WHITEHEAD GROUPS OF LOOP GROUP SCHEMES OF NULLITY ONE

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Abstract. We define and study the Whitehead group of isotropic (almost) simple simply connected group schemes over Laurent polynomial rings $k[t^{\pm 1}]$, where k is a field of characteristic 0. Our motivation for doing this comes from infinite dimensional Lie theory.
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1. Introduction

Let $\mathbf{L}$ be an isotropic absolutely almost simple simply connected algebraic group over a field k. Soulé and Margaux [Mg, Cor. 3.6] have given a precise description of the Whitehead group of the *constant* group scheme $\mathbf{L} \times_k \mathbb{A}^1_k$ over an affine line $\mathbb{A}^1_k$. The main purpose of this note is to replace the affine line by the punctured affine line $\mathrm{Spec}(R)$ with $R = k[t^{\pm 1}]$, and then study the structure of the Whitehead group of an arbitrary simple simply connected group schemes over R. We emphasize that our groups need not be constant, i.e. that they need not come from k-groups by base change to R.

Recall that for such an $\mathbf{L}$, the Whitehead group $W(k, \mathbf{L})$ is the quotient $W(k, \mathbf{L}) = \mathbf{L}(k)/\mathbf{L}(k)^+$ where $\mathbf{L}(k)^+$ stands for the subgroup of $\mathbf{L}(k)$ generated by the k-points of the unipotent radicals $R_u(\mathbf{Q})(k)$ and $R_u(\mathbf{Q}^-)(k)$ of two opposite proper parabolic subgroups $\mathbf{Q}$ and $\mathbf{Q}^-$ of $\mathbf{L}$. It is worth noticing that $\mathbf{L}(k)^+$ is automatically a normal subgroup of $L(k)$, hence $W(k, \mathbf{L})$ carries a natural group structure. Furthermore, it is known that $W(k, \mathbf{L})$ does not depend on the choice of a pair of opposite parabolic subgroups $\mathbf{Q}$ and $\mathbf{Q}^-$.

Our motivation for looking at this problem for group schemes over R comes from infinite dimensional Lie theory (where groups and algebras over Laurent polynomial rings play a prominent role. See [CGP1], [CGP2], [GP1] and [GP2] for details). Recall that for proving conjugacy of Cartan subalgebras in extended affine Lie algebras (which are infinite dimensional Lie

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algebras over a ground field k) it is important to describe their automorphism groups.

Given an extended affine Lie algebra E we can consider its corresponding centreless core $\widetilde{E}$ (from which E can be reconstructed). We can view $\widetilde{E}$ as either a Lie algebra (infinite dimensional) over k or over its centroid R. It is known that $\widetilde{E}$ is a "simple" Lie algebra over R[1] and that R is isomorphic to a Laurent polynomial ring in finitely many variables. One can prove that there exists a natural homomorphism $\mathrm{Aut}(E) \to \mathrm{Aut}_k(\widetilde{E})$ and one would like to show that it is surjective. The group $\mathrm{Aut}_k(\widetilde{E})$ is generated by its subgroups $\mathrm{Aut}_R(\widetilde{E})$ and $\mathrm{Aut}_k(R)$. The connected component of the group $\mathrm{Aut}_R(\widetilde{E})$ is a simple adjoint group scheme G over R. Elements of unipotent radicals of parabolic subgroups of G can be written using exponential map and the same exponential map provides us with their liftings to $\mathrm{Aut}(E)$. This is why it is important to understand how far the subgroup of $G(R)$ generated by unipont elements is from the group $G(R)$ itself.

The concept of Whitehead group for arbitrary (isotropic simple simply connected) group schemes has not been defined in the literature yet. Defining this object (under certain assumptions) is our first task. Consider a connected ring R and a simple simply connected group scheme $\mathfrak{G}$ over R.[2] We will throughout assume that $\mathfrak{G}$ is isotropic, i.e. that it has a closed subgroup isomorphic to the multiplicative group $\mathbf{G}_{m,R}$ over R, or equivalently it has a proper parabolic subgroup.

We admit that, even under these assumptions, it is not clear a priori what the "correct" definition of Whitehead groups should be. Mimicking the field case, we could take a pair $(\mathfrak{P}, \mathfrak{P}^-)$ of opposite proper parabolic subgroup schemes of $\mathfrak{G}$ and define $W_{\mathfrak{P}}(R, \mathfrak{G})$ as before, i.e. as the set of left (right) cosets of $\mathfrak{G}(R)$ modulo the "elementary" subgroup $E_{\mathfrak{P}}(R)$ of $\mathfrak{G}(R)$ generated by $R_u(\mathfrak{P})(R)$ and $R_u(\mathfrak{P}^-)(R)$. Note that $W_{\mathfrak{P}}(R, \mathfrak{G})$ depends only on $\mathfrak{P}$, but not on the choice of an opposite parabolic subgroup $\mathfrak{P}^-$ [PS, §1], and that $W_{\mathfrak{P}}(R, \mathfrak{G})$ has no group structure in general case [Su].

Since we wish to work with a normal subgroup of $\mathfrak{G}(R)$, we will consider the normal subgroup $E_{\mathfrak{P}}^{st}(R)$ of $\mathfrak{G}(R)$ generated by $E_{\mathfrak{P}}(R)$ and we call the quotient group

$$W_{\mathfrak{P}}^{st}(R, \mathfrak{G}) = \mathfrak{G}(R)/E_{\mathfrak{P}}^{st}(R)$$

the *stable Whitehead group of* $\mathfrak{G}$ *relative to* $\mathfrak{P}$.

If $\mathfrak{G}$ contains a split torus $\mathbf{G}_{m,R}^2 = \mathbf{G}_{m,R} \times \mathbf{G}_{m,R}$ or, more generally, if the fibers $\mathfrak{G} \times_R (R/\mathfrak{m})$ are of relative (split) rank ≥ 2 for all maximal ideals $\mathfrak{m}$ of R it is known that $E_{\mathfrak{P}}(R) = E_{\mathfrak{P}}^{st}(R)$ and that this normal subgroup does

[1]This is true except for a well-understood family of absolute type A given by quantum tori with "generic" entries.

[2]We remind the reader that this means that, for all $x \in \mathrm{Spec}(R)$, the geometric fiber $\mathfrak{G}_{\overline{x}}$ of $\mathfrak{G}$ is an (almost) simple and simply connected algebraic group over the corresponding algebraic closure of residue field $\overline{k(x)}$. Because the base is assumed to be connected, the type of this simply connected group is unique (i.e. independent of x).

not depend on the choice of $\mathfrak{P}$ (Petrov-Stavrova [PS, Theorem 1]). Hence in this case the stable Whitehead group is an invariant of $\mathfrak{G}$.

We assume henceforth that k is a field of characteristic 0, and let $R = k[t^{\pm 1}]$ be the corresponding ring of Laurent polynomials. Let $\mathbf{G}$ be a split almost simple simply connected algebraic group over k and let $\mathfrak{G}$ be a twisted form of the R-group $\mathbf{G} \times_k R$, namely $\mathfrak{G}$ is a group scheme over R such that $\mathfrak{G} \times_R S \simeq \mathbf{G} \times_k S$ for some faithfully flat and finitely presented extension S/R. Thus $\mathfrak{G}$ corresponds to a torsor over R under $\mathbf{Aut}(\mathbf{G})$. Since $\mathbf{Aut}(\mathbf{G})$ is smooth, we may assume that S/R is étale.

Recall that according to [GP2, Theorem 5.1] and (*ibid*, Cor. 6.3) any such $\mathfrak{G}$ is loop reductive. For the definitions of loop cocycles, loop group schemes and their properties we refer to § 2. Here we recall only that $\mathfrak{G}$ can be realized as a twist of $\mathbf{G} \times_k R$ by a loop cocycle η and that η gives rise to a connected reductive algebraic group $\mathbf{H}$ over k and a closed immersion $\mathbf{H} \times_k R \hookrightarrow \mathfrak{G}$. In many cases one can think of $\mathbf{H}$ as being a "maximal" constant subgroup scheme of $\mathfrak{G}$.

We can now state our main result.

1.1. **Theorem.** (1) *The group $\mathfrak{G}(R)$ is generated by $\mathbf{H}(k)$ and $E^{st}_{\mathfrak{P}}(R)$.*

(2) *Assume that $\mathfrak{G}$ is quasi-split. Then $W^{st}_{\mathfrak{P}}(R, \mathfrak{G}) = 1$.*

The interest of the result is for twisted non-split group schemes. Indeed the split case follows from a general result of Steinberg on Chevalley groups over euclidean rings [St1, Cor. 3, p. 115].

Under the conditions of the Petrov–Stavrova's result quoted above we have a stronger result.

1.2. **Corollary.** *Assume that for each closed point s of* Spec(R)*, the $k(s)$–algebraic group $\mathfrak{G} \times_R k(s)$ is of relative rank ≥ 2.*

(i) *$\mathfrak{G}(R) = \mathbf{H}(k) \cdot E_{\mathfrak{P}}(R)$;*

(ii) *If $\mathfrak{G}$ is quasi-split, then $E_{\mathfrak{P}}(R) = \mathfrak{G}(R)$.*

(iii) *If k is algebraically closed, then $E_{\mathfrak{P}}(R) = \mathfrak{G}(R)$.*

Note that if k is algebraically closed, it is known [P2] that $\mathfrak{G}$ is quasi-split. Assertion (iii) is then a consequence of (ii). In turn, assertion (ii) is a consequence of Theorem 1.1, part (2) and the above quoted result of Petrov–Stavrova asserting that $E_{\mathfrak{P}}(R) = E^{st}_{\mathfrak{P}}(R)$.

For k non algebraically closed, our result stated in Theorem 1.1 is not enough to compute precisely the stable Whitehead group $W^{st}_{\mathfrak{P}}(R, \mathfrak{G})$ in general, but we have some conjectures as to its nature. Let $R = k[t^{\pm 1}]$.

1.3. **Conjecture.** (1) $E_{\mathfrak{P}}(R) = E^{st}_{\mathfrak{P}}(R)$;

(2) *The natural maps*

$$W^{st}_{\mathfrak{P}}(R, \mathfrak{G}) \to W(K, \mathfrak{G} \times_R K) \to W(F, \mathfrak{G} \times_R F)$$

are group isomorphisms where $K = k(t)$ and $F = k((t))$ is the completion of K at zero.

(3) *The map* $\mathbf{H}(k) \to \mathfrak{G}(R)$ *induces an isomorphism* $\mathbf{H}(k)/\mathcal{R} \simeq W^{st}_{\mathfrak{P}}(R, \mathfrak{G})$.

Assertion (1) can be rephrased by saying that $E_{\mathfrak{P}}(R)$ is a normal subgroup of $\mathfrak{G}(R)$. In assertion (3), $\mathcal{R}$ stands for the R-equivalence relation for the k–group $\mathbf{H}$.

We can see then the quasi-split case as providing some evidence for the conjecture (see Theorem 6.3).

Notation and conventions. Throughout this work k denotes a field of characteristic 0 and $\overline{k} = k_s$ an algebraic (separable) closure of k. We also set $R = k[t^{\pm 1}]$, the ring of Laurent polynomials attached to k, and let $K = k(t)$ denote its fraction field.

By a k–group we will simply mean a group scheme over $\mathrm{Spec}(k)$. Similarly for an R–group.

We will use bold roman characters, e.g. $\mathbf{G}$, $\mathbf{g}$, to denote k–groups and their Lie algebras. The notation $\mathfrak{G}$ and $\mathfrak{g}$ will be reserved for R–groups (which are usually not obtained from a k–group by base change) and their Lie algebras. The (relative) rank of a reductive R-group $\mathfrak{G}$ is the maximum of the ranks of its split subtori $\mathfrak{T} \subset \mathfrak{G}$

2. The loop setting

For the reader's convenience we recall the definition of loop group schemes. Throughout this section $\mathfrak{X}$ will denote a connected noetherian scheme over k, and $\mathbf{G}$ a k–group which is locally of finite presentation.[3]

2.1. **The algebraic fundamental group.** Fix a geometric point a of $\mathfrak{X}$ i.e. a morphism $a : \mathrm{Spec}(\Omega) \to \mathfrak{X}$ where Ω is an algebraically closed field.

Let $\mathfrak{X}_{fet}$ be the category of finite étale covers of $\mathfrak{X}$, and F the covariant functor from $\mathfrak{X}_{fet}$ to the category of finite sets given by

$$F(\mathfrak{X}') = \{\text{geometric points of } \mathfrak{X}' \text{ above } a).$$

That is, $F(\mathfrak{X}')$ consists of all morphisms $a' : \mathrm{Spec}\,(\Omega) \to \mathfrak{X}'$ for which the diagram

$$\begin{tikzcd} & \mathfrak{X}' \arrow[d] \\ \mathrm{Spec}\,(\Omega) \arrow[ur, "a'"] \arrow[r, "a"] & \mathfrak{X} \end{tikzcd}$$

commutes. The group of automorphisms of the functor F is called the *algebraic fundamental group of* $\mathfrak{X}$ *at* a, and is denoted by $\pi_1(\mathfrak{X}, a)$. The functor F is *pro-representable*: there exists a directed set I, objects $(\mathfrak{X}_i)_{i \in I}$ of $\mathfrak{X}_{fet}$, surjective morphisms $\varphi_{ij} \in \mathrm{Hom}_{\mathfrak{X}}(\mathfrak{X}_j, \mathfrak{X}_i)$ for $i \leq j$ and geometric points $a_i \in F(\mathfrak{X}_i)$ such that $a_i = \varphi_{ij} \circ a_j$, and the canonical map $f : \varinjlim \mathrm{Hom}_{\mathfrak{X}}(\mathfrak{X}_i, \mathfrak{X}') \to F(\mathfrak{X}')$ is bijective.

[3]The case most relevant to our work is that of the group of automorphisms of a reductive k–group.

Since the $\mathfrak{X}_i$ are finite and étale over $\mathfrak{X}$ the morphisms φ_{ij} are affine. Thus the inverse limit

$$\mathfrak{X}^{sc} = \varprojlim \mathfrak{X}_i$$

exists in the category of schemes over $\mathfrak{X}$ [EGA4, § 8.2]. For any scheme $\mathfrak{X}'$ over $\mathfrak{X}$ we thus have a canonical map

$$\mathrm{Hom}_{Pro-\mathfrak{X}}(\mathfrak{X}^{sc}, \mathfrak{X}') \stackrel{\mathrm{def}}{=} \varinjlim \mathrm{Hom}_{\mathfrak{X}}(\mathfrak{X}_i, \mathfrak{X}') \simeq F(\mathfrak{X}')$$

obtained by considering the canonical morphisms $\varphi_i : \mathfrak{X}^{\mathrm{sc}} \to \mathfrak{X}_i$.

In computing $\mathfrak{X}^{sc} = \varprojlim \mathfrak{X}_i$ we may replace $(\mathfrak{X}_i)_{i\in I}$ by any cofinal family. This allows us to assume that the $\mathfrak{X}_i$ are (connected) Galois, i.e. the $\mathfrak{X}_i$ are connected and the (left) action of $\mathrm{Aut}_{\mathfrak{X}}(\mathfrak{X}_i)$ on $F(\mathfrak{X}_i)$ is transitive. We then have

$$F(\mathfrak{X}_i) \simeq \mathrm{Hom}_{Pro-\mathfrak{X}}(\mathfrak{X}^{sc}, \mathfrak{X}_i) \simeq \mathrm{Hom}_{\mathfrak{X}}(\mathfrak{X}_i, \mathfrak{X}_i) = \mathrm{Aut}_{\mathfrak{X}}(\mathfrak{X}_i).$$

Thus $\pi_1(\mathfrak{X}, a)$ can be identified with the group $\varprojlim \mathrm{Aut}_{\mathfrak{X}}(\mathfrak{X}_i)^{opp}$. Each group $\mathrm{Aut}_{\mathfrak{X}}(\mathfrak{X}_i)$ is finite, and this endows $\pi_1(\mathfrak{X}, a)$ with the structure of a profinite topological group.

Suppose now that our $\mathfrak{X}$ is a geometrically connected k–scheme. We will denote $\mathfrak{X} \times_k \overline{k}$ by $\overline{\mathfrak{X}}$. Fix a geometric point $\overline{a} : \mathrm{Spec}(\overline{k}) \to \overline{\mathfrak{X}}$. Let a (resp. b) be the geometric point of $\mathfrak{X}$ [resp. $\mathrm{Spec}(k)$] given by the composite maps $a : \mathrm{Spec}(\overline{k}) \xrightarrow{\overline{a}} \overline{\mathfrak{X}} \to \mathfrak{X}$ [resp. $b : \mathrm{Spec}(\overline{k}) \xrightarrow{\overline{a}} \overline{\mathfrak{X}} \to \mathrm{Spec}(k)$]. Then by [SGA1, Théorème IX.6.1]

$$\pi_1\big(\mathrm{Spec}(k), b\big) \simeq \mathrm{Gal}(k) := \mathrm{Gal}(\overline{k}/k)$$

and the sequence

$$1 \to \pi_1(\overline{\mathfrak{X}}, \overline{a}) \to \pi_1(\mathfrak{X}, a) \to \mathrm{Gal}(k) \to 1 \tag{2.1.1}$$

is exact.

2.2. **Example.** Assume that $\mathfrak{X} = \mathrm{Spec}\,(R)$ where $R = k[t^{\pm 1}]$ is the Laurent polynomial ring with coefficients in k. The simply connected cover R^{sc} of R is

$$R_\infty = \varinjlim \overline{R}_m$$

with $\overline{R}_m = k_s[t^{\pm\frac{1}{m}}]$. The "evaluation at 1" provides a geometric point that we denote by a. The algebraic fundamental group is best described as

$$\pi_1(\mathfrak{X}, a) = \widehat{\mathbb{Z}}(1) \rtimes \mathrm{Gal}\,(k). \tag{2.2.1}$$

where $\widehat{\mathbb{Z}}(1)$ denotes the abstract group $\varprojlim_m \boldsymbol{\mu}_m(\overline{k})$ equipped with the natural action of the absolute Galois group $\mathrm{Gal}(k)$.

2.3. **Loop torsors and groups.** Because of the universal nature of $\mathfrak{X}^{sc}$ we have a natural group homomorphism

$$\mathbf{G}(\overline{k}) \longrightarrow \mathbf{G}(\mathfrak{X}^{sc}). \tag{2.3.1}$$

The group $\pi_1(\mathfrak{X}, a)$ acts on $\overline{k}$, hence on $\mathbf{G}(\overline{k})$, via the group homomorphism $\pi_1(\mathfrak{X}, a) \to \mathrm{Gal}\,(k)$. This action is continuous, and together with (2.3.1) yields a map

$$H^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(\overline{k})\big) \to H^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(\mathfrak{X}^{sc})\big).$$

2.4. **Definition.** *A torsor $\mathfrak{E}$ over $\mathfrak{X}$ under $\mathbf{G}$ is called a* loop torsor *if its isomorphism class $[\mathfrak{E}]$ in $H^1_{\acute{e}t}(\mathfrak{X}, \mathbf{G})$ belongs to the image of the composite map*

$$H^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(\overline{k})\big) \to H^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(\mathfrak{X}^{sc})\big) \subset H^1_{\acute{e}t}(\mathfrak{X}, \mathbf{G}).$$

We will denote by $H^1_{loop}(\mathfrak{X}, \mathbf{G})$ the subset of $H^1_{\acute{e}t}(\mathfrak{X}, \mathbf{G})$ consisting of classes of loop torsors. They are given by (continuous) cocycles in the image of the natural map $Z^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(\overline{k})\big) \to Z^1_{\acute{e}t}(\mathfrak{X}, \mathbf{G})$, which we call *loop cocycles.*

2.5. **Geometric and arithmetic part of a loop cocycle.** We assume henceforth that our geometric point a lies above a k-rational point of $\mathfrak{X}$. This provides (see §3.3 of [GP2] for details) an action of $\mathrm{Gal}\,(k)$ on $\pi_1(\overline{\mathfrak{X}}, \overline{a})$ and natural splitting of the exact sequence (2.1.1). Thus

$$\pi_1(\mathfrak{X}, a) = \pi_1(\overline{\mathfrak{X}}, \overline{a}) \rtimes \mathrm{Gal}\,(k).$$

By means of this decomposition we can think of loop cocycles as being comprised of a geometric and an arithmetic part, as we now explain.

Let $\eta \in Z^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(\overline{k})\big)$. The restriction $\eta_{|\mathrm{Gal}(k)}$ is called the *arithmetic part* of η and it is denoted by η^{ar}. It is easily seen that η^{ar} is in fact a cocycle in $Z^1\big(\mathrm{Gal}(k), \mathbf{G}(\overline{k})\big)$. If η is fixed in our discussion, we will at times denote the cocycle η^{ar} by the more traditional notation z. In particular, for $s \in \mathrm{Gal}(k)$ we write z_s instead of η^{ar}_s.

Next we consider the restriction of η to $\pi_1(\overline{\mathfrak{X}}, \overline{a})$ that we denote by η^{geo} and called the *geometric part* of η.

We thus have a map

$$\begin{array}{ccccc} \Theta \ : Z^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(\overline{k})\big) & \longrightarrow & Z^1\big(\mathrm{Gal}(k), \mathbf{G}(\overline{k})\big) \times \mathrm{Hom}\,\big(\pi_1(\overline{\mathfrak{X}}, \overline{a}), \mathbf{G}(\overline{k})\big) \\ \eta & \mapsto & \big(\quad \eta^{ar} \quad , \quad \eta^{geo} \quad \big) \end{array}$$

The group $\mathrm{Gal}(k)$ acts on $\pi_1(\overline{\mathfrak{X}}, \overline{a})$ by conjugation. On $\mathbf{G}(\overline{k})$, the Galois group $\mathrm{Gal}(k)$ acts on two different ways. There is the natural action arising for the action of $\mathrm{Gal}(k)$ on $\overline{k}$, and there is also the twisted action given by the cocycle $\eta^{ar} = z$. Following standard practice to view the abstract group $\mathbf{G}(\overline{k})$ as a $\mathrm{Gal}(k)$–module with the twisted action by z we write ${}_z\mathbf{G}(\overline{k})$.

2.6. **Lemma.** *The map Θ described above yields a bijection between $Z^1\big(\pi_1(\mathfrak{X}, a), \mathbf{G}(\overline{k})\big)$ and couples (z, η^{geo}) with $z \in Z^1\big(\mathrm{Gal}(k), \mathbf{G}(\overline{k})\big)$ and $\eta^{geo} \in \mathrm{Hom}_{\mathrm{Gal}(k)}\,\big(\pi_1(\overline{\mathfrak{X}}, \overline{a}), {}_z\mathbf{G}(\overline{k})\big)$.*

Proof. See Lemma 3.7 of [GP2]. □

2.7. **Remark.** Assume that $R = k[t^{\pm 1}]$ and $\mathfrak{X} = \mathrm{Spec}(R)$. It is easy to verify that η^{geo} arises from a unique k–group homomorphism

$$ {}_\infty\boldsymbol{\mu} = \varprojlim \boldsymbol{\mu}_m \to {}_z\mathbf{G} $$

2.8. **Loop reductive groups.** Let $\mathfrak{H}$ be a reductive group scheme over $\mathfrak{X}$. Since $\mathfrak{X}$ is connected, for all $x \in \mathfrak{X}$ the geometric fibers $\mathfrak{H}_{\overline{x}}$ are reductive group schemes of the same "type" [SGA3, XXII, 2.3]. By Demazure's theorem there exists a unique split reductive group $\mathbf{H}_0$ over k such that $\mathfrak{H}$ is a twisted form (in the étale topology of $\mathfrak{X}$) of $\mathfrak{H}_0 = \mathbf{H}_0 \times_k \mathfrak{X}$. We will call $\mathbf{H}_0$ the *Chevalley k–form of* $\mathfrak{H}$. The $\mathfrak{X}$–group $\mathfrak{H}$ corresponds to a torsor $\mathfrak{E}$ over $\mathfrak{X}$ under the group scheme $\mathbf{Aut}(\mathfrak{H}_0)$, namely $\mathfrak{E} = \mathbf{Isom}_{gr}(\mathfrak{H}_0, \mathfrak{H})$. We recall that $\mathbf{Aut}(\mathfrak{H}_0)$ is representable by a smooth and separated group scheme over $\mathfrak{X}$ by [SGA3, XXII, 2.3]. It is well-known that $\mathfrak{H}$ is then the contracted product $\mathfrak{E} \wedge^{\mathbf{Aut}(\mathfrak{H}_0)} \mathfrak{H}_0$ (see [DG] III §4 n°3 for details).

2.9. **Definition.** *We say that a group scheme $\mathfrak{H}$ over $\mathfrak{X}$ is* loop reductive *if it is reductive and if $\mathfrak{E}$ is a loop torsor.*

2.10. **Example.** Let $R = k[t^{\pm 1}]$. According to [GP2] every reductive group scheme $\mathfrak{G}$ over R is loop reductive. Thus $\mathfrak{G}$ is isomorphic to the twist ${}_\eta(\mathbf{G} \times_k R)$ of its Chevalley form $\mathbf{G}$ by a cocycle

$$ \eta : \pi_1(R) \to \mathbf{Aut}(\mathbf{G})(R^{sc}) $$

which takes values in the subgroup $\mathbf{Aut}(\mathbf{G})(k_s)$ of $\mathbf{Aut}(\mathbf{G})(R^{sc})$.

3. Preliminaries

We keep the notation of the previous section. In particular, $R = k[t^{\pm 1}]$ is the ring of Laurent polynomials over a field k and $\mathfrak{G}$ is a twisted R–form of an almost simple split simply connected group $\mathbf{G}$ by a loop cocycle η.

3.1. **The subgroup H.** We fix a Killing couple $(\mathbf{B}, \mathbf{T})$ of $\mathbf{G}$ and denote by Δ the associated Dynkin diagram. For each subset I of Δ, we let $\mathbf{T}_I = \Big(\bigcap\limits_{\alpha \in I} \ker(\alpha) \Big)^0$, $\mathbf{L}_I = \mathbf{Z}_{\mathbf{G}}(\mathbf{T}_I)$ and we denote by $\mathbf{P}_I$ and $\mathbf{P}_I^-$ the standard parabolic subgroups attached to I.

Let I_t be the Tits index of $\mathfrak{G}$. By the Witt-Tits decomposition [GP2, § 8.2], we can assume additionally that η takes value in

$$ \mathbf{Aut}_{I_t}(\mathbf{G})(k_s) := \mathbf{Aut}(\mathbf{G}, \mathbf{P}_{I_t}, \mathbf{L}_{I_t})(k_s). $$

Then the twisted subgroup scheme $\mathfrak{P} = {}_\eta(\mathbf{P}_{I_t} \times_k R)$ is a minimal parabolic subgroup of $\mathfrak{G}$ and we may consider the above defined subgroup $E_{\mathfrak{P}}(R)$ of $\mathfrak{G}(R)$.

Furthermore, since loop cocycles define "toral classes" [GP2, §6.1], we can also assume that η takes values in

$$ \mathbf{Aut}_{I_t}(\mathbf{G}, \mathbf{T})(k_s) := \mathbf{Aut}(\mathbf{G}, \mathbf{P}_{I_t}, \mathbf{L}_{I_t}, \mathbf{T})(k_s). $$

Any such cocycle factorizes at finite level: there exists an integer $m \geq 1$, a finite Galois extension $\widetilde{k}$ of k containing $\boldsymbol{\mu}_m(k_s)$ such that η is represented by a cocycle

$$\Gamma \to \mathbf{Aut}_{I_t}(\mathbf{G},\mathbf{T})(\widetilde{k}),$$

still denoted by η, where

$$\Gamma := \mathrm{Gal}(\widetilde{k}[t^{\pm\frac{1}{m}}]/R) = \boldsymbol{\mu}_m(\widetilde{k}) \rtimes \mathrm{Gal}(\widetilde{k}/k).$$

Let $\mathbf{Y}$ be a quasi-projective k–variety equipped with a left action of the k–group $\mathbf{Aut}_{I_t}(\mathbf{G},\mathbf{T})$. Then we can twist the k-variety $\mathbf{Y}$ by η^{ar} [Se1, I.5.3] and the R-scheme $\mathbf{Y}_R = \mathbf{Y} \times_k R$ by η [M, I.5.3] The composition of natural maps

$$\eta^{geo} : \boldsymbol{\mu}_m \longrightarrow \mathbf{Aut}_{I_t}({}_{\eta^{ar}}\mathbf{G}, {}_{\eta^{ar}}\mathbf{T}) \longrightarrow \mathbf{Aut}({}_{\eta^{ar}}\mathbf{Y})$$

gives a natural action of $\boldsymbol{\mu}_m$ on the variety ${}_{\eta^{ar}}\mathbf{Y}$ and we denote by $({}_{\eta^{ar}}\mathbf{Y})^{\eta^{geo}}$ the corresponding subvariety consisting of fixed points. It follows immediately from the twisting procedure that we now have a natural closed immersion

$$j : ({}_{\eta^{ar}}\mathbf{Y})^{\eta^{geo}} \times_k R \hookrightarrow {}_{\eta}(\mathbf{Y}_R).$$

This observation applied to $\mathbf{Y} = \mathbf{G}$ gives rise to the k-algebraic group $\mathbf{H} = ({}_{\eta^{ar}}\mathbf{G})^{\eta^{geo}}$ and the closed immersion $j : \mathbf{H} \times_k R \hookrightarrow \mathfrak{G}$. Note that according to Steinberg's connectedness theorem [St2, Theorem 8.1], $\mathbf{H}$ is a (connected) reductive group. Thus we have natural embeddings $\mathbf{H}(k) \hookrightarrow \mathbf{H}(R) \hookrightarrow \mathfrak{G}(R)$.

3.2. **Example.** Assume that $\mathfrak{G}$ is a quasi-split form of $\mathbf{G} = \mathbf{SL}_{n+1}$ which does not come from k. The corresponding twisting cocycle η is determined by the following data: a quadratic étale extension $S = R(\sqrt{u})$, $u \in R^\times$, of R over which $\mathfrak{G}$ becomes split, and an outer automorphism σ of $\mathbf{SL}_{n+1}$ of order 2. Since $\mathfrak{G}$ is not obtained from a k-group by base change we have $u = at$ for some $a \in k^\times$, hence $\eta^{ar} = 1$. It follows that $\mathbf{H} = \mathbf{SL}_{n+1}^{\langle\sigma\rangle}$.

We now note that the twisting data is determined uniquely (up to equivalence) by the quadratic extension S/R. Indeed, any other choice of an outer automorphism of $\mathbf{SL}_{n+1}$ of order 2 gives rise to a loop cocycle η' which is equivalent to η (because any two quasi-split R-forms of $\mathbf{SL}_{n+1}$ which are split over S are isomorphic over R (see [SGA3, XXIV 3.11])).

It is known that if $n+1$ is even, say $2l$, then over the algebraic closure $\overline{k}$ there are two conjugacy classes of outer automorphisms of order two. They are of the form $x \to (x^{-1})^\tau$, $x \in \mathbf{SL}_{n+1}$ where τ is either a symplectic involution of the matrix algebra M_{n+1} or an orthogonal involution. In the first case $\mathbf{H} \simeq \mathbf{Sp}_l$ and in the second case $\mathbf{H} \simeq \mathbf{SO}_{n+1}$. If $n+1$ is odd then τ is automatically an orthogonal involution and hence $\mathbf{H} \simeq \mathbf{SO}_{n+1}$. Note that in the orthogonal case we may choose τ to be split, hence $\mathbf{H} = \mathbf{SO}_{n+1}$ is also split.

3.3. **Parabolic subgroups of $\mathfrak{G}$.** We now come to the description of parabolic subgroups of our R–group $\mathfrak{G}$. Recall that by our construction $\mathfrak{P} = {}_\eta(\mathbf{P}_{I_t} \times_k R)$ is a minimal parabolic subgroup of $\mathfrak{G}$. The k–group

$$\mathbf{Aut}_{I_t}(\mathbf{G}, \mathbf{T}) = \mathbf{Aut}(\mathbf{G}, \mathbf{P}_{I_t}, \mathbf{L}_{I_t}, \mathbf{T})$$

is a subgroup of $\mathbf{Aut}(\mathbf{G}, \mathbf{T})$ and hence we have a natural mapping

$$\mathbf{Aut}_{I_t}(\mathbf{G}, \mathbf{T}) \to \mathbf{Aut}(\Delta).$$

Pushing the 1-cocycle η by this map, we get a homomorphism

$$\eta_\Delta : \Gamma \to \mathbf{Aut}(\Delta)$$

called sometimes the "star action". As in the field case, the parabolic subgroups of $\mathfrak{G}$ containing $\mathfrak{P}$ are of the form ${}_\eta(\mathbf{P}_I \times_k R)$ for I running over the subsets of Δ containing I_t and stable under the action η_Δ. These are called the "standard parabolic subgroups" of $\mathfrak{G}$.

3.4. **Proposition.** *Let $\mathfrak{Q}$ be a parabolic subgroup of $\mathfrak{G}$. Then $\mathfrak{Q}$ is $\mathfrak{G}(R)$–conjugate to a unique standard parabolic subgroup of $\mathfrak{G}$.*

Proof. The unicity is clear since the standard parabolic subgroups provide distinct conjugacy classes.

Assume first that the parabolic subgroup $\mathfrak{Q}$ is minimal and choose a Levi R–subgroup $\mathfrak{M}$ of $\mathfrak{Q}$ (its existence is granted by [SGA3, XXVI.2.3]). Then $\mathfrak{M}$ is a loop reductive group scheme by [GP2, Theorem 5.1]. Furthermore Theorem 15.1 of [CGP2] asserts that the couple $(\mathfrak{Q}, \mathfrak{M})$ is $\mathfrak{G}(R)$-conjugate to $(\mathfrak{P} = {}_\eta(\mathbf{P}_{I_t} \times_k R), {}_\eta(\mathbf{L}_{I_t} \times_k R))$.

In the general case, let $\mathfrak{Q}_{min} \subset \mathfrak{Q}$ be a minimal parabolic subgroup of $\mathfrak{G}$. By the preceding case, we can assume that $\mathfrak{Q}_{min} = \mathfrak{P}$, so that $\mathfrak{Q}$ is a standard parabolic subgroup of $\mathfrak{G}$. □

3.5. **Corollary.** $\mathfrak{G}(K) = \mathfrak{P}(K)\, \mathfrak{G}(R)$.

Proof. The set $(\mathfrak{G}/\mathfrak{P})(R)$ parametrizes R–parabolic subgroups of $\mathfrak{G}$ which are locally conjugate for the étale topology to $\mathfrak{P}$ [SGA3, XXVI.3.20]. Proposition 3.4 says that $\mathfrak{G}(R)$ acts transitively on $(\mathfrak{G}/\mathfrak{P})(R)$, hence we have a natural bijection

$$\mathfrak{G}(R)/\mathfrak{P}(R) \xrightarrow{\sim} (\mathfrak{G}/\mathfrak{P})(R).$$

Similarly, by a theorem of Borel-Tits [BT65, th. 4.13] we have

$$\mathfrak{G}(K)/\mathfrak{P}(K) \xrightarrow{\sim} (\mathfrak{G}/\mathfrak{P})(K).$$

But $\dim(R) = 1$ and $\mathfrak{G}/\mathfrak{P}$ is a projective R–scheme, hence $(\mathfrak{G}/\mathfrak{P})(R) = (\mathfrak{G}/\mathfrak{P})(K)$. This implies that the natural embedding $\mathfrak{G}(R) \hookrightarrow \mathfrak{G}(K)$ induces a bijection

$$\mathfrak{G}(R)/\mathfrak{P}(R) \xrightarrow{\sim} \mathfrak{G}(K)/\mathfrak{P}(K)$$

and the result follows. □

3.6. **Subgroups attached to roots.** By our construction ${}_\eta(\mathbf{T}_{I_t} \times_k R)$ is the centre of the Levi subgroup ${}_\eta(\mathbf{L}_{I_t} \times_k R)$ of the minimal parabolic subgroup $\mathfrak{P}$ of $\mathfrak{G}$. Let $\mathfrak{S}$ be the maximal split subtorus of ${}_\eta(\mathbf{T}_{I_t} \times_k R)$ and $\widehat{\mathfrak{S}}$ be its character group. The torus $\mathfrak{S}$ is also maximal split in $\mathfrak{G}$, since otherwise $\mathfrak{P}$ would not be a minimal parabolic subgroup of $\mathfrak{G}$. Note that $\mathfrak{S}_F$ is still maximal split over F [GP2, Cor. 7.4.3]. The torus $\mathfrak{S}$ acts by the adjoint representation on $\mathfrak{g} = \mathrm{Lie}(\mathfrak{G})(R)$ and we have the decomposition [SGA3, XXIV.6]

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Psi(\mathfrak{G},\mathfrak{S})} \mathfrak{g}_\alpha$$

where $\Psi(\mathfrak{G},\mathfrak{S}) \subset \widehat{\mathfrak{S}}$ is the set of roots of $\mathfrak{G}$ with respect to $\mathfrak{S}$. This decomposition can be extended to F and since $\mathfrak{S}_F$ is maximal split over F, $\Psi(\mathfrak{G},\mathfrak{S})$ is nothing but the relative root system of $\mathfrak{G}_F$ with respect to $\mathfrak{S}_F$.

The minimal parabolic subgroup $\mathfrak{P}$ of $\mathfrak{G}$ defines a basis Δ_0 of the root system $\Psi(\mathfrak{G},\mathfrak{S})$ and an order on $\Psi(\mathfrak{G},\mathfrak{S})$. We denote by $\mathfrak{W} = N_{\mathfrak{G}}(\mathfrak{S})/\mathfrak{Z}_{\mathfrak{G}}(\mathfrak{S})$ the relative Weyl group. It is a finite constant R–group, say $\mathfrak{W} \simeq W_R$, and for each root $\alpha \in \Psi(\mathfrak{G},\mathfrak{S})$ we denote by $w_\alpha \in W = \mathfrak{W}(R)$ the associated reflection.

For each simple root $\alpha \in \Delta_0$ we denote by $\mathfrak{P}_\alpha = \mathfrak{P}_\alpha^+$ (resp. $\mathfrak{P}_\alpha^-$) the parabolic subgroup scheme [SGA3, XXVI.6.1] of $\mathfrak{G}$ such that its Lie algebra $\mathrm{Lie}(\mathfrak{P}_\alpha^+)(R)$ (resp. $\mathrm{Lie}(\mathfrak{P}_\alpha^-)(R)$) is generated by $\mathfrak{g}_0$, $\mathfrak{g}_\beta$ for all positive (resp. negative) roots β and all $\beta \in \Psi(\mathfrak{G},\mathfrak{S}) \cap \mathbb{Q}_{<0} \cdot \alpha$ (resp. $\beta \in \Psi(\mathfrak{G},\mathfrak{S}) \cap \mathbb{Q}_{>0} \cdot \alpha$). These two parabolic subgroups are opposite and share the common Levi subgroup $\mathfrak{Z}_\alpha = \mathfrak{P}_\alpha \cap \mathfrak{P}_\alpha^-$. We denote by $\mathfrak{G}_\alpha = D(\mathfrak{Z}_\alpha)$ the derived subgroup of $\mathfrak{Z}_\alpha$ [SGA3, XXII.6]. It is well-known that $\mathfrak{G}_\alpha$ is a semisimple R–group scheme.

3.7. **Lemma.** (1) $\mathfrak{S}_\alpha = (\mathfrak{S} \cap \mathfrak{G}_\alpha)^0$ *is a maximal split torus of the R–group $\mathfrak{G}_\alpha$. Furthermore, $\mathfrak{S}_\alpha$ is of rank one.*

(2) *If $\mathfrak{G}$ is simply connected (resp. quasi-split), so is $\mathfrak{G}_\alpha$.*

Proof. (1) The F–group $\mathfrak{G}_{\alpha,F}$ is of relative rank one and $\mathfrak{S}_{\alpha,F}$ is its maximal split subtorus over F. A fortiori, $\mathfrak{S}_\alpha$ is a maximal split torus of $\mathfrak{G}_\alpha$ of rank 1.

(2) The fact that the derived group of a Levi subgroup of a parabolic subgroup scheme is simply connected can be checked on the geometric fibers. The problem is thus reduced to the case when the base scheme is (the spectrum of) an algebraically closed field. We can now apply [SS, Cor. 5.8].

Next we assume that $\mathfrak{G}$ is quasi-split. Then $\mathfrak{P}$ is a Borel subgroup of $\mathfrak{G}$. Since $\mathfrak{Z}_\alpha$ is a Levi subgroup of $\mathfrak{P}_\alpha$, the intersection $\mathfrak{Z}_\alpha \cap \mathfrak{P}$ is a Borel subgroup of $\mathfrak{Z}_\alpha$ [SGA3, XXVI.1.20]. Thus $\mathfrak{Z}_\alpha$ is quasi-split and so is its derived group $\mathfrak{G}_\alpha$. $\square$

4. Proof of Theorem 1.1 (1)

Recall that $\mathfrak{G}$ is assumed to be almost simple simply connected. The proof that we provide is a refinement of the methods used by Soulé and Margaux. It is based on a delicate analysis of group actions on Bruhat-Tits euclidean (affine) buildings. At first glance, since $\mathrm{Spec}(k[t^{\pm 1}])$ is obtained from the projective line $\mathbb{P}^1$ by deleting 0 and ∞, it would seem more natural to look at the action of $\mathfrak{G}(R)$ on the twin building $\mathcal{B}(\mathfrak{G}_{k((t))}) \times \mathcal{B}(\mathfrak{G}_{k((t^{-1}))})$ as it was done in [CGP1] and [GP2, §5.1]. This action is indeed well-understood in the split case (Abramenko [A]) but, as it turns out, it would appear to be ill-suited to deal with the twisted case. We work then separately with the building at infinity $\mathcal{B}_- = \mathcal{B}(\mathfrak{G}_{k((t^{-1}))})$ and the building at zero $\mathcal{B}_+ = \mathcal{B}(\mathfrak{G}_{k((t))})$.

4.1. **Buildings.** Let $\widetilde{O}_- = \widetilde{k}[[t^{-\frac{1}{m}}]]$, $\widetilde{F}_- = \widetilde{k}((t^{-\frac{1}{m}}))$ and $F_- = k((t^{-1}))$ We denote by $\widetilde{\mathcal{B}}_-$ the Bruhat–Tits euclidean building of $\mathfrak{G}_{\widetilde{F}_-} = \mathbf{G} \times_k \widetilde{F}_-$ [BT2]. It is equipped with a natural action of

$$\Gamma = \mathrm{Gal}\big(\widetilde{k}[t^{\pm\frac{1}{m}}]/k[t^{\pm 1}]\big) = \mathrm{Gal}\big(\widetilde{F}_-/F_-\big)$$

and $\mathbf{Aut}(\mathbf{G}(\widetilde{F}_-))$. This allows us to consider the twisted action of Γ on $\widetilde{\mathcal{B}}_-$ given by

$$\gamma \star x = \eta(\gamma) \cdot \gamma(x) \quad \gamma \in \Gamma,\, x \in \widetilde{\mathcal{B}}_-.$$

There exists a natural embedding $\mathcal{B}_- \hookrightarrow \widetilde{\mathcal{B}}_-$ and the Bruhat-Tits-Rousseau's descent theorem [Ro, V.1] states that $\mathcal{B}_- \xrightarrow{\sim} (\widetilde{\mathcal{B}}_-)^{\star\Gamma}$ where the notation ${}_\star\Gamma$ means that we take the fixed points with respect to the twisted action of Γ.

The hyperspecial group $\mathbf{G}(\widetilde{O}_-)$ of $\mathbf{G}(\widetilde{F}_-)$ fixes a unique point $\widetilde{\phi}_-$ of $\widetilde{\mathcal{B}}_-$ [BT1, §9.1.9.c]. This point $\widetilde{\phi}_-$ is ${}_\star\Gamma$–stable because η takes values in $\mathbf{Aut}\big(\mathbf{G}(k_s)\big)$, hence it descends to a point ϕ_- of $\mathcal{B}_-$.

Similarly, we set $\widetilde{O}_+ = \widetilde{k}[[t^{\frac{1}{m}}]]$, $\widetilde{F}_+ = \widetilde{k}((t^{\frac{1}{m}}))$ and the corresponding buildings (resp. origins) will be denoted $\mathcal{B}_+$ and $\widetilde{\mathcal{B}}_+$ (resp. ϕ_+ and $\widetilde{\phi}_+$).

The first step of the proof of Theorem 1.1 (1) is to see why it suffices to deal with points of $\mathfrak{G}(R)$ having only a pole at infinity.

4.2. **Reduction to a single pole.** We shall use here the preliminaries considered in § 3.1. Consider the twisted action of Γ on the abstract groups $\mathbf{G}(\widetilde{k}[t^{\frac{1}{m}}])$ and $\mathbf{G}(\widetilde{k})$ by η. Since

$$\mathbf{G}(\widetilde{k}) \subset \mathbf{G}(\widetilde{k}[t^{\frac{1}{m}}]) \subset \mathbf{G}(\widetilde{k}[t^{\pm\frac{1}{m}}])$$

it follows that

$$\mathbf{H}(k) = H^0(\Gamma, {}_\eta\mathbf{G}(\widetilde{k})) \subset H^0\big(\Gamma, {}_\eta\mathbf{G}(\widetilde{k}[t^{\frac{1}{m}}])\big) \subset H^0\big(\Gamma, {}_\eta\mathbf{G}(\widetilde{k}[t^{\pm\frac{1}{m}}])\big) = \mathfrak{G}(R).$$

Clearly, the evaluation of $t^{\frac{1}{m}}$ at 0

$$ev_0 : \mathbf{G}(\widetilde{k}[t^{\frac{1}{m}}]) \to \mathbf{G}(\widetilde{k})$$

is a Γ-equivariant map (for the twisted action). It induces a map

$$H^0\big(\Gamma, {}_\eta\mathbf{G}(\widetilde{k}[t^{\frac{1}{m}}])\big) \to H^0(\Gamma, {}_\eta\mathbf{G}(\widetilde{k})) = \mathbf{H}(k).$$

The composition of the natural mappings

$$\mathbf{H}(k) \hookrightarrow H^0\big(\Gamma, {}_\eta\mathbf{G}(\widetilde{k}[t^{\frac{1}{m}}])\big) \longrightarrow H^0(\Gamma, {}_\eta\mathbf{G}(\widetilde{k})) = \mathbf{H}(k)$$

is the identity map, hence

$$H^0\big(\Gamma, {}_\eta\mathbf{G}(\widetilde{k}[t^{\frac{1}{m}}])\big) = \Theta \rtimes \mathbf{H}(k),$$

where

$$\Theta = \mathrm{Ker}\Big(H^0\big(\Gamma, {}_\eta\mathbf{G}(\widetilde{k}[t^{\frac{1}{m}}])\big) \to \mathbf{H}(k)\Big)$$

which can also be thought of as $H^0(\Gamma, {}_\eta\widetilde{\Theta})$ with

$$\widetilde{\Theta} = \mathrm{Ker}\Big(\mathbf{G}(\widetilde{k}[t^{\frac{1}{m}}]) \xrightarrow{ev_0} \mathbf{G}(\widetilde{k})\Big).$$

4.3. **Lemma.** (1) $\mathfrak{G}\big(k((t))\big)^+$ *is an open (in the t-adic topology) normal subgroup of* $\mathfrak{G}\big(k((t))\big)$.

(2) $\mathfrak{G}\big(k((t))\big) = \mathbf{H}(k) \cdot \mathfrak{G}\big(k((t))\big)^+$.

(3) $\mathfrak{G}(R) = \mathbf{H}(k)\,\Theta\, E_{\mathfrak{P}}(R)$.

Proof. (1) The group $\mathfrak{G}\big(k((t))\big)^+$ is normal non-central in $\mathfrak{G}\big(k((t))\big)$, it is then open by a result of Riehm, see [PR, §3.1, th. 3]. The above reference is for p-adic fields but it also works here since the implicit function theorem holds [Se2, LG III, §10.2].

(2) Let $\mathfrak{H}$ be the canonical Bruhat-Tits $k[[t]]$-group scheme associated to ϕ_+ [BT2]. Its generic fiber is $\mathfrak{G}_{k((t))} = {}_\eta(\mathbf{G} \times_k k((t)))$ and we have

$$\mathfrak{H}(k[[t]]) = \mathrm{Stab}_{\mathfrak{G}(k((t)))}(\phi_+).$$

By Galois descent, this is also

$$\mathrm{Stab}_{\mathfrak{G}(k((t)))}(\phi_+) = \mathrm{Stab}_{\mathbf{G}(\widetilde{k}((t^{\frac{1}{m}})))}(\widetilde{\phi}_+)^{*\Gamma} = \mathbf{G}\big(\widetilde{k}[[t^{\frac{1}{m}}]]\big)^{*\Gamma}.$$

In particular, arguing as above we obtain that the specialization map at $t^{\frac{1}{m}} = 0$ defines a surjective map $ev_0 : \mathfrak{H}(k[[t]]) \to \mathbf{H}(k)$ and the decomposition

$$\mathfrak{H}(k[[t]]) = J \rtimes \mathbf{H}(k) \tag{4.3.1}$$

where J is the kernel of ev_0.

4.4. **Claim.** $\mathfrak{G}\big(k((t))\big) = \mathfrak{H}(k[[t]]) \cdot \mathfrak{G}\big(k((t))\big)^+$.

We need to recall the construction of $\mathfrak{H}$ done in [BT2, §5] which starts with the construction of a smooth $k[[t]]$-group scheme $\mathfrak{Z}$ with generic fiber $Z = Z_{\mathfrak{G}_{k((t))}}\big(\mathfrak{S}_{k((t))}\big)$ [BT2, §5.2.1] such that

$$\mathfrak{Z}\big(k[[t]]\big) = \Big\{ g \in Z\big(k((t))\big) \;\mid\; \chi(g) \in k[[t]]^\times \;\forall \chi \in \mathrm{Hom}_{k((t))-gp}(Z, \mathbb{G}_{m,k((t))}) \Big\}.$$

The group $\mathfrak{Z}$ is a $k[[t]]$-subgroup scheme of $\mathfrak{H}$ and is connected (*ibid*, 5.2.3 and 5.2.5). Hence $\mathfrak{Z}\big(k[[t]]\big)$ is a subgroup of $\mathfrak{H}(k[[t]])$, and it is enough to show that $\mathfrak{Z}\big(k[[t]]\big)$ generates the quotient $\mathfrak{G}\big(k((t))\big)/\mathfrak{G}\big(k((t))\big)^+$.

Let $\mathcal{C}_+$ be a chamber of the apartment $\mathcal{A}(\mathfrak{S}_{k((t))})$ associated to the maximal split torus $\mathfrak{S}_{k((t))}$ of $\mathfrak{G}_{k((t))}$. We know that $\mathcal{C}_+$ is a fundamental simplicial domain for the action of $\mathfrak{G}\big(k((t))\big)$ on $\mathcal{B}_+$ ([BT2, prop. 5.2.12] and [BT1, cor. 2.1.6]), so that $\mathfrak{G}\big(k((t))\big)$ is generated by the $\mathrm{Stab}_{\mathfrak{G}\left(k((t))\right)}(x)$ for x running over the points of the closure of $\mathcal{C}_+$ according to [So1].

Each point $x \in \mathcal{A}(\mathfrak{S}_{k((t))})$ defines a canonical $k[[t]]$-group scheme $\mathfrak{G}_x$ such that $\mathfrak{G}_x\big(k[[t]]\big) = \mathrm{Stab}_{\mathfrak{G}\left(k((t))\right)}(x)$. Since $\mathfrak{G}_{k((t))}$ is simply connected, we know that $\mathfrak{G}_x$ is connected [BT2, 5.2.9], hence the group $\mathfrak{G}_x\big(k[[t]]\big)$ is generated by its subgroup $\mathfrak{Z}\big(k[[t]]\big)$ and $\mathfrak{G}_x\big(k[[t]]\big) \cap \mathfrak{G}\big(k((t))\big)^+$ (*ibid*, 5.2.4). In other words, the images of $\mathfrak{G}_x\big(k[[t]]\big)$ and $\mathfrak{Z}\big(k[[t]]\big)$ coincide in the quotient $\mathfrak{G}\big(k((t))\big)/\mathfrak{G}\big(k((t))\big)^+$. Given that the above holds for all x belonging to the closure of $\mathcal{C}_+$, we get that $\mathfrak{Z}\big(k[[t]]\big)$ generates $\mathfrak{G}\big(k((t))\big)/\mathfrak{G}\big(k((t))\big)^+$, whence the Claim.

It remains then to show that the image of the subgroup J of $\mathfrak{H}(k[[t]])$ in $W\big(k((t)), \mathfrak{G}\big)$ is also trivial. This group had been looked at in great detail in the appendix of [CGP2] where we called it the pro-unipotent radical of the group $\mathfrak{H}(k[[t]])$. We showed that it is an open subgroup of $\mathfrak{H}(k[[t]])$ and then also of $\mathfrak{G}(k((t)))$. By means of the congruence filtration, it admits a fundamental system of neighbourhoods (J_n) where the J_n's are closed and open normal subgroups such that J_n/J_{n+1} is a finite dimensional k-vector space for each $n \geq 0$.

By Part (1), there exists an integer $N \geq 1$ such that $J_N \subset \mathfrak{G}\big(k((t))\big)^+$. On the other hand there exists an integer d such that each element of $W\big(k((t)), \mathfrak{G}_{k((t))}\big) \simeq \mathfrak{G}\big(k((t))\big)/\mathcal{R}$ is of exponent dividing d [G, Rem. 7.6]. Here $\mathfrak{G}\big(k((t))\big)/\mathcal{R}$ is the group of $\mathcal{R}$-equivalence classes. Arguing by induction on n we easily get that J^d maps onto J/J_n where J^d stands for the (normal) subgroup of J generated by the d-powers. Therefore the image of $J = J^d \cdot J_N$ in $W\big(k((t)), \mathfrak{G}_{k((t))}\big)$ is trivial. In other words $J \subset \mathfrak{G}\big(k((t))\big)^+$, hence decomposition (4.3.1) yields

$$\mathfrak{G}\big(k((t))\big) = \mathbf{H}(k) \cdot \mathfrak{G}\big(k((t))\big)^+.$$

(3) Let $g \in \mathfrak{G}(R)$. Viewing g as an element of $\mathfrak{G}(k((t)))$ and multiplying it on the left by a suitable element of $\mathbf{H}(k)$, we may assume without loss of generality that $g \in \mathfrak{G}\big(k((t))\big)^+$. Let $u_1, \dots, u_n \in R_u(\mathfrak{P})\big(k((t))\big)$ and $v_1, \dots, v_n \in R_u(\mathfrak{P}^-)(k((t)))$ be such that $g = u_1\, v_1\, u_2\, v_2 \, \dots \, u_n\, v_n$. Recall that the group

$$\mathfrak{H}(k[[t]]) = H^0\big(\Gamma, {}_\eta\mathbf{G}(\widetilde{k}[[t^{\frac{1}{m}}]])\big)$$

is an open (in the analytic topology) subgroup of

$$H^0\big(\Gamma, {}_\eta\mathbf{G}(\widetilde{k}((t^{\frac{1}{m}})))\big) = \mathfrak{G}(k((t))).$$

Since the unipotent radicals $R_u(\mathfrak{P})$ and $R_u(\mathfrak{P}^-)$ of $\mathfrak{P}$ and $\mathfrak{P}^-$ are split $R_u(\mathfrak{P})(R)$ and $R_u(\mathfrak{P}^-)(R)$ are dense in $R_u(\mathfrak{P})(k((t)))$ and $R_u(\mathfrak{P}^-)(k((t)))$ respectively. Choose

$$u_1', \dots, u_n' \in R_u(\mathfrak{P})(R) \quad \text{and} \quad v_1', \dots, v_n' \in R_u(\mathfrak{P}^-)(R)$$

such that

$$g_0 := (u_1\, v_1\, u_2\, v_2\, \dots\, u_n\, v_n)(u_1'\, v_1'\, u_2'\, v_2'\, \dots\, u_n'\, v_n')^{-1} \in H^0\big(\Gamma, {}_\eta\mathbf{G}(\widetilde{k}[[t^{\frac{1}{m}}]])\big).$$

Then $g = g_0\, u_1'\, v_1'\, u_2'\, v_2'\, \dots\, u_n'\, v_n'$. Evidently

$$g_0 \in H^0\big(\Gamma, {}_\eta\mathbf{G}(\widetilde{k}[t^{\frac{1}{m}}])\big) = \Theta \rtimes \mathbf{H}(k) = \mathbf{H}(k) \cdot \Theta$$

(because g and all u_i', v_i' are in $\mathfrak{G}(R)$). Thus $g \in \mathbf{H}(k) \cdot \Theta \cdot E_{\mathfrak{P}}(R)$. $\square$

4.5. **Buildings and fundamental domains.** We work in this subsection with the buildings $\widetilde{\mathcal{B}}_-$ and $\mathcal{B}_-$ at infinity.

Let $\widetilde{\mathcal{A}}_- = \widetilde{\phi}_- + (\widehat{\mathbf{T}})^0 \otimes_{\mathbb{Z}} \mathbb{R}$ be the apartment of $\widetilde{\mathcal{B}}_-$ associated to the maximal split torus $\mathbf{T} \times_k \widetilde{F}_-$ of $\mathbf{G} \times_k \widetilde{F}_-$. The canonical pairing $\langle\, ,\, \rangle : (\widehat{\mathbf{T}})^0 \times \widehat{\mathbf{T}} \to \mathbb{Z}$ extends to

$$\langle\, ,\, \rangle : ((\widehat{\mathbf{T}})^0 \otimes_{\mathbb{Z}} \mathbb{R}) \times (\widehat{\mathbf{T}} \otimes_{\mathbb{Z}} \mathbb{R}) \to \mathbb{R}.$$

We then define the *sector* (quartier)

$$\widetilde{\mathcal{Q}} := \widetilde{\phi}_- + \widetilde{D} \ \text{ where } \ \widetilde{D} := \big\{ v \in \widehat{\mathbf{T}}^0 \otimes_{\mathbb{Z}} \mathbb{R} \ \mid \ \langle v, b \rangle \geq 0 \ \ \forall\, b \in \Delta \big\}.$$

The following is a quite formal consequence of Soulé's theorem [So2].

4.6. **Lemma.** (a) *For each point $x \in \widetilde{\mathcal{Q}}$, the decomposition $\mathbf{G}\big(\widetilde{k}[t^{\frac{1}{m}}]\big) = \widetilde{\Theta} \rtimes \mathbf{G}(\widetilde{k})$ induces a decomposition*

$$\mathrm{Stab}_{\widetilde{\Theta}}(x) \rtimes \mathrm{Stab}_{\mathbf{G}(\widetilde{k})}(x) \simeq \mathrm{Stab}_{\mathbf{G}\left(\widetilde{k}[t^{\frac{1}{m}}]\right)}(x)$$

(b) *The set $\mathbf{G}(\widetilde{k}) \cdot \widetilde{\mathcal{Q}}$ is a simplicial fundamental domain for the action of the group $\widetilde{\Theta}$ on $\widetilde{\mathcal{B}}_-$.*

Proof. (a) We need to show that the natural injective map

$$\mathrm{Stab}_{\widetilde{\Theta}}(x) \rtimes \mathrm{Stab}_{\mathbf{G}(\widetilde{k})}(x) \to \mathrm{Stab}_{\mathbf{G}\left(\widetilde{k}[t^{\frac{1}{m}}]\right)}(x)$$

is onto. If $x = \widetilde{\phi}_-$, we have seen that

$$\mathbf{G}(\widetilde{k}) = \mathrm{Stab}_{\mathbf{G}(\widetilde{k})}(x) = \mathrm{Stab}_{\mathbf{G}\left(\widetilde{k}[t^{\frac{1}{m}}]\right)}(x),$$

so we may assume that $x \neq \widetilde{\phi}_-$. We recall that $\mathrm{Stab}_{\mathbf{G}\left(\widetilde{k}[t^{\frac{1}{m}}]\right)}(x)$ is also the isotropy (or fixator) of the half line $[x[$ in $\widetilde{\mathcal{Q}}$ of direction $\overrightarrow{\widetilde{\phi}_- x}$. Write $x = \widetilde{\phi}_- + d$ with $d \in \widetilde{D}$ and put

$$I_x = \Big\{ \alpha \in \Delta \ \mid \ \langle\, d, \alpha\, \rangle = 0 \Big\}.$$

¿From [Mg, Prop. 2.5.(3)], we know that

$$\mathrm{Fix}_{\mathbf{G}\left(\widetilde{k}[t^{\frac{1}{m}}]\right)}([x[) \ \subset \ \mathbf{U}_{I_x}\Big(\widetilde{k}[t^{\frac{1}{m}}]\Big) \rtimes \mathbf{L}_{I_x}(\widetilde{k}).$$

Since $\mathbf{L}_{I_x}(\widetilde{k}) \subset \mathrm{Fix}_{\mathbf{G}\left(\widetilde{k}[t^{\frac{1}{m}}]\right)}([x[)$ (*ibid*, lemma 2.3.(1)), it suffices to look at the group $\mathrm{Stab}_{\mathbf{U}_{I_x}\left(\widetilde{k}[t^{\frac{1}{m}}]\right)}(x)$.

4.7. **Claim.** *$\mathbf{U}_{I_x}(\widetilde{k})$ fixes x.*

The claim yields that $\mathrm{Stab}_{\mathbf{U}_{I_x}\left(\widetilde{k}[t^{\frac{1}{m}}]\right)}(x)$ is generated by $\mathrm{Stab}_{\mathbf{U}_{I_x}(\widetilde{k})}(x)$ and $\mathrm{Stab}_{\widetilde{\Theta}}(x) \cap \mathbf{U}_{I_x}\Big(\widetilde{k}[t^{\frac{1}{m}}]\Big)$. Let us prove the Claim. The group $\mathbf{U}_{I_x}(\widetilde{k})$ is generated by the $\mathbf{U}_\alpha(\widetilde{k})$ with $\alpha \in \Delta$ satisfying $\langle\, d, \alpha\,\rangle = 0$. But $\mathbf{U}_\alpha(\widetilde{k})$ fixes $[x[$ for each such α (subgroups of type (II) in [Mg, page 396]) whence the Claim.

(b) Soulé's theorem states that $\widetilde{\mathcal{Q}}$ is a simplicial fundamental domain for the action of the group $\mathbf{G}(\widetilde{k}[t^{\frac{1}{m}}])$ on $\widetilde{\mathcal{B}}_-$. In particular, we have

$$\widetilde{\mathcal{B}}_- = \mathbf{G}(\widetilde{k}[t^{\frac{1}{m}}]) \cdot \widetilde{\mathcal{Q}} = \Big(\widetilde{\Theta} \rtimes \mathbf{G}(\widetilde{k})\Big) \cdot \widetilde{\mathcal{Q}} = \widetilde{\Theta} \cdot (\mathbf{G}(\widetilde{k}) \cdot \widetilde{\mathcal{Q}}).$$

It remains to show that two points of $\mathbf{G}(\widetilde{k}) \cdot \widetilde{\mathcal{Q}}$ which are in the same orbit of $\widetilde{\Theta}$ are necessary equal. Let $y_1 = g_1\, x_1,\ y_2 = g_2\, x_2 \in \mathbf{G}(\widetilde{k}) \cdot \widetilde{\mathcal{Q}}$ be such that $g g_1 x_1 = g_2 x_2$ with $g \in \widetilde{\Theta}$. By Soulé's theorem we have $x_1 = x_2$. Let $x = x_1 = x_2$. Then

$$g_2^{-1} g g_1 = (g_2^{-1} g g_2)(g_2^{-1} g_1) \in \mathrm{Stab}_{\mathbf{G}(\widetilde{k}[t^{\frac{1}{m}}])}(x).$$

Note that $\widetilde{\Theta}$ is a normal subgroup in $\mathbf{G}(\widetilde{k}[t^{\frac{1}{m}}])$. By applying (a) we get that $g_2^{-1} g_1 \in \mathrm{Stab}_{\mathbf{G}(\widetilde{k})}(x)$. We conclude that

$$y_1 = g_1 \cdot x = (g_2\, (g_2^{-1}\, g_1)) \cdot x \ = g_2 \cdot x = y_2,$$

as required. □

The major reason to use the groups Θ and $\widetilde{\Theta}$ is because they afford the following very precise control on their subgroups stabilizing points on the building $\widetilde{\mathcal{B}}_-$.

4.8. **Lemma.** *Let x be a point of $\widetilde{\mathcal{B}}_-$.*

(1) *There exists a split unipotent $\widetilde{k}$–group $\mathbf{V}$ such that $\mathbf{V}(\widetilde{k}) = \mathrm{Stab}_{\widetilde{\Theta}}(x)$.*

(2) *Assume that $x \in \mathcal{B}_-$. Then $H^1\big(\Gamma, {}_\eta(\mathrm{Stab}_{\widetilde{\Theta}}(x))\big) = 1$.*

Proof. (1) By Lemma 4.6 we can assume that $x = q \in \widetilde{\mathcal{Q}}$. Let $q = \widetilde{\phi}_- + d$ with $d \in \widetilde{D}$ and put

$$I_q = \Big\{ \alpha \in \Delta \mid \langle\, d, \alpha \,\rangle = 0 \Big\}.$$

¿From [Mg, Prop. 2.5.(3)], we know that

$$\mathrm{Stab}_{\mathbf{G}(\widetilde{k}[t^{\frac{1}{m}}])}(q) \subset \mathbf{U}_{I_q}\big(\widetilde{k}\big[t^{\frac{1}{m}}\big]\big) \rtimes \mathbf{L}_{I_q}(\widetilde{k}) \subset \mathbf{P}_{I_q}\big(\widetilde{k}\big[t^{\frac{1}{m}}\big]\big)$$

where $\mathbf{U}_{I_q}$ is the unipotent radical of $\mathbf{P}_{I_q}$. Since

$$\widetilde{\Theta} = \mathrm{Ker}\Big(\mathbf{G}(\widetilde{k}[t^{\frac{1}{m}}]) \xrightarrow{ev_0} \mathbf{G}(\widetilde{k})\Big)$$

it follows that

$$\mathrm{Stab}_{\widetilde{\Theta}}(q) \subset \mathbf{U}_{I_q}(\widetilde{k}[t^{\frac{1}{m}}]). \tag{4.8.1}$$

Let $\Phi^+_{I_q} = \{\alpha \in \Phi(\mathbf{G}, \mathbf{T}) \mid \langle\, d, \alpha \,\rangle > 0\}$. Clearly, every $\alpha \in \Phi^+_{I_q}$ is a positive root (because $q \in \widetilde{\mathcal{Q}}$) and the corresponding root subgroup $\mathbf{U}_\alpha$ is contained in $\mathbf{U}_{I_q}$. By [So2, end of §1.1] there exist non-negative integers $m_{\alpha,q}$, $\alpha \in \Phi^+_{I_q}$ such that

$$\mathrm{Stab}_{\mathbf{U}_{I_q}(\widetilde{k}[t^{\frac{1}{m}}])}(q) = \Big\langle \mathbf{U}_\alpha(f) \;\mid \alpha \in \Phi^+_{I_q},\; f \in \widetilde{k}[t^{\frac{1}{m}}],\, \deg(f) \leq m_{\alpha,q} \Big\rangle.$$

Therefore $\mathrm{Stab}_{\mathbf{U}_{I_q}(\widetilde{k}[t^{\frac{1}{m}}])}(q)$ is the group of $\widetilde{k}$-points of a connected split unipotent group, say $\mathbf{W}$. But

$$\mathrm{Stab}_{\widetilde{\Theta}}(q) = \ker\Big(\mathrm{Stab}_{\mathbf{U}_{I_q}(\widetilde{k}[t^{\frac{1}{m}}])}(q) \to \mathbf{U}_{I_q}(\widetilde{k})\Big),$$

hence $\mathrm{Stab}_{\widetilde{\Theta}}(q) = \mathbf{V}(\widetilde{k})$ where $\mathbf{V}$ is a subgroup of $\mathbf{W}$ given by the additional condition $f(0) = 0$. Evidently $\mathbf{V}$ is split unipotent (because $\mathrm{char}(k) = 0$).

(2) Let $x \in \mathcal{B}_-$. Since it is fixed by the twisted action of Γ, $\mathbf{V}(\widetilde{k})$ is Γ–stable with respect to the twisted action of Γ by η. So we may view $\mathbf{V}(\widetilde{k})$ as a Γ-module. The descending central sequence for $\mathbf{V}$ provides a filtration

$$\mathbf{1} \subset \mathbf{V}_n \subset \cdots \subset \mathbf{V}_1 \subset \mathbf{V}_0 = \mathbf{V}$$

where $\mathbf{V}_i$ are $\widetilde{k}$–subgroups of $\mathbf{V}$ such that $\mathbf{V}_i/\mathbf{V}_{i+1}$ is isomorphic to $\mathbf{G}^{r_i}_{a,\widetilde{k}}$. Taking $\widetilde{k}$-points we get a filtration

$$1 \subset \mathbf{V}_n(\widetilde{k}) \subset \cdots \subset \mathbf{V}_1(\widetilde{k}) \subset \mathbf{V}_0(\widetilde{k}) = \mathbf{V}(\widetilde{k})$$

which is the descending central sequence of $\mathbf{V}(\widetilde{k}) = \mathrm{Stab}_{\widetilde{\Theta}}(x)$ satisfying $\mathbf{V}_i(\widetilde{k})/\mathbf{V}_{i+1}(\widetilde{k}) \cong (\widetilde{k})^{r_i}$ for $i = 0, .., n-1$. All terms of the last filtration are characteristic subgroups, so that they are stable with respect to the twisted action of Γ on $\mathbf{V}(\widetilde{k})$. Since $\mathrm{char}(\widetilde{k}) = 0$ the abelian group $\mathbf{V}_i(\widetilde{k})/\mathbf{V}_{i+1}(\widetilde{k})$ is infinitely divisible. It follows that

$$H^1\big(\Gamma,\, {}_\eta(\mathbf{V}_i(\widetilde{k})/\mathbf{V}_{i+1}(\widetilde{k}))\big) = 0$$

for $i = 0, ..., n-1$. By dévissage, we get

$$1 = H^1\big(\Gamma,\, {}_\eta(\mathbf{V}(\widetilde{k}))\big) = H^1\big(\Gamma,\, {}_\eta(\mathrm{Stab}_{\widetilde{\Theta}}(x))\big),$$

as required. □

4.9. **Descent argument.** Since η takes value in $\mathbf{Aut}_{I_t}(\mathbf{G},\mathbf{T})(\widetilde{k})$, it follows that $\mathbf{G}(\widetilde{k})$ and $\mathbf{G}(\widetilde{k}) \cdot \widetilde{\mathcal{Q}}$ are stable under the twisted action of Γ. We put $\mathcal{J} = \Big(\mathbf{G}(\widetilde{k}) \,.\, \widetilde{\mathcal{Q}}\Big)^{\star\Gamma}$.

4.10. **Lemma.** (1) *$\mathcal{J}$ is a contractible subcomplex of $\mathcal{B}_-$.*
(2) *$\mathcal{J}$ is a fundamental simplicial domain for the action of Θ on $\mathcal{B}_-$.*

Proof. (1) We are given a point $x \in \mathcal{J}$ and have to show that the facet F_x in $\mathcal{B}_-$ associated to x is included in $\mathcal{J}$. By descent theory of Bruhat-Tits building [Ro, V.1] we have $F_x = (\widetilde{F}_x)^{\star\Gamma} = \widetilde{F}_x \cap \mathcal{B}_-$ where $\widetilde{F}_x$ stands for the facet of $\widetilde{\mathcal{B}}_-$ attached to x. Since $\mathbf{G}(\widetilde{k}) \cdot \widetilde{\mathcal{Q}}$ is a subcomplex of $\widetilde{\mathcal{B}}_-$, we have $\widetilde{F}_x \subset \mathbf{G}(\widetilde{k}) \cdot \widetilde{\mathcal{Q}}$. Taking the Γ–invariants we conclude $F_x \subset \mathcal{J}$.

The contractibility can be established as follows. Given $x \in \mathcal{J}$, the segment $[\phi_-, x]$ is included in $\mathcal{J}$. So the restriction of the standard contraction of $\widetilde{\mathcal{B}}_-$ to $\widetilde{\phi}_-$ [BT1, Prop. 7.4.20.(v)] induces a contraction of $\mathcal{J}$ to ϕ_-.

(2) According to Lemma 4.6, $\mathbf{G}(\widetilde{k}) \cdot \widetilde{\mathcal{Q}}$ is a fundamental domain for the action of the group $\widetilde{\Theta}$ on $\widetilde{\mathcal{B}}_-$, hence two distinct points of $\mathcal{J}$ are not in the same orbit of $\widetilde{\Theta}$. It remains to show that any point $x \in \mathcal{B}_-$ is conjugate under Θ to a point of $\mathcal{J}$. Again by Lemma 4.6, there exists a unique point $q \in \mathbf{G}(\widetilde{k}) \cdot \widetilde{\mathcal{Q}}$ and $g \in \widetilde{\Theta}$ such that $x = g \cdot q$. Since x, $\widetilde{\Theta}$ and $\mathbf{G}(\widetilde{k}) \cdot \widetilde{\mathcal{Q}}$ are Γ–stable, by unicity we get that $q \in \mathcal{J}$. For each $\gamma \in \Gamma$, from $\gamma \star x = x$ we have

$$x = \gamma \star x = (\gamma \star g) \cdot q = g \cdot q.$$

Hence $\gamma \mapsto a_\gamma = g^{-1}\,(\gamma \star g)$ is a 1-cocycle for Γ with value in ${}_\eta\big(\mathrm{Stab}_{\widetilde{\Theta}}(q)\big)$.

According to Lemma 4.8 (2), we have $H^1(\Gamma, {}_\eta\big(\mathrm{Stab}_{\widetilde{\Theta}}(q)\big) = 1$. Hence there exists $g_0 \in \mathrm{Stab}_{\widetilde{\Theta}}(q)$ such that

$$a_\gamma = g^{-1}\,(\gamma \star g) = g_0^{-1}\,(\gamma \star g_0)$$

for each $\gamma \in \Gamma$. We put $g' = g g_0^{-1} \in H^0(\Gamma, {}_\eta\widetilde{\Theta}) = \Theta$. Then

$$x = g \cdot q = g' \cdot (g_0 \cdot q) = g' \cdot q \in \Theta \cdot \mathcal{J}$$

as desired. □

Thus, the subspace $\mathcal{J}$ of $\mathcal{B}_-$ is contractible, hence connected and simply connected. Since it is a fundamental simplicial domain for the action of Θ on $\mathcal{B}_-$, it follows that Θ is generated by the subgroups $\mathrm{Stab}_\Theta(x)$ for x running over $\mathcal{J}$ [So1]. Recall that according to Lemma 4.3 (2) we have $\mathfrak{G}(R) = \mathbf{H}(k)\,\Theta\, E_{\mathfrak{P}}(R)$. The following lemma completes the proof of Theorem 1.1 (1).

4.11. **Lemma.** *Let $x \in \mathcal{J}$. Then there exists $g_x \in \mathfrak{G}(R)$ such that*

$$\mathrm{Stab}_\Theta(x) \subset g_x\, R_u(\mathfrak{P})(R)\, g_x^{-1}.$$

Proof. We write $x \in \mathcal{J} \subset \mathbf{G}(\widetilde{k})\cdot\widetilde{\mathcal{Q}}$ in the form $x = g\cdot q$ where $q = \phi_- + d \in \widetilde{\mathcal{Q}}$, $d \in \widetilde{D}$ and $g \in \mathbf{G}(\widetilde{k})$. If $q = \phi_-$, then $x = \phi_-$ and $\mathrm{Stab}_{\mathbf{G}(\widetilde{k}[t^{\frac{1}{m}}])}(\phi) = \mathbf{G}(\widetilde{k})$. This implies that $\mathrm{Stab}_{\widetilde{\Theta}}(x) = 1$ and a fortiori $\mathrm{Stab}_\Theta(x) = 1$ and there is nothing to do.

We now assume that $d \neq 0$ and as in Lemma 4.8 (1) consider the set of roots

$$I_q = \Big\{ \alpha \in \Delta \ \mid\ \langle\, d, \alpha\, \rangle = 0 \Big\}.$$

Recall that by (4.8.1)

$$\mathrm{Stab}_{\widetilde{\Theta}}(q) \ \subset\ \mathbf{U}_{I_q}(\widetilde{k}[t^{\frac{1}{m}}]) \ \subset\ \mathbf{P}_{I_q}(\widetilde{k}[t^{\frac{1}{m}}]).$$

Since $x = g\cdot q$, we get

$$\text{(4.11.1)}\quad \mathrm{Stab}_{\widetilde{\Theta}}(x) \subset (g\mathbf{U}_{I_q}g^{-1})(\widetilde{k}[t^{\frac{1}{m}}]) \subset (g\mathbf{P}_{I_q}g^{-1})(\widetilde{k}[t^{\frac{1}{m}}]) = \mathbf{Q}(\widetilde{k}[t^{\frac{1}{m}}]).$$

where $\mathbf{Q}$ stands for the $\widetilde{k}$–parabolic subgroups $\mathbf{Q} = g\mathbf{P}_{I_q}g^{-1}$ of $\mathbf{G}_{\widetilde{k}}$.

We next want to show that $\mathbf{Q}$ is stable with respect to the twisted action of Γ.

4.12. **Claim.** $\eta(\Gamma) \subset \mathbf{Aut}(\mathbf{G},\mathbf{Q})(\widetilde{k})$.

We need here the link $\mathcal{L}_{\phi_-}$ which is nothing but the (combinatorial) spherical building $\mathcal{S}(\mathbf{G}_{\widetilde{k}})$ (see [Mg, §2.4]) and which is equipped with the twisted action of Γ (because ϕ_- is Γ–stable). For $n >> 0$ the point $q_n = \phi_- + \frac{d}{n}$ belongs to $\mathcal{L}_{\widetilde{\phi}_-}$ and defines then a $\widetilde{k}$–parabolic subgroup of $\mathbf{G}$ which turns out to be $\mathbf{P}_{I_q,\widetilde{k}}$ because $I_q = I_{q_n}$. Similarly the point $x_n = \phi_- + \frac{g\,.\,d}{n}$ of $\mathcal{L}_{\widetilde{\phi}_-}$ defines the $\widetilde{k}$–parabolic subgroups $\mathbf{Q}$.

Note that by construction $x = g\cdot q = \phi_- + g\cdot d$ is ${}_\star\Gamma$–stable. Since ϕ_- is also ${}_\star\Gamma$–stable, so is $g\cdot d$ and hence so is x_n. Therefore the $\widetilde{k}$–parabolic subgroup $\mathbf{Q}$ of $\mathbf{G}$ is preserved by the twisted action of Γ, whence the claim.

Recall that $\mathfrak{G}$ is the twist of $\mathbf{G}_R = \mathbf{G}\times_k R$ by the 1-cocycle η. By the claim, we see that the parabolic group $\mathbf{Q}\times_k R$ of $\mathbf{G}_R$ defines by twisting an parabolic subgroup $\mathfrak{Q}$ of $\mathfrak{G}$. By Lemma 3.4, there exists $g_x \in \mathbf{G}(R)$ such that $\mathfrak{P} \subset g_x^{-1}\,\mathfrak{Q}\,g_x$ of $\mathfrak{G}$. Clearly, $g_x^{-1}\,R_u(\mathfrak{Q})\,g_x \subset R_u(\mathfrak{P})$. The inclusion (4.11.1) reads then as

$$\text{(4.12.1)}\quad \mathrm{Stab}_{\widetilde{\Theta}}(x) \subset R_u(\mathfrak{Q})(\widetilde{k}[t^{\frac{\pm 1}{m}}]) \ \subset\ g_xR_u(\mathfrak{P})(\widetilde{k}[t^{\pm\frac{1}{m}}])g_x^{-1}.$$

Taking the fixed points under ${}_\star\Gamma$ yields the inclusion

$$\mathrm{Stab}_\Theta(x) \subset R_u(\mathfrak{Q})(R) \subset\ g_x\, R_u(\mathfrak{P})(R)\, g_x^{-1}.$$

This concludes the proof. □

5. Steinberg's method

We maintain the notation introduced in § 3.6. To finish the proof of part (2) of Theorem 1.1 we first reduce it to the case of groups of (relative) rank 1 and then we use the decomposition of $\mathfrak{G}(R)$ in part (1). Our reduction to quasi-split groups of rank 1 is based, to some extent, on Steinberg's method in [St2, § 8].

5.1. **Proposition.** *Let $w \in W$ and write $w = w_{\alpha_1} \cdots w_{\alpha_l}$ with $\alpha_1, \cdots, \alpha_l \in \Delta_0$. Then*

$$\mathfrak{P}(K) w \mathfrak{P}(K) \subset \mathfrak{P}(K)\, \mathfrak{G}_{\alpha_1}(R) \cdots \mathfrak{G}_{\alpha_l}(R).$$

Proof. We reason by induction on l. Let first $l = 1$. Then we have

$$\mathfrak{P}(K) w_\alpha \mathfrak{P}(K) \subset \mathfrak{P}(K) w_\alpha \mathfrak{P}(K) \cup \mathfrak{P}(K) = \mathfrak{P}(K)\, \langle w_\alpha \rangle\, \mathfrak{P}(K) = \mathfrak{P}_\alpha(K)$$

according to Tits' system properties [Bor, §21.16]. We denote by $\mathrm{rad}^u(\mathfrak{P}_\alpha)$ the unipotent radical of $\mathfrak{P}_\alpha$. According to [SGA3, XXVI.1.20], we have $\mathfrak{P} = \mathrm{rad}^u(\mathfrak{P}_\alpha) \rtimes \mathfrak{Q}$ where $\mathfrak{Q}$ is a parabolic subgroup of $\mathfrak{Z}_\alpha$. Since $\mathfrak{P}_\alpha = \mathrm{rad}^u(\mathfrak{P}_\alpha) \rtimes \mathfrak{Z}_\alpha$, it follows that we have an isomorphism $\mathfrak{Z}_\alpha / \mathfrak{Q} \xrightarrow{\sim} \mathfrak{P}_\alpha / \mathfrak{P}$ (both quotients are representable). Then $(\mathfrak{Z}_\alpha/\mathfrak{Q})_K \xrightarrow{\sim} (\mathfrak{P}_\alpha/\mathfrak{P})_K$ is the K–variety of parabolic subgroups of $(\mathfrak{Z}_\alpha)_K$ of the same type as $\mathfrak{Q}_K$. According to [SGA3, XXVI.1.19], it is also the K–variety of parabolic subgroups of the derived group $(\mathfrak{G}_\alpha)_K$ of $(\mathfrak{Z}_\alpha)_K$ of the same type as the K–parabolic subgroup $\mathfrak{Q}_K \cap (\mathfrak{G}_\alpha)_K$ of $(\mathfrak{G}_\alpha)_K$. Now we apply Borel-Tits' theorem [BT65, th. 4.13] which states that $\mathfrak{G}_\alpha(K)$ acts transitively on the K-points of that variety. In other words, the map $\mathfrak{G}_\alpha(K) \to (\mathfrak{Z}_\alpha/\mathfrak{Q})(K) \xrightarrow{\sim} (\mathfrak{P}_\alpha/\mathfrak{P})(K)$ is surjective, so that $\mathfrak{P}_\alpha(K) = \mathfrak{G}_\alpha(K) \cdot \mathfrak{P}(K)$. By taking the opposite decomposition we conclude that

$$\mathfrak{P}(K) w_\alpha \mathfrak{P}(K) \subset \mathfrak{P}_\alpha(K) \subset \mathfrak{P}(K)\, \mathfrak{G}_\alpha(K).$$

It remains to check that $\mathfrak{P}(K)\, \mathfrak{G}_\alpha(K) \subset \mathfrak{P}(K)\, \mathfrak{G}_\alpha(R)$. Let $\mathfrak{Q}_\alpha = \mathfrak{P} \cap \mathfrak{G}_\alpha$. It is a parabolic subgroup of the R–group $\mathfrak{G}_\alpha$. According to Proposition 3.5, we have $\mathfrak{G}_\alpha(K) = \mathfrak{Q}_\alpha(K)\, \mathfrak{G}_\alpha(R)$. Therefore

$$\mathfrak{P}(K)\, \mathfrak{G}_\alpha(K) = \mathfrak{P}(K)\, \mathfrak{Q}_\alpha(K)\, \mathfrak{G}_\alpha(R) = \mathfrak{P}(K)\, \mathfrak{G}_\alpha(R).$$

Let now $l \geq 2$ and write $w = w_{\alpha_1} w'$. Then we have

$$\begin{aligned}
\mathfrak{P}(K)\, w\, \mathfrak{P}(K) &= \mathfrak{P}(K)\, w_{\alpha_1} w'\, \mathfrak{P}(K) \\
&\subset \big(\mathfrak{P}(K)\, w_{\alpha_1}\big)\, \big(\mathfrak{P}(K)\, w'\, \mathfrak{P}(K)\big) \\
&\subset \mathfrak{P}(K)\, w_{\alpha_1}\, \mathfrak{P}(K)\, \mathfrak{G}_{\alpha_2}(R) \cdots \mathfrak{G}_{\alpha_l}(R) \quad [\text{ Induction}] \\
&\subset \mathfrak{P}(K)\, \mathfrak{G}_{\alpha_1}(R)\, \mathfrak{G}_{\alpha_2}(R) \cdots \mathfrak{G}_{\alpha_l}(R) \quad [\text{ Case } l = 1]
\end{aligned}$$

as required. □

¿From the Bruhat decomposition $\mathfrak{G}(K) = \bigcup_{w \in W} \mathfrak{P}(K) w \mathfrak{P}(K)$, we get the following.

5.2. **Corollary.** *$\mathfrak{G}(R)$ is generated by $\mathfrak{P}(R)$ and the $\mathfrak{G}_\alpha(R)$ for all roots $\alpha \in \Delta_0$.*

Proof. Let $g \in \mathfrak{G}(R)$. Viewing g as an element of $\mathfrak{G}(K)$ we can write it in the form $g = p_1 w p_2$ where $p_1, p_2 \in \mathfrak{P}(K)$ and $w \in W$. By Proposition 5.1 we have $g = p_1 w p_2 = p_3 g_1 \cdots g_l$ where $p_3 \in \mathfrak{P}(K)$ and $g_1, \dots, g_l \in \mathfrak{G}_{\alpha_i}(R)$, $\alpha_i \in \Delta_0$. Since all g_i and g are in $\mathfrak{G}(R)$ we conclude that $p_3 \in \mathfrak{P}(R)$. □

5.3. **Application to the quasi-split case.** We assume here additionally that $\mathfrak{G}$ is quasi-split. In this case $\mathfrak{B} = \mathfrak{P}$ is a Borel subgroup of $\mathfrak{G}$.

5.4. **Lemma.** *$\mathfrak{B}(R) \subset \langle E_{\mathfrak{B}}(R), \mathfrak{G}_\alpha(R) \rangle$ where α runs over Δ_0.*

Proof. Recall that $\mathfrak{B} = R_u(\mathfrak{B}) \rtimes \mathfrak{T}$, where $\mathfrak{T}$ is the twist of $\mathbf{T}$ by η, so that we need to deal with $\mathfrak{T}(R)$ only. But $\mathfrak{T}$ has a decomposition [SGA3, XXIV.3.13]

$$\mathfrak{T} \xrightarrow{\sim} \prod_{\alpha \in \Delta_0} \prod_{R_\alpha/R} (\mathbf{G}_{m,R_\alpha})$$

where the R_α are connected étale covers of R. It remains to note that each summand $\prod_{R_\alpha/R}(\mathbf{G}_{m,R_\alpha})$ is a maximal torus of $\mathfrak{G}_\alpha$, hence the statement follows. □

We can now easily finish the proof of Theorem 1.1 (2). By Lemma 3.7, $\mathfrak{G}_\alpha$ is semisimple simply connected and quasi-split of relative rank 1. Hence according to Lemma 5.4 and Corollary 5.2 we may assume without loss of generality that $\mathfrak{G} = \mathfrak{G}_\alpha$ is a quasi-split group scheme of relative rank 1. The following cases can occur.

$\mathfrak{G}_\alpha$ is of absolute type A_1. Then $\mathfrak{G}_\alpha = \mathbf{SL}_2$. Since R is euclidean $\mathfrak{G}_\alpha(R)$ is generated by "elementary matrices".

$\mathfrak{G}_\alpha$ is of absolute type A_1^2. Then $\mathfrak{G}_\alpha = \prod_{R'/R} \mathbf{SL}_{2,R'}$ where R'/R is the unique (connected) quadratic étale extension of R which splits $\mathfrak{G}_\alpha$. It is known that R' is a Laurent polynomial ring [GP2, Lemma 2.8]. Since $\mathfrak{G}_\alpha(R) = \mathbf{SL}_2(R')$ we are reduced to the previous case.

$\mathfrak{G}_\alpha$ is of absolute type A_1^3. The argument is similar.

$\mathfrak{G}_\alpha$ is of absolute type A_2. Let S/R be the quadratic (connected) étale extension splitting $\mathfrak{G}_\alpha$.

Denote by τ the non-trivial automorphism of S over R. Since $\mathfrak{G}_\alpha$ is quasi-split over R and split over S it admits a realization $\mathfrak{G}_\alpha \simeq \mathbf{SU}(f)$ where $f = x\tau(y) + z\tau(z)$, $x, y, z \in S$, is a 3-dimensional hermitian form on $V = S \oplus S \oplus S$. Let $W \subset V$ be the submodule spanned by first two components of V. Obviously, $\mathbf{SU}(f|_W) \simeq \mathbf{SL}_2$. Therefore $\mathbf{SU}(f|_W)(R)$ is contained in the subgroup $E_{\mathfrak{B}}(R)$ of $\mathbf{SU}(f)(R)$. In particular, $\mathbf{T}(R) \subset E_{\mathfrak{B}}(R)$ where $\mathbf{T} \simeq \mathbf{G}_{m,R}$ is a split R–torus consisting of matrices of the form

$$\begin{pmatrix} x & 0 & 0 \\ 0 & x^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Consider now the group $\mathbf{H}$. By Example 3.2, $\mathbf{H} \simeq \mathbf{SO}_3$. Let $\mathbf{T}' \subset \mathbf{H}$ be a split maximal k-torus. Since the simply connected covering of $\mathbf{H}$ is $\mathbf{SL}_2$ the group of k-points of $\mathbf{H}$ is generated by unipotent elements and $\mathbf{T}'(k)$. The two maximal split tori $\mathbf{T}'_R$ and $\mathbf{T}$ of the R–group $\mathfrak{G}_\alpha$ are conjugate under the action of $\mathfrak{G}_\alpha(R)$ by [CGP2]. Since $\mathbf{T}(R) \subset E_{\mathfrak{B}}(R)$, we get $\mathbf{H}(k) \subset \mathbf{H}(R) \subset E^{st}_{\mathfrak{B}}(R)$. The application of part (1) of Theorem 1.1 completes the proof of Theorem 1.1 (2).

6. Remarks on Conjecture 1.3

The following lemma shows that in parts (2) and (3) of Conjecture 1.3 one needs to prove injectivity only.

6.1. **Lemma.** (1) *The map* $W^{st}_{\mathfrak{P}}(R, \mathfrak{G}) \to W(F, \mathfrak{G}_F)$ *is surjective.*

(2) *The map* $W^{st}_{\mathfrak{P}}(R, \mathfrak{G}) \to W(K, \mathfrak{G}_K)$ *is surjective.*

Proof. (1) This follows from Lemma 4.3 (2) and Theorem 1.1 (1).

(2) This is an argument of strong approximation. Let $g \in \mathfrak{G}(K)$ and let Σ be the set of closed points of $\mathrm{Spec}(R)$ where g is not regular. For each point $x \in \Sigma$, we denote by R_x the local ring at x; R_x is a DVR and we denote by $\widehat{R}_x$ its completion and by $\widehat{K}_x$ its fraction field. By [G, Lemme 4.5], we have $\mathfrak{G}(\widehat{K}_x) = \mathfrak{G}(\widehat{K}_x)^+ \cdot \mathfrak{G}(\widehat{R}_x)$ for each $x \in \Sigma$. In particular, we can decompose

$$g = g_x^+ \, g_x, \quad g_x \in \mathfrak{G}(\widehat{R}_x), \;\; g_x^+ \in \mathfrak{G}(\widehat{K}_x)^+,$$

for each $x \in \Sigma$. Let $\mathrm{Spec}(R') = \mathrm{Spec}(R) \setminus \Sigma$. By (loc. cit., Lemme 4.6), the group $\mathfrak{G}(R') \cap \mathfrak{G}(K)^+$ is dense in $\prod_{x \in \Sigma} \mathfrak{G}(\widehat{K}_x)^+$. Since $\prod_{x \in \Sigma} \mathfrak{G}(\widehat{R}_x)$ is open in $\prod_{x \in \Sigma} \mathfrak{G}(\widehat{K}_x)$, there exists $g' \in \mathfrak{G}(R') \cap \mathfrak{G}(K)^+$ such that

$$(g')^{-1} \, g_x^+ \in \mathfrak{G}(\widehat{R}_x), \quad \forall x \in \Sigma.$$

It follows that $(g')^{-1} \, g \in \mathfrak{G}(\widehat{R}_x)$ for each $x \in \Sigma$, so that $(g')^{-1} \, g \in \mathfrak{G}(R)$. Thus $[g] \in W(K, G_K)$ is in the image of the map $W^{st}_{\mathfrak{P}}(R, \mathfrak{G}) \to W(K, \mathfrak{G}_K)$. $\square$

6.2. **Remarks.** (a) If Conjecture 1.3 (2) holds, the stable Whitehead group $W^{st}_{\mathfrak{P}}(R, \mathfrak{G})$ has finite exponent.

(b) Since $W(K, \mathfrak{G} \times_R K) \to \mathfrak{G}(K)/\mathcal{R}$ is an isomorphism by [G, 7.2] we have well-defined maps

$$\mathbf{H}(k)/\mathcal{R} \to \mathbf{H}(K)/\mathcal{R} \to \mathfrak{G}(K)/\mathcal{R}.$$

So if part (2) of the conjecture holds, it would imply that the map $\mathbf{H}(k) \to W^{st}_{\mathfrak{P}}(R, \mathfrak{G})$ induces a well defined map $\mathbf{H}(k)/\mathcal{R} \to W^{st}_{\mathfrak{P}}(R, \mathfrak{G})$.

We summarize here cases that support our conjecture.

6.3. **Theorem.** *Parts (2), (3) of Conjecture 1.3 hold in the following cases.*

(i) k *is algebraically closed.*

(ii) $\mathfrak{G}$ *is constant, i.e.* $\eta^{geo} = 1$.

Proof. Case (i). By [P2], $\mathfrak{G}$ is quasi-split, hence we have

$$W_{\mathfrak{P}}^{st}(R,\mathfrak{G}) = W(K,\mathfrak{G}\times_k K) = W(F,\mathfrak{G}\times_k F) = 1.$$

Also, $\mathbf{H}(k)/\mathcal{R} = 1$ and the assertion is therefore clear.

Case (ii). We have $\mathbf{H} = {}_{\eta^{ar}}\mathbf{G}$ and $\mathfrak{G} = \mathbf{H}\times_k R$. Then we have a well-defined map $W(k,\mathbf{H}) \to W_{\mathfrak{P}}^{st}(R,\mathfrak{G})$. It fits in a sequence of maps

$$W(k,\mathbf{H}) \to W_{\mathfrak{P}}^{st}(R,\mathfrak{G}) \to W(K,\mathfrak{G}_K) \to W(F,\mathfrak{G}_F).$$

The first map is surjective by Theorem 1.1 (1), the second and the third are surjective as well by Lemma 6.1. But the composite map $W(k,\mathbf{H}) \to W(F,\mathbf{H}) \cong W(F,\mathfrak{G}_F)$ is an isomorphism [G, Theorem 7.3], so all the maps above are isomorphisms as well. Thus part (2) of the conjecture holds for $\mathfrak{G}$. Part (3) follows from the fact that the natural map $W(k,\mathbf{H}) \to \mathbf{H}(k)/\mathcal{R}$ is an isomorphism (*ibid*, Theorem 7.2). $\square$

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