MOTIVIC DECOMPOSITION OF CERTAIN SPECIAL LINEAR GROUPS

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Abstract. We compute the motive of the algebraic group $G = \mathbf{SL}_1(D)$ for a central simple algebra D of prime degree over a perfect field. As an application we determine certain motivic cohomology groups and differentials in the motivic spectral sequence of G.

1. Introduction

In this paper we study the motive in the triangulated category of geometric mixed effective motives $\mathit{DM}_{gm}^{eff}(F)$ over a perfect field F and the motivic cohomology of the algebraic group $\mathbf{SL}_1(D)$ of reduced norm 1 elements in a central simple algebra D of prime degree l.

In [18], A. Suslin computed the K-cohomology groups of the (split) special linear group $\mathbf{SL}_n$ and the symplectic groups $\mathbf{Sp}_{2n}$ using higher Chern classes in K-cohomology. O. Pushin in [15] constructed higher Chern classes in motivic cohomology and found decompositions of the motives of the groups $\mathbf{SL}_n$ and $\mathbf{GL}_n$ into direct sums of Tate motives. S. Biglari computed in [1] the motives of certain split reductive groups over $\mathbb{Q}$. In particular, he showed that

$$M(\mathbf{SL}_n)_{\mathbb{Q}} \simeq \coprod_{i=0}^{n-1} \mathit{Sym}^i\big(\mathbb{Q}(2)[3] \oplus \mathbb{Q}(3)[5] \oplus \cdots \oplus \mathbb{Q}(n)[2n-1]\big). \tag{1.1}$$

A. Huber and B. Kahn determined the motives over $\mathbb{Z}$ of split reductive groups in [9].

The motives of non-split algebraic groups are more complicated. The slices of the slice filtration of the motive $M\big(\mathbf{GL}_l(D)\big)$ for a division algebra D of prime degree were computed by E. Shinder in [17].

In this paper we study the motive of the group $G = \mathbf{SL}_1(D)$, where D is a central simple algebra of a prime degree l. As a warm-up, consider the simplest case $l = 2$. The variety of G is then an open subscheme of a 3-dimensional projective isotropic quadric X given by the homogeneous quadratic equation $\mathrm{Nrd} = t^2$, where Nrd is the reduced norm form of D. The surface $Y = X \setminus G$, given by $\mathrm{Nrd} = 0$, is isomorphic to $S \times S$, where S is the Severi-Brauer variety

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of D (a conic curve in the case $l = 2$). Computing the motives of X and Y as in [17, §4], we get an exact triangle

$$M(G) \to \mathbb{Z} \oplus M(S)(1)[2] \oplus \mathbb{Z}(3)[6] \to M(S)(1)[2] \oplus M(S)(2)[4] \to M(G)[1].$$

Canceling out the summands $M(S)(1)[2]$, we obtain an isomorphism

$$M(G) \simeq \mathbb{Z} \oplus N(2)[3], \tag{1.2}$$

where the motive N is defined by the exact triangle

$$\mathbb{Z}(1)[2] \to M(S) \to N \to \mathbb{Z}(1)[3]$$

with the first morphism dual to the canonical one $M(S) \to \mathbb{Z}$.

In the general case, when l is an arbitrary prime, since the group G and its motive are split over a field extension of degree l, the torsion part of motivic cohomology of G is l-torsion. We work over the coefficient ring $\mathbb{Z}\big[\frac{1}{(l-1)!}\big]$, just inverting insignificant integers.

As in the case $l = 2$, the motive of G can be computed out of motive of the Severi-Brauer variety S of the algebra D. Let N be the motive defined by the exact triangle

$$\mathbb{Z}(l-1)[2l-2] \to M(S) \to N \to \mathbb{Z}(l-1)[2l-1].$$

As $(l-1)!$ is invertible in the coefficient ring, one can define symmetric $\mathsf{Sym}^i(M)$ and alternating powers $\mathsf{Alt}^i(M)$ of any motive M for $i = 0, 1, \ldots, l-1$. The main result of the paper is the following theorem generalizing (1.1) and (1.2) (see Theorem 11.1).

Theorem. *Let D be a central simple algebra of prime degree l over a perfect field F. Then there is an isomorphism*

$$M\big(\mathbf{SL}_1(D)\big) \xrightarrow{\sim} \coprod_{i=0}^{l-1} \mathsf{Sym}^i\big(N(2)[3]\big) = \coprod_{i=0}^{l-1} \big(\mathsf{Alt}^i N\big)(2i)[3i]$$

in the category $\mathsf{DM}_{gm}^{eff}(F)$ of motives over F with coefficients in $\mathbb{Z}\big[\frac{1}{(l-1)!}\big]$.

The most difficult part of the proof is the construction of a morphism $M(G) \to N(2)[3]$ in $\mathsf{DM}_{gm}^{eff}(F)$. The main players of the proof are the groups $H^{3,2}(G) \simeq \mathbb{Z}$ and the Chow group $\mathrm{CH}^{l+1}(G) = H^{2l+2,l+1}(G) \simeq \mathbb{Z}/l\mathbb{Z}$ (when D is not split). These groups are related by a pair of homomorphisms

$$H^{3,2}(G) \leftarrow \mathrm{Hom}\big(M(G), N(2)[3]\big) \to H^{2l+2,l+1}(G). \tag{1.3}$$

We prove that there is a morphism $M(G) \to N(2)[3]$ with the images in (1.3) generating the two side cyclic groups. This is done in Section 10.

Using Theorem 1 and the exact triangle (Corollary 6.5)

$$(\mathsf{Alt}^{i-1}N)(l-1)[2l-2] \to \mathsf{Alt}^i M(S) \to (\mathsf{Alt}^i N) \to (\mathsf{Alt}^{i-1}N)(l-1)[2l-1],$$

we can compute inductively the motivic cohomology of G. As an application, in Section 12 we compute the motivic cohomology $H^{p,q}(G)$ with $2q - p \leq 1$. We also compute certain differentials in the motivic spectral sequence of G.

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2. Motivic cohomology

The base field F is assumed to be perfect. We fix a prime integer l and work over the *coefficient ring* $\mathbf{Z}$ that is either the ring of integers $\mathbb{Z}$ or $\mathbb{Z}\big[\frac{1}{(l-1)!}\big]$, or the localization $\mathbb{Z}_{(l)}$ of $\mathbb{Z}$ by the prime ideal generated by l. Note that $\mathbf{Z}/l\,\mathbf{Z} = \mathbb{Z}/l\mathbb{Z}$.

We write $\mathit{DM}(F) := \mathit{DM}^{eff}_{gm}(F)$ for the *triangulated category of (geometric mixed effective) motives* with coefficients in $\mathbf{Z}$ (see [19]). If p and $q \geq 0$ are integers, $\mathbf{Z}(q)[p]$ denotes the *Tate motive* and $M(X)$ the motive of a smooth variety X over F. We have $M(\operatorname{Spec} F) = \mathbf{Z} := \mathbf{Z}(0)[0]$. For a motive M and an integer $q \geq 0$, we write $M(q)$ for $M \otimes \mathbf{Z}(q)$.

For a motive M in $\mathit{DM}(F)$ define the *motivic cohomology* by

$$H^{p,q}(M) := \operatorname{Hom}\big(M, \mathbf{Z}(q)[p]\big),$$

where Hom is taken in the category $\mathit{DM}(F)$. If X is a smooth variety, simply write $H^{p,q}(X)$ for $H^{p,q}\big(M(X)\big)$. We have

$$H^{p,q}(X) = 0 \quad \text{if} \quad p > 2q \quad \text{or} \quad p > q + \dim(X). \tag{2.1}$$

In particular, $H^{p,q}(F) = 0$ if $p > q$. Moreover, $H^{p,p}(F) = K^M_p(F)$, the Milnor K-groups of F (see [12, Lecture 5]).

The bi-graded group $\coprod_{p,q} H^{p,q}(X)$ has a natural structure of a graded commutative ring (with respect to p, [12, Theorem 15.9]).

There is a canonical isomorphism between $H^{2p,p}(X)$ and the *Chow groups* $\mathrm{CH}^p(X)$ of (rational equivalence) classes of algebraic cycles on a smooth variety X of codimension p ([12, Lecture 18]). We also write $\mathrm{CH}^p(M) := H^{2p,p}(M)$ for every motive M.

The *cancelation* theorem (see [21]) states that the canonical morphism

$$\operatorname{Hom}(M, N) \xrightarrow{\sim} \operatorname{Hom}\big(M(1), N(1)\big)$$

is an isomorphism for every two motives M and N.

The natural functor from the category of smooth projective varieties over F to $\mathit{DM}(F)$ extends uniquely to a canonical functor from the category $\mathit{Chow}(F)$ of Chow motives over F to $\mathit{DM}(F)$ (see [19, Proposition 2.1.4]). The motives in $\mathit{DM}(F)$ coming from $\mathit{Chow}(F)$ are called *pure* motives.

Let M be any motive and X a smooth projective variety of pure dimension d over F. The two canonical morphisms (given by the diagonal of X in the category of Chow motives)

$$\mathbf{Z}(d)[2d] \to M(X \times X) \to \mathbf{Z}(d)[2d]$$

together with the cancelation theorem define the two mutually inverse isomorphisms (see [9, Appendix B])

$$\operatorname{Hom}\big(M, M(X)\big) \rightleftarrows \operatorname{Hom}\big(M \otimes M(X), \mathbf{Z}(d)[2d]\big) = \mathrm{CH}^d\big(M \otimes M(X)\big). \tag{2.2}$$

In particular, if Y is another smooth projective variety, then

$$\operatorname{Hom}_{DM(F)}\big(M(Y), M(X)\big) \simeq \operatorname{CH}^d(Y \times X) = \operatorname{Hom}_{Chow(F)}\big(M(Y), M(X)\big).$$

We say that a motive M is *of degree* d if M is a direct summand of a motive of the form $M(X)(q)[p]$ with $2q-p=d$, where X is a smooth projective variety. The pure motives are of degree 0. The following statement is an immediate consequence of (2.1).

Lemma 2.3. *Let M and N be motives of degree d and e respectively. If $d > e$, then* $\operatorname{Hom}(M, N) = 0$. □

The *coniveau* spectral sequence for a smooth variety X over F,

$$E_1^{p,q} = \coprod_{x \in X^{(p)}} H^{q-p,n-p} F(x) \Rightarrow H^{p+q,n}(X),$$

where $X^{(p)}$ is the set of points in X of codimension p, yields isomorphisms

$$H^{i+n,n}(X) \simeq A^i(X, K_n) \quad \text{when} \quad n - i \leq 2$$

with the K-*cohomology* groups $A^i(X, K_n)$ defined in [16].

If X is a variety over F, we write X_{sep} for the variety $X \otimes_F F_{\mathrm{sep}}$ over a separable closure F_{sep} of F.

3. Severi-Brauer varieties

Let D be a central simple algebra of degree n over F and S the *Severi-Brauer* variety $\mathrm{SB}(D)$ of right ideals in D of rank n. This is a smooth projective variety of dimension $n-1$ (see [6]). If D is *split*, i.e., $D = \operatorname{End}(V)$ for an n-dimensional vector space V over F, then S is isomorphic to the projective space $\mathbb{P}(V)$. Therefore, in the split case,

$$M(S) \simeq \mathbf{Z} \oplus \mathbf{Z}(1)[2] \oplus \cdots \oplus \mathbf{Z}(l-1)[2l-2].$$

Let $I \to S$ be the *tautological* vector bundle of rank n (with the fiber over a right ideal the ideal itself). We have $D = \operatorname{End}(I)^{op} = \operatorname{End}(I^\vee)$, where $I^\vee$ is the vector bundle dual to I. In the split case, when $S = \mathbb{P}(V)$,

$$I = V^\vee \otimes L_t = \operatorname{Hom}(V, L_t),$$

where $L_t \to \mathbb{P}(V)$ is the tautological line bundle. The sheaf of sections of L_t is $O(-1)$.

In the split case, when $S = \mathbb{P}(V)$, let $s \in \operatorname{CH}^1(S)$ be the class of a hyperplane section. We have

$$\operatorname{CH}^i\big(\mathbb{P}(V)\big) = \begin{cases} \mathbb{Z}s^i, & i = 0, 1, \ldots, n-1; \\ 0, & \text{otherwise.} \end{cases}$$

The ring $\operatorname{End}\big(M(S)\big) = \operatorname{CH}^{n-1}(S \times S)$ is canonically isomorphic to the product $\mathbf{Z}^n$ of n copies of $\mathbf{Z}$ with the idempotents $s^i \times s^{n-1-i}$.

In the non-split case we have the following statement (see [13, Corollary 8.7.2]):

Lemma 3.1. *When D is a division algebra of prime degree l, the natural map* $\mathrm{CH}^*(S) \to \mathrm{CH}^*(S_{\mathrm{sep}})$ *is injective and it identifies the Chow group of S as follows*

$$\mathrm{CH}^i(S) = \begin{cases} \mathbf{Z}\,1, & i = 0; \\ l\,\mathbf{Z}\,s^i, & i = 1, \dots, l-1; \\ 0, & \textit{otherwise.} \end{cases} \qquad \square$$

Any of the two projections $p : S \times S \to S$ is the projective bundle of $I^\vee$, i.e., $S \times S = \mathbb{P}_S(I^\vee)$. Let L be the tautological line bundle of this projective bundle. By the projective bundle theorem [19, Proposition 3.5.1], we have:

$$\mathrm{CH}^{l-1}(S \times S) = \mathrm{CH}^{l-1}(S) \cdot 1 \oplus \mathrm{CH}^{l-2}(S) \cdot \xi \oplus \cdots \oplus \mathrm{CH}^0(S) \cdot \xi^{l-1}, \tag{3.2}$$

where ξ is the first Chern class of L in $\mathrm{CH}^1(S \times S)$. Consider the composition

$$\mathit{type} : \mathrm{CH}^{l-1}(S \times S) = \operatorname{End} M(S) \to \operatorname{End} M(S_{\mathrm{sep}}) \xrightarrow{\sim} \mathbf{Z}^l .$$

Proposition 3.3. *Let D be a division algebra of degree l and $S = \mathrm{SB}(D)$. Then the ring homomorphism*

$$\mathit{type} : \operatorname{End} M(S) \to \mathbf{Z}^l$$

is injective. Its image consists of all tuples $(a_1, a_2, \dots, a_l)$ such that

$$a_1 \equiv a_2 \equiv \cdots \equiv a_l \pmod{l}.$$

Proof. It follows from Lemma 3.1 and (3.2) that type is injective and $[\mathrm{Im}(\mathit{type}) : l\,\mathbf{Z}^l] = l$. Therefore, the identity in $\mathbf{Z}^l$ and $l\,\mathbf{Z}^l$ generate $\mathrm{Im}(\mathit{type})$. $\square$

We will also need the following lemma.

Lemma 3.4. *Let M_1 and M_2 be direct sum of shifts of $M(S)$ (with arbitrary coefficients) and $f : M_1 \to M_2$ a morphism in $\mathit{DM}(F)$. If f is an isomorphism over a field extension, then f is also an isomorphism.*

Proof. Write M_1 and M_2 as direct sums of the homogeneous (degree k) components $M_1^{(k)}$ and $M_2^{(k)}$ respectively. By Lemma 2.3, the morphism f is given by a triangular matrix with the diagonal terms $f_k : M_1^{(k)} \to M_2^{(k)}$. By assumption, the matrix is invertible over a splitting field L, hence all f_k are isomorphisms over L. Note that f_k is a shift of a morphism of pure motives that are direct sums of shifts of $M(S)$. By [4, Corollary 92.7], all f_k are isomorphisms. Therefore, the triangular matrix is invertible and hence f is an isomorphism. $\square$

4. The motive N

Let D be a central simple algebra of prime degree l over F and $S = \mathrm{SB}(D)$. The motive N is defined by the triangle

$$\mathbf{Z}(l-1)[2l-2] \to M(S) \xrightarrow{\kappa} N \xrightarrow{\varepsilon} \mathbf{Z}(l-1)[2l-1] \tag{4.1}$$

in $\mathit{DM}(F)$ with the first morphism of pure motives given by the identity in $\mathrm{CH}^0(S)$. We have

$$N_{\mathrm{sep}} \simeq \mathbf{Z} \oplus \mathbf{Z}(1)[2] \oplus \cdots \oplus \mathbf{Z}(l-2)[2l-4], \tag{4.2}$$

therefore, $\mathrm{Hom}\big(M(S_{\mathrm{sep}}), N_{\mathrm{sep}}\big) \simeq \mathbf{Z}^{l-1}$.

Consider the map

$$\mathit{type} : \mathrm{Hom}(M(S), N) \to \mathrm{Hom}\big(M(S_{\mathrm{sep}}), N_{\mathrm{sep}}\big) \simeq \mathbf{Z}^{l-1}\,.$$

For example, $\mathit{type}(\kappa) = (1, 1, \ldots, 1)$.

Proposition 4.3. *Let D be a division algebra of degree l and $S = \mathrm{SB}(D)$. Then the homomorphism*

$$\mathit{type} : \mathrm{Hom}(M(S), N) \to \mathbf{Z}^{l-1}$$

is injective. Its image consists of all tuples $(a_1, a_2, \ldots, a_{l-1})$ such that

$$a_1 \equiv a_2 \equiv \cdots \equiv a_{l-1} \quad (\mathit{mod}\ l).$$

Proof. Let $\varphi \in \mathrm{Ker}(\mathit{type})$. The triangle (4.1) yields an exact sequence

$$\mathrm{CH}^{l-1}(S) \to \mathrm{End}\, M(S) \to \mathrm{Hom}(M(S), N) \to H^{2l-1,l-1}(S).$$

The last term is zero as $2l-1 > 2(l-1)$. Therefore, $\varphi = \kappa \circ \sigma$ for some $\sigma \in \mathrm{End}\, M(S)$. By assumption, $\mathit{type}(\sigma) = (0, \ldots, 0, a)$, where $a \equiv 0$ modulo l in view of Proposition 3.3. Then σ comes from $\mathrm{CH}^{l-1}(S) = l\,\mathbf{Z}$ by Lemma 3.1 and hence $\varphi = 0$. This proves injectivity. The second statement follows from Proposition 3.3. □

Lemma 4.4. *There is an isomorphism*

$$N \otimes M(S) \simeq M(S) \oplus M(S)(1)[2] \oplus \cdots \oplus M(S)(l-2)[2l-4].$$

In particular, $N \otimes M(S)$ is a pure motive.

Proof. The triangle (4.1) is split after tensoring with $M(S)$. Indeed, the morphism $M(S)(l-1)[2l-2] \to M(S) \otimes M(S)$ has a left inverse given by the class of the diagonal in $\mathrm{CH}^{2l-2}(S \times S \times S)$. □

Lemma 4.5. *We have $\mathrm{CH}^i(N) = 0$ if $i > l$.*

Proof. In the exact sequence induced by (4.1)

$$H^{2i-2l+1,i-l+1}(F) \to \mathrm{CH}^i(N) \to \mathrm{CH}^i(S)$$

the first and the last terms are trivial as $2i-2l+1 > i-l+1$ and $\dim(S) < l$. □

Since $\mathrm{Hom}\big(\mathbf{Z}(q)[p], \mathbf{Z}\big) = 0$ if $q > 0$, the natural morphism $M(S) \to \mathbf{Z}$ factors uniquely through a morphism $\nu : N \to \mathbf{Z}$.

5. Higher Chern classes

Let X be a smooth variety. The higher Chern classes with values in motivic cohomology were constructed in [15]:

$$c_{j,i} : K_j(X) \to H^{2i-j,i}(X).$$

We will be using the classes

$$c_i := c_{1,i+1} : K_1(X) \to H^{2i+1,i+1}(X).$$

Proposition 5.1 ([17, §4.1])**.** *Let L be a vector bundle over a smooth variety X and $\alpha \in K_1(X)$. Then*

$$c_i\big(\alpha \cdot [L]\big) = \sum_{j=0}^{i} (-1)^j \binom{i}{j} c_{i-j}(\alpha) h^j,$$

where $h \in \mathrm{CH}^1(X) = H^{2,1}(X)$ is the first (classical) Chern class of L.

Let $E \to X$ be a vector bundle of rank n. We write $\mathbf{SL}(E)$ for the group scheme over X of determinant 1 automorphisms of E.

Let a be the generic element of $\mathbf{SL}(E)$ (see [18, §4]). We also write a for the corresponding element in $K_1\big(\mathbf{SL}(E)\big)$. We have $c_0(a) = 0$ since $\det(a) = 1$. For a sequence $\mathbf{i} = (i_1, i_2, \dots, i_k)$ with $1 \le i_1 < i_2 < \cdots < i_k \le n-1$, set $d_{\mathbf{i}} = i_1 + i_2 + \cdots + i_k$ and $e_{\mathbf{i}} = k$. Let

$$c_{\mathbf{i}}(\alpha) := c_{i_1}(\alpha) c_{i_2}(\alpha) \cdots c_{i_k}(\alpha) \in H^{2d_{\mathbf{i}}+e_{\mathbf{i}}, d_{\mathbf{i}}+e_{\mathbf{i}}}\big(\mathbf{SL}(E)\big).$$

Proposition 5.2. *Let $E \to X$ be a vector bundle of rank n. Then the $H^{*,*}(X)$-module $H^{*,*}\big(\mathbf{SL}(E)\big)$ is free with basis $\{c_{\mathbf{i}}(\alpha)\}$ over all sequences $\mathbf{i}$.*

Proof. This follows from [15, Proposition 3]. □

Write $\widetilde{c}_{\mathbf{i}}(\alpha)$ for the composition

$$M\big(\mathbf{SL}(E)\big) \xrightarrow{\mathrm{diag}} M\big(\mathbf{SL}(E)\big) \otimes M\big(\mathbf{SL}(E)\big) \xrightarrow{j \otimes c_{\mathbf{i}}(\alpha)} M(X)\big(d_{\mathbf{i}} + e_{\mathbf{i}}\big)\big[2d_{\mathbf{i}} + e_{\mathbf{i}}\big],$$

where $j : M\big(\mathbf{SL}(E)\big) \to M(X)$ is the canonical morphism. The following corollary is deduced from Proposition 5.2 the same way as in [17, Proposition 4.2].

Corollary 5.3. (cf. [17, Proposition 4.4]) The morphisms $\widetilde{c}_{\mathbf{i}}(\alpha)$ yield an isomorphism

$$M\big(\mathbf{SL}(E)\big) \xrightarrow{\sim} \coprod_{\mathbf{i}} M(X)\big(d_{\mathbf{i}} + e_{\mathbf{i}}\big)\big[2d_{\mathbf{i}} + e_{\mathbf{i}}\big]. \qquad \square$$

Remark 5.4. The natural composition

$$M\big(\mathbf{SL}(E)\big) \xrightarrow{\widetilde{c}_{\mathbf{i}}(\alpha)} M(X)\big(d_{\mathbf{i}} + e_{\mathbf{i}}\big)\big[2d_{\mathbf{i}} + e_{\mathbf{i}}\big] \to \mathbf{Z}\big(d_{\mathbf{i}} + e_{\mathbf{i}}\big)\big[2d_{\mathbf{i}} + e_{\mathbf{i}}\big]$$

coincides with $c_{\mathbf{i}}(\alpha)$.

Corollary 5.5. *There is a canonical isomorphism*

$$M(\mathbf{SL}_n) \simeq \coprod_{\mathbf{i}} \mathbf{Z}\big(d_{\mathbf{i}} + e_{\mathbf{i}}\big)\big[2d_{\mathbf{i}} + e_{\mathbf{i}}\big]. \qquad \square$$

Let $G = \mathbf{SL}(E)$ where $E \to X$ is a vector bundle of rank n over a smooth variety X. Consider the grading on $M(G)$ with respect to the value $e(\mathbf{i})$:

$$M(G)^{(k)} := \coprod_{e_{\mathbf{i}}=k} M(X)\big(d_{\mathbf{i}} + k\big)\big[2d_{\mathbf{i}} + k)\big]$$

for $k = 0, 1, \dots, n-1$. Thus,

$$M(G) = \coprod_{k=0}^{n-1} M(G)^{(k)}$$

and each motive $M(G)^{(k)}$ has degree k.

Example 5.6. In the split case, we have a natural isomorphism $M(\mathbf{SL}_l)^{(1)} \simeq N(2)[3]$.

Let D be a central simple algebra of prime degree l over F and $G = \mathbf{SL}_1(D)$. Let S be the Severi-Brauer variety of D.

Corollary 5.3 yields

Corollary 5.7. *There is a canonical isomorphism*

$$M(G \times S) \simeq \coprod_{\mathbf{i}} M(S)\big(d_{\mathbf{i}} + e_{\mathbf{i}}\big)\big[2d_{\mathbf{i}} + e_{\mathbf{i}}\big].$$

In particular, $\mathrm{CH}^*(G \times S) \simeq \mathrm{CH}^*(S)$*.* $\square$

It follows from Corollary 5.5 that $M(G_{\mathrm{sep}})^{(1)} \simeq N_{\mathrm{sep}}(2)[3]$ and therefore $\mathrm{Hom}\big(M(G_{\mathrm{sep}}), M(S_{\mathrm{sep}})(2)[3]\big)$ is naturally isomorphic to $\mathbf{Z}^{l-1}$. Consider the map

$$\mathsf{type} : \mathrm{Hom}\big(M(G), M(S)(2)[3]\big) \to \mathrm{Hom}\big(M(G_{\mathrm{sep}}), M(S_{\mathrm{sep}})(2)[3]\big) \simeq \mathbb{Z}^{l-1}.$$

By (2.2) and Corollary 5.7, we have

$$\mathrm{Hom}\big(M(G), M(S)(2)[3]\big) = H^{2l+1,l+1}(G \times S) = \coprod_{i=1}^{l-1} \mathrm{CH}^i(S) c_{l-i}(\alpha). \tag{5.8}$$

Lemma 3.1 and (5.8) yield the following proposition.

Proposition 5.9. *Let D be a division algebra of degree l and $S = \mathrm{SB}(D)$. Then the homomorphism*

$$\mathsf{type} : \mathrm{Hom}\big(M(G), M(S)(2)[3]\big) \to \mathbf{Z}^{l-1}$$

is injective and $\mathrm{Im}(\mathsf{type}) = l\,\mathbf{Z}^{l-1}$*.* $\square$

We will need the Chow groups of G that were computed in [10].

Proposition 5.10. *Let D be a central division algebra of prime degree l and $G = \mathbf{SL}_1(D)$. There is an element $h \in \mathrm{CH}^{l+1}(G)$ such that*

$$\mathrm{CH}^*(G) = \mathbf{Z}\cdot 1 \oplus (\mathbf{Z}/l\,\mathbf{Z})h \oplus (\mathbf{Z}/l\,\mathbf{Z})h^2 \oplus \cdots \oplus (\mathbf{Z}/l\,\mathbf{Z})h^{l-1}. \qquad \square$$

Recall that $D \simeq \mathrm{End}(I^\vee)$ and $G \times S \simeq \mathbf{SL}(I^\vee)$, where I is the tautological vector bundle over S of rank n (see Section 3).

Suppose that the algebra D is split. We can compare the generic matrices $\bar{\alpha}$ in $G = \mathbf{SL}_l$ and α in $G \times S = \mathbf{SL}(I^\vee)$. The bundle $I^\vee \otimes L_t$ over $G \times S$ is trivial, hence

$$\bar{\alpha} \times S = \alpha \otimes L_t \quad \text{in} \quad K_1(G \times S).$$

We have the Chern classes $c_i(\alpha) \in H^{2i+1,i+1}(G \times S)$ and $c_i := c_i(\bar{\alpha}) \in H^{2i+1,i+1}(G)$. We also write c_i for its image in $H^{2i+1,i+1}(G \times S)$ under the pull-back map given by the projection $G \times S \to G$.

By Proposition 5.1, we have

$$c_i = \sum_{j=0}^{i-1} \binom{i}{j} c_{i-j}(\alpha) s^j \quad \text{in} \quad H^{2i+1,i+1}(G \times S) \tag{5.11}$$

for all $i = 1, 2 \ldots, l-1$, since the first Chern class of L_t is equal to $-s$, where $s \in \mathrm{CH}^1(S)$ is the class of a hyperplane section, and $c_0(\alpha) = 0$ as $\det(\alpha) = 1$. In particular, $c_1 = c_1(\alpha)$.

The group $H^{3,2}(G) = A^1(G, K_2)$ is infinite cyclic with a canonical generator, and this group does not change under field extensions. (This is true for every absolutely simple simply connected group, see [5, Part II, §9].) Therefore, we can write $H^{3,2}(G) = \mathbb{Z}c_1$ viewing c_1 as a generator of $H^{3,2}(G)$.

6. Symmetric and alternating powers

We consider motives with coefficients in $\mathbf{Z} = \mathbb{Z}\big[\frac{1}{(l-1)!}\big]$ in this section. Let $i = 0, 1, \ldots, l-1$. The symmetric group Σ_i acts naturally on the i-th tensor power $M^{\otimes i}$ of a motive M. The elements

$$\tau_i = \frac{1}{i!}\sum_{\sigma \in \Sigma_i} \sigma \quad \text{and} \quad \rho_i = \frac{1}{i!}\sum_{\sigma \in \Sigma_i} \mathrm{sgn}(\sigma)\sigma$$

are idempotents in the group ring of Σ_i. The motives $\mathsf{Sym}^i(M) := (M, \tau_i)$ and $\mathsf{Alt}^i(M) := (M, \rho_i)$, that are split off M by the projectors τ_i and ρ_i, are called the *i-th symmetric power* and *i-th alternating power of M* respectively. We have $\mathsf{Sym}^0(M) = \mathbb{Z} = \mathsf{Alt}^0(M)$ and $\mathsf{Sym}^1(M) = M = \mathsf{Alt}^1(M)$.

We will need the following properties of symmetric and alternating powers.

Proposition 6.1 ([1, Proposition 2.3])**.** *Let M and N be two motives. Then*

(1) $\mathsf{Sym}^i\big(M[1]\big) \simeq (\mathsf{Alt}^i M)[i]$ *and* $\mathsf{Alt}^i\big(M[1]\big) \simeq (\mathsf{Sym}^i M)[i]$,
(2) $\mathsf{Sym}^i\big(M(q)\big) \simeq (\mathsf{Sym}^i M)(iq)$,
(3) $\mathsf{Sym}^i(M \oplus N) = \coprod_{k+m=i} \mathsf{Sym}^k(M) \otimes \mathsf{Sym}^m(N)$ *and similarly for* Alt.

Corollary 6.2. *We have*

$$\mathsf{Sym}^i\big(\mathbf{Z}(q)[p]\big) \simeq \begin{cases} \mathbf{Z}(iq)[ip], & \text{if } p \text{ is even;} \\ 0, & \text{if } p > 1 \text{ is odd.} \end{cases} \qquad \square$$

Example 6.3. Let N be the motive $\mathbf{Z} \oplus \mathbf{Z}(1)[2] \oplus \cdots \oplus \mathbf{Z}(l-2)[2l-4]$ (see (4.2)). Then

$$\mathsf{Sym}^k\big(N(2)[3]\big) = (\mathsf{Alt}^k N)(2k)[3k] = \coprod_{e_{\mathbf{i}}=k} \mathbf{Z}\big(d_{\mathbf{i}} + k\big)\big[2d_{\mathbf{i}} + k\big],$$

with the notation from Section 5.

Proposition 6.4 ([8, Proposition 15])**.** *Let $X \to Y \to Z \to X[1]$ be an exact triangle. Then there are sequences of morphisms*

$$\mathsf{Alt}^i X = T_0 \to T_1 \to \cdots \to T_i = \mathsf{Alt}^i Y,$$

$$\mathsf{Sym}^i X = V_0 \to V_1 \to \cdots \to V_i = \mathsf{Sym}^i Y$$

and exact triangles

$$T_{j-1} \to T_j \to \mathsf{Alt}^{i-j} X \otimes \mathsf{Alt}^j Z \to T_{j-1}[1],$$

$$V_{j-1} \to V_j \to \mathsf{Sym}^{i-j} X \otimes \mathsf{Sym}^j Z \to V_{j-1}[1]$$

for every $j = 1, 2, \ldots, i$.

Assuming that $\mathsf{Alt}^k X = 0$ for $k > 1$, we get an exact triangle

$$X \otimes \mathsf{Alt}^{i-1} Z \to \mathsf{Alt}^i Y \to \mathsf{Alt}^i Z \to \big(X \otimes \mathsf{Alt}^{i-1} Z\big)[1].$$

Applying this to the exact triangle (4.1), we have the following proposition.

Corollary 6.5. *There is an exact triangle*

$$(\mathsf{Alt}^{i-1} N)(l-1)[2l-2] \to \mathsf{Alt}^i M(S) \to \mathsf{Alt}^i N \to (\mathsf{Alt}^{i-1} N)(l-1)[2l-1]. \ \square$$

This proposition will be used in Section 12 to compute inductively the motivic cohomology of $\mathsf{Alt}^i N$.

The pure motive $\mathsf{Alt}^i M(S)$ is a direct summand of $M(S^i)$ and the latter is a direct sum of shifts of the motive $M(S)$. If D is a division algebra, the motive $M(S)$ is indecomposable [11, Corollary 2.22]. When the coefficient ring $\mathbf{Z}$ is the local ring $\mathbb{Z}_{(l)}$, by uniqueness of the decomposition [3, Corollary 35], $\mathsf{Alt}^i M(S)$ is a pure motive that is a direct sum of pure shifts of $M(S)$. Moreover, since in the split case

$$\mathsf{Alt}^i M(S_{\mathrm{sep}}) = \mathbf{Z}\big((i(i-1)/2\big)\big[i(i-1)\big] \oplus \text{terms with higher shifts},$$

we must have

$$\mathsf{Alt}^i M(S) = M(S)\big((i(i-1)/2\big)\big[i(i-1)\big] \oplus \text{terms with higher shifts}. \tag{6.6}$$

7. Split case

We are going to prove the main theorem in the split case. Let $G = \mathbf{SL}_l$ with prime l. We have

$$N = \mathbf{Z} \oplus \mathbf{Z}(1)[2] \oplus \mathbf{Z}(2)[4] \oplus \cdots \oplus \mathbf{Z}(l-2)[2l-4].$$

In fact, N is a direct summand of the motive of $S = \mathbb{P}^{l-1}$:

$$M(S) = N \oplus \mathbf{Z}(l-1)[2l-2].$$

The Chern classes $c_1(\bar{a}), c_2(\bar{a}), \ldots, c_{l-1}(\bar{a})$ with values in the motivic cohomology of G, where $\bar{a}$ is the generic matrix in $\mathbf{SL}_l$, define a morphism

$$\varphi_1 : M(G) \to N(2)[3].$$

For every $i = 0, 1, \ldots, l-1$, consider the composition

$$\varphi_i : M(G) \xrightarrow{\text{diag}} M(G^i) \xrightarrow{\varphi^i} N(2)[3]^{\otimes i} \to \mathsf{Sym}^i\big(N(2)[3]\big),$$

where the first morphism is given by the diagonal embedding.

Proposition 7.1. *In the split case $G = \mathbf{SL}_l$, the morphism*

$$\varphi = (\varphi_i) : M(G) \to \coprod_{i=0}^{l-1} \mathsf{Sym}^i\big(N(2)[3]\big)$$

is an isomorphism.

Proof. By Corollary 5.5, Example 6.3 and Lemma 2.3, for every $i = 0, 1, \ldots, l-1$, the morphism φ_i is zero on $M(G)^{(j)}$ for $j > i$ and yields an isomorphism

$$M(G)^{(i)} \xrightarrow{\sim} \mathsf{Sym}^i\big(N(2)[3]\big).$$

The result follows, as φ is given by an invertible triangular matrix. $\square$

8. Compactifications of G

Let D be a central division algebra over F and $G = \mathbf{SL}_1(D)$. By [2, §6.2] and [10, §6], G admits a smooth projective $G \times G$-equivariant compactification X. In other words, X is a projective variety equipped with an action of $G \times G$ and containing G as an open orbit on which the group $G \times G$ acts by the left-right translations. The motive of X is split (i.e., $M(X)$ is a direct sum of Tate motives) over any field extension that splits D by [10, Theorem 6.5].

By Proposition 5.10, the group $\mathrm{CH}^{l+1}(G)$ is cyclic of order l. Choose a generator $h \in \mathrm{CH}^{l+1}(G)$ and let $\bar{h} \in \mathrm{CH}^{l+1}(X)$ be any element such that $\bar{h}|_G = h$. Set

$$R = \mathbb{Z} \oplus \mathbb{Z}(l+1)[2l+2] \oplus \mathbb{Z}(2l+2)[4l+4] \oplus \cdots \oplus \mathbb{Z}(l^2-1)[2l^2-2]. \tag{8.1}$$

Proposition 8.2 ([10, §8]). *When $\mathbf{Z} = \mathbb{Z}_{(l)}$, the morphism $M(X) \to R$ defined by the powers of $\bar{h}$ has a right inverse. Moreover, we have $M(X) \simeq R \oplus T$ for a pure motive T that is a direct sum of pure shifts of $M(S)$, where $S = \mathrm{SB}(D)$.*

Note that the composition $\pi : M(G) \to M(X) \to R$ is given by the powers

$$h^i \in \mathrm{CH}^{i(p+1)}(G) = \mathrm{Hom}\big(M(G), \mathbb{Z}(i(p+1)[2i(p+1)]\big).$$

9. The morphism θ

In this section we construct a morphism $\theta : M(S)(2)[3] \to M(G)$.

As $M(G_{\mathrm{sep}})^{(1)} \simeq N_{\mathrm{sep}}(2)[3]$ by Example 5.6, there is a natural isomorphism

$$\mathrm{Hom}\big(M(S_{\mathrm{sep}})(2)[3], M(G_{\mathrm{sep}})^{(1)}\big) \simeq \mathbf{Z}^{l-1}.$$

Projecting $M(G_{\mathrm{sep}})$ onto $M(G_{\mathrm{sep}})^{(1)}$, we get a composition

$$\mathsf{type} : \mathrm{Hom}\big(M(S)(2)[3], M(G)\big) \to \mathrm{Hom}\big(M(S_{\mathrm{sep}})(2)[3], M(G_{\mathrm{sep}})\big) \to \mathbf{Z}^{l-1}.$$

By Corollary 5.7, there is a canonical morphism

$$\theta : M(S)(2)[3] \to M(G \times S) \to M(G),$$

where the first morphism is the projection to the $\mathbf{i}$-component for $\mathbf{i} = (1)$ and the second morphism is given by the projection $G \times S \to G$.

Note that by definition of θ, the composition of $M(S)(2)[3] \to M(G \times S)$ with $\widetilde{c}_k(\alpha) : M(G \times S) \to M(S)(k+1)[2k+1]$ is zero for $k > 1$. In view of (5.11) and Remark 5.4, the composition

$$M(S_{\mathrm{sep}})(2)[3] \xrightarrow{\theta_{\mathrm{sep}}} M(G_{\mathrm{sep}}) \xrightarrow{c_i} \mathbf{Z}(i+1)[2i+1]$$

for $i = 1, 2, \dots, l-1$ coincides with $i = \binom{i}{i-1}$ times the composition

$$M(S_{\mathrm{sep}})(2)[3] \to M(G_{\mathrm{sep}} \times S_{\mathrm{sep}}) \xrightarrow{c_1 s^{i-1}} \mathbf{Z}(i+1)[2i+1].$$

The latter is equal to the morphism $s^{i-1} : M(S_{\mathrm{sep}})(2)[3] \to \mathbf{Z}(i+1)[2i+1]$ that is the identity on the summand $\mathbf{Z}(i+1)[2i+1]$. It follows that

$$\mathsf{type}(\theta) = (1, 2, \dots, l-1).$$

10. A key lemma

Let D be a central simple algebra of degree l and $S = \mathrm{SB}(D)$.

Lemma 10.1. *Let Y be a variety over F such that D is split over the residue field $F(y)$ for every $y \in Y$. Then the push-forward homomorphism $\mathrm{CH}_j(Y \times S) \to \mathrm{CH}_j(Y)$ is surjective for every j.*

Proof. Let $y \in Y$ be a point of dimension j. As S is split over $F(y)$, there is a $F(y)$-rational point $y' \in Y \times S$ in the fiber of the projection $q : Y \times S \to Y$ over y. We have $[y] = q_*([y'])$. $\square$

Lemma 10.2. *Let U be a smooth variety such that D is split over $F(u)$ for every $u \in U$. Then $H^{2j+1,j}(M(U) \otimes N) = 0$ for every j.*

Proof. The last term in the exact sequence induced by the triangle (4.1)

$$H^{2j,j}(U \times S) \to H^{2k,k}(U) \to H^{2j+1,j}(M(U) \otimes N) \to H^{2j+1,j}(U \times S),$$

where $k = j - l + 1$, is zero as $2j + 1 > 2j$. The first map is surjective by Lemma 10.1. □

Let Y be a closed subvariety of a smooth variety X. We define the motive $M_Y(X)$ by the triangle

$$M(X \setminus Y) \to M(X) \to M_Y(X) \to M(X \setminus Y)[1]. \tag{10.3}$$

Lemma 10.4. *Let X be a smooth irreducible variety and $Y \subset X$ an equidimensional closed subvariety such that S is split over $F(y)$ for every $y \in Y$. Then $H^{2i+1,i}\big(M_Y(X) \otimes N\big) = 0$ for every i.*

Proof. We proceed by induction on $\dim(Y)$. Choose a closed subset $Z \subset Y$ of pure codimension 1 such that $Y \setminus Z$ is smooth. In the exact triangle

$$M_{Y \setminus Z}(X \setminus Z) \to M_Y(X) \to M_Z(X) \to M_{Y \setminus Z}(X \setminus Z)[1]$$

the first term is isomorphic to $M(Y \setminus Z)(c)[2c]$ for $c = \operatorname{codim}_X(Y)$ since $Y \setminus Z$ is smooth by [19, Proposition 3.5.4]. In the exact sequence

$$H^{2i+1,i}(M_Z(X) \otimes N) \to H^{2i+1,i}(M_Y(X) \otimes N) \to H^{2i+1,i}(M_{Y \setminus Z}(X \setminus Z) \otimes N)$$

the last tern is isomorphic to $H^{2i-2c+1,i-c}\big(M(Y \setminus Z) \otimes N\big)$ which is zero by Lemma 10.2, and the first term is zero by induction. □

Let X be a smooth $G \times G$-equivariant compactification of $G = \mathbf{SL}_1(D)$ (see Section 8). Set $Y := X \setminus G$. By [10, Lemma 7.1], D is split by $F(y)$ for every $y \in Y$. Applying Lemma 10.4 to the exact cohomology sequence for the exact triangle (10.3) we get the following corollary.

Corollary 10.5. *The natural homomorphism*

$$\mathrm{CH}^i(M(X) \otimes N) \to \mathrm{CH}^i(M(G) \otimes N)$$

is surjective for every i. □

Lemma 10.6. *The natural homomorphism*

$$\mathrm{CH}^{l+1}(R \otimes N) \to \mathrm{CH}^{l+1}(M(G) \otimes N)$$

induced by $\pi : M(G) \to R$ is surjective.

Proof. The group $\mathrm{CH}^{l+1}(M(G) \otimes N)$ is l-torsion as it is split over a splitting field. Therefore, we may assume that $\mathbf{Z} = \mathbb{Z}_{(l)}$. Recall (see Section 8) that $M(X) = R \oplus T$, where R is defined in (8.1) and the pure motive T is a direct sum of shifts of $M(S)$. Moreover, π is the composition $M(G) \to M(X) \to R$.

Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{CH}^{l+1}(R\otimes N) \arrow[r, "\alpha"] \arrow[dr] & \mathrm{CH}^{l+1}(M(G)\otimes N) & \mathrm{CH}^{2l}(M(G)\otimes N\otimes S) \arrow[l] \\
 & \mathrm{CH}^{l+1}(M(X)\otimes N) \arrow[u, two heads, "\beta"] & \mathrm{CH}^{2l}(M(X)\otimes N\otimes S) \arrow[l] \arrow[u] \\
 & \mathrm{CH}^{l+1}(T\otimes N) \arrow[u] & \mathrm{CH}^{2l}(T\otimes N\otimes S), \arrow[l, two heads, "\gamma"'] \arrow[u]
\end{tikzcd}$$

where γ (and the two homomorphisms above γ) are induced by the morphism $\mathbf{Z}(l-1)[2l-2] \to M(S)$. By Corollary 10.5, β is surjective. The pure motive T is a direct sum of shifts of $M(S)$, so is $T \otimes N$ by Lemma 4.4. It follows from Lemma 10.1 that γ is surjective.

Again by Lemma 4.4,

$$\mathrm{CH}^{2l}(M(G)\otimes N\otimes S) = \coprod_{i=0}^{l-2} \mathrm{CH}^{2l-i}(G\times S) = 0.$$

as $\mathrm{CH}^j(G \times S) = 0$ for $j \geq l$ in view of Corollary 5.7. Recall that $M(X) = R \oplus T$. By diagram chase, α is surjective. □

Consider the following key diagram:

$$\begin{tikzcd}
\mathrm{Hom}\big(M(G), M(S)(2)[3]\big) \arrow[r, "\xi"] \arrow[d, "\wr"] & \mathrm{Hom}\big(M(G), N(2)[3]\big) \arrow[r] \arrow[d, "\rho"] & \mathrm{CH}^{l+1}(G) \arrow[r] \arrow[d, "\sigma"] & 0 \\
H^{2l+1,l+1}(G\times S) \arrow[r, "\tau"] & H^{3,2}(G) \arrow[r] & \mathrm{CH}^{l+1}(M(G)\otimes N) \arrow[r] & 0.
\end{tikzcd}$$

The rows of the diagram are induced by the exact triangle (4.1). The left vertical homomorphism is an isomorphism by (2.2). The rows are exact since $\mathrm{CH}^{l+1}(G \times S) \simeq \mathrm{CH}^{l+1}(S) = 0$ by Corollary 5.7. The morphisms ρ and σ are induced by the morphism $\nu : N \to \mathbf{Z}$ (see Section 4).

The diagram is commutative. Indeed, both compositions in the right square take a morphism $\varphi : M(G) \to N(2)[3]$ to $(\varepsilon(2)[3] \circ \varphi) \otimes \nu$. (The morphisms ε and ν are defined in Section 4.) Both compositions in the left square take a morphism $\psi : M(G) \to M(S)(2)[3]$ to $\nu(2)[3] \circ \psi$.

Now we can prove the following key lemma.

Lemma 10.7. *The homomorphism induced by the morphism $\nu : N \to \mathbb{Z}$*

$$\sigma : \mathrm{CH}^{l+1}(G) \to \mathrm{CH}^{l+1}(M(G)\otimes N)$$

is an isomorphism.

Proof. In the commutative diagram

$$\begin{tikzcd} \mathrm{CH}^{l+1}(R) \arrow[r] \arrow[d] & \mathrm{CH}^{l+1}(R \otimes N) \arrow[d] \\ \mathrm{CH}^{l+1}(G) \arrow[r] & \mathrm{CH}^{l+1}(M(G) \otimes N) \end{tikzcd}$$

the right vertical map (induced by π) is surjective by Lemma 10.6. We have $\mathrm{CH}^{l+1}(R) = \mathbf{Z}$ and by Lemma 4.5, $\mathrm{CH}^{l+1}(R \otimes N) = \mathrm{CH}^0(N) = \mathbf{Z}$, hence the top map is an isomorphism. It follows that the bottom map in the diagram is surjective. If D is split, the group $\mathrm{CH}^{l+1}(G)$ is trivial an we are done.

Suppose D is a division algebra. Since $H^{3,2}(G) = \mathbf{Z}\, c_1$ (see Section 5), by Proposition 5.9, the image of τ in the key diagram is equal to $l\,\mathbf{Z}\, c_1$. It follows that $\mathrm{CH}^{l+1}(M(G) \otimes N)$ is a cyclic group of order l. The group $\mathrm{CH}^{l+1}(G)$ is also cyclic of order l by Proposition 5.10. The statement follows from the surjectivity of σ. □

It follows from Corollary 5.5 and Example 5.6 that $\mathrm{Hom}\big(M(G_{\mathrm{sep}}), N_{\mathrm{sep}}(2)[3]\big)$ is naturally isomorphic to $\mathbf{Z}^{l-1}$. Consider the map

$$\mathit{type} : \mathrm{Hom}\big(M(G), N(2)[3]\big) \to \mathrm{Hom}\big(M(G_{\mathrm{sep}}), N_{\mathrm{sep}}(2)[3]\big) \simeq \mathbf{Z}^{l-1}\,.$$

Proposition 10.8. *Let D be a division algebra of degree l and $S = \mathrm{SB}(D)$. Then the homomorphism*

$$\mathit{type} : \mathrm{Hom}\big(M(G), N(2)[3]\big) \to \mathbf{Z}^{l-1}$$

is injective. Its image consists of all tuples $(a_1, a_2, \dots, a_{l-1})$ such that

$$a_1 \equiv 2a_2 \equiv \cdots \equiv (l-1)a_{l-1} \pmod{l}.$$

Proof. Let $\beta \in \mathrm{Hom}\big(M(G), N(2)[3]\big)$ have zero type. We have $\rho(\beta) = kc_1$ with k the first component of the type of β. Hence $k = 0$. It follows from Lemma 10.7 that the image of β in $\mathrm{CH}^{l+1}(G)$ is trivial, t.e., $\beta = \xi(\gamma)$ for some $\gamma \in \mathrm{Hom}\big(M(G), M(S)(2)[3]\big)$ with $\mathit{type}(\gamma) = 0$. By Proposition 5.9, $\gamma = 0$. This proves the injectivity of *type*.

Take any $\beta \in \mathrm{Hom}\big(M(G), N(2)[3]\big)$. We have

$$\mathit{type}(\beta) = (a_1, a_2, a_3, \dots, a_{l-1})$$

for some $a_i \in \mathbb{Z}$. Composing β with $\theta : M(S)(2)[3] \to M(G)$ (see Section 9) we get a morphism $M(S)(2)[3] \to N(2)[3]$ of type $\big(a_1, 2a_2, 3a_3, \dots (l-1)a_{l-1}\big)$. By Proposition 4.3, we have $a_1 \equiv 2a_2 \equiv \cdots \equiv (l-1)a_{l-1}$ modulo l.

By Proposition 5.9, the image of the map *type* contains $l\,\mathbf{Z}^{l-1}$. To finish the proof it suffices to find a β such that $\mathit{type}(\beta)$ is not divisible by l. By Lemma 10.7 and diagram chase, the map ρ is surjective. Hence there is a morphism $\beta : M(G) \to N(2)[3]$ such that the composition of β with $N(2)[3] \to \mathbb{Z}(2)[3]$ coincides with c_1, i.e., $\mathit{type}(\beta) = (1, \dots)$. □

Remark 10.9. If $\alpha \in \operatorname{Hom}\big(M(G), N(2)[3]\big)$ is such that $\mathit{type}(\alpha)$ is not divisible by l, then $\rho(\alpha)$ is not divisible by l in $H^{3,2}(G) = \mathbf{Z}\, c_1$, and hence by Lemma 10.7, the image of α in $\mathrm{CH}^{l+1}(G)$ is not zero if D is not split.

11. Main theorem

Now the coefficient ring is $\mathbf{Z} = \mathbb{Z}\big[\frac{1}{(l-1)!}\big]$. By Proposition 10.8, there is a unique morphism $\beta_1 : M(G) \to N(2)[3]$ with

$$\mathit{type}(\beta_1) = \big(1^{-1}, 2^{-1}, \dots, (l-1)^{-1}\big).$$

For every $i = 0, 1, \dots, l-1$ we have a composition

$$\beta_i : M(G) \xrightarrow{diag} M(G^i) \xrightarrow{\beta^i} N(2)[3]^{\otimes i} \to \mathit{Sym}^i\big(N(2)[3]\big).$$

Theorem 11.1. *Let D be a central simple algebra of prime degree l over a perfect field F. Then the morphism*

$$\beta = (\beta_i) : M\big(\mathbf{SL}_1(D)\big) \to \coprod_{i=0}^{l-1} \mathit{Sym}^i\big(N(2)[3]\big) = \coprod_{i=0}^{l-1} \big(\mathit{Alt}^i N\big)(2i)[3i]$$

in the category $\mathit{DM}(F)$ of motives over F with coefficients in $\mathbb{Z}\big[\frac{1}{(l-1)!}\big]$ is an isomorphism.

Proof. We first prove the theorem in the split case. The morphisms β : $M(G) \to N(2)[3]$ and $\varphi : M(G) \to N(2)[3]$ of type $(1, 1, \dots, 1)$ defined in Section 7 differ by an automorphism of $N(2)[3]$ of type $(1, 2, \dots, l-1)$. Therefore, the statement follows from Proposition 7.1.

Assume that D is a division algebra. We show next that $1_{M(S)} \otimes \beta$ is an isomorphism. By Corollary 5.7, the motive $M(S) \otimes M(G) = M(S \times G)$ is a direct sum of shifts of $M(S)$. The motive $M(S) \otimes N$ is a direct sum of shifts of $M(S)$ by Lemma 4.4, hence so is $M(S) \otimes (\mathit{Alt}^i N)$. By the first part of the proof, β is an isomorphisms over a splitting field, hence so is $1_{M(S)} \otimes \beta$. By Lemma 3.4, $1_{M(S)} \otimes \beta$ is an isomorphism. It follows that

$$1_{M(S^i)} \otimes \beta \quad \text{is an isomorphism for every} \quad i > 0. \tag{11.2}$$

We embed the category $\mathit{DM}(F)$ into a larger triangulated category $\mathit{DM}_-^{eff}(F)$ of *motivic complexes* with coefficients in $\mathbf{Z}$ as a full subcategory (see [19]).

Let $\check{C}(S)$ be the motive in $\mathit{DM}_-^{eff}(F)$ associated with the simplicial scheme given by the powers of S (see [20, Appendix B]). Using the exact triangle in the proof of [20, Proposition 8.1] we see from (11.2) that $1_{\check{C}(S)} \otimes \beta$ is an isomorphism.

It follows from Remark 10.9 that the composition

$$M(G) \xrightarrow{\beta_1} N(2)[3] \xrightarrow{\varepsilon(2)[3]} \mathbf{Z}(l+1)[2l+2]$$

represents a nontrivial element $h \in \mathrm{CH}^{l+1}(G)$. Therefore, for every $i = 0, 1, \dots, l-1$, the composition

$$M(G) \xrightarrow{\beta_i} \mathit{Sym}^i\big(N(2)[3]\big) \xrightarrow{\delta_i} \mathit{Sym}^i\big(\mathbf{Z}(l+1)[2l+2]\big) = \mathbf{Z}\big(i(l+1)\big)[2i(l+1)],$$

where $\delta_i = \mathsf{Sym}^i\big(\varepsilon(2)[3]\big)$, is equal to h^i. By Section 8, we have a commutative diagram

$$\begin{tikzcd} M(G) \arrow[r, "\beta"] \arrow[d, "\alpha"] & \coprod_{i=0}^{l-1} \mathsf{Sym}^i\big(N(2)[3]\big) \arrow[d, "\delta"] \\ M(X) \arrow[r, "\gamma"] & R, \end{tikzcd}$$

where X is a smooth compactification of G and $\delta = \coprod \delta_i$.

Consider the motive $\widetilde{C}(S)$ in $\mathsf{DM}_{-}^{eff}(F)$ defined by the exact triangle

$$\widetilde{C}(S) \to \check{C}(S) \to \mathbf{Z} \to \widetilde{C}(S)[1]. \tag{11.3}$$

We also have an exact triangle

$$M(G) \to M(X) \to M_Y(X) \to M(G)[1],$$

where $Y = X \setminus G$. The algebra D is split by the residue field $F(y)$ for every $y \in Y$ by [10, Lemma 7.1]. Hence, by [17, Lemma 3.4], $\widetilde{C}(S) \otimes M_Y(X) = 0$. Therefore, $1_{\widetilde{C}(S)} \otimes \alpha$ is an isomorphism.

By Proposition 8.2, $M(X) \simeq R \oplus T$, where T is a direct sum of shifts of $M(S)$ if $\mathbf{Z} = \mathbb{Z}_{(l)}$. Since $\widetilde{C}(S) \otimes T = 0$, we have $1_{\widetilde{C}(S)} \otimes \gamma$ is an isomorphism when $\mathbf{Z} = \mathbb{Z}_{(l)}$. As $\widetilde{C}(S)$ vanishes over a splitting field of D of degree l, $1_{\widetilde{C}(S)} \otimes \gamma$ is an isomorphism when $\mathbf{Z} = \mathbb{Z}\big[\frac{1}{(l-1)!}\big]$.

By Proposition 6.4 applied to the exact triangle (4.1), there is a Postnikov tower connecting $\mathsf{Sym}^i\big(N(2)[3]\big)$ and

$$\mathsf{Sym}^i\big(\mathbf{Z}(l+1)[2l+2]\big) = \mathbf{Z}\big(i(l+1)\big)[2i(l+1)]$$

with "factors" divisible by $M(S)$. Since $\widetilde{C}(S) \otimes M(S) = 0$, the morphism

$$1_{\widetilde{C}(S)} \otimes \mathsf{Sym}^i\varepsilon : \widetilde{C}(S) \otimes \mathsf{Sym}^i\big(N(2)[3]\big) \xrightarrow{\sim} \widetilde{C}(S)\big(i(l+1)\big)[2i(l+1)]$$

Is an isomorphism. Therefore, $1_{\widetilde{C}(S)} \otimes \delta$ is an isomorphism.

It follows from the commutativity of the diagram that $1_{\widetilde{C}(S)} \otimes \beta$ is an isomorphism. Finally, by 5-lemma applied to the exact triangle (11.3), the morphism β is an isomorphism. □

12. Applications

As an application of Theorem 11.1, we compute certain motivic cohomology of G. The Chow groups $\mathrm{CH}^i(G) = H^{2i,i}(G)$ are given in Proposition 5.10. In Theorem 12.4 below we compute the groups $H^{2i+1,i+1}(G)$.

The following Lemma is an immediate application of the exact triangle in Corollary 6.5.

Lemma 12.1. *If $p > 2q$, then*

$$H^{p,q}\big(\mathsf{Alt}^{i-1}N\big) \simeq H^{p+2l-1,q+l-1}\big(\mathsf{Alt}^{i}N\big). \qquad \square$$

We compute the Chow groups of N.

Lemma 12.2. *We have*

$$\mathrm{CH}^i(N) = \begin{cases} \mathbf{Z}, & \text{if } i = 0; \\ l\,\mathbf{Z}, & \text{if } i = 1, 2, \dots, l-2; \\ F^\times / \operatorname{Nrd}(D^\times), & \text{if } i = l; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We may assume that $\mathbf{Z} = \mathbb{Z}$. Using (4.1) we get $\mathrm{CH}^i(N) \simeq \mathrm{CH}^i(S)$ for $i \le l-2$ and apply Lemma 3.1. In the exact sequence

$$0 \to \mathrm{CH}^{l-1}(N) \to \mathrm{CH}^{l-1}(S) \to \mathrm{CH}^0(\mathbf{Z})$$

the last map is injective again by Lemma 3.1, hence $\mathrm{CH}^{l-1}(N) = 0$. In the exact sequence

$$H^{2l-1,l}(S) \to H^{1,1}(F) \to \mathrm{CH}^l(N) \to 0$$

the first map is isomorphic to $A^{l-1}(S, K_l) \to K_1^M(F) = F^\times$ and its image is equal to $\operatorname{Nrd}(D^\times)$ since the image is generated by the norms from finite field extensions that split D. By Lemma 4.5, $\mathrm{CH}^i(N) = 0$ if $i > l$. $\square$

Lemma 12.3. *We have*

$$H^{2i+1,i}(\mathsf{Alt}^2 N) = \begin{cases} \mathbf{Z}/l\,\mathbf{Z}, & \text{if } i = l-1; \\ F^\times / \operatorname{Nrd}(D^\times), & \text{if } i = 2l-1; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Using the triangle in Corollary 6.5, we get an exact sequence

$$\mathrm{CH}^i\big(\mathsf{Alt}^2 M(S)\big) \to \mathrm{CH}^{i-l+1}(N) \to H^{2i+1,i}(\mathsf{Alt}^2 N) \to 0.$$

The middle group is trivial if $i < l-1$, $l = 2l-2$ and $l > 2l-1$ by Lemma 12.2. The first map in the sequence is surjective in the split case since $\mathsf{Alt}^2 N$ is pure and $2i+1 > 2i$. As $\mathrm{CH}^{i-l+1}(N) = l\,\mathbf{Z}$ for $i = l, l+1, \dots, 2l-3$ by Lemma 12.2, the first map is also surjective in general for these values of i. If $i = 2l-1$, the first group is trivial as $\mathsf{Alt}^2 M(S)$ is a direct summand of $M(S \times S)$ and $\dim(S \times S) = 2l-2$.

Finally consider the case $i = l-1$. We may assume that $\mathbf{Z} = \mathbb{Z}_{(l)}$. As

$$\mathsf{Alt}^2 M(S) = M(S)(1)[2] \oplus M(S)(3)[6] \oplus \cdots \oplus M(S)(l-2)[2l-4],$$

we have

$$\mathrm{CH}^{l-1}\big(\mathsf{Alt}^2 M(S)\big) = \mathrm{CH}^1(S) \oplus \mathrm{CH}^3(S) \oplus \cdots \oplus \mathrm{CH}^{l-2}(S).$$

This is divisible by l when going to the split case by Lemma 3.1. Whence the case $i = l-1$. $\square$

Lemmas 12.1, 12.2 and 12.3 then yield

Theorem 12.4. *Let D be a central division algebra of degree l over F. Then*

$$H^{2i+1,i+1}\big(\mathbf{SL}_1(D)\big) = \begin{cases} F^\times, & \text{if } i = 0;\\ \mathbb{Z}c_1, & \text{if } i = 1;\\ l\mathbb{Z}c_i, & \text{if } i = 2, 3, \dots, l-1;\\ \mathbb{Z}/l\mathbb{Z}, & \text{if } i = k(l+1)+1 \text{ for } k = 1, \dots, l-2;\\ F^\times/\operatorname{Nrd}(D^\times), & \text{if } i = k(l+1) \text{ for } k = 1, \dots, l-1;\\ 0, & \text{otherwise.} \end{cases} \qquad \square$$

Let $G = \mathbf{SL}_1(D)$. Note that the cup-product maps

$$F^\times \otimes \mathbb{Z}/l\mathbb{Z} = H^{1,1}(F) \otimes \operatorname{CH}^{k(l+1)}(G) \to H^{2k(l+1)+1,k(l+1)+1}(G) = F^\times/\operatorname{Nrd}(D^\times)$$

are natural surjections for $k = 1, \dots, l-1$.

Consider the motivic spectral sequence for G when D is not split (see [7]):

$$E_2^{p,q} = H^{p-q,-q}(G) \Rightarrow K_{-p-q}(G).$$

The K-groups of G were computed in [18, Theorem 6.1]. In particular, $K_0(G) = \mathbb{Z}$ and $K_1(G) = K_1(F) \oplus K_0(D) \oplus K_0(D^{op}) \simeq F^\times \oplus 3\mathbb{Z} \oplus 3\mathbb{Z}$. It follows that the zero-diagonal limit terms $E_\infty^{p,-p}$ are trivial if $p \neq 0$. On the other hand, by Proposition 5.10, we have

$$E_2^{p,-p} = \operatorname{CH}^p(G) = \begin{cases} \mathbb{Z}, & \text{if } p = 0;\\ \mathbb{Z}/l\mathbb{Z}, & \text{if } p = i(l+1) \text{ for } i = 1, \dots, l-1;\\ 0, & \text{otherwise.} \end{cases}$$

Note that in the split case all the differentials coming to the zero diagonal are trivial. It follows that in the general case such differential are l-torsion. By [14, Theorem 3.4], nontrivial differentials coming to the zero diagonal can appear only on pages $E_s^{*,*}$ with $l-1$ dividing $s-1$. It follows that the nonzero differentials appear only on page $E_l^{*,*}$ and they are

$$d : E_l^{1+k(l+1),-2-k(l+1)} \to E_l^{(k+1)(l+1),-(k+1)(l+1)} = \mathbb{Z}/l\mathbb{Z} \tag{12.5}$$

for $k = 0, 1, \dots, l-2$. These maps are all surjective and are isomorphisms for $k > 0$, thus "clearing" the zero diagonal and partially the first diagonal. The other differentials on the l-th page coming to the first diagonal are the cup-products with $F^\times$ of the differentials (12.5). Nontrivial $E_\infty^{*,*}$-terms on the first diagonal are $F^\times$ and $l\mathbb{Z}$ ($l-1$ times).

Below is a fragment of the third page of the spectral sequence when $l = 3$.

$$
\begin{tikzcd}
\mathbb{Z} & & & & & & & & \\
F^{\times} & 0 & & & & & & & \\
* & \mathbb{Z} \arrow[ddrrr] & 0 & & & & & & \\
& * \arrow[ddrrr] & 3\mathbb{Z} & 0 & & & & & \\
& & * & 0 & \mathbb{Z}/3\mathbb{Z} & & & & \\
& & & * & F^{\times}/N & 0 & & & \\
& & & & * & \mathbb{Z}/3\mathbb{Z} \arrow[ddrrr] & 0 & & \\
& & & & & * \arrow[ddrrr] & 0 & 0 & \\
& & & & & & * & 0 & \mathbb{Z}/3\mathbb{Z} \\
& & & & & & & * & F^{\times}/N
\end{tikzcd}
$$

It follows from Theorem 12.4 that the Chern classes $c_2, c_3, \ldots, c_{l-1}$ (which are defined in the split case) are not defined over F if D is not split. (Recall that c_1 is always defined over F.) We will show that the product $c_1 c_2 \cdots c_k$ is defined over F for all $k = 1, 2, \ldots, l-1$.

Lemma 12.6. *For every $i = 1, 2, \ldots, l-1$, if $q < i(i-1)/2$, the group $H^{p,q}(\mathsf{Alt}^i N)$ is trivial for every p.*

Proof. Induction on k. We may assume that $\mathbf{Z} = \mathbb{Z}_{(l)}$. The basic triangle (6.5) yields an exact sequence

$$H^{p-2l+1,q-l+1}(\mathsf{Alt}^{i-1} N) \to H^{p,q}(\mathsf{Alt}^i N) \to H^{p,q}\big(\mathsf{Alt}^i M(S)\big).$$

The first term is trivial by induction as $q - l + 1 < (i-1)(i-2)/2$. The last group is zero by (6.6). $\square$

Theorem 12.7. *Let $G = \mathbf{SL}_1(D)$ for a central simple algebra D of prime degree l. Then the product of Chern classes $c_1 c_2 \cdots c_k$ is defined over F for all $k = 1, 2, \ldots, l-1$.*

Proof. We may assume that $\mathbf{Z} = \mathbb{Z}_{(l)}$. The product $c_1c_2\cdots c_k$ belongs to $H^{(k+1)^2-1,(k+1)(k+2)/2-1}(G)$. Consider the following direct summand of this group (see Theorem 11.1):

$$H^{(k+1)^2-1,(k+1)(k+2)/2-1}\big((\mathsf{Alt}^k N)(2k)[3k]\big) = \mathrm{CH}^{(k^2-k)/2}\big(\mathsf{Alt}^k N\big).$$

The basic triangle (6.5) yields an exact sequence

$$H^{k^2-k-2l+1,(k^2-k)/2-l+1}\big(\mathsf{Alt}^{k-1}N\big) \to \mathrm{CH}^{(k^2-k)/2}\big(\mathsf{Alt}^k N\big) \to \\ \mathrm{CH}^{(k^2-k)/2}\big(\mathsf{Alt}^k M(S)\big) \to \mathrm{CH}^{(k^2-k)/2-l+1}\big(\mathsf{Alt}^{k-1}N\big).$$

The side terms are trivial by Lemma 12.6. The third term is isomorphic to $H^{0,0}(S) = \mathbf{Z}$ by (6.6). Therefore, the group $\mathrm{CH}^{(k^2-k)/2}\big(\mathsf{Alt}^k N\big)$ contains an element representing $c_1c_2\cdots c_k$ over a splitting field. □

References

[1] S. Biglari, *Motives of reductive groups*, Amer. J. Math. **134** (2012), no. 1, 235–257.

[2] M. Brion and S. Kumar, *Frobenius splitting methods in geometry and representation theory*, Progress in Mathematics, vol. 231, Birkhäuser Boston Inc., Boston, MA, 2005.

[3] V. Chernousov and A. Merkurjev, *Motivic decomposition of projective homogeneous varieties and the Krull-Schmidt theorem*, Transform. Groups **11** (2006), no. 3, 371–386.

[4] R. Elman, N. Karpenko, and A. Merkurjev, *The algebraic and geometric theory of quadratic forms*, American Mathematical Society, Providence, RI, 2008.

[5] R. Garibaldi, A. Merkurjev, and Serre J.-P., *Cohomological invariants in galois cohomology*, American Mathematical Society, Providence, RI, 2003.

[6] Ph. Gille and T. Szamuely, *Central simple algebras and Galois cohomology*, Cambridge Studies in Advanced Mathematics, vol. 101, Cambridge University Press, Cambridge, 2006.

[7] D. Grayson, *The motivic spectral sequence*, Handbook of K-theory. Vol. 1, Springer, Berlin, 2005, pp. 39–69.

[8] V. Guletskiĭ, *Finite-dimensional objects in distinguished triangles*, J. Number Theory **119** (2006), no. 1, 99–127.

[9] A. Huber and B. Kahn, *The slice filtration and mixed Tate motives*, Compos. Math. **142** (2006), no. 4, 907–936.

[10] N. Karpenko and A. Merkurjev, *Motivic decomposition of compactifications of certain group varieties*, arXiv:1402.5520 (2014).

[11] N. Karpenko, *Upper motives of algebraic groups and incompressibility of Severi-Brauer varieties*, J. Reine Angew. Math. **677** (2013), 179–198.

[12] C. Mazza, V. Voevodsky, and C. Weibel, *Lecture notes on motivic cohomology*, Clay Mathematics Monographs, vol. 2, American Mathematical Society, Providence, RI, 2006.

[13] A. Merkurjev and A. Suslin, *K-cohomology of Severi-Brauer varieties and the norm residue homomorphism*, Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), no. 5, 1011–1046, 1135–1136.

[14] A. Merkurjev, *Adams operations and the Brown-Gersten-Quillen spectral sequence*, Quadratic forms, linear algebraic groups, and cohomology, Dev. Math., vol. 18, Springer, New York, 2010, pp. 305–313.

[15] O. Pushin, *Higher Chern classes and Steenrod operations in motivic cohomology*, K-Theory **31** (2004), no. 4, 307–321.

[16] M. Rost, *Chow groups with coefficients*, Doc. Math. **1** (1996), No. 16, 319–393 (electronic).

[17] E. Shinder, *On the motive of the group of units of a division algebra*, arXiv:1202.3090 (2014).
[18] A. Suslin, *K-theory and K-cohomology of certain group varieties*, Algebraic K-theory, Adv. Soviet Math., vol. 4, Amer. Math. Soc., Providence, RI, 1991, pp. 53–74.
[19] V. Voevodsky, *Triangulated categories of motives over a field*, Cycles, transfers, and motivic homology theories, Ann. of Math. Stud., vol. 143, Princeton Univ. Press, Princeton, NJ, 2000, pp. 188–238.
[20] V. Voevodsky, *Motivic cohomology with $\mathbf{Z}/2$-coefficients*, Publ. Math. Inst. Hautes Études Sci. (2003), no. 98, 59–104.
[21] V. Voevodsky, *Cancellation theorem*, Doc. Math. (2010), no. Extra volume: Andrei A. Suslin sixtieth birthday, 671–685.

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