WITT KERNELS OF QUADRATIC FORMS FOR MULTIQUADRATIC EXTENSIONS IN CHARACTERISTIC 2

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Abstract. Let F be a field of characteristic 2 and let K/F be a purely inseparable extension of exponent 1. We show that the extension is excellent for quadratic forms. Using the excellence we recover and extend results by Aravire and Laghribi who computed generators for the kernel $W_q(K/F)$ of the natural restriction map $W_q(F) \to W_q(K)$ between the Witt groups of quadratic forms of F and K, respectively, where K/F is a finite multiquadratic extension of separability degree at most 2.

1. Introduction

Throughout this article, we will only consider fields of characteristic 2. Let K be a finite multiquadratic extension of a field F of separability degree at most 2, in other words, $K = F(\sqrt{a_1}, \ldots, \sqrt{a_n})$, or $K = F(\sqrt{a_1}, \ldots, \sqrt{a_n}, \wp^{-1}(b))$, $a_i, b \in F^*$, where $\wp^{-1}(b)$ is a root of $X^2 + X + b$. In [1], Aravire and Laghribi computed the kernel $W_q(K/F)$ of the natural map (induced by scalar extension) $W_q(F) \to W_q(K)$ between the Witt groups of nonsingular quadratic forms over F and K, respectively. They show that

$$W_q(K/F) = \sum_{i=1}^{n} \langle 1, a_i \rangle_b \otimes W_q(F) \ ,$$

in the purely inseparable case, and

$$W_q(K/F) = W(F) \otimes [1, b] + \sum_{i=1}^{n} \langle 1, a_i \rangle_b \otimes W_q(F)$$

in the case of separability degree 2, where $\langle 1, a_i \rangle_b$ is understood to be a binary bilinear form in its diagonal notation, $[a, b]$ represents the nonsingular quadratic form $ax^2 + xy + by^2$, and $W_q(F)$ is considered as a module over the Witt ring of nonsingular symmetric bilinear forms $W(F)$. The proof of Aravire and Laghribi uses differential forms. Actually, they prove more, namely they determine the kernel of the restriction map in Kato's cohomology $H_2^n(F) \to H_2^n(K)$ and then deduce the result on the Witt kernels by using Kato's theorem [7] that $H_2^*(F)$ is naturally isomorphic to the graded Witt module of quadratic forms over F. For more details, we refer the reader to Aravire and Laghribi's article [1] and the references there.

The purpose of the present paper is to give a new more elementary proof of these results on the Witt kernels. Our approach is completely different. On the

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one hand, we show less since we don't get the results on the kernels for the graded Witt modules. On the other hand, we show more. Recall that an extension K/F is called excellent (for quadratic forms) if for any quadratic form q over F, the anisotropic part of q over K is defined over F, i.e. there exists an anisotropic form φ over F with $(q_K)_{\mathrm{an}} \cong \varphi_K$. We will show that the extension K/F is excellent if it is purely inseparable of exponent 1, i.e. if $K^2 \subseteq F$. Using this, we can determine the Witt kernel for the compositum of such an exponent 1 extension with the function field of a quadratic or bilinear Pfister form. Note that the case $K = F(\sqrt{a_1}, \cdots, \sqrt{a_n}, \wp^{-1}(b))$ can be interpreted as the compositum of a finite exponent 1 extension with the function field of a 1-fold quadratic Pfister form, thus it also becomes a special case of our result.

Our main result is the following:

Main Theorem. *Let K/F be a field extension such that $K^2 \subset F$.*

(i) *The extension K/F is excellent for quadratic forms.*

(ii)

$$W_q(K/F) = \sum_{t \in K^{*2}} \langle 1, t \rangle_b \otimes W_q(F) \ .$$

In the next section, we introduce some basic terminology on bilinear and quadratic forms in characteristic 2. In the third section, we prove the main theorem, and in the fourth section, we provide some remarks concerning our main result and we generalize and extend it.

2. Terminology and definitions

For all undefined terminology on quadratic and bilinear forms, in particular in characteristic 2, we refer to [2] and [6]. We assume throughout that F is a field of characteristic 2. Bilinear forms are always assumed to be symmetric, and underlying vector spaces of bilinear and quadratic forms are always finite-dimensional.

Let (B, V) be a bilinear form over an F-vector space V. The radical is defined to be $\mathrm{rad}(B) = \{x \in V \mid B(x, V) = 0\}$, and B is said to be nonsingular if $\mathrm{rad}(B) = 0$. We define the value sets $D_F(B) = \{B(x, x) \mid x \in V \setminus \{0\}\}$, $D_F^0(B) = D_F(B) \cup \{0\}$ and $D_F^*(B) = D_F(B) \cap F^*$. B is called isotropic if $D_F(B) = D_F^0(B)$ and anisotropic if $D_F(B) = D_F^*(B)$. One has the usual notions of isometry $\cong$, orthogonal sum $\perp$ and tensor product $\otimes$ of bilinear forms. In the sequel, we always assume bilinear forms to be nonsingular.

A 2-dimensional isotropic bilinear form is called a metabolic plane, in which case one can always find a basis such that the Gram matrix with respect to that basis is of the shape

$$\begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix}$$

for some $a \in F$. If $a = 0$, this is called a hyperbolic plane. A bilinear form B is said to be metabolic (resp. hyperbolic) if it is the orthogonal sum of metabolic (resp. hyperbolic) planes. It is not difficult to see that a form B is hyperbolic iff $D_F(B) = \{0\}$.

If the Gram matrix of a form B with respect to a certain basis is a diagonal matrix with entries a_i, $1 \le i \le n = \dim(B)$, then we write $B \cong \langle a_1, \cdots, a_n \rangle_b$. A diagonalization exists iff B is not hyperbolic.

A bilinear form B can be decomposed as $B \cong B_{\text{an}} \perp B_m$ with B_{an} anisotropic and B_m metabolic. B_{an} is uniquely determined up to isometry, but generally not B_m.

We call two bilinear forms B and B' Witt equivalent if $B_{\text{an}} \cong B'_{\text{an}}$. The equivalence classes together with addition induced by $\perp$ and multiplication induced by $\otimes$ define the Witt ring of F denoted by $W(F)$.

An n-fold bilinear Pfister form (or bilinear n-Pfister for short) is a form of type $\langle\langle a_1, \ldots, a_n\rangle\rangle_b := \langle 1, a_1\rangle_b \otimes \ldots \otimes \langle 1, a_n\rangle_b$ for some $a_i \in F^*$. A bilinear Pfister form π is round, i.e., $\pi \cong x\pi$ for all $x \in D_F(\pi)^*$, and π is isotropic iff π is metabolic.

Now let (q, V) be a quadratic form over an F-vector space V, with associated bilinear form $B_q(x, y) = q(x + y) + q(x) + q(y)$. One defines the radical $\text{rad}(q) = \text{rad}(B_q)$ and calls q nonsingular if $\text{rad}(q) = 0$ and totally singular if $\text{rad}(q) = V$. Isometry $\cong$ and orthogonal sum $\perp$ are defined in the usual way, and we define $D_F(q) = \{q(x) \mid x \in V \setminus \{0\}\}$, $D^0_F(q) = D_F(q) \cup \{0\}$ and $D^*_F(q) = D_F(q) \cap F^*$, and we call q isotropic resp. anisotropic if $D_F(q) = D^0_F(q)$ resp. $D_F(q) = D^*_F(q)$. A nonsingular quadratic form can always be written with respect to a suitable basis as $ax^2 + xy + by^2$ with $a, b \in F$, and we denote this form by $[a, b]$. Totally singular forms are exactly the diagonal forms $a_1x_1^2 + \ldots + a_nx_n^2$, for which we write $\langle a_1, \ldots, a_n\rangle$. A hyperbolic plane $\mathbb{H}$ is a 2-dimensional nonsingular isotropic quadratic form and one has $\mathbb{H} \cong [0, 0]$, and a quadratic form is called hyperbolic if it is an orthogonal sum of hyperbolic planes. The form $[a, b]$ is hyperbolic iff $ab \in \wp(F) = \{c^2 + c \mid c \in F\}$.

Each quadratic form q has a decomposition $q \cong q_r \perp q_s$ with q_r nonsingular and q_s totally singular. In fact, q_s is nothing but the restriction of q to $\text{rad}(q)$ and it is thus uniquely determined, whereas q_r is generally not uniquely determined up to isometry, but $\dim(q_r)$ is uniquely determined. This decomposition can be refined as follows:

$$q \cong \underbrace{\mathbb{H} \perp \ldots \perp \mathbb{H}}_{k \text{ times}} \perp q_0 \perp \langle a_1, \ldots, a_m\rangle \perp \langle\underbrace{0, \ldots, 0}_{\ell \text{ times}}\rangle$$

such that q_0 is nonsingular and $q_{\text{an}} = q_0 \perp \langle a_1, \ldots, a_m\rangle$ is anisotropic. In this decomposition, k is uniquely determined and called the Witt index $i_W(q)$ of q, ℓ is uniquely determined and called the defect $i_d(q)$ of q, q_{an} is uniquely determined up to isometry and called the anisotropic part of q, and $(k \times \mathbb{H}) \perp q_{\text{an}}$ is also uniquely determined up to isometry and called the nondefective part of q.

If (q, V) and (q', V') are quadratic forms defined on F-vector spaces V and V', respectively, then we say that q' dominates q, $q \prec q'$, it there exists an injective F-linear map $t : V \to V'$ with $q'(tx) = q(x)$ for all $x \in V$.

We call two quadratic forms q, q' Witt equivalent, $q \sim q'$, if $q_{\text{an}} \cong q'_{\text{an}}$. The classes of nonsingular quadratic forms together with addition induced by the orthogonal sum form the Witt group $W_q(F)$ of quadratic forms. $W_q(F)$ has a natural structure as $W(F)$-module, essentially given by scaling $\langle a\rangle_b \otimes q = aq$.

An n-fold quadratic Pfister form (or quadratic n-Pfister for short) is a form of type $\langle\langle a_1, \ldots, a_n]] := \langle\langle a_1, \ldots, a_{n-1}\rangle\rangle_b \otimes [1, a_n]$ for some $a_1, \ldots, a_{n-1} \in F^*$, $a_n \in F$. A quadratic Pfister form π is round, i.e., $\pi \cong x\pi$ for all $x \in D_F(\pi)^*$, and π is isotropic iff π is hyperbolic.

If φ is a (quadratic or bilinear) form over F and if K/F is a field extension, then we write $\varphi_K := \varphi \otimes K$ for the form obtained by scalar extension. This induces a natural homomorphism $W(F) \to W(K)$ resp. $W_q(F) \to W_q(K)$ whose kernel will be denoted by $W(K/F)$ resp. $W_q(K/F)$. In analogy to the definition of excellence of field extensions in the theory of quadratic forms in characteristic $\neq 2$ as defined in [3], we say that K/F is excellent for quadratic resp. bilinear forms if for any quadratic resp. bilinear form φ over F there exists a form ψ over F with $(\varphi_K)_{\mathrm{an}} \cong \psi_K$, in other words, the anisotropic part of φ over K is defined over F.

3. Witt kernels and excellence for extensions of exponent one

Let us now turn to the proof of the main theorem. Throughout this section, K/F will be a field extension with $K^2 \subset F$. We define

$$J_{K/F} = \sum_{t \in K^{*2}} \langle 1, t\rangle_b \otimes W_q(F) \ .$$

Remark 3.1. In [5], it was shown that a purely inseparable exponent 1 extension K/F is excellent for bilinear forms and that the bilinear Witt kernel $W(K/F)$ is generated by $\{\langle 1, t\rangle_b \,|\, t \in K^{*2}\}$. Hence, one readily has $J_{K/F} = W(K/F) \otimes W_q(F)$.

Lemma 3.2. *$J_{K/F} \subset W_q(K/F)$.*

Proof. If $t \in K^{*2}$, then clearly $(\langle 1, t\rangle_b)_K \cong (\langle 1, 1\rangle_b)_K$ is metabolic and hence, for any $q \in \langle 1, t\rangle_b \otimes W_q(F)$ we have $q_K = 0 \in W_q(K)$. □

Lemma 3.3. (Cf. [6, Lemma 2.2].) *Let $q \cong q_r \perp q_s$ be a quadratic form over F with q_r nonsingular and $q_s \cong \langle a_1, \ldots, a_n\rangle$ totally singular, not all $a_i = 0$, and let K/F be any field extension. Then there exist $1 \leq r \leq n$ and $1 \leq i_1 < i_2 < \ldots < i_t \leq n$ such that for $q'_s \cong \langle a_{i_1}, \ldots, a_{i_r}\rangle$, one has $((q_s)_K))_{\mathrm{an}} \cong (q'_s)_K$. and $(q_K)_{\mathrm{an}} \cong \varphi \perp (q'_s)_K$ for some nonsingular φ over K. In particular, every field extension is excellent for totally singular quadratic forms.*

Let us restate the main theorem in more detail.

Theorem 3.4. *Let $q \cong q_r \perp q_s$ be an anisotropic quadratic form over F with q_r nonsingular and q_s totally singular. Then there exists a nonsingular form q'_r over F with $\dim q'_r \leq \dim q_r$, a totally singular form $q'_s \prec q_s$ over F, and a form $\psi \in J_{K/F}$ such that*

(i) *$(q_K)_{\mathrm{an}} \cong (q'_r \perp q'_s)_K$, and*
(ii) *$q \perp \psi \sim q'_r \perp q_s$.*

In particular, K/F is excellent and $W_q(K/F) = J_{K/F}$.

Proof. First note that the result on the Witt kernel follows by considering the case $\dim q_s = 0$ and $q_K \sim 0$ in which case $\dim q'_r = 0$ as well and thus $q \cong \psi$.

To prove (i) and (ii), write $q \cong q_r \perp q_s$ with q_r nonsingular of dimension $2m$, and q_s totally singular of dimension ℓ, and let $n = \dim q = 2m + \ell$.

If q_K is anisotropic, there is nothing to show. Also, one readily sees that by Lemma 3.3, we may assume that if $\dim q_s > 0$, then $(q_s)_K$ is anisotropic.

So let us assume that q_K is isotropic, i.e. $\dim(q_K)_{\mathrm{an}} < n$, and $(q_s)_K$ is anisotropic. In particular, $m \geq 1$. If $m = 1$ and $\ell = 0$, then $q \cong a[1, b]$ for some $b \notin \wp(F)$ and some $a \in F^*$, and q_K being isotropic means that $a[1, b]_K \cong \mathbb{H}$ and hence $b \in \wp(K)$ and thus $F(\wp^{-1}(b)) \subset K$, a contradiction to K/F being purely inseparable.

Hence, we may assume $m \geq 1$ and $n = 2m + \ell \geq 3$.

To prove the theorem, it suffices to construct a form $\beta \in J_{K/F}$ such that for $\widetilde{q} = (q \perp \beta)_{\mathrm{an}}$ one has $\dim \widetilde{q} < \dim q$. Then we know that $q_K \sim \widetilde{q}_K$ and we conclude by a simple induction on m.

Let us write

$$q \cong \underbrace{[a_1, b_1] \perp \ldots \perp [a_m, b_m]}_{q_r} \perp \underbrace{\langle c_1, \ldots, c_\ell \rangle}_{q_s} .$$

Since q_K is isotropic, there exist $x_i, y_i, z_j \in K$, $1 \le i \le m$, $1 \le j \le \ell$, with

$$0 = \sum_{i=1}^{m} (a_i x_i^2 + x_i y_i + b_i y_i^2) + \sum_{j=1}^{\ell} c_j z_j^2 \tag{3.1}$$

and not all x_i, y_i, z_j equal to 0.

Suppose there exists $i \in \{1, \ldots, m\}$ with $x_i = y_i = 0$, or $j \in \{1, \ldots, \ell\}$ with $z_j = 0$. Then write $q \cong \widehat{q} \perp [a_i, b_i]$ resp. $q \cong \widehat{q} \perp \langle c_i \rangle$. Thus, $\dim \widehat{q} < \dim q$ and $\widehat{q}_K$ is isotropic. We then may proceed by working with $\widehat{q}$ rather than q.

So we may assume that for each $1 \le i \le m$ we have $x_i \ne 0$ or $y_i \ne 0$, and that for each $1 \le j \le \ell$ we have $z_j \ne 0$. Without loss of generality, there exists $1 \le k \le m$ with $x_i \ne 0$ for $1 \le i \le k$, and $x_{k+1} = \ldots = x_m = 0$. In particular, $y_i \ne 0$ for $k+1 \le i \le m$.

Note that $K^2 \subseteq F$, so $x_i^2, y_i^2, z_j^2 \in F$, so, if $x_i \ne 0$, $\pi_i \cong \langle 1, x_i^2 \rangle_b \otimes [a_i, b_i] \in J_{K/F}$ and $[a_i, b_i] \perp \pi_i \sim [a_i x_i^2, \frac{b_i}{x_i^2}]$. Put

$$q_1 := \left[a_1 x_1^2, \frac{b_1}{x_1^2}\right] \perp \ldots \perp \left[a_k x_k^2, \frac{b_k}{x_k^2}\right] \perp [a_{k+1}, b_{k+1}] \perp \ldots \perp [a_m, b_m] \perp q_s .$$

Then $q_1 \sim q \perp \pi_1 \perp \ldots \perp \pi_k$, $\dim q = \dim q_1$ and $q_K \sim (q_1)_K$. Since we are interested in the anisotropic part of q over K and since q_1 differs from q by a form in $J_{K/F} \subseteq W_q(K/F)$, and since

$$a_i x_i^2 + x_i y_i + b_i y_i^2 = (a_i x_i^2) \cdot 1^2 + 1 \cdot (x_i y_i) + \frac{b_i}{x_i^2} (x_i y_i)^2 ,$$

we may assume furthermore, by replacing q by q_1, that in Eq. 3.1 and by reassigning the letters a_i, b_i, we have $x_1 = \ldots = x_k = 1$, so together with the other assumptions

$$0 = \sum_{i=1}^{k} (a_i + y_i + b_i y_i^2) + \sum_{i=k+1}^{m} b_i y_i^2 + \sum_{j=1}^{\ell} c_j z_j^2 .$$

Now using repeatedly $[t, u] \perp [v, w] \cong [t + v, u] \perp [v, u + w]$, we get

$$[a_1, b_1] \perp \ldots \perp [a_k, b_k] \cong [a_1 + \ldots + a_k, b_1] \perp [a_2, b_1 + b_2] \perp \ldots \perp [a_k, b_1 + b_k] . \tag{3.2}$$

Note that furthermore

$$\begin{aligned} 0 \;=\; & (a_1 + \ldots + a_k) + (y_1 + \ldots + y_k) + b_1 (y_1 + \ldots + y_k)^2 \\ & + (b_1 + b_2) y_2^2 + \ldots + (b_1 + b_k) y_k^2 + \textstyle\sum_{i=k+1}^{m} b_i y_i^2 + \sum_{j=1}^{\ell} c_j z_j^2 , \end{aligned} \tag{3.3}$$

so in view of Eqs. 3.2 and 3.3, we may assume $k = 1$, i.e. in Eq. 3.1, we have $x_1 = 1$, $x_2 = \ldots = x_m = 0$. We also still may assume that $y_i \ne 0$ for $2 \le i \le m$ or else we could again omit those terms $[a_i, b_i]$ with $i \ge 2$ and $y_i = 0$. By adding forms in $J_{K/F}$ similarly as above, we may furthermore assume $y_2 = \ldots = y_m = 1$, so that Eq. 3.1 becomes

$$0 = a_1 + y_1 + b_1 y_1^2 + \sum_{i=1}^{k} b_i + \sum_{j=1}^{\ell} c_j z_j^2 . \tag{3.4}$$

Note that, again similarly as before,

(3.5)
$$[a_2,b_2]\perp\ldots\perp[a_m,b_m]\cong[a_2,b_2+\ldots+b_m]\perp[a_2+a_3,b_3]\perp\ldots\perp[a_2+a_m,b_m]$$

Eqs. 3.4 and 3.5 together imply that in Eq. 3.1, we may assume $m\leq 2$, $x_1=1$ and $y_2=1$ (if $m=2$), still with $2m+\ell\geq 3$.

Suppose $\ell=0$. Then $m=2$ and we are in the case $q\cong[a_1,b_1]\perp[a_2,b_2]$ and there exists $y\in K$ with $a_1+y+b_1y^2+b_2=0$. But $y^2\in F$, therefore also $y\in F$ and we have that q is also isotropic over F, a contradiction.

Suppose now $\ell>0$ and $m=2$. So $q\cong[a_1,b_1]\perp[a_2,b_2]\perp\langle c_1,\ldots c_\ell\rangle$ and Eq. 3.1 becomes with the above assumptions $0=a_1+y+b_1y^2+b_2+c_1z_1^2+\ldots+c_\ell z_\ell^2$ with $y,z_i\in K$. Recall that by assumption all $z_i\in K^*$, hence $z_i^2\in F^*$. Then put
$$\tau_i:=\langle 1,z_i^2\rangle_b\otimes[a_2z_i^2,c_i]\cong[a_2z_i^2,c_i]\perp[c_iz_i^2,a_2]\in J_{K/F}$$
and we have
$$[a_2,d]\perp\langle c_i\rangle\perp\tau_i\sim[a_2,d+c_iz_i^2]\perp\langle c_i\rangle$$
and hence, with $\tau\cong\tau_1\perp\ldots\perp\tau_\ell\in J_{K/F}$, we get
$$[a_2,b_2]\perp q_s\perp\tau\sim[a_2,b_2+c_1z_1^2+\ldots+c_\ell z_\ell^2]\perp q_s\ .$$
Put $b_2'=b_2+c_1z_1^2+\ldots+c_\ell z_\ell^2$. This shows that for $q_2:=[a_1,b_1]\perp[a_2,b_2']\perp q_s$ we have $q\perp\tau\sim q_2$, hence $q_K\sim(q_2)_K$ and (with the same y as before) $a_1+y+b_1y^2+b_2'=0$, so q_2 is isotropic, and $\widetilde{q}:=(q_2)_{\mathrm{an}}$ is the desired form with $\dim\widetilde{q}<\dim q_2=\dim q$.

Suppose finally that $\ell>0$ and $m=1$, so $q\cong[a_1,b_1]\perp\langle c_1,\ldots c_\ell\rangle$ and Eq. 3.1 becomes with the above assumptions $0=a_1+y+b_1y^2+c_1z_1^2+\ldots+c_\ell z_\ell^2$ with $y,z_i\in K$.

If $y=0$, put
$$\rho_i:=\langle 1,z_i^2\rangle_b\otimes[b_1z_i^2,c_i]\cong[b_1z_i^2,c_i]\perp[b_1,c_iz_i^2]\in J_{K/F}\ ,$$
and we have
$$[d,b_1]\perp\langle c_i\rangle\perp\rho_i\sim[d+c_iz_i^2,b_1]\perp\langle c_i\rangle\ ,$$
and with $\rho:=\rho_1\perp\ldots\perp\rho_\ell\in J_{K/F}$, we get similarly as before
$$q\perp\rho\sim[\underbrace{a_1+c_1z_1^2+\ldots+c_\ell z_\ell^2}_{0},b_1]\perp q_s\sim q_s\ .$$
and $\widetilde{q}:=q_s$ is the desired form.

If $y\neq 0$, put
$$\nu_i:=\big\langle 1,\frac{z_i^2}{y^2}\big\rangle_b\otimes\big[a_1\frac{z_i^2}{y^2},c_i\big]\cong\big[a\frac{z_i^2}{y^2},c_i\big]\perp\big[a_1,c_i\frac{z_i^2}{y^2}\big]\in J_{K/F}\ ,$$
and we have
$$[a_1,d]\perp\langle c_i\rangle\perp\nu_i\sim\big[a_1,d+c_i\frac{z_i^2}{y^2}\big]\perp\langle c_i\rangle\ ,$$
and with $\nu:=\nu_1\perp\ldots\perp\nu_\ell\in J_{K/F}$ and $b_1':=b_1+c_1\frac{z_1^2}{y^2}+\ldots+c_\ell\frac{z_\ell^2}{y^2}$ we get
$$q\perp\nu\sim[a_1,b_1']\perp q_s$$
and (with the same y as before) $a_1+y+b_1'y^2=0$, i.e. $[a_1,b_1']$ is isotropic and hence hyperbolic, and $q\perp\nu\sim q_s$, and again $\widetilde{q}:=q_s$ is the desired form. This completes the proof. □

4. Some corollaries and remarks

We are now interested in extensions that are obtained by composing purely inseparable exponent 1 extensions with function fields of Pfister forms, and in the Witt kernels of these new extensions.

If π is an n-fold quadratic Pfister form over F ($n \geq 1$), then let $F(\pi)$ be the function field of the projective quadric X_π given by the equation $\pi = 0$.

If $B = \langle\langle a_1, \ldots, a_n \rangle\rangle_b$, $a_i \in F^*$, is an n-fold bilinear Pfister form with associated totally singular quadratic form $q(x) = B(x,x)$, i.e. $q = \langle\langle a_1, \ldots, a_n \rangle\rangle$, then there exist $k \leq n$ and $1 \leq i_1 < \ldots < i_k \leq n$ with $q_{\mathrm{an}} \cong \langle\langle a_{i_1}, \ldots, a_{i_k} \rangle\rangle$ (see, e.g., [6, §8]). Put $B_0 = \langle\langle a_{i_1}, \ldots, a_{i_k} \rangle\rangle_b$. We define the function field $F(B) := F(q)$. Then $F(q)$ is a purely transcendental extension of $F(q_{\mathrm{an}}) = F(B_0)$ (see, e.g., [4, Remark 7.4(iii)]). Since anisotropic forms stay anisotropic over purely transcendental extensions, we have that $W_q(F(B)/F) = W_q(F(B_0)/F)$. Note that if $k = 0$, i.e. $q_{\mathrm{an}} \cong \langle 1 \rangle$ and thus $B_0 \cong \langle 1 \rangle_b$, we have $F(B_0) = F$ and hence $W_q(F(B)/F) = W_q(F(B_0)/F) = 0$.

We have the following results on Witt kernels for function fields of Pfister forms.

Proposition 4.1. *Let $n \geq 1$ and let π is an n-fold quadratic Pfister form over F, or let $B = \langle\langle a_1, \ldots, a_n \rangle\rangle_b$, $a_i \in F^*$, be an n-fold bilinear Pfister form with associated totally singular quadratic form $q(x) = B(x,x)$ such that $\dim q_{\mathrm{an}} \geq 2$, i.e. $[F^2(a_1, \ldots, a_n) : F^2] > 1$.*

(i) ([8, Th. 1.4(2)] or [2, Cor. 23.6].) *Let $\varphi \in W_q(F(\pi)/F)$ be anisotropic, then there exists a bilinear form β with $\varphi \cong \beta \otimes \pi$. In particular, $W_q(F(\pi)/F) = W(F) \otimes \pi$.*

(ii) ([8, Th. 1.4(3)].) *Let φ be an anisotropic form in $W_q(F(B)/F)$, then there exists a nonsingular quadratic form τ with $\varphi \cong B_0 \otimes \tau$. In particular, $W_q(F(B)/F) = B_0 \otimes W_q(F)$.*

Remark 4.2. Note that for $b \in F \setminus \wp(F)$, $\pi = [1, b]$ is an anisotropic 1-fold quadratic Pfister form and $F(\pi) = F(\wp^{-1}(b))$, and for $a \in F \setminus F^2$, $B = \langle 1, a \rangle_b$ is an anisotropic 1-fold bilinear Pfister form and $F(B) = F(\sqrt{a})$, so the kernels in Proposition 4.1 are nothing but the well known ones for separable resp. inseparable quadratic extensions.

Corollary 4.3. *Let $n \geq 1$ and let π be an n-fold quadratic Pfister form over F. Let K be a purely inseparable extension of F of exponent 1. Let $L = K(\pi)$. Then $W_q(L/F) = W_q(K/F) + WF \otimes \pi$.*

Proof. Clearly, $W_q(K/F) + WF \otimes \pi \subseteq W_q(L/F)$. Conversely, let φ be a form over F with $\varphi \in W_q(L/F)$. By the excellence property of K/F, there exists a form ψ over F with $(\varphi_K)_{\mathrm{an}} \cong \psi_K$. If $\dim \psi = 0$ then $\varphi \in W_q(K/F)$ and we are done. So suppose $\dim \psi > 0$. Since $0 = \varphi_L = \psi_L \in W_q(L)$, it follows by Proposition 4.1 that there exists a bilinear form β over L with $\psi_L \cong \beta \otimes \pi_L$. By the excellence of K/F, we may in fact assume that β is already defined over F, see [3, Prop. 2.11] (the argument there works also in characteristic 2), so we may put $\psi \cong \beta \otimes \pi$. But then $\varphi - \beta \otimes \pi \in W_q(K/F)$ and hence $\varphi \in W_q(K/F) + WF \otimes \pi$ as desired. □

Lemma 4.4. *Let K/F be a purely inseparable extension of exponent 1, let B be a bilinear Pfister form over F and let ρ be a nonsingular quadratic form over K such that $B_K \otimes \rho$ is defined over F, i.e. there exists a quadratic form φ over F with $\varphi_K \cong B_K \otimes \rho$. Then there exists a nonsingular quadratic form ρ_0 over F with $\varphi_K \cong B_K \otimes \rho \cong (B \otimes \rho_0)_K$.*

Proof. This is essentially [9, Th. 1], except that there K was assumed to be an inseparable quadratic extension. But the proof works in exactly the same way by using the fact that for any $b \in K$, one has that $[1, b]$ is defined over F since $[1, b] \cong [1, b^2]$ over K with $b^2 \in F$ since $K^2 \subset F$. □

Corollary 4.5. *Let K be a purely inseparable extension of F of exponent 1. Let $n \geq 1$ and let $B = \langle\langle a_1, \ldots, a_n \rangle\rangle_b$, $a_i \in F^*$, be an n-fold bilinear Pfister form over F with associated totally singular quadratic form $q = \langle\langle a_1, \ldots, a_n \rangle\rangle$, and let $k \leq n$ and $1 \leq i_1 < \ldots < i_k \leq n$ be such that $(q_K)_{\mathrm{an}} \cong \langle\langle a_{i_1}, \ldots, a_{i_k} \rangle\rangle_K$. Put $B_0 = \langle\langle a_{i_1}, \ldots, a_{i_k} \rangle\rangle_b$. Let $L = K(B)$. If $\dim B_0 = 1$ then $W_q(L/F) = W_q(K/F)$. If $\dim B_0 > 1$ then $W_q(L/F) = W_q(K/F) + B_0 \otimes W_q(F)$.*

Proof. By the remarks preceding Proposition 4.1, we have that $K(B)/K(B_0)$ is purely transcendental, hence $W_q(L/F) = W_q(K(B_0)/F)$. We are done if $\dim B_0 = 1$ as then $K(B_0) = K$. So let us assume $\dim B_0 > 1$. Clearly, $W_q(K/F) + B_0 \otimes W_q(F) \subseteq W_q(L/F)$. Conversely, let φ be a form over F with $\varphi \in W_q(L/F) = W_q(K(B_0)/F)$. By the excellence property of K/F, there exists a form ψ over F with $(\varphi_K)_{\mathrm{an}} \cong \psi_K$. If $\dim \psi = 0$ then $\varphi \in W_q(K/F)$ and we are done. So suppose $\dim \psi > 0$. Since $0 = \varphi_{K(B_0)} = \psi_{K(B_0)} \in W_q(K(B_0))$, it follows from Proposition 4.1 that there exists a nonsingular form τ over K with $\psi_K \cong (B_0)_K \otimes \tau$. By Lemma 4.4, we may assume that τ is already defined over F, hence we may put $\psi \cong B_0 \otimes \tau$. But then $\varphi - B_0 \otimes \tau \in W_q(K/F)$ and hence $\varphi \in W_q(K/F) + B_0 \otimes W_q(F)$ as desired. □

Theorem 3.4 essentially says that $W_q(K/F)$ is additively generated by forms $\langle 1, t \rangle_b \otimes [c, d]$ with $t \in K^{*2}$ and $c, d \in F$. We next show that we don't need all $t \in K^{*2}$ for this to be true. Recall that a subset $T \subset F$ is called 2-independent if for all finite subsets $\{t_1, \ldots, t_n\} \subseteq T$ with $n \in \mathbb{N}$ and $t_i \neq t_j$ for $i \neq j$, one has

$$[F(\sqrt{t_1}, \ldots, \sqrt{t_n}) : F] = [F^2(t_1, \ldots, t_n) : F^2] = 2^n \ .$$

A 2-independent set T with $F^2(T) = F$ is called a 2-basis of F.

If K/F is a purely inseparable exponent 1 extension, then clearly there exists a 2-independent set $T \subseteq F$ with $K = F(\sqrt{t} \,|\, t \in T)$.

Corollary 4.6. *Let K/F be a purely inseparable exponent 1 extension, and let $T \subseteq F$ be a 2-independent set such that $K = F(\sqrt{t} \,|\, t \in T)$. Then*

$$W_q(K/F) = \sum_{t \in T} \langle 1, t \rangle_b \otimes W_q(F) \ .$$

Proof. Obviously, $\sum_{t \in T} \langle 1, t \rangle_b \otimes W_q(F) \subseteq W_q(K/F)$ as $t \in T$ implies $t \in K^{*2}$.

For the reverse inclusion, let $q \in W_q(K/F)$ be anisotropic. It is clear that there exists already a finite subset $T' \subset T$ such that for $K' = F(\sqrt{t} \,|\, t \in T')$, we have $q \in W_q(K'/F)$. By invoking Proposition 4.1 and Remark 4.2 together with Corollary 4.5, an easy induction on the cardinality $|T'|$ shows that $W_q(K'/F) = \sum_{t \in T'} \langle 1, t \rangle_b \otimes W_q(F)$. Hence $q \in W_q(K'/F) \subseteq \sum_{t \in T} \langle 1, t \rangle_b \otimes W_q(F)$ as desired. □

Corollary 4.6 together with Corollary 4.3 (in the case $n = 1$, see Remark 4.2) immediately imply the result by Aravire and Laghribi [1] mentioned in the introduction.

Corollary 4.7. *Let* $K = F(\sqrt{a_1}, \dots, \sqrt{a_n})$*,* $a_i \in F^*$*, and* $L = K(\wp^{-1}(b))$ *for some* $b \in F$*. Then*

$$\begin{array}{rcl} W_q(K/F) & = & \sum_{i=1}^n \langle 1, a_i \rangle_b \otimes W_q(F) \\ W_q(L/F) & = & W(F) \otimes [1, b] + \sum_{i=1}^n \langle 1, a_i \rangle_b \otimes W_q(F) \end{array}$$

Remark 4.8. A different way of deriving the previous corollary for $W_q(K/F)$ from our Theorem 3.4 is as follows. It suffices to show that each form of type $\langle 1, t \rangle_b \otimes [c, d]$, $t \in K^{*2}$, $c, d \in F$, can be written in $W_q(F)$ as a sum of forms of type $\langle 1, a_i \rangle_b \otimes q_i$ for suitable nonsingular quadratic forms q_i over F.

Now $t \in F^2(a_1, \dots, a_n)$, so for each $I \subseteq \{1, \dots, n\}$ there exist $x_I \in F$ such that $t = \sum_{I \subset \{1,\dots,n\}} (\prod_{i \in I} a_i) x_I^2$. The desired result now follows by a straightforward induction on n using the following relations in $W_q(F)$, where we assume $w \in F$, $u, v, x \in F^*$ with $u + v \neq 0$ in the third relation:

- $\langle 1, 1 \rangle_b \otimes [1, w] = 0$;
- $\langle 1, ux^2 \rangle_b \otimes [1, w] = \langle 1, u \rangle_b \otimes [1, w]$;
- $\langle 1, u + v \rangle_b \otimes [1, w] = \langle 1, u \rangle_b \otimes [1, \frac{wu}{u+v}] + \langle 1, v \rangle_b \otimes [1, \frac{wv}{u+v}]$;
- $\langle 1, uv \rangle_b \otimes [1, w] = \langle 1, u \rangle_b \otimes [1, w] + \langle 1, v \rangle_b \otimes [u, \frac{w}{u}]$.

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