedding of F–algebras with involution $\varepsilon : [a)_F \longrightarrow (A, \sigma)$ such that $\mathrm{Trd}_A(\varepsilon(c)x) = (c + \tau(c))f(x)$ for all $c \in F_a$ and $x \in \mathrm{Sym}(A, \sigma)$. Therefore by (7.2), $\varepsilon(F_a)$ is contained in a σ–stable quaternion algebra Q Brauer equivalent to A and $(A, \sigma) \simeq \mathrm{Ad}(\varphi) \otimes (Q, \sigma|_Q)$ for some non-alternating symmetric form φ over F. There exists an $\ell \in F_a$ such that $\ell^2 = \ell + a$ and $\tau(\ell) = 1 - \ell$. For $u = \varepsilon(\ell) \in Q$ we obtain that $u^2 = u + a$ and $\sigma(u) = 1 - u$. Then for all $s_1 \in \mathrm{Sym}(\mathrm{Ad}(\varphi))$ and $s_2 \in \mathrm{Sym}(Q, \sigma|_Q)$ we have

$$f(s_1 \otimes s_2) = \frac{\mathrm{Trd}_A(\varepsilon(\ell) \cdot (s_1 \otimes s_2))}{\ell + \tau(\ell)} = \mathrm{Trd}_B(s_1) \cdot \mathrm{Trd}_Q(u \cdot s_2),$$

where $B = \mathrm{End}_F(V)$ for the underlying vector space V of the form φ. It follows from (4.4) that

$$(A, \sigma, f) \simeq \mathrm{Ad}(\varphi) \otimes (Q, \sigma|_Q, \mathrm{Trd}_Q^{\sigma|_Q, u})\,.$$

Finally, since Q is Brauer equivalent to A and thus to $[a,b)_F$ and since $(Q,\sigma|_Q)$ is of the same type as (A,σ), we obtain that $(Q,\sigma|_Q,\mathrm{Trd}_Q^{\sigma|_Q,u})\simeq [a\,\|\,b)_F$. $\square$

8. Hyperbolicity over the function field of a conic

We are now almost ready to prove our main result.

8.1. **Lemma.** *Let $a\in F$ with $4a\neq -1$, $b\in F^\times$ and $Q=[a,b)_F$. Let φ be a symmetric bilinear form over F such that $(\varphi\otimes\langle\langle a]])_{F(Q)}$ is hyperbolic. Then $\mathrm{Ad}(\varphi)\otimes[a\,\|\,b)_F$ is hyperbolic.*

Proof. If the quadratic form $\langle\langle a]]$ is isotropic then it is hyperbolic, and then the conclusion follows by (6.6). We may therefore assume that $\langle\langle a]]$ is anisotropic.

Suppose now first that $\varphi\otimes\langle\langle a]]$ is anisotropic. As this form becomes hyperbolic over $F(Q)$, we then obtain by (5.3) that there exists a symmetric bilinear form ψ over F such that $\varphi\otimes\langle\langle a]]\simeq\psi\otimes\langle\langle b,a]]\simeq\langle\langle b\rangle\rangle\otimes\psi\otimes\langle\langle a]]$. By (6.5) we obtain that

$$\mathrm{Ad}(\varphi)\otimes[a\,\|\,b)_F\simeq \mathrm{Ad}(\psi)\otimes\mathrm{Ad}(\langle\langle b\rangle\rangle)\otimes[a\,\|\,b)_F\,.$$

As $\mathrm{Ad}(\langle\langle b\rangle\rangle)\otimes[a\,\|\,b)_F$ is hyperbolic by (5.8) we conclude that $\mathrm{Ad}(\varphi)\otimes[a\,\|\,b)_F$ is hyperbolic by (4.8).

In general, we apply (2.1) to obtain symmetric bilinear forms φ_1 and φ_2 over F such that

$$\varphi\otimes\langle\langle a]]\simeq\varphi_1\otimes\langle\langle a]]\perp\varphi_2\otimes\langle\langle a]]$$

and where $\varphi_1\otimes\langle\langle a]]$ is anisotropic and $\varphi_2\otimes\langle\langle a]]$ is hyperbolic. By (6.5) we have that

$$\mathrm{Ad}(\varphi)\otimes[a\,\|\,b)_F\simeq\mathrm{Ad}(\varphi_1\perp\varphi_2)\otimes[a\,\|\,b)_F\,.$$

By the previous case, if φ_1 is nontrivial, then $\mathrm{Ad}(\varphi_1)\otimes[a\,\|\,b)_F$ is hyperbolic. Moreover, by (6.6), if φ_2 is non-trivial, then $\mathrm{Ad}(\varphi_2)\otimes[a\,\|\,b)_F$ is hyperbolic. Hence, using (4.9) we conclude that $\mathrm{Ad}(\varphi)\otimes[a\,\|\,b)_F$ is hyperbolic. $\square$

8.2. **Theorem.** *Let (A,σ,f) be an F–algebra with quadratic pair such that A is Brauer equivalent to the F–quaternion algebra Q. Then (A,σ,f) is hyperbolic if and only if $(A,\sigma,f)_{F(Q)}$ is hyperbolic.*

Proof. Clearly, if (A,σ,f) is hyperbolic, then so is $(A,\sigma,f)_{F(Q)}$. Assume now that $(A,\sigma,f)_{F(Q)}$ is hyperbolic. We fix $a\in F$ with $4a\neq -1$ and $b\in F^\times$ such that $Q\simeq[a,b)_F$. Then Q splits over F_a, and hence $F_a(Q)/F_a$ is a rational function field. Note that A_{F_a} is split and therefore $(A,\sigma,f)_{F_a}$ is adjoint to a quadratic form over F_a. Since $(A,\sigma,f)_{F_a(Q)}$ is hyperbolic, we obtain using (4.6) and (2.4) that $(A,\sigma,f)_{F_a}$ is hyperbolic. Using (7.3) we conclude that

$$(A,\sigma,f)\simeq\mathrm{Ad}(\varphi)\otimes[a\,\|\,b)_F$$

for a symmetric bilinear form φ over F. Since $[a\,\|\,b)_{F(Q)}\simeq\mathrm{Ad}(\langle\langle a]]_{F(Q)})$, it follows using (4.5) that

$$\mathrm{Ad}(\varphi\otimes\langle\langle a]])_{F(Q)}\simeq(A,\sigma,f)_{F(Q)}\,.$$

Hence, $(\varphi\otimes\langle\langle a]])_{F(Q)}$ is hyperbolic by (4.6). We conclude by (8.1) that (A,σ,f) is hyperbolic. $\square$

8.3. **Remark.** As mentioned in the introduction, the methods of [16, (3.3)] and [6] can be more directly adapted to give the characteristic 2 analogue of our main result in terms of generalised quadratic pairs (see [2]). It is shown in [15] that the groups and exact sequences required in the arguments of [16, (3.3)] and [6] have characteristic free analogues for Witt groups of generalised quadratic forms. Using these exact sequences, the arguments from [16, (3.3)] and [6] carry over to this wider setting. This together with the correspondence between generalised quadratic forms and algebras with quadratic pair (see [11]) yields our result (8.2).

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References

[1] A. Albert. *Structure of Algebra*, volume 24 of *Colloquium Publications*. Am. Math. Soc., 1968.
[2] A. Bak. On modules with quadratic form. *Algebraic K-theory and its geometric applications. Lecture notes in math.*, 108:55–66, 1969.
[3] E. Bayer-Fluckiger, D.B. Shapiro, and J.-P. Tignol. Hyperbolic involutions. *Math. Z.*, 214:461–476, 1993.
[4] K. J. Becher. A proof of the Pfister Factor Conjecture. *Invent. Math.*, 173(1):1–6, 2008.
[5] G. Berhuy, C. Frings, and J.-P. Tignol. Galois cohomology of the classical groups over imperfect fields. *J. Pure Appl. Algebra*, 211:307–341, 2007.
[6] I. Dejaiffe. Formes antihermitiennes devenant hyperboliques sur un corps de dèploiment. *C.R. Acad. Sci., Paris, Sèr. I, Math.*, 332:105–108, 2001.
[7] A. Dolphin. Metabolic involutions. *Journal of Algebra*, 336(1):286–300, 2011.
[8] A. Dolphin. Decomposition of algebras with involution in characteristic 2. *Journal of Pure and Applied Algebra*, 217(9):1620–1633, 2013.
[9] A. Dolphin. The Pfister Factor Conjecture for quadratic pairs. *Preprint*, 2013.
[10] R. Elman, N. Karpenko, and A. Merkurjev. *The Algebraic and Geometric Theory Quadratic Forms*, volume 56 of *Colloq. Publ., Am. Math. Soc.* Am. Math. Soc., 2008.
[11] M.A. Elomary and J.-P. Tignol. Classification of quadratic forms over skew fields of characteristic 2. *Journal of Algebra*, 240:366–392, 2001.
[12] N.A. Karpenko. *Hyperbolicity of orthogonal involutions Doc. Math.*, Extra volume: Andrei A. Suslin sixtieth birthday: 371–392, 2010.
[13] M.-A. Knus. *Quadratic and Hermitian Forms over Rings*, volume 294 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, 1991.
[14] M.-A. Knus, A.S. Merkurjev, M. Rost, and J.-P. Tignol. *The Book of Involutions*, volume 44 of *Colloq. Publ., Am. Math. Soc.* Am. Math. Soc., 1998.
[15] M.-A. Knus and O. Villa. Quadratic quaternion forms, involutions and triality. *Doc. Math. Quadratic Forms LSU*, pages 201–218, 2001.
[16] R. Parimala, R. Sridharan, and V. Suresh. Hermitian analogue of a theorem of Springer. *J. Algebra*, 243:780–789, 2001.
[17] R. Pierce. *Associative Algebras*. Graduate texts in mathematics. Springer-Verlag, 1982.
[18] W. Scharlau. *Quadratic and hermitian forms*, volume 270 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, 1985.

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