INVARIANTS OF DEGREE 3 AND TORSION IN THE CHOW GROUP OF A VERSAL FLAG

ALEXANDER MERKURJEV, ALEXANDER NESHITOV, AND KIRILL ZAINOULLINE

Abstract. We prove that the group of normalized cohomological invariants of degree 3 modulo the subgroup of semidecomposable invariants of a semisimple split linear algebraic group G is isomorphic to the torsion part of the Chow group of codimension 2 cycles of the respective versal G-flag. In particular, if G is simple, we show that this factor group is isomorphic to the group of indecomposable invariants of G. As an application, we construct nontrivial cohomological classes for indecomposable central simple algebras.

1. Introduction

Let G be a split semisimple linear algebraic group over a field F. The purpose of the present paper is to relate together three different topics: the *geometry* of twisted G-flag varieties, the theory of *cohomological invariants* of G and the *representation theory* of G.

As for the first, let U/G be a *classifying space* of G in the sense of Totaro, that is U is an open G-invariant subset in some representation of G with $U(F) \neq \emptyset$ and $U \to U/G$ is a G-torsor. Consider the generic fiber U^{gen} of U over U/G. It is a G-torsor over the quotient field K of U/G called the *versal* G-torsor [7, Ch. I, §5]. We denote by X^{gen} the respective flag variety U^{gen}/B over K, where B is a Borel subgroup of G, and call it the *versal* flag. The variety X^{gen} can be viewed as the 'most twisted' form of the 'most complicated' G-flag variety and, hence, is the most natural object to study. In particular, understanding its geometry via studying the *Chow group* $\mathrm{CH}(X^{\text{gen}})$ of algebraic cycles modulo the rational equivalence relation, leads to understanding the geometry of all other G-flag varieties.

Recall that the group $\mathrm{CH}(X)$ of a twisted flag variety X has been a subject of intensive investigations for decades: starting with fundamental results by Grothendieck, Demazure, Berstein-Gelfand-Gelfand in 70's describing its *free part* and numerous recent results by Baek, Karpenko, Peyre, Zhong and many others (including the authors of the present paper) providing information about its *torsion part*.

Our second ingredient, the theory of cohomological invariants, has been mainly inspired by the works of J.-P. Serre and M. Rost. Given a field extension L/F and a positive integer d we consider the Galois cohomology group $H^{d+1}(L, \mathbb{Q}/\mathbb{Z}(d))$

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denoted by $H^{d+1}(L,d)$. Following [7, Ch. II, §1] a degree d *cohomological invariant* is a natural transformation of functors

$$a\colon H^1(-,G)\to H^d(-,d-1)$$

on the category of field extensions over F. We denote the group of degree d invariants by $\mathrm{Inv}^d(G,d-1)$.

Following [15, §1] invariant a is called *normalized* if it sends trivial torsor to zero. We denote the subgroup of normalized invariants by $\mathrm{Inv}^d(G,d-1)_{\mathrm{norm}}$. Invariant a is called *decomposable* if it is given by a cup-product with an invariant of degree 2. We denote the subgroup of decomposable invariants by $\mathrm{Inv}^3(G,2)_{\mathrm{dec}}$. The factor group $\mathrm{Inv}^3(G,2)_{\mathrm{norm}}/\mathrm{Inv}^3(G,2)_{\mathrm{dec}}$ is denoted by $\mathrm{Inv}^3(G,2)_{\mathrm{ind}}$ and is called the group of *indecomposable* invariants. This group has been studied by Garibaldi, Kahn, Levine, Rost, Serre and others in the simply-connected case. In recent work [15] it was shown how to compute it in general using new results on motivic cohomology obtained in [16]. In particular, it was computed for all adjoint split groups in [15] and for split simple groups in [2].

In the present paper we introduce a new subgroup of *semi-decomposable* invariants $\mathrm{Inv}^3(G,2)_{\mathrm{sdec}}$ which consists of invariants $a\in\mathrm{Inv}^3(G,2)_{\mathrm{norm}}$ such that for every field extension L/F and a G-torsor Y over L

$$a(Y)=\sum_{i\text{ finite}}\phi_i\cup b_i(Y)\ \text{ for some } \phi_i\in L^\times \text{ and } b_i\in\mathrm{Inv}^2(G,1)_{\mathrm{norm}}.$$

This plays an important role in our computations and applications. Observe that by definition $\mathrm{Inv}^3(G,2)_{\mathrm{dec}}\subseteq\mathrm{Inv}^3(G,2)_{\mathrm{sdec}}\subseteq\mathrm{Inv}^3(G,2)_{\mathrm{norm}}$.

As for the last ingredient, the representation theory, recall that the classical *character map* identifies the representation ring of G with the subring $\mathbb{Z}[T^*]^W$ of W-invariant elements of the integral group ring $\mathbb{Z}[T^*]$, where W is the Weyl group which acts naturally on the group of characters T^* of a split maximal torus T of G. Let $(\widetilde{I}^W)$ denote the ideal generated by classes of augmented representations of the simply-connected cover $\tilde{G}$ of G, i.e. by augmented W-invariant elements of the group ring of the weight lattice of $\tilde{G}$.

Our main result says that

Theorem. *Let G be a split semisimple linear algebraic group over a field F and let X^{gen} denote the associated versal flag. There is a short exact sequence*

$$0\to\tfrac{\mathrm{Inv}^3(G,2)_{\mathrm{sdec}}}{\mathrm{Inv}^3(G,2)_{\mathrm{dec}}}\to\mathrm{Inv}^3(G,2)_{\mathrm{ind}}\to\mathrm{CH}^2(X^{\mathrm{gen}})_{\mathrm{tors}}\to 0,$$

together with a group isomorphism $\frac{\mathrm{Inv}^3(G,2)_{\mathrm{sdec}}}{\mathrm{Inv}^3(G,2)_{\mathrm{dec}}}\simeq\frac{c_2((\widetilde{I}^W)\cap\mathbb{Z}[T^*])}{c_2(\mathbb{Z}[T^*]^W)}$*, where c_2 is the second* Chern class *map of* [15, §3c].

In addition, if G is simple, then $\mathrm{Inv}^3(G,2)_{\mathrm{sdec}}=\mathrm{Inv}^3(G,2)_{\mathrm{dec}}$*, so there is an isomorphism* $\mathrm{Inv}^3(G,2)_{\mathrm{ind}}\simeq\mathrm{CH}^2(X^{\mathrm{gen}})_{\mathrm{tors}}$.

Observe that if G is not simple, then $\mathrm{Inv}^3(G,2)_{\mathrm{sdec}}$ does not necessarily coincide with $\mathrm{Inv}^3(G,2)_{\mathrm{dec}}$ (see Example 3.1).

The nature of our result suggests that it should have applications in several directions, e.g. for cohomological invariants and algebraic cycles on twisted flag varieties.

For instance, since the group $\mathrm{Inv}^3(G,2)_{\mathrm{ind}}$ has been computed for all simple split groups in [15] and [2], it immediately gives computation of the torsion part of $\mathrm{CH}^2(X^{\mathrm{gen}})$, hence, extending previous results by [13] and [22]. As another straightforward consequence, using the coincidence $\mathrm{Inv}^3(G,2)_{\mathrm{sdec}} = \mathrm{Inv}^3(G,2)_{\mathrm{dec}}$ we construct non-trivial cohomological classes for indecomposable central simple algebras, hence, answering questions posed in [8] and [1].

The paper is organized as follows: In section 2 we construct an exact sequence relating the groups of invariants with the torsion part of the Chow group, hence, proving the first part of the theorem. In section 3 we compute this exact sequence case by case for all simple groups, hence, proving the second part. In the last section we discuss applications.

2. Semi-decomposable invariants and the Chow group

Let G be a split semisimple linear algebraic group over a field F. We fix a split maximal torus T of G and a Borel subgroup B containing T. Consider the T-equivariant structure map $U \to \operatorname{Spec} F = pt$, where U is the open G-invariant subset from the introduction.

Characteristic maps and classes. By [5] the induced pullback on T-equivariant Chow groups $\mathrm{CH}_T(pt) \to \mathrm{CH}_T(U)$ is an isomorphism. Since $\mathrm{CH}_T(U) \simeq \mathrm{CH}(U/T) \simeq \mathrm{CH}(U/B)$ and $\mathrm{CH}_T(pt)$ can be identified with the symmetric algebra $\mathrm{Sym}(T^*)$ of the group of characters of T, it gives an isomorphism

$$\mathfrak{c}^{\mathrm{CH}}\colon \mathrm{Sym}(T^*) \xrightarrow{\simeq} \mathrm{CH}(U/B). \tag{1}$$

Similarly, by the homotopy invariance and localization property of the equivariant K-theory [18, Theorems 8 and 11] the induced pull-back on T-equivariant K-groups gives a surjection

$$\mathfrak{c}^{\mathrm{K}_0}\colon \mathbb{Z}[T^*] \twoheadrightarrow K_0(U/B),$$

where the integral group ring $\mathbb{Z}[T^*]$ can be identified with $K_T(pt)$ and $K_0(U/B) \simeq K_0(U/T) \simeq K_T(U)$.

Let $\tau^i(X)$ denote the i-th term of the *topological filtration* on K_0 of a smooth variety X and let $\tau^{i/i+1}$, $i \geq 0$ denote its i-th subsequent quotient. Let I denote the augmentation ideal of $\mathbb{Z}[T^*]$.

2.1. **Lemma.** *The map $\mathfrak{c}^{\mathrm{K}_0}$ induces isomorphisms on subsequent quotients*

$$I^i/I^{i+1} \xrightarrow{\simeq} \tau^{i/i+1}(U/B), \qquad \textit{for } 0 \leq i \leq 2,$$

and, its restriction $\mathfrak{c}^{\mathrm{K}_0}\colon I^2 \to \tau^2(U/B)$ is surjective.

Proof. By [6, Ex. 15.3.6] the Chern class maps induce isomorphisms $c_i\colon \tau^{i/i+1}(X) \xrightarrow{\simeq} \mathrm{CH}^i(X)$ for $0 \leq i \leq 2$. Since the Chern classes commute with pullbacks and $\mathrm{Sym}^i(T^*) \simeq I^i/I^{i+1}$, the isomorphisms then follow from (1).

Finally, since $I/I^2 \simeq \tau^{1/2}(U/B)$, $\mathfrak{c}^{\mathrm{K}_0}(x) \in \tau^2(U/B)$ implies that $x \in I^2$. $\square$

Consider the natural inclusion of the versal flag $\imath\colon X^{\mathrm{gen}} = U^{\mathrm{gen}}/B \hookrightarrow U/B$. Since $\imath$ is a limit of open embeddings, by the *localization property* of Chow groups, the induced pullback gives surjections

$$\imath^{\mathrm{CH}}\colon \mathrm{CH}^i(U/B) \twoheadrightarrow \mathrm{CH}^i(X^{\mathrm{gen}}).$$

Moreover, the induced pullback in K-theory restricted to τ^i also gives surjections

$$\imath^{\mathrm{K}_0}\colon \tau^i(U/B) \twoheadrightarrow \tau^i(X^{\mathrm{gen}}).$$

Indeed, by definition $\tau^i(X^{\mathrm{gen}})$ is generated by the classes $[\mathcal{O}_Z]$ for closed subvarieties Z of X^{gen} with $\operatorname{codim} Z \geqslant i$ and each $[\mathcal{O}_Z]$ is the pullback of the element $[\mathcal{O}_{\bar{Z}}]$ in $\tau^i(U/B)$, where $\bar{Z}$ is the closure of Z inside U/B.

Let L be a splitting field of the versal torsor U^{gen}. According to [10, Thm. 4.5] composites

$$\mathrm{Sym}(T^*) \xrightarrow{\mathfrak{c}^{\mathrm{CH}}} \mathrm{CH}(U/B) \xrightarrow{\imath^{\mathrm{CH}}} \mathrm{CH}(X^{\mathrm{gen}}) \xrightarrow{res} \mathrm{CH}(X^{\mathrm{gen}}_L) \quad \text{and}$$

$$\mathbb{Z}[T^*] \xrightarrow{\mathfrak{c}^{\mathrm{K}_0}} K_0(U/B) \xrightarrow{\imath^{\mathrm{K}_0}} K_0(X^{\mathrm{gen}}) \xrightarrow{res} K_0(X^{\mathrm{gen}}_L)$$

give the classical characteristic maps for the Chow groups and for the K-groups respectively (here we identify the rightmost groups with the Chow group and the K-group of the split flag G/B respectively). Restricting the latter to I^2 and τ^2 we obtain the map

$$\mathfrak{c}\colon I^2 \xrightarrow{\mathfrak{c}^{\mathrm{K}_0}} \tau^2(U/B) \xrightarrow{\imath^{\mathrm{K}_0}} \tau^2(X^{\mathrm{gen}}) \xrightarrow{res} \tau^2(X^{\mathrm{gen}}_L) = \tau^2(G/B).$$

From this point on, we denote by $\mathfrak{c}^{\mathrm{CH}}$, $\imath^{\mathrm{CH}}$ and by $\mathfrak{c}^{\mathrm{K}_0}$, $\imath^{\mathrm{K}_0}$ the respective restrictions to Sym^2, CH^2 and I^2, τ^2.

Let Λ be the weight lattice. Consider the integral group ring $\mathbb{Z}[\Lambda]$. Let $\widetilde{I}$ denote its augmentation ideal. The Weyl group W acts naturally on $\mathbb{Z}[\Lambda]$. Let $(\widetilde{I}^W)$ denote the ideal generated by W-invariant elements in $\widetilde{I}$.

2.2. **Lemma.** *The kernel of the composite* $I^2 \overset{\mathfrak{c}^{\mathrm{K}_0}}{\to} \tau^2(U/B) \overset{\imath^{\mathrm{K}_0}}{\to} \tau^2(X^{\mathrm{gen}})$ *is* $(\widetilde{I}^W) \cap I^2$.

Proof. By the results of Panin [21], $K_0(X^{\mathrm{gen}})$ is the direct sum of $K_0(A_i)$ for some central simple algebras A_i over K. So $K_0(X^{\mathrm{gen}})$ is a free abelian group and, hence, the restriction $\tau^2(X^{\mathrm{gen}}) \to \tau^2(X^{\mathrm{gen}}_L)$ is injective. Therefore, $\ker(\imath^{\mathrm{K}_0} \circ \mathfrak{c}^{\mathrm{K}_0})$ coincides with the kernel of the characteristic map $\mathfrak{c}\colon I^2 \to \tau^2(G/B)$. Since $\mathfrak{c}$ factors as $I^2 \hookrightarrow \widetilde{I}^2 \to \tau^2(G/B)$ and the kernel of the second map is $(\widetilde{I}^W) \cap \widetilde{I}^2$ by the theorem of Steinberg [24], we get $\ker \mathfrak{c} = (\widetilde{I}^W) \cap I^2$ (here we used that $\mathbb{Z}[T^*] \cap \widetilde{I}^i = I^i$). $\square$

Consider the second Chern class map $c_2\colon \tau^2(U/B) \to \mathrm{CH}^2(U/B)$.

2.3. **Lemma.** *We have* $c_2(\ker \imath^{\mathrm{K}_0}) = \ker \imath^{\mathrm{CH}}$.

Proof. Consider the diagram

$$\begin{tikzcd}
\tau^3(U/B) \arrow[r] \arrow[d, "\imath^{\mathrm{K}_0}|_{\tau^3}"'] & \tau^2(U/B) \arrow[r, "c_2"] \arrow[d, "\imath^{\mathrm{K}_0}"] & \mathrm{CH}^2(U/B) \arrow[r] \arrow[d, "\imath^{\mathrm{CH}}"] & 0 \\
\tau^3(X^{\mathrm{gen}}) \arrow[r] & \tau^2(X^{\mathrm{gen}}) \arrow[r, "c_2"] & \mathrm{CH}^2(X^{\mathrm{gen}}) \arrow[r] & 0
\end{tikzcd}$$

Its vertical maps are surjective and the rows are exact by [6, Ex. 15.3.6]. The result then follows by the diagram chase. $\square$

Consider the composite $\mathfrak{c}_2\colon I^2 \xrightarrow{\mathfrak{c}^{\mathrm{K}_0}} \tau^2(U/B) \xrightarrow{c_2} \mathrm{CH}^2(U/B)$. Observe that it coincides with the Chern class map defined in [15, §3c].

2.4. **Lemma.** *We have* $\ker \imath^{\mathrm{CH}} = \mathfrak{c}_2((\widetilde{I}^W) \cap I^2)$.

Proof. Since $\mathfrak{c}^{\mathrm{K}_0}$ is surjective by lemma 2.1, we have by lemma 2.2 and 2.3
$$\mathfrak{c}_2((\widetilde{I}^W)\cap I^2)=\mathfrak{c}_2(\ker(\imath^{\mathrm{K}_0}\circ\mathfrak{c}^{\mathrm{K}_0}))=c_2(\ker\imath^{\mathrm{K}_0})=\ker\imath^{\mathrm{CH}}.\qquad\square$$

Following [15] we denote
$$\mathrm{Dec}(G):=(\mathfrak{c}^{\mathrm{CH}})^{-1}\circ\mathfrak{c}_2(\mathbb{Z}[T^*]^W).$$
And we set
$$\mathrm{SDec}(G):=(\mathfrak{c}^{\mathrm{CH}})^{-1}\circ\mathfrak{c}_2((\widetilde{I}^W)\cap\mathbb{Z}[T^*]).$$
Since the action of W on Λ is essential, i.e. $\Lambda^W=0$, we have $(\widetilde{I}^W)\subseteq\widetilde{I}^2$. Therefore, for any $x\in(\widetilde{I}^W)$ we have $x\equiv x'\bmod\widetilde{I}^3$ and, hence, $\mathfrak{c}_2(x)=\mathfrak{c}_2(x')$ for some $x'\in\mathbb{Z}[\Lambda]^W$, where $\mathbb{Z}[\Lambda]^W$ is the subring of W-invariants. So there are inclusions
$$\mathrm{Dec}(G)\subseteq\mathrm{SDec}(G)\subseteq\mathrm{Sym}^2(T^*)^W.\tag{2}$$

2.5. **Lemma.** *We have* $\mathrm{CH}^2(X^{\mathrm{gen}})\simeq\mathrm{Sym}^2(T^*)/\mathrm{SDec}(G)$.

Proof. By (1) and lemma 2.4 we have
$$\mathrm{CH}^2(X^{\mathrm{gen}})\simeq\mathrm{CH}^2(U/B)/\mathfrak{c}_2((\widetilde{I}^W)\cap I^2)\simeq\mathrm{Sym}^2(T^*)/\mathrm{SDec}(G).\qquad\square$$

2.6. **Corollary.** *We have* $\mathrm{CH}^2(X^{\mathrm{gen}})_{\mathrm{tors}}\simeq\mathrm{Sym}^2(T^*)^W/\mathrm{SDec}(G)$.

Proof. By the lemma it remains to show that
$$(\mathrm{Sym}^2(T^*)/\mathrm{SDec}(G))_{\mathrm{tors}}=\mathrm{Sym}^2(T^*)^W/\mathrm{SDec}(G).$$
Indeed, suppose that $x\in\mathrm{Sym}^2(T^*)$ and $nx\in\mathrm{SDec}(G)$. Then nx lies in $\mathrm{Sym}^2(T^*)^W$ by (2). So for every $w\in W$ we have $n(wx-x)=0$. Since $\mathrm{Sym}^2(T^*)$ has no torsion, $x\in\mathrm{Sym}^2(T^*)^W$. Conversely, let $x\in\mathrm{Sym}^2(T^*)^W$. Since the second Chern class map $c_2\colon I^2\to\mathrm{Sym}^2(T^*)$ is surjective, there is a preimage $y\in I^2$ of x. Take $y'=\sum_{w\in W}w\cdot y\in\mathbb{Z}[T^*]^W\subseteq(\widetilde{I}^W)\cap\mathbb{Z}[T^*]$. Since c_2 is W-equivariant and coincides with the composite $(\mathfrak{c}^{\mathrm{CH}})^{-1}\circ\mathfrak{c}_2$, we get $(\mathfrak{c}^{\mathrm{CH}})^{-1}\circ\mathfrak{c}_2(y')=|W|\cdot x\in\mathrm{SDec}(G)$. $\square$

Cohomological Invariants. For a smooth F-scheme X let $\mathcal{H}^3(2)$ denote the Zariski sheaf on X associated to a presheaf $W\mapsto H^3_{\text{ét}}(W,\mathbb{Q}/\mathbb{Z}(2))$. The Bloch-Ogus-Gabber theorem (see [4] and [11]) implies that its group of global sections $H^0_{\mathrm{Zar}}(X,\mathcal{H}^3(2))$ is a subgroup in $H^3(F(X),2)$.

Consider the versal G-torsor U^{gen} over the quotient field K of the classifying space U/G. By [3, Thm. A] the map $\Theta\colon\mathrm{Inv}^3(G,2)\to H^3(K,2)$ defined by $\Theta(a):=a(U^{\mathrm{gen}})$ gives an inclusion
$$\mathrm{Inv}^3(G,2)\hookrightarrow H^0_{\mathrm{Zar}}(U/G,\mathcal{H}^3(2)).$$

2.7. **Lemma.** *We have* $a(U^{\mathrm{gen}})\in\ker[H^3(K,2)\to H^3(K(X^{\mathrm{gen}}),2)]$ *for any* $a\in\mathrm{Inv}^3(G,2)_{\mathrm{norm}}$.

Proof. Consider the composite $q\colon\operatorname{Spec}K(U^{\mathrm{gen}})\to U^{\mathrm{gen}}\to U/G$. Observe that the pullback q^* factors as
$$q^*\colon H^0_{\mathrm{Zar}}(U/G,\mathcal{H}^3(2))\to H^3(K(X^{\mathrm{gen}}),2)\to H^3(K(U^{\mathrm{gen}}),2).$$
Since $U^{\mathrm{gen}}\to X^{\mathrm{gen}}$ is a B-torsor, $K(U^{\mathrm{gen}})$ is purely transcendental over $K(X^{\mathrm{gen}})$, so the last map of the composite is injective. Since the U^{gen} becomes trivial over $K(U^{\mathrm{gen}})$, we have $q^*(a(U^{\mathrm{gen}}))=a(U^{\mathrm{gen}}\times_K K(U^{\mathrm{gen}}))=0$. Therefore, $a(U^{\mathrm{gen}})\in\ker[H^3(K,2)\to H^3(K(X^{\mathrm{gen}}),2)]$. $\square$

2.8. **Lemma.** *Let $Y \to \operatorname{Spec} L$ be a G-torsor and $X = Y/B$. Let L^{sep} denote the separable closure of L, Γ_L its Galois group and $X^{sep} = X \times_L L^{sep}$. Then the Γ_L action on $\operatorname{Pic} X^{sep}$ is trivial.*

Proof. It follows by [19, Prop. 2.2]. □

The Tits map. Consider a short exact sequence of F-group schemes

$$1 \to C \to \widetilde{G} \xrightarrow{\pi} G \to 1.$$

Given a character $\chi \in C^*$ of the center and a field extension L/F consider the *Tits map* [25, §4,5]

$$\alpha_{\chi,L} \colon H^1(L,G) \xrightarrow{\partial} H^2(L,C) \xrightarrow{\chi_*} H^2(L,\mathbb{G}_m),$$

where ∂ is the connecting homomorphism (if C is non-smooth, we replace it by $\mathbb{G}_m$ and G by the respective push-out as in [7, II, Example 2.1]). This gives rise to a cohomological invariant β_χ of degree two

$$\beta_\chi \colon Y \mapsto \alpha_{\chi,L}(Y) \quad \text{for every } G\text{-torsor } Y \in H^1(L,G).$$

[3, Theorem 2.4] shows that the assignment $\chi \to \beta_\chi$ provides an isomorphism $C^* \to \operatorname{Inv}^2(G,1)$.

For a G-torsor Y over L there is an exact sequence studied in [17], [22] and [7, II, Thm. 8.9]:

$$A^1((Y/B)^{sep}, K_2)^\Gamma \xrightarrow{\rho} \ker[H^3(L,2) \to H^3(L(Y/B),2)] \xrightarrow{\delta_Y} \operatorname{CH}^2(Y/B).$$

The multiplication map $L^{sep} \otimes \operatorname{CH}^1(Y/B)^{sep} \to A^1((Y/B)^{sep}, K_2)$ is an isomorphism. By lemma 2.8 we obtain an exact sequence

$$L \otimes \Lambda \xrightarrow{\rho_Y} \ker[H^3(L,2) \to H^3(L(Y/B),2)] \xrightarrow{\delta_Y} \operatorname{CH}^2(Y/B). \tag{3}$$

According to [17] the map ρ_Y acts as follows:

$$\rho_Y(\phi \otimes \lambda) = \phi \cup \beta_{\overline{\lambda}}(Y), \quad \text{where } \phi \in L^\times,\ \lambda \in \Lambda \text{ and}$$

$\overline{\lambda}$ denotes the image of λ in $\Lambda/T^* = C^*$.

We define the subgroup $\operatorname{Inv}^3(G,2)_{\mathrm{sdec}}$ of *semi-decomposable* invariants as follows:

2.9. **Definition.** An invariant $a \in \operatorname{Inv}^3(G,2)_{\mathrm{norm}}$ is called semi-decomposable, if there is a finite set $b_i \in \operatorname{Inv}^2(G,1)_{\mathrm{norm}}$ such that for every field extension L/F and a torsor $Y \in H^1(L,G)$ we have

$$a(Y) = \sum_i \phi_i \cup b_i(Y) \text{ for some } \phi_i \in L^\times.$$

Observe that by definition, we have

$$\operatorname{Inv}^3(G,2)_{\mathrm{dec}} \subseteq \operatorname{Inv}^3(G,2)_{\mathrm{sdec}} \subseteq \operatorname{Inv}^3(G,2)_{\mathrm{norm}}$$

and $a \in \operatorname{Inv}^3(G,2)_{\mathrm{sdec}}$ if and only if $a(Y) \in \operatorname{im}(\rho_Y) = \ker(\delta_Y)$ for every torsor Y.

2.10. **Lemma.** *We have $a \in \operatorname{Inv}^3(G,2)_{\mathrm{sdec}}$ if and only if $a(U^{\mathrm{gen}}) \in \ker(\delta_{U^{\mathrm{gen}}})$.*

Proof. If a is a semi-decomposable invariant, then $a(U^{\mathrm{gen}}) = \sum_{\chi \in C^*} \phi_\chi \cup \beta_\chi(U^{\mathrm{gen}})$ lies in the image of $\rho_{U^{\mathrm{gen}}}$, hence, $\delta_{U^{\mathrm{gen}}}(a(U^{\mathrm{gen}})) = 0$. On the other hand, let a be a degree 3 invariant such that $\delta_{U^{\mathrm{gen}}}(a(U^{\mathrm{gen}})) = 0$ and let Y be a G-torsor over a field extension L/F. We show that $\delta_Y(a(Y)) = 0$.

We may assume that L is infinite (replacing L by $L(t)$ if needed). Choose a rational point $y \in (U/G)_L$ such that Y is isomorphic to the fiber of $U \to U/G$ over y. Let R be the completion of the regular local ring $\mathcal{O}_{(U/G)_L,y}$ and let $\hat{K}$ be its quotient field. The ring R is a regular local ring with residue field L. By the theorem of Grothendieck Y_R is a pullback of Y via the projection $\operatorname{Spec} R \to \operatorname{Spec} L(y)$. Then the G-torsors $Y_{\hat{K}}$ and $U^{\mathrm{gen}}_{\hat{K}}$ over $\hat{K}$ are isomorphic. We have

$$\delta_Y(a(Y))_{\hat{K}} = \delta_{Y_{\hat{K}}}(a(Y_{\hat{K}})) = \delta_{U^{\mathrm{gen}}_{\hat{K}}}(a(U^{\mathrm{gen}}_{\hat{K}})) = \delta_{U^{\mathrm{gen}}}(a(U^{\mathrm{gen}}))_{\hat{K}} = 0.$$

The restriction $\mathrm{CH}^2(Y/B) \to \mathrm{CH}^2((Y/B)_{\hat{K}})$ is injective, since it is split by the specialization map with respect to a system of local parameters of R. Therefore, $\delta_Y(a(Y)) = 0$ for every Y, hence, a is semi-decomposable. $\square$

Now we are ready to prove the first part of the main theorem:

2.11. **Theorem.** *The map $\delta_{U^{\mathrm{gen}}}$ induces a short exact sequence*

$$0 \longrightarrow \tfrac{\mathrm{Inv}^3(G,2)_{\mathrm{sdec}}}{\mathrm{Inv}^3(G,2)_{\mathrm{dec}}} \longrightarrow \mathrm{Inv}^3(G,2)_{\mathrm{ind}} \xrightarrow{g} \mathrm{CH}^2(X^{\mathrm{gen}})_{\mathrm{tors}} \longrightarrow 0,$$

and there is a group isomorphism

$$\tfrac{\mathrm{Inv}^3(G,2)_{\mathrm{sdec}}}{\mathrm{Inv}^3(G,2)_{\mathrm{dec}}} \simeq \tfrac{c_2((\widetilde{I}^W)\cap\mathbb{Z}[T^*])}{c_2(\mathbb{Z}[T^*]^W)}.$$

Proof. Consider the following diagram. The rows are exact sequences given by [12, Thm. 1.1] and vertical arrows are pullbacks:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{CH}^2(U/G) \arrow[r] \arrow[d] & \mathbb{H}^4_{\text{ét}}(U/G,\mathbb{Z}(2)) \arrow[r] \arrow[d] & H^0_{\mathrm{Zar}}(U/G,\mathcal{H}^3(2)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{CH}^2(U/B) \arrow[r] \arrow[d, "\imath^{\mathrm{CH}}"'] & \mathbb{H}^4_{\text{ét}}(U/B,\mathbb{Z}(2)) \arrow[r] \arrow[d] & H^0_{\mathrm{Zar}}(U/B,\mathcal{H}^3(2)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{CH}^2(X^{\mathrm{gen}}) \arrow[r] & \mathbb{H}^4_{\text{ét}}(X^{\mathrm{gen}},\mathbb{Z}(2)) \arrow[r] & H^0_{\mathrm{Zar}}(X^{\mathrm{gen}},\mathcal{H}^3(2)) \arrow[r] & 0
\end{tikzcd}$$

Since $F(U/B) = K(X^{\mathrm{gen}})$, lemma 2.7 implies that the composite

$$\mathrm{Inv}^3(G,2)_{\mathrm{norm}} \to H^0_{\mathrm{Zar}}(U/G,\mathcal{H}^3(2)) \to H^0_{\mathrm{Zar}}(U/B,\mathcal{H}^3(2))$$

is zero. By the diagram chase there is a homomorphism

$$\mathrm{Inv}^3(G,2)_{\mathrm{norm}} \to \mathrm{CH}^2(U/B)/\,\mathrm{CH}^2(U/G).$$

The map $X^{\mathrm{gen}} \to U/B \to U/G$ factors as $X^{\mathrm{gen}} \to \operatorname{Spec} K \to U/G$, hence the composite of pullbacks $\mathrm{CH}^2(U/G) \to \mathrm{CH}^2(U/B) \xrightarrow{\imath^{\mathrm{CH}}} \mathrm{CH}^2(X^{\mathrm{gen}})$ coincides with the composite $\mathrm{CH}^2(U/G) \to \mathrm{CH}^2(\operatorname{Spec} K) \to \mathrm{CH}^2(X^{\mathrm{gen}})$ which is zero. This gives a homomorphism $g\colon \mathrm{Inv}^3(G,2)_{\mathrm{norm}} \to \mathrm{CH}^2(U/B)/\,\mathrm{CH}^2(U/G) \to \mathrm{CH}^2(X^{\mathrm{gen}})$ which by the proof of theorem of B. Kahn (see [7, II, §8, 8.1-8.5]) factors through the map $\delta_{U^{\mathrm{gen}}}$ of (3). By [15, 3.9] the map g also factors through $\mathrm{Inv}^3(G,2)_{\mathrm{ind}} \xrightarrow{\simeq}$

$\frac{\mathrm{Sym}^2(T^*)^W}{\mathrm{Dec}(G)}$. So there is a commutative diagram

$$\begin{tikzcd} \mathrm{Inv}^3(G,2)_{\mathrm{norm}} \arrow[r, "g"] \arrow[d] & \mathrm{CH}^2(X^{\mathrm{gen}})_{\mathrm{tors}} \\ \frac{\mathrm{Sym}^2(T^*)^W}{\mathrm{Dec}(G)} \arrow[r] & \frac{\mathrm{Sym}^2(T^*)^W}{\mathrm{SDec}(G)} \arrow[u, "\simeq"', "\mathrm{Cor.}\ 2.6"] \end{tikzcd} \quad . \tag{4}$$

The bottom row of (4) gives a short exact sequence

$$0 \to \frac{\mathrm{SDec}(G)}{\mathrm{Dec}(G)} \to \frac{\mathrm{Sym}^2(T^*)^W}{\mathrm{Dec}(G)} \to \mathrm{CH}^2(X^{\mathrm{gen}})_{\mathrm{tors}} \to 0.$$

Lemma 2.10 and composite (3) give an exact sequence

$$0 \to \mathrm{Inv}^3(G,2)_{\mathrm{sdec}} \to \mathrm{Inv}^3(G,2)_{\mathrm{norm}} \xrightarrow{g} \mathrm{CH}^2(X^{\mathrm{gen}})_{\mathrm{tors}}.$$

Combining these together and factoring modulo $\mathrm{Inv}^3(G,2)_{\mathrm{dec}}$ we obtain an isomorphism

$$\frac{\mathrm{Inv}^3(G,2)_{\mathrm{sdec}}}{\mathrm{Inv}^3(G,2)_{\mathrm{dec}}} \cong \frac{\mathrm{SDec}(G)}{\mathrm{Dec}(G)}. \qquad \square$$

3. Semi-decomposable invariants vs. decomposable invariants

In this section we prove case by case that the groups of decomposable $\mathrm{Inv}^3(G,2)_{\mathrm{dec}}$ and semi-decomposable $\mathrm{Inv}^3(G,2)_{\mathrm{sdec}}$ invariants coincide for all split simple G, hence, proving the second part of our main theorem. More precisely, we show that

$$\mathrm{Dec}(G) = \mathfrak{c}_2(\mathbb{Z}[T^*]^W) = \mathfrak{c}_2((\widetilde{I}^W) \cap \mathbb{Z}[T^*]) = \mathrm{SDec}(G) \text{ in } \mathrm{Sym}^2(T^*)^W$$

(here we denote $(\mathfrak{c}^{\mathrm{CH}})^{-1} \circ \mathfrak{c}_2$ simply by $\mathfrak{c}_2$). Observe that in the simply connected case $\mathrm{Sym}^2(\Lambda)^W = \mathbb{Z}q$, where q corresponds to the normalized Killing form from [9, §1B], and $\mathrm{Dec}(G) \subseteq \mathrm{SDec}(G) \subseteq \mathrm{SDec}(\widetilde{G}) = \mathrm{Dec}(\widetilde{G})$.

3.1. **Example.** If G is not simple, then $\mathrm{Dec}(G) \neq \mathrm{SDec}(G)$ in general. Indeed, consider a quadratic form q of degree 4 with trivial discriminant (it corresponds to a $\mathbf{SO}_4$-torsor). According to [7, Example 20.3] there is an invariant given by $q \mapsto \alpha \cup \beta \cup \gamma$, where α is represented by q and $\langle\langle \beta, \gamma \rangle\rangle = \langle \alpha \rangle q$ is the 2-Pfister form. By definition this invariant is semi-decomposable (this fact was pointed to us by Vladimir Chernousov). Since it is non-trivial over an algebraic closure of F, it is not decomposable.

3.1. *Adjoint groups of type A_n $(n \geq 1)$, B_n $(n \geq 2)$, C_n $(n \geq 3,\ 4 \nmid n)$, D_n $(n \geq 5,\ 4 \nmid n)$, E_6, E_7 and special orthogonal groups of type D_n $(n \geq 4)$.*

For classical adjoint types we have $\mathrm{Inv}^3(G,2)_{\mathrm{norm}} = \mathrm{Inv}^3(G,2)_{\mathrm{dec}}$ by [15, §4b], so we immediately obtain $\mathrm{Inv}^3(G,2)_{\mathrm{dec}} = \mathrm{Inv}^3(G,2)_{\mathrm{sdec}}$. For exceptional types by [7, p.135] and [15, §4b] we have $\mathrm{Dec}(G) = \mathrm{Dec}(\widetilde{G}) = 6\mathbb{Z}q$ for E_6 and $\mathrm{Dec}(G) = \mathrm{Dec}(\widetilde{G}) = 12\mathbb{Z}q$ for E_7. For special orthogonal groups $G = \mathbf{SO}_{2n}$ by [7, §15] we have $\mathrm{Dec}(\mathbf{SO}_{2n}) = \mathrm{Dec}(\mathbf{Spin}_{2n}) = 2\mathbb{Z}q$ (here $\tilde{G} = \mathbf{Spin}_{2n}$), hence, $\mathrm{Dec}(G) = \mathrm{SDec}(G)$.

3.2. *Non-adjoint groups of type A_{n-1} ($n \geq 4$).*

Let p be a prime integer and $G = \mathbf{SL}_{p^s}/\boldsymbol{\mu}_{p^r}$ for some integers $s \geq r > 0$. If p is odd, we set $k = \min\{r, s-r\}$ and if $p = 2$ we assume that $s \geq r+1$ and set $k = \min\{r, s-r-1\}$. It is shown in [2, §4] that the group $\mathrm{Inv}^3(G,2)_{\mathrm{ind}}$ is cyclic of order p^k. On the other hand, by [13, Example 4.15] if X is the Severi-Brauer variety of a generic algebra A^{gen}, then $\mathrm{CH}^2(X)_{\mathrm{tors}}$ is also a cyclic group of order p^k. The canonical morphism $X^{\mathrm{gen}} \to X$ is an iterated projective bundle, hence, $\mathrm{CH}^2(X^{\mathrm{gen}})_{\mathrm{tors}} \simeq \mathrm{CH}^2(X)_{\mathrm{tors}}$ is a cyclic group of order p^k. It follows from the exact sequence of theorem 2.11 that $\mathrm{Inv}^3(G,2)_{\mathrm{sdec}} = \mathrm{Inv}^3(G,2)_{\mathrm{dec}}$.

More generally, let $G = \mathbf{SL}_n/\boldsymbol{\mu}_m$, where $m \mid n$. Let p^s and p^r be the highest powers of a prime integer p dividing n and m respectively. Consider the canonical homomorphism $H = \mathbf{SL}_{p^s}/\boldsymbol{\mu}_{p^r} \to G$. We claim that it induces an isomorphism between the p-primary component of $\mathrm{Inv}^3(G,2)_{\mathrm{ind}}$ and the group $\mathrm{Inv}^3(H,2)_{\mathrm{ind}}$.

Indeed, let $H' = \mathbf{SL}_n/\boldsymbol{\mu}_{p^r}$. It follows from [2, Theorem 4.1] that the natural homomorphism $\mathrm{Inv}^3(H',2)_{\mathrm{ind}} \to \mathrm{Inv}^3(H,2)_{\mathrm{ind}}$ is an isomorphism. Thus, it suffices to show that the pull-back map for the canonical surjective homomorphism $H' \to G$ with kernel $\boldsymbol{\mu}_t$, where $t := m/p^r$ is relatively prime to p, induces an isomorphism between the p-primary component of $\mathrm{Inv}^3(G,2)_{\mathrm{ind}}$ and $\mathrm{Inv}^3(H',2)_{\mathrm{ind}}$. Let $\Lambda \subset \Lambda'$ be the character groups of maximal tori of G and H' respectively. The factor group Λ'/Λ is isomorphic to $\boldsymbol{\mu}_t^* = \mathbb{Z}/t\mathbb{Z}$. Since the functor $\Lambda \mapsto \frac{\mathrm{Sym}^2(\Lambda)^W}{\mathrm{Dec}(\Lambda)}$ is quadratic in Λ, the kernel and the cokernel of the homomorphism

$$\mathrm{Inv}^3(G,2)_{\mathrm{ind}} = \frac{\mathrm{Sym}^2(\Lambda)^W}{\mathrm{Dec}(\Lambda)} \to \frac{\mathrm{Sym}^2(\Lambda')^W}{\mathrm{Dec}(\Lambda')} = \mathrm{Inv}^3(H',2)_{\mathrm{ind}}$$

are killed by t^2. As t is relatively prime to p, the claim follows.

Since the p-primary component of $\mathrm{CH}(X^{\mathrm{gen}})_{\mathrm{tors}}$ and the group $\mathrm{CH}(X_H^{\mathrm{gen}})_{\mathrm{tors}}$ are isomorphic by [13, Prop. 1.3] (here X_H^{gen} denotes the versal flag for H), we obtain that $\mathrm{Inv}^3(G,2)_{\mathrm{ind}} \simeq \mathrm{CH}(X^{\mathrm{gen}})_{\mathrm{tors}}$ and, therefore, by the exact sequence of theorem 2.11 $\mathrm{Inv}^3(G,2)_{\mathrm{sdec}} = \mathrm{Inv}^3(G,2)_{\mathrm{dec}}$.

3.3. *Adjoint groups of type C_{4m} ($m \geq 1$).*

By [15, §4b] we have $\mathrm{Sym}^2(T^*)^W = \mathbb{Z}q$ and $\mathrm{Dec}(G) = \mathfrak{c}_2(\mathbb{Z}[T^*]^W) = 2\mathbb{Z}q$. We want to show that $\mathfrak{c}_2(x) \in 2\mathbb{Z}q$ for every element $x \in (\widetilde{I}^W) \cap \mathbb{Z}[T^*]$.

Given a weight $\chi \in \Lambda$ we denote by $W(\chi)$ its W-orbit and we define $\widehat{e^\chi} := \sum_{\lambda \in W(\chi)} (1 - e^{-\lambda})$. By definition, the ideal $(\widetilde{I}^W)$ is generated by elements $\{\widehat{e^{\omega_i}}\}_{i=1..4m}$ corresponding to the fundamental weights ω_i. An element x can be written as

$$x = \sum_{i=1}^{4m} n_i \widehat{e^{\omega_i}} + \delta_i \widehat{e^{\omega_i}}, \quad \text{ where } n_i \in \mathbb{Z} \text{ and } \delta_i \in \widetilde{I}. \tag{5}$$

Similar to [26, §3] consider a ring homomorphism $f\colon \mathbb{Z}[\Lambda] \to \mathbb{Z}[\Lambda/T^*]$ induced by taking the quotient $\Lambda \to \Lambda/T^* = C^*$. We have $\Lambda/T^* \simeq \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}[\Lambda/T^*] = \mathbb{Z}[y]/(y^2 - 2y)$, where $y = f(e^{\omega_1} - 1)$. Observe that C^* is W-invariant.

By definition, $f(I) = 0$, so $f(x) = 0$. Since $\omega_i \in T^*$ for all even i, $f(\widehat{e^{\omega_i}}) = y$ for all odd i and $f(\delta_i) \in f(\widetilde{I}) = (y)$, we get

$$0 = f(x) = \sum_{i \text{ is odd}} n_i d_i y + m_i d_i y^2 = (\sum_{i \text{ is odd}} n_i + 2m_i) d_i y,$$

where $m_i \in \mathbb{Z}$ and $d_i = 2^i\binom{4m}{i}$ is the cardinality of $W(\omega_i)$, which implies that $(\sum_{i \text{ is odd}} n_i + 2m_i)d_i = 0$. Dividing this sum by the g.c.d. of all d_i's and taking the result modulo 2 (here one uses the fact $\frac{n}{g.c.d.(n,k)} \mid \binom{n}{k}$), we obtain that the coefficient n_1 in the presentation (5) has to be even.

We now compute $\mathfrak{c}_2(x)$. Let $\Lambda = \mathbb{Z}e_1 \oplus \ldots \oplus \mathbb{Z}e_{4m}$. The root lattice is given by $T^* = \{\sum a_i e_i \mid \sum a_i \text{ is even}\}$ and

$$\omega_1 = e_1,\ \omega_2 = e_1 + e_2,\ \omega_3 = e_1 + e_2 + e_3, \ldots, \omega_{4m} = e_1 + \ldots + e_{4m}.$$

By [9, §2] we have $\mathfrak{c}_2(x) = \sum_{i=1}^{4m} n_i \mathfrak{c}_2(\widehat{e^{\omega_i}})$ and $\mathfrak{c}_2(\widehat{e^{\omega_i}}) = N(\widehat{e^{\omega_i}})q$, where

$$N(\sum a_j e^{\lambda_j}) = \tfrac{1}{2}\sum a_j \langle \lambda_j, \alpha^\vee\rangle^2 \text{ for a fixed long root } \alpha.$$

If we set $\alpha = 2e_{4m}$, then $\langle \lambda, \alpha^\vee\rangle = (\lambda, e_{4m})$ and

$$N(\widehat{e^{\omega_i}}) = \tfrac{1}{2}\sum_{\lambda\in W(\omega_i)} \langle \lambda, \alpha^\vee\rangle^2 = \tfrac{1}{2}\sum_{\lambda\in W(\omega_i)} (\lambda, e_{4m})^2 = 2^{i-1}\binom{4m-1}{i-1}$$

which is even for $i \geq 2$ (here we used the fact that the Weyl group acts by permutations and sign changes on $\{e_1, \ldots, e_{4m}\}$). Since n_1 is even, we get that $\mathfrak{c}_2(x) \in 2\mathbb{Z}q$.

3.4. *Half-spin and adjoint groups of type D_{2m} ($m \geq 2$).*

We first treat the half-spin group $G = \mathbf{HSpin}_{8m}$. As in the C_n-case all even fundamental weights are in T^* and all odd fundamental weights correspond to a generator of $\Lambda/T^* \simeq \mathbb{Z}/2\mathbb{Z}$. Therefore, the map $f\colon \mathbb{Z}[\Lambda] \to \mathbb{Z}[\Lambda/T^*]$ applied to the element $x = \sum_{i=1}^{2m} n_i\widehat{e^{\omega_i}} + \delta_i\widehat{e^{\omega_i}}$ gives the same equality $(\sum_{i \text{ is odd}} n_i + 2m_i)d_i = 0$, where $m_i \in \mathbb{Z}$, $d_i = 2^i\binom{2m}{i}$ for $i \leq 2m-2$ and $d_{2m-1} = 2^{2m-1}$. Dividing by the g.c.d. of d_i's and taking modulo 2 we obtain that n_1 is even if $m > 2$ and $n_1 + n_3$ is even if $m = 2$.

We now compute $\mathfrak{c}_2(x)$. Take a long root $\alpha = e_{2m-1} + e_{2m}$. Then $(\alpha, \alpha) = 2$ and $\langle \lambda, \alpha^\vee\rangle = (\lambda, e_{2m-1}) + (\lambda, e_{2m})$. For $i \leq 2m-2$ we have

$$N(\widehat{e^{\omega_i}}) = \sum_{\lambda\in W(\omega_i)} (\lambda, e_{2m})^2 + (\lambda, e_{2m})(\lambda, e_{2m-1}) = \sum_{\lambda\in W(\omega_i)} (\lambda, e_{2m})^2 = 2^i\binom{2m-1}{i-1},$$

and $N(\widehat{e^{\omega_{2m-1}}}) = N(\widehat{e^{\omega_{2m}}}) = 2^{2m-3}$ (here we used the fact that W acts by permutations and even sign changes).

Finally, if $m > 2$, we obtain $\mathfrak{c}_2(x) = \sum_i n_i N(\widehat{e^{\omega_i}})q \in 4\mathbb{Z}q$, where $4\mathbb{Z}q = \mathrm{Dec}(\mathbf{HSpin}_{4m})$ by [2, §5]. If $m = 2$, then $N(\widehat{e^{\omega_4}}) = 2$, hence, $\mathfrak{c}_2(x) \in 2\mathbb{Z}q$, where $2\mathbb{Z}q = \mathrm{Dec}(\mathbf{HSpin}_8)$ again by [2, §5].

If $m > 2$, for the adjoint group $G = \mathbf{PGO}_{8m}$ by [15, §4] and the respective half-spin case we obtain

$$4\mathbb{Z}q = \mathrm{Dec}(\mathbf{PGO}_{8m}) \subseteq \mathrm{SDec}(\mathbf{PGO}_{8m}) \subseteq \mathrm{SDec}(\mathbf{HSpin}_{8m}) = 4\mathbb{Z}q.$$

If $G = \mathbf{PGO}_8$, direct computations (see [20]) show that $\mathrm{Dec}(G) = \mathrm{SDec}(G)$.

4. Applications

Observe that $H^3(F, \mathbb{Z}/n\mathbb{Z}(2))$ is the n-th torsion part of $H^3(F, 2)$ for every n and $H^3(F, \mathbb{Z}/n\mathbb{Z}(2)) = H^3(F, \mu_n^{\otimes 2})$ if $\mathrm{char}(F)$ does not divide n.

4.1. *Type C_n.* Let $G = \mathbf{PGSp}_{2n}$ be the split projective symplectic group. For a field extension L/F, the set $H^1(L, G)$ is identified with the set of isomorphism classes of central simple L-algebras A of degree $2n$ with a symplectic involution σ (see [14, §29]). A decomposable invariant of G takes an algebra with involution (A, σ) to the cup-product $\phi \cup [A]$ for a fixed element $\phi \in F^\times$. In particular, decomposable invariants of G are independent of the involution.

Suppose that $4 \mid n$. It is shown in [15, Theorem 4.6] that the group of indecomposable invariants $\mathrm{Inv}^3(G, 2)_{\mathrm{ind}}$ is cyclic of order 2. If $\mathrm{char}(F) \neq 2$, Garibaldi, Parimala and Tignol constructed in [8, Theorem A] a degree 3 cohomological invariant Δ_{2n} of the group G with coefficients in $\mathbb{Z}/2\mathbb{Z}$. They showed that if $a \in A$ is a σ-symmetric element of $A^\times$ and $\sigma' = \mathrm{Int}(a) \circ \sigma$, then

$$\Delta_{2n}(A, \sigma') = \Delta_{2n}(A, \sigma) + \mathrm{Nrp}(a) \cup [A], \tag{6}$$

where Nrp is the pfaffian norm. In particular, Δ_{2n} does depend on the involution and therefore, the invariant Δ_{2n} is not decomposable. Hence the the class of Δ_{2n} in $\mathrm{Inv}^3(G, 2)_{\mathrm{ind}}$ is nontrivial.

It follows from (6) that the class $\Delta_{2n}(A) \in \frac{H^3(L,\mathbb{Z}/2\mathbb{Z})}{L^\times \cup [A]}$ of $\Delta_{2n}(A, \sigma)$ depends only on the L-algebra A of degree $2n$ and exponent 2 but not on the involution. Since $\Delta_{2n}(A, \sigma)$ is not decomposable, it is not semi-decomposable by our main theorem. The latter implies that $\Delta_{2n}(A)$ is *nontrivial generically*, i.e. there is a central simple algebra A of degree $2n$ over a field extension of F with exponent 2 such that $\Delta_{2n}(A) \neq 0$. This answers a question raised in [8]. (See [1, Remark 4.10] for the case $n = 4$.)

4.2. *Type A_{n-1}.* Let $G = \mathbf{SL}_n / \boldsymbol{\mu}_m$, where n and m are positive integers such that n and m have the same prime divisors and $m \mid n$. Given a field extension L/F the natural surjection $G \to \mathbf{PGL}_n$ yields a map

$$\alpha : H^1(L, G) \to H^1(L, \mathbf{PGL}_n) \subset \mathrm{Br}(L)$$

taking a G-torsor Y over L to the class of a central simple algebra $A(Y)$ of degree n and exponent dividing m. By definition, a decomposable invariant of G is of the form $Y \mapsto \phi \cup [A(Y)]$ for a fixed $\phi \in F^\times$.

The map $\mathbf{SL}_m \to \mathbf{SL}_n$ taking a matrix M to the tensor product $M \otimes I_{n/m}$ with the identity matrix, gives rise to a group homomorphism $\mathbf{PGL}_m \to G$. The induced homomorphism (see [15, Theorem 4.4])

$$\varphi : \mathrm{Inv}^3(G, 2)_{\mathrm{norm}} \to \mathrm{Inv}^3(\mathbf{PGL}_m, 2)_{\mathrm{norm}} = F^\times / F^{\times m}$$

is a splitting of the inclusion homomorphism

$$F^\times / F^{\times m} = \mathrm{Inv}^3(G, 2)_{\mathrm{dec}} \hookrightarrow \mathrm{Inv}^3(G, 2)_{\mathrm{norm}}.$$

Collecting descriptions of p-primary components of $\mathrm{Inv}^3(G, 2)_{\mathrm{ind}}$ (see 3.2) we get

$$\mathrm{Inv}^3(G, 2)_{\mathrm{ind}} \simeq \tfrac{m}{k}\mathbb{Z}q / m\mathbb{Z}q, \qquad \text{where } k = \begin{cases} \gcd(\frac{n}{m}, m), & \text{if } \frac{n}{m} \text{ is odd;} \\ \gcd(\frac{n}{2m}, m), & \text{if } \frac{n}{m} \text{ is even.} \end{cases} \tag{7}$$

Let $\Delta_{n,m}$ be a (unique) invariant in $\mathrm{Inv}^3(G, 2)_{\mathrm{norm}}$ such that its class in $\mathrm{Inv}^3(G, 2)_{\mathrm{ind}}$ corresponds to $\frac{m}{k} q + m\mathbb{Z}q$ and $\varphi(\Delta_{n,m}) = 0$. Note that the order of $\Delta_{n,m}$ in $\mathrm{Inv}^3(G, 2)_{\mathrm{norm}}$ is equal to k. Therefore, $\Delta_{n,m}$ takes values in $H^3(-, \mathbb{Z}/k\mathbb{Z}(2)) \subset H^3(-, 2)$.

Fix a G-torsor Y over F and consider the twists ${}^Y G$ and $\mathbf{SL}_1(A(Y))$ by Y of the groups G and $\mathbf{SL}_n$ respectively. The group $F^\times$ acts transitively on the fiber over $A(Y)$ of the map α. If $\phi \in F^\times$, we write ${}^\phi Y$ for the corresponding element in the fiber. By (7) the image of $\Delta_{n,m}$ under the natural composition

$$\mathrm{Inv}^3(G,2)_{\mathrm{norm}} \simeq \mathrm{Inv}^3({}^Y G,2)_{\mathrm{norm}} \longrightarrow \mathrm{Inv}^3(\mathbf{SL}_1(A(Y)),2)_{\mathrm{norm}}$$

is a $\frac{m}{k}$-multiple of the Rost invariant. Recall that the Rost invariant takes the class of ϕ in $F^\times/\operatorname{Nrd}(A(Y)^\times) = H^1(F, \mathbf{SL}_1(A(Y)))$ to the cup-product $\phi \cup [A(Y)] \in H^3(F,2)$. So we get

$$\Delta_{n,m}({}^\phi Y) - \Delta_{n,m}(Y) \in F^\times \cup \tfrac{m}{k}[A(Y)]. \tag{8}$$

Given a central simple L-algebra A of degree n and exponent dividing m, we define an element

$$\Delta_{n,m}(A) \in \frac{H^3(L,\mathbb{Z}/k\mathbb{Z}(2))}{L^\times \cup \frac{m}{k}[A]}$$

as follows. Choose a G-torsor Y over L with $A(Y) \simeq A$ and set $\Delta_{n,m}(A)$ to be the class of $\Delta_{n,m}(Y)$ in the factor group. It follows from (8) that $\Delta_{n,m}(A)$ is independent of the choice of Y.

4.1. **Proposition.** *Let A be a central simple L-algebra of degree n and exponent dividing m. Then the order of $\Delta_{n,m}(A)$ divides k. If A is a generic algebra, then the order of $\Delta_{n,m}(A)$ is equal to k.*

Proof. If k' is a proper divisor of k, then the multiple $k'\Delta_{n,m}$ is not decomposable. By our theorem $k'\Delta_{n,m}$ is not semi-decomposable and, hence, $k'\Delta_{n,m}(A) \neq 0$. $\square$

4.2. **Example.** Let A be a central simple F-algebra of degree $2n$ divisible by 8 and exponent 2. Choose a symplectic involution σ on A. The group $\mathbf{PGSp}_{2n}$ is a subgroup of $\mathbf{SL}_{2n}/\boldsymbol{\mu}_2$, hence, if $\mathrm{char}(F) \neq 2$, the restriction of the invariant $\Delta_{2n,2}$ on $\mathbf{PGSp}_{2n}$ is the invariant $\Delta_{2n}(A,\sigma)$ considered in subsection 4.1. It follows that $\Delta_{2n,2}(A) = \Delta_{2n}(A)$ in the group $H^3(F,\mathbb{Z}/2\mathbb{Z})/(F^\times \cup [A])$.

The class $\Delta_{n,m}$ is trivial on decomposable algebras:

4.3. **Proposition.** *Let n_1, n_2, m be positive integers such that m divides n_1 and n_2. Let A_1 and A_2 be two central simple algebras over F of degree n_1 and n_2 respectively and of exponent dividing m. Then $\Delta_{n_1 n_2, m}(A_1 \otimes_F A_2) = 0$.*

Proof. The tensor product homomorphism $\mathbf{SL}_{n_1} \times \mathbf{SL}_{n_2} \to \mathbf{SL}_{n_1 n_2}$ yields a homomorphism

$$\mathrm{Sym}^2(T^*_{n_1 n_2}) \to \mathrm{Sym}^2(T^*_{n_1}) \oplus \mathrm{Sym}^2(T^*_{n_2}),$$

where T_{n_1}, T_{n_2} and $T_{n_1 n_2}$ are maximal tori of respective groups. The image of the canonical Weyl-invariant generator $q_{n_1 n_2}$ of $\mathrm{Sym}^2(T^*_{n_1 n_2})$ is equal to $n_2 q_{n_1} + n_1 q_{n_2}$. Since n_1 and n_2 are divisible by m, the pull-back of the invariant $\Delta_{n_1 n_2, m}$ under the homomorphism $(\mathbf{SL}_{n_1}/\boldsymbol{\mu}_m) \times (\mathbf{SL}_{n_2}/\boldsymbol{\mu}_m) \to \mathbf{SL}_{n_1 n_2}/\boldsymbol{\mu}_m$ is trivial. $\square$

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Alexander Merkurjev, University of California at Los Angeles
E-mail address: merkurev@math.ucla.edu

Alexander Neshitov, Steklov Mathematical Institute, St.Petersburg / University of Ottawa
E-mail address: neshitov@yandex.ru

Kirill Zainoulline, University of Ottawa
E-mail address: kirill@uottawa.ca