A NUMERICAL INVARIANT FOR LINEAR REPRESENTATIONS OF FINITE GROUPS

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Abstract. We study the notion of essential dimension for a linear representation of a finite group. In characteristic zero we relate it to the canonical dimension of certain products of Weil transfers of generalized Severi-Brauer varieties. We then proceed to compute the canonical dimension of a broad class of varieties of this type, extending earlier results of the first author. As a consequence, we prove analogues of classical theorems of R. Brauer and O. Schilling about the Schur index, where the Schur index of a representation is replaced by its essential dimension. In the last section we show that essential dimension of representations can behave in rather unexpected ways in the modular setting.

1. Introduction

Let K/k be a field extension, G be a finite group of exponent e, and $\rho\colon G \to \mathrm{GL}_n(K)$ be a non-modular representation of G whose character takes values in k. (Here "non-modular" means that $\mathrm{char}(k)$ does not divide $|G|$.) A theorem of Brauer says that if k contains a primitive eth root of unity ζ_e then ρ is defined over k, i.e., ρ is K-equivalent to a representation $\rho'\colon G \to \mathrm{GL}_n(k)$; see, e.g. [30, §12.3]. If $\zeta_e \notin k$, we would like to know "how far" ρ is from being defined over k. In the case, where ρ is absolutely irreducible, a classical answer to this question is given by the Schur index of ρ, which is the smallest degree of a finite field extension l/k such that ρ is defined over l. Some background material on the Schur index and further references can be found in Section 2.

In this paper we introduce and study another numerical invariant, the essential dimension $\mathrm{ed}(\rho)$, which measures "how far" ρ is from being defined over k in a different way. Here ρ is not assumed to be irreducible; for the definition of $\mathrm{ed}(\rho)$, see Section 6. In Section 8 we show that the maximal value of $\mathrm{ed}(\rho)$, as ρ ranges over representations with a fixed character $\chi\colon G \to k$, which we denote by $\mathrm{ed}(\chi)$, can be expressed as the canonical

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dimension of a certain product of Weil transfers of generalized Severi-Brauer varieties. We use this to show that $\mathrm{ed}(\rho) \leq |G|^2/4$ for any n, k, and K/k in Section 9 and to prove a variant of a classical theorem of Brauer in Section 10. In Section 11 we compute the canonical dimension of a broad class of Weil transfers of generalized Severi-Brauer varieties, extending earlier results of the first author from [18] and [20]. This leads to a formula for the essential p-dimension of an irreducible character in terms of its decomposition into absolutely irreducible components; see Corollary 12.3. As an application we prove a variant of a classical theorem of Schilling in Section 13. In the last section we show that $\mathrm{ed}(\rho)$ can be unexpectedly large in the non-modular setting.

2. Notation and representation-theoretic preliminaries

Throughout this paper G will denote a finite group of exponent e, k a field, $\overline{k}$ an algebraic closure of k, K and F field extensions of k, ζ_d a primitive dth root of unity, ρ a finite-dimensional representation of G, and χ a character of G. In this section we will assume that $\mathrm{char}(k)$ does not divide G.

2a. **Characters and character values.** A function $\chi\colon G \to k$ is said to be *a character* of G, if χ is the character of some representation $\rho\colon G \to \mathrm{GL}_n(K)$ for some field extension K/k.

If $\chi\colon G \to \overline{k}$ is a character, and F/k is a field, we set

$$F(\chi) := F(\chi(g)) \,|\, g \in G) \subset F(\zeta_e).$$

Since $F(\zeta_e)$ is an abelian extension of F, so is $F(\chi)$. Moreover, $F(\chi)$ is stable under automorphisms $F(\zeta_e)/F$, i.e., independent of the choice of the eth root of unity ζ_e in $\overline{F}$.

Lemma 2.1. *(a) Let $\chi, \chi'\colon G \to \overline{k}$ be characters and F/k be a field extension. Then*

(a) every automorphism $h \in \mathrm{Gal}(F(\chi)/F)$ leaves $k(\chi)$ invariant.

(b) If χ and χ' are conjugate over F then they are conjugate over k.

(c) Suppose k is algebraically closed in F. Then the converse to part (b) also holds. That is, if χ, χ' are conjugate over k then they are conjugate over F.

Proof. (a) It is enough to show that $h(\chi(g)) \subset F(\chi)$ for every $g \in G$. Since the sequence of Galois groups

$$1 \to \mathrm{Gal}(F(\zeta_e)/F(\chi)) \to \mathrm{Gal}(F(\zeta_e)/F) \to \mathrm{Gal}(F(\chi)/F) \to 1$$

is exact, h can be lifted to an element of $\mathrm{Gal}(F(\zeta_e)/F)$. By abuse of notation, we will continue to denote it by h. The eigenvalues of $\rho(g)$ are of the form $\zeta_e^{i_1}, \dots, \zeta_e^{i_n}$ for some integers $i_1, \dots, i_n$. The automorphism h sends ζ_e to another primitive eth root of unity ζ_e^j for some integer j. Then

$$h(\chi(g)) = h(\zeta_e^{i_1} + \dots + \zeta_e^{i_n}) = \zeta_e^{ji_1} + \dots + \zeta_e^{ji_n} = \chi(g^j) \in F(\chi)\,,$$

as desired.

(b) is an immediate consequence of (a).

(c) If k is algebraically closed in F, then the homomorphism

$$\mathrm{Gal}(F(\chi_1)/F) \to \mathrm{Gal}(k(\chi_1)/k)$$

given by $\sigma \mapsto \sigma_{|k(\chi)}$ is surjective; see [25, Theorem VI.1.12]. $\square$

2b. **The envelope of a representation.** If $\rho\colon G \to \mathrm{GL}_n(F)$ is a representation over some field F/k, we define the k-*envelope* $\mathrm{Env}_k(\rho)$ as the k-linear span of $\rho(G)$.

Lemma 2.2. *$\mathrm{Env}_k(s \cdot \rho)$ is k-isomorphic $\mathrm{Env}_k(\rho)$ for any integer $s \geqslant 1$.*

Proof. The diagonal embedding $\mathrm{M}_n(F) \hookrightarrow \mathrm{M}_n(F) \times \cdots \times \mathrm{M}_n(F)$ (s times) induces an isomorphism between $\mathrm{Env}_k(\rho)$ and $\mathrm{Env}_k(s \cdot \rho)$. $\square$

Lemma 2.3. *Assume the character χ of $\rho\colon G \to \mathrm{GL}_n(F)$ is k-valued. Then the natural homomorphism $\mathrm{Env}_k(\rho) \otimes_k F \to \mathrm{Env}_F(\rho)$ is an isomorphism of F-algebras.*

Proof. It suffices to show that if $\rho(g_1), \dots, \rho(g_r)$ are linearly dependent over F for some elements $g_1, \dots, g_r \in G$, then they are linearly dependent over k. Indeed, suppose

$$a_1\rho(g_1) + \cdots + a_r\rho(g_r) = 0$$

in $\mathrm{M}_n(F)$ for some $a_1, \dots, a_r \in F$, such that $a_i \neq 0$ for some i. Then

$$\mathrm{tr}((a_1\rho(g_1) + \cdots + a_r\rho(g_r)) \cdot \rho(g)) = 0$$

for every $g \in G$, which simplifies to

$$a_1\chi(g_1g) + \cdots + a_r\chi(g_rg) = 0.$$

The homogeneous linear system

$$x_1\chi(g_1g) + \cdots + x_r\chi(g_rg) = 0$$

in variables $x_1, \dots, x_r$ has coefficients in k and a non-trivial solution in F. Hence, it has a non-trivial solution $b_1, \dots, b_r$ in k, and we get that

$$\mathrm{tr}((b_1\rho(g_1) + \cdots + b_r\rho(g_r)) \cdot \rho(g)) = 0$$

for every $g \in G$.

Note that $\mathrm{Env}_k(\rho)$ is, by definition, a homomorphic image of the group ring $k[G]$. Hence, $\mathrm{Env}_k(\rho)$ is semisimple and consequently, the trace form in $\mathrm{Env}_k(\rho)$ is non-degenerate. It follows that the elements $\rho(g_1), \dots, \rho(g_r)$ are linearly dependent over k, as desired. $\square$

2c. **The Schur index.** Suppose K/k is a field extension, and $\rho_1\colon G \to \mathrm{GL}_n(K)$ is an absolutely irreducible representation with character $\chi_1\colon G \to K$. By taking $F = \overline{K}$ in Lemma 2.3, one easily deduces that $\mathrm{Env}_{k(\chi_1)}(\rho_1)$ is a central simple algebra of degree n over $k(\chi_1)$. The index of this algebra is called *the Schur index* of ρ_1. We will denote it by $m_k(\rho_1)$.

In the sequel we will need the following properties of the Schur index.

Theorem 2.4. *Let K be a field, G be a finite group such that $\mathrm{char}(K)$ does not divide $|G|$, and $\rho\colon G \to \mathrm{GL}_n(K)$ be an irreducible representation. Denote the character of ρ by χ.*

(a) Over the algebraic closure $\overline{K}$, ρ decomposes as

$$\rho_{\overline{K}} \simeq m(\rho_1 \oplus \cdots \oplus \rho_r), \tag{2.5}$$

where $\rho_1, \dots, \rho_r$ are pairwise non-isomorphic irreducible representations of G defined over $\overline{K}$, and m is their common Schur index $m_K(\rho_1) = \cdots = m_K(\rho_r)$.

(b) Let $\chi_i\colon G \to \overline{K}$ *be the character of* ρ_i *for* $i = 1, \ldots, r$*. Then* $K(\chi_1) = \cdots = K(\chi_r)$ *is an abelian extension of* K *of degree* r*. Moreover,* $\mathrm{Gal}(K(\chi_1)/K)$ *transitively permutes* $\chi_1, \ldots, \chi_r$*.*

(c) Conversely, every irreducible representation $\rho_1\colon G \to \mathrm{GL}_1(\overline{K})$ *occurs as an irreducible component of a unique* K*-irreducible representation* $\rho\colon G \to \mathrm{GL}_n(K)$*, as in* (2.5)*.*

(d) The center Z *of* $\mathrm{Env}_K(\rho)$ *is* K*-isomorphic to* $K(\chi_1) = K(\chi_2) = \cdots = K(\chi_r)$*.* $\mathrm{Env}_K(\rho)$ *is a central simple algebra over* Z *of index* m*.*

(e) The multiplicity of ρ_1 *in any representation of* G *defined over* K *is a multiple of* $m_K(\rho_1)$*. Consequently,* $m_K(\rho_1)$ *divides* $m_k(\rho_1)$ *for any field extension* K/k*.*

(f) m *divides* $\dim(\rho_1) = \cdots = \dim(\rho_r)$*.*

Proof. See [12, Theorem 74.5] for parts (a)-(d), and [11, Corollary 74.8] for parts (e) and (f). □

As a consequence of Theorem 2.4, we obtain the following

Corollary 2.6. *Let* K/k *be a field extension,* $\rho\colon G \to \mathrm{GL}_n(K)$ *be a representation, whose character takes values in* k*, and*

$$\rho = d_1\rho_1 \oplus \cdots \oplus d_r\rho_r$$

be the irreducible decomposition of ρ *over the algebraic closure* $\overline{K}$*. Then* ρ *can be realized over* k *(i.e.,* ρ *is* K*-equivalent to a representation* $\rho'\colon G \to \mathrm{GL}_n(k)$*) if and only if the Schur index* $m_k(\rho_i)$ *divides* d_i *for every* $i = 1, \ldots, r$*.* □

3. Preliminaries on essential and canonical dimension

3a. **Essential dimension.** Let $\mathcal{F} : \mathbf{Fields}_k \to \mathbf{Sets}$ be a covariant functor, where $\mathbf{Fields}_k$ is the category of field extensions of k and $\mathbf{Sets}$ is the category of sets. We think of the functor $\mathcal{F}$ as specifying the type of algebraic objects under consideration, $\mathcal{F}(K)$ as the set of algebraic objects of this type defined over K, and the morphism $\mathcal{F}(i)\colon \mathcal{F}(K) \to \mathcal{F}(L)$ associated to a field extension

$$k \subset K \overset{i}{\hookrightarrow} L \tag{3.1}$$

as "base change". For notational simplicity, we will write $\alpha_L \in \mathcal{F}(L)$ instead of $\mathcal{F}(i)(\alpha)$,

Given a field extension L/K, as in (3.1), an object $\alpha \in \mathcal{F}(L)$ is said to *descend* to K if it lies in the image of $\mathcal{F}(i)$. The *essential dimension* $\mathrm{ed}(\alpha)$ is defined as the minimal transcendence degree of K/k, where α descends to K. The essential dimension $\mathrm{ed}(\mathcal{F})$ of the functor $\mathcal{F}$ is the supremum of $\mathrm{ed}(\alpha)$ taken over all $\alpha \in \mathcal{F}(K)$ and all K.

Usually $\mathrm{ed}(\alpha) < \infty$ for every $\alpha \in \mathcal{F}(K)$ and every K/k; see [7, Remark 2.7]. On the other hand, $\mathrm{ed}(\mathcal{F}) = \infty$ in many cases of interest; for example, see Proposition 14.1 below.

The essential dimension $\mathrm{ed}_p(\alpha)$ of α at a prime integer p is defined as the minimal value of $\mathrm{ed}(\alpha_{L'})$, as L' ranges over all finite field extensions L'/L such that p does not divide the degree $[L' : L]$. The essential dimension $\mathrm{ed}_p(\mathcal{F})$ is then defined as the supremum of $\mathrm{ed}_p(\alpha)$ as K ranges over all field extensions of k and α ranges over $\mathcal{F}(K)$.

For generalities on essential dimension, see [2, 27, 28, 7].

3b. **Canonical dimension.** An interesting example of a covariant functor $\mathbf{Fields}_k \to \mathbf{Sets}$ is the "detection functor" $\mathcal{D}_X$ associated to an algebraic k-variety X. For a field extension K/k, we define

$$\mathcal{D}_X(K) := \begin{cases} \text{a one-element set, if } X \text{ has a } K\text{-point, and} \\ \emptyset, \text{ otherwise.} \end{cases}$$

If $k \subset K \overset{i}{\hookrightarrow} L$ then $0 \leqslant |\mathcal{D}_X(K)| \leqslant |\mathcal{D}_X(L)| \leqslant 1$. Thus there is a unique morphism of sets $\mathcal{D}_X(K) \to \mathcal{D}_X(L)$, which we define to be $\mathcal{D}_X(i)$.

The essential dimension (respectively, the essential p-dimension) of the functor $\mathcal{D}_X$ is called the *canonical dimension* of X (respectively, *the canonical p-dimension* of X) and is denoted by $\mathrm{cd}(X)$ (respectively, $\mathrm{cd}_p(X)$). If X is smooth and projective, then $\mathrm{cd}(X)$ (respectively, $\mathrm{cd}_p(X)$) equals the minimal dimension of the image of a rational self-map $X \dashrightarrow X$ (respectively, of a correspondence $X \rightsquigarrow X$ of degree prime to p). In particular,

$$0 \leqslant \mathrm{cd}_p(X) \leqslant \mathrm{cd}(X) \leqslant \dim(X) \tag{3.2}$$

for any prime p. If $\mathrm{cd}(X) = \dim(X)$, we say that X is incompressible. If $\mathrm{cd}_p(X) = \dim(X)$, we say that X is p-incompressible. For details on the notion of canonical dimension for algebraic varieties, we refer the reader to [27, §4].

We will say that smooth projective varieties X and Y defined over K are *equivalent* if there exist rational maps $X \dashrightarrow Y$ and $Y \dashrightarrow X$. Similarly, we will say that X and Y are p-equivalent for a prime integer p, if there exist correspondences $X \rightsquigarrow Y$ and $Y \rightsquigarrow X$ of degree prime to p.

Lemma 3.3. *(a) If X and Y are equivalent, then $\mathrm{cd}(X) = \mathrm{cd}(Y)$ and $\mathrm{cd}_p(X) = \mathrm{cd}_p(Y)$ for any p.*

(b) If X and Y are p-equivalent for some p, then $\mathrm{cd}_p(X) = \mathrm{cd}_p(Y)$.

Proof. (a) Let K/k be a field extension. By Nishimura's lemma, X has a K-point if and only if so does Y; see [29, Proposition A.6]. Thus the detection functors $\mathcal{D}_X$ and $\mathcal{D}_Y$ are isomorphic, and $\mathrm{cd}(X) = \mathrm{ed}(\mathcal{D}_X) = \mathrm{ed}(\mathcal{D}_Y) = \mathrm{cd}(Y)$. Similarly, $\mathrm{cd}_p(X) = \mathrm{cd}_p(Y)$.

For a proof of part (b) see [23, Lemma 3.6 and Remark 3.7]. □

4. Balanced algebras

Let Z/k be a Galois field extension, and A be a central simple algebra over Z. Given $\alpha \in \mathrm{Gal}(Z/k)$, we will denote the "conjugate" Z-algebra $A \otimes_Z Z$, where the tensor product is taken via $\alpha\colon Z \to Z$, by ${}^{\alpha}A$. We will say that A is *balanced over* k if ${}^{\alpha}A$ is Brauer-equivalent to a tensor power of A for every $\alpha \in \mathrm{Gal}(Z/k)$.

Note that A is balanced, if the Brauer class of A descends to k: ${}^{\alpha}A$ is then isomorphic to A for any α. In this section we will consider another family of balanced algebras.

Let K/k be a field extension, $\rho\colon G \to \mathrm{GL}_n(K)$ be an irreducible representation whose character χ is k-valued. Recall from Theorem 2.4 that $\mathrm{Env}_k(\rho)$ is a central simple algebra over $Z \simeq k(\chi_1) = \cdots = k(\chi_n)$.

Proposition 4.1. *$\mathrm{Env}_k(\rho)$ is balanced over k.*

Proof. Recall from [33, p. 14] that a cyclotomic algebra B/Z is a central simple algebra of the form
$$B = \bigoplus_{g\in \mathrm{Gal}(Z(\zeta)/Z)} Z(\zeta)u_g\,,$$
where ζ is a root of unity, $Z(\zeta)$ is a maximal subfield of B, and the basis elements u_g are subject to the relations
$$u_g x = g(x)u_g \text{ and } u_g u_h = \beta(g,h)u_{gh} \quad \text{for every } x \in Z(\zeta) \text{ and } g,h \in \mathrm{Gal}(Z(\zeta)/Z).$$
Here $\beta\colon G\times G \to Z(\zeta)^*$ is a 2-cocycle whose values are powers of ζ. Following the notational conventions in [33], we will write $B := (\beta, Z(\zeta)/Z)$.

By [33, Corollary 3.11], $\mathrm{Env}_k(\rho)$ is Brauer-equivalent to some cyclotomic algebra B/Z, as above. Thus it suffices to show that every cyclotomic algebra is balanced over k, i.e., ${}^{\alpha}B$ is Brauer-equivalent to a power of B over Z for every $\alpha \in \mathrm{Gal}(Z/k)$.

By Theorem 2.4(d), Z is k-isomorphic to $k(\chi_1)$, which is, by definition a subfield of $k(\zeta_e)$, where e is the exponent of G. Thus there is a root of unity ϵ such that
$$Z(\zeta) \subset k(\zeta,\zeta_e) = k(\epsilon)$$
and both ζ and ζ_e are powers of ϵ. Note that $k(\epsilon)/k$ is an abelian extension, and the sequence of Galois groups
$$1 \to \mathrm{Gal}(k(\epsilon)/Z) \to \mathrm{Gal}(k(\epsilon)/k) \to \mathrm{Gal}(Z/k) \to 1$$
is exact. In particular, every $\alpha \in \mathrm{Gal}(Z/k)$ can be lifted to an element of $\mathrm{Gal}(k(\epsilon)/k)$, which we will continue to denote by α. Then $\alpha(\epsilon) = \epsilon^t$ for some integer t. Since ζ is a power of ϵ, and each $\beta(g,h)$ is a power of ζ, we have
$$\alpha(\beta(g,h)) = \beta(g,h)^t \text{ for every } g,h \in \mathrm{Gal}(Z(\zeta)/k). \tag{4.2}$$
We claim that ${}^{\alpha}B$ is Brauer-equivalent to $B^{\otimes t}$ over Z. Indeed, since
$$B = (\beta, Z(\zeta)/Z),$$
we have ${}^{\alpha}B = (\alpha(\beta), Z(\zeta)/Z)$. By (4.2), ${}^{\alpha}B = (\alpha(\beta), Z(\zeta)/Z) = (\beta^t, Z(\zeta)/Z)$, and $(\beta^t, Z(\zeta)/Z)$ is Brauer-equivalent to $B^{\otimes t}$, as desired. □

5. Generalized Severi-Brauer varieties and Weil transfers

Suppose Z/k is a finite Galois field extension and A is a central simple algebra over Z. For $1 \leqslant m \leqslant \deg(A)$, we will denote by $\mathrm{SB}(A,m)$ the generalized Severi-Brauer variety (or equivalently, the twisted Grassmannian) of $(m-1)$-dimensional subspaces in $\mathrm{SB}(A)$. The Weil transfer $R_{Z/k}(\mathrm{SB}(A,m))$ is a smooth projective absolutely irreducible k-variety of dimension $[Z:k]\cdot m\cdot(\deg(A)-m)$. For generalities on the Weil transfer, see, e.g., [15].

Proposition 5.1. *Let Z, k and A be as above, $X := R_{Z/k}(\mathrm{SB}(A,m))$ for some $1 \leqslant m \leqslant \deg(A)$, and K/k be a field extension.*

(a) Write $K_Z := K\otimes_k Z$ as a direct product $K_1\times\cdots\times K_s$, where $K_1/Z,\dots,K_s/Z$ are field extensions. Then X has a K-point if and only if the index of the central simple algebra $A_{K_i} := A\otimes_Z K_i$ divides m for every $i = 1,\dots,s$.

(b) Assume that m divides $\operatorname{ind}(A)$, A is balanced and $K = k(X)$ is the function field of X. Then $K_Z = K \otimes_k Z$ is a field, and $A \otimes_k K \simeq A \otimes_Z K_Z$ is a central simple algebra over K_Z of index m.

Proof. First note that $A \otimes_k K \simeq A \otimes_Z K_Z$.

(a) By the definition of the Weil transfer, $X = R_{Z/k}(\mathrm{SB}(A, m))$ has a K-point if and only if $\mathrm{SB}(A, m)$ has a K_Z-point or equivalently, if and only if $\mathrm{SB}(A, m)$ has a K_i-point for every $i = 1, \dots, s$. On the other hand, by [4, Proposition 3], $\mathrm{SB}(A, m)$ has a K_i-point if and only if the index of A_{K_i} divides m.

(b) Since X is absolutely irreducible (see, e.g. [4, Lemma 3]), K_Z is Z-isomorphic to the function field of the Z-variety

$$X_Z := X \times_{\operatorname{Spec}(k)} \operatorname{Spec}(Z) = \prod_{\alpha \in \operatorname{Gal}(Z/k)} \mathrm{SB}({}^{\alpha}A, m)\,,$$

see [5, §2.8]. Set $F := Z(\mathrm{SB}(A, m))$. By [31, Corollary 1],

$$\operatorname{ind}(A \otimes_Z F) = m\,.$$

Since A is balanced, i.e., each algebra ${}^{\alpha}A$ is a power of A, $\operatorname{ind}({}^{\alpha}A \otimes_Z F)$ divides m for every $\alpha \in \operatorname{Gal}(Z/k)$. By [4, Proposition 3], each $\mathrm{SB}({}^{\alpha}A, m)_F$ is rational over F. Thus the natural projection of Z-varieties

$$X_Z = \prod_{\alpha \in \operatorname{Gal}(Z/k)} \mathrm{SB}({}^{\alpha}A, m) \to \mathrm{SB}(A, m)$$

induces a purely transcendental extension of function fields $F \hookrightarrow K_Z$. Consequently,

$$\operatorname{ind}(A \otimes_Z K_Z) = \operatorname{ind}(A \otimes_Z F) = m\,,$$

as claimed. □

6. The essential dimension of a representation

Let us now fix a finite group G and an arbitrary field k, and consider the covariant functor

$$\mathbf{Rep}_{G,k} : \mathbf{Fields}_k \to \mathbf{Sets}$$

defined by $\mathbf{Rep}_{G,k}(K) := \{K\text{-isomorphism classes of representations } G \to \mathrm{GL}_n(K)\}$ for every field K/k. Here $n \geqslant 1$ is allowed to vary.

The essential dimension $\operatorname{ed}(\rho)$ of a representation $\rho\colon G \to \mathrm{GL}_n(K)$ is defined by viewing ρ as an object in $\mathbf{Rep}_{G,k}(K)$, as in Section 3. That is, $\operatorname{ed}(\rho)$ is the smallest transcendence degree of an intermediate field $k \subset K_0 \subset K$ such that ρ is K-equivalent to a representation $\rho'\colon G \to \mathrm{GL}_n(K_0)$. To illustrate this notion, we include an example, where $\operatorname{ed}(\rho)$ is positive and two elementary lemmas.

Example 6.1. Let $\mathbb{H} = (-1, -1)$ be the algebra of Hamiltonian quaternions over $k = \mathbb{R}$, i.e., the 4-dimensional $\mathbb{R}$-algebra given by two generators i, j, subject to relations, $i^2 = j^2 = -1$ and $ij = -ji$. The multiplicative subgroup $G = \{\pm 1, \pm i, \pm j, \pm ij\}$ of $\mathbb{H}$ is the quaternion group of order 8. Let $K = \mathbb{R}(\mathrm{SB}(\mathbb{H}))$, where $\mathrm{SB}(\mathbb{H})$ denotes the Severi-Brauer variety of $\mathbb{H}$. The representation $\rho\colon G \hookrightarrow \mathbb{H} \hookrightarrow \mathbb{H} \otimes_{\mathbb{R}} K \simeq \mathrm{M}_2(K)$ is easily seen to be absolutely irreducible. We claim that $\operatorname{ed}(\rho) = 1$. Indeed, $\operatorname{trdeg}_{\mathbb{R}}(F) = 1$, for any intermediate extension $\mathbb{R} \subset F \subset K$, unless $F = \mathbb{R}$. On the other hand, ρ cannot descend to $\mathbb{R}$, because $\mathrm{Env}_{\mathbb{R}}(\rho) = \mathbb{H}$, and thus $m_{\mathbb{R}}(\rho) = \mathrm{ind}(\mathbb{H}) = 2$ by Theorem 2.4(d). □

Lemma 6.2. *Let G be a finite group, K/k be a field, $\rho_i\colon G \to \mathrm{GL}_{n_i}(K)$ be representations of G over K (for $i = 1, \dots, s$) and $\rho \simeq a_1\rho_1 \oplus \cdots \oplus a_s\rho_s$, where $a_1, \dots, a_s \geqslant 1$ are integers. Then $\mathrm{ed}(\rho) \leqslant \mathrm{ed}(\rho_1) + \cdots + \mathrm{ed}(\rho_s)$.*

Proof. Suppose ρ_i descends to an intermediate field $k \subset K_i \subset K$, where $\mathrm{trdeg}_k(K_i) = \mathrm{ed}(\rho_i)$. Let K_0 be the subfield of K generated by $K_1, \dots, K_s$. Then ρ descends to K_0 and $\mathrm{ed}(\rho) \leqslant \mathrm{trdeg}_k(K_0) \leqslant \mathrm{trdeg}_k(K_1) + \cdots + \mathrm{trdeg}_k(K_s) = \mathrm{ed}(\rho_1) + \cdots + \mathrm{ed}(\rho_s)$. □

Lemma 6.3. *Let $k \subset K$ be fields, G be a finite group, and $\rho\colon G \to \mathrm{GL}_n(K)$ be a representation. Let $k' := k(\chi) \subset K$, where χ is the character of ρ. Then the essential dimension of ρ is the same, whether we consider it as an object on $\mathbf{Rep}_{K,k}$ or $\mathbf{Rep}_{K,k'}$.*

Proof. If ρ descends to an intermediate field $k \subset F \subset K$, then F automatically contains k'. Moreover, $\mathrm{trdeg}_k(F) = \mathrm{trdeg}_{k'}(F)$. The rest is immediate from the definition. □

Lemma 6.4. *Assume that $\mathrm{char}(k)$ does not divide $|G|$ and the Schur index $m_k(\lambda)$ equals 1 for every absolutely irreducible representation $\lambda\colon G \to \mathrm{GL}_n(K)$, where K contains k. (In fact, it suffices to only consider $K = \overline{k}$.) Then $\mathrm{ed}(\rho) = 0$ for any representation $\rho\colon G \to \mathrm{GL}_n(L)$ over any field L/k. In other words, $\mathrm{ed}(\mathbf{Rep}_{G,k}) = 0$.*

Proof. Let χ be the character of ρ and $k' := k(\chi)$. By Theorem 2.4(e), $m_{k'}(\lambda) = 1$ for every absolutely irreducible representation $\lambda\colon G \to \mathrm{GL}_n(K)$ of G. By Lemma 6.3 we may replace k by $k' = k(\chi)$ and thus assume that χ is k-valued. Corollary 2.6 now tells us that ρ descends to k. □

Remark 6.5. The condition of Lemma 6.4 is always satisfied if $\mathrm{char}(k) > 0$; see [12, Theorem 74.9]. This tells us that for non-modular representations the notion of essential dimension is only of interest when $\mathrm{char}(k) = 0$. The situation is drastically different in the modular setting; see Section 14.

7. Irreducible characters

In view of Remark 6.5, we will now assume that $\mathrm{char}(k) = 0$. In this setting there is a tight connection between representations and characters.

Lemma 7.1. *Suppose F_1/k, F_2/k are field extensions, and*

$$\rho_1\colon G \to \mathrm{GL}_n(F_1), \quad \rho_2\colon G \to \mathrm{GL}_n(F_2)$$

are representations of a finite group G, with the same character $\chi\colon G \to k$. Then the k-algebras $\mathrm{Env}_k(\rho_1)$ and $\mathrm{Env}_k(\rho_2)$ are isomorphic.

Proof. Let F/k be a field containing both F_1 and F_2. Then ρ_1 and ρ_2 are equivalent over F, because they have the same character. Thus $\mathrm{Env}_k(\rho_1)$ and $\mathrm{Env}_k(\rho_2)$ are conjugate inside $\mathrm{M}_n(F)$. □

Given a representation $\rho\colon G \to \mathrm{GL}_n(F)$, with a k-valued character $\chi\colon G \to k$, Lemma 7.1 tells us that, up to isomorphism, the k-algebra $\mathrm{Env}_k(\rho)$ depends only on χ and not on the specific choice of F and ρ. Thus we may denote this algebra by $\mathrm{End}_k(\chi)$.

If ρ is absolutely irreducible (and the character χ is not necessarily k-valued), it is common to write $m_k(\chi)$ for the index of $\mathrm{End}_{k(\chi)}(\chi)$ instead of $m_k(\rho)$.

Let $\chi\colon G \to k$ be a character of G. Write

$$\chi = \sum_{i=1}^{r} m_i \chi_i \,, \tag{7.2}$$

where $\chi_1, \dots, \chi_r\colon G \to \overline{k}$ are absolutely irreducible and $m_1, \dots, m_r$ are non-negative integers. Since χ is k-valued, $m_i = m_j$ whenever χ_i and χ_j are conjugate over k.

Lemma 7.3. *Let $\chi = \sum_{i=1}^{r} m_i \chi_i\colon G \to k$ be a character of G, as in* (7.2). *Then the following are equivalent.*

(a) χ is the character of a K-irreducible representation $\rho\colon G \to \mathrm{GL}_n(K)$ for some field extension K/k.

(b) $\chi_1, \dots, \chi_r$ form a single $\mathrm{Gal}(k(\chi_1)/k)$-orbit and $m_1 = \cdots = m_r$ divides $m_k(\chi_1) = \cdots = m_k(\chi_r)$.

Proof. (a) $\Longrightarrow$ (b): By Theorem 2.4(a) and (b), $\chi = m(\chi_1 + \cdots + \chi_r)$, where $\chi_1, \dots, \chi_r$ are absolutely irreducible characters transitively permuted by $\mathrm{Gal}(K(\chi_1)/K)$, and $m = m_K(\chi_1) = \cdots = m_K(\chi_r)$. By Lemma 2.1(b), $\chi_1, \dots, \chi_r$ are also transitively permuted by $\mathrm{Gal}(k(\chi_1)/k)$. Moreover, by Theorem 2.4(e), m divides $m_k(\chi_1) = \cdots = m_k(\chi_r)$.

(b) $\Longrightarrow$ (a): Let K be the function field of the Weil transfer variety $R_{Z/k}(\mathrm{SB}(A, m))$, where A is the underlying division algebra and Z is the center of $\mathrm{Env}_k(\chi)$. Since the variety $R_{Z/k}(\mathrm{SB}(A, m))$ is absolutely irreducible, k is algebraically closed in K. Lemma 2.1(c) now tells us that $\chi_1, \dots, \chi_r$ are conjugate over K. By Theorem 2.4(c) there exists an irreducible K-representation ρ whose character is $m_K(\chi_1)(\chi_1 + \cdots + \chi_r)$. It remains to show that $m_K(\chi_1) = m$. Indeed,

$$m_K(\chi_1) = \mathrm{ind}(\mathrm{Env}_K(\chi)) = \mathrm{ind}(\mathrm{Env}_k(\chi) \otimes_k K) = m \,.$$

Here the first equality follows from Theorem 2.4(d), the second from Lemma 2.3, and the third from Proposition 5.1(b). □

We will say that a character $\chi\colon G \to k$ is *irreducible over* k if it satisfies the equivalent conditions of Lemma 7.3.

8. The essential dimension of a character

In this section we will assume that $\mathrm{char}(k) = 0$ and consider subfunctors

$$\mathbf{Rep}_\chi : \mathbf{Fields}_k \to \mathbf{Sets}$$

of $\mathbf{Rep}_{G,k}$ given by

$$\mathbf{Rep}_\chi(K) := \{K\text{-isomorphism classes of representations } \rho\colon G \to \mathrm{GL}_n(K) \text{ with character } \chi\}$$

for every field K/k. Here $\chi\colon G \to k$ is a fixed character and $n = \chi(1_G)$. The assumption that χ takes values in k is natural in view of Lemma 6.3, and the assumption that $\mathrm{char}(k) = 0$ in view of Remark 6.5. Since any two K-representations with the same character are equivalent, $\mathbf{Rep}_\chi(K)$ is either empty or has exactly one element. We will say that χ *can be realized over* K/k if $\mathbf{Rep}_\chi(K) \neq \emptyset$. In particular, $\mathbf{Rep}_\chi$ and $\mathbf{Rep}_{\chi'}$ are isomorphic if and only if χ and χ' can be realized over the same fields K/k.

Definition 8.1. Let $\chi\colon G \to k$ be a character of a finite group G and p be a prime integer. We will refer to the essential dimension of $\mathbf{Rep}_\chi$ as the *essential dimension of* χ and will denote this number by $\mathrm{ed}(\chi)$. Similarly for the essential p-dimension:

$$\mathrm{ed}(\chi) := \mathrm{ed}(\mathbf{Rep}_\chi) \text{ and } \mathrm{ed}_p(\chi) := \mathrm{ed}_p(\mathbf{Rep}_\chi).$$

We will say that characters χ and λ of G, are *disjoint* if they have no common absolutely irreducible components.

Lemma 8.2. *(a) If the characters $\chi, \lambda\colon G \to k$ are disjoint then $\mathbf{Rep}_{\chi+\lambda} \simeq \mathbf{Rep}_\chi \times \mathbf{Rep}_\lambda$.*

(b) Suppose a character $\chi = \sum_{i=1}^s m_i\chi_i\colon G \to k$ is as in (7.2). Set $\chi' := \sum_{i=1}^r m'_i\chi_i$, where m'_i is the greatest common divisor of m_i and $m_k(\chi_i)$. Then $\mathbf{Rep}_\chi \simeq \mathbf{Rep}_{\chi'}$.

Proof. Let K be a field extension of k.

(a) By Corollary 2.6, $\chi + \lambda$ can be realized over K if and only if both χ and λ can be realized over K.

(b) By Corollary 2.6

(i) χ can be realized over K if and only if

(ii) $m_K(\chi_i)$ divides m_i, for every $i = 1, \dots, s$.

By Theorem 2.4(e), $m_K(\chi_i)$ divides $m_k(\chi)$. Thus (ii) is equivalent to

(iii) $m_K(\chi_1) = \cdots = m_K(\chi_s)$ divides m'.

Applying Corollary 2.6 one more time, we see that (iii) is equivalent to

(iv) χ' can be realized over K.

In summary, χ can be realized over K if and only if χ' can be realized over K, as desired. □

As we observed above, $\mathbf{Rep}_\chi(K)$ has at most one element for every field K/k. In other words, $\mathbf{Rep}_\chi$ is a *detection functor* in the sense of [21] or [27, Section 4a]. We saw in Section 3b that to every algebraic variety X defined over k, we can associate the detection functor $\mathcal{D}_X$, where $\mathcal{D}_X(K)$ is either empty or has exactly one element, depending on whether or not X has a K-point. Given a character $\chi\colon G \to k$, it is thus natural to ask if there exists a smooth projective k-variety X_χ such that the functors $\mathbf{Rep}_\chi$ and $\mathcal{D}_{X_\chi}$ are isomorphic. The rest of this section will be devoted to showing that this is, indeed, always the case. We begin by defining X_χ.

Definition 8.3. (a) Let G be a finite group and $\chi := m(\chi_1 + \cdots + \chi_r)\colon G \to k$ be an irreducible character of G, where $\chi_1, \dots, \chi_r$ are $\mathrm{Gal}(k(\chi_1)/k)$-conjugate absolutely irreducible characters, and $m \geqslant 1$ divides $m_k(\chi_1) = \cdots = m_k(\chi_r)$. We define the k-variety X_χ as the Weil transfer $R_{Z/k}(\mathrm{SB}(A_\chi, m))$, where Z is the center and A_χ is the underlying division algebra of $\mathrm{Env}_k(\chi)$.

(b) More generally, suppose $\chi := \lambda_1 + \cdots + \lambda_s$, where $\lambda_1, \dots, \lambda_s\colon G \to k$ are pairwise disjoint and irreducible over k. Then we define $X_\chi := X_{\lambda_1} \times_k \cdots \times_k X_{\lambda_r}$, where each X_{λ_i} is a Weil transfer of a generalized Severi-Brauer variety, as in part (a).

Theorem 8.4. *Let G be a finite group and $\chi := \lambda_1 + \cdots + \lambda_s$ be a character, where*

$$\lambda_1, \ldots, \lambda_s \colon G \to k$$

are pairwise disjoint and irreducible over k. Let X_χ be the k-variety, as in Definition 8.3. Then the functors $\mathbf{Rep}_\chi$ and $\mathcal{D}_{X_\chi}$ are isomorphic. Consequently $\mathrm{ed}(\chi) = \mathrm{cd}(X_\chi)$ and $\mathrm{ed}_p(\chi) = \mathrm{cd}_p(X_\chi)$ for any prime p.

Proof. In view of Lemma 8.2(a) we may assume that χ is irreducible over k, i.e., $s = 1$ and $\chi = \lambda_1$. Write $\chi := m(\chi_1 + \cdots + \chi_r)$, where $\chi_1, \ldots, \chi_r \colon G \to \overline{k}$ are the absolutely irreducible components of χ. Let K/k be a field extension. By Corollary 2.6 the following conditions are equivalent:

(i) $\mathbf{Rep}_\chi(K) \neq \emptyset$, i.e., χ can be realized over K,

(ii) $m_K(\chi_j)$ divides m for $j = 1, \ldots, r$.

Note that while the characters $\chi_1, \ldots, \chi_r$ are conjugate over k, they may not be conjugate over K. Denote the orbits of $\mathrm{Gal}(\overline{K}/K)$-action on $\chi_1, \ldots, \chi_r$ by $\mathcal{O}_1, \ldots, \mathcal{O}_t$, and set $\mu_i := \sum_{\chi_j \in \mathcal{O}_i} \chi_j$, so that $\chi = m(\mu_1 + \cdots + \mu_t)$.

Denote the center of the central simple algebra $\mathrm{Env}_k(\chi)$ by Z. Write $K_Z := K \otimes_k Z$ as a direct product $K_1 \times \cdots \times K_s$, where $K_1/Z, \ldots, K_s/Z$ are field extensions, as in Proposition 5.1. By Lemma 2.3,

$$\mathrm{Env}_K(\chi) \simeq \mathrm{Env}_k(\chi) \otimes_Z K \simeq \mathrm{Env}_k(\chi) \otimes_Z K_Z \simeq (\mathrm{Env}_K(\chi) \otimes_Z K_1) \times \cdots \times (\mathrm{Env}_K(\chi) \otimes_Z K_s)\,, \tag{8.5}$$

where $\simeq$ denotes isomorphism of K-algebras. On the other hand, since $\mu_1, \ldots, \mu_t$ are K-valued characters,

$$\mathrm{Env}_K(\chi) \simeq \mathrm{Env}_K(m\mu_1) \times \cdots \times \mathrm{Env}_K(m\mu_t)\,. \tag{8.6}$$

Suppose $\chi_j \in \mathcal{O}_i$. Then by Lemma 2.2 $\mathrm{Env}_K(m\mu_i) \simeq \mathrm{Env}_K(\mu_i) \simeq \mathrm{Env}_K(m_K(\chi_j)\mu_i)$, and by Theorem 2.4(d), $\mathrm{Env}_K(m_K(\chi_j)\mu_i)$ is a central simple algebra of index $m_K(\chi_j)$. Comparing (8.5) and (8.6), we conclude that $s = t$, and after renumbering $K_1, \ldots, K_s$, we may assume that K_i is the center of $\mathrm{Env}_K(m\mu_i)$. Thus (ii) is equivalent to

(iii) the index of $\mathrm{Env}_K(m\mu_i) \simeq \mathrm{Env}_K(\chi) \otimes_Z K_i$ divides m for every $i = 1, \ldots, s$.

By Proposition 5.1(a), (iii) is equivalent to

(iv) X_χ has a K-point, i.e., $\mathcal{D}_{X_\chi}(K) \neq \emptyset$.

The equivalence of (i) and (iv) shows that the functors $\mathbf{Rep}_\chi$ and $\mathcal{D}_{X_\chi}$ are isomorphic. Now

$$\mathrm{ed}(\chi) \overset{\mathrm{def}}{=} \mathrm{ed}(\mathbf{Rep}_\chi) = \mathrm{ed}(\mathcal{D}_{X_\chi}) \overset{\mathrm{def}}{=} \mathrm{cd}(X_\chi)$$

and similarly for the essential dimension at p. □

Remark 8.7. Theorem 8.4 can, in fact, be applied to an arbitrary k-valued character $\chi \colon G \to k$. Indeed, the character χ' of Lemma 8.2(b) is a sum of pairwise disjoint k-irreducible characters. Thus $\mathbf{Rep}_\chi \simeq \mathbf{Rep}_{\chi'}$ by Lemma 8.2, and $\mathbf{Rep}_{\chi'} \simeq \mathcal{D}_{X_{\chi'}}$ by Theorem 8.4.

9. Upper bounds

In this section we will, once again, assume that $\mathrm{char}(k) = 0$. Combining Theorem 8.4 with the inequality (3.2), we obtain the following

Corollary 9.1. *Let G be a finite group and $\chi = m(\chi_1 + \dots + \chi_r)\colon G \to k$ be an irreducible character over k, as in Section 7. Then* $\mathrm{ed}(\chi) \leqslant \dim(X_\chi) = rm(m_k(\chi_1) - m)$. □

We are now in a position to prove upper bounds on the essential dimension of an arbitrary representation of a finite group G.

Proposition 9.2. *Let $\rho\colon G \to \mathrm{GL}_n(K)$ be a representation of a finite group G over a field K/k.*

(a) If ρ is K-irreducible (but not necessarily absolutely irreducible), then $\mathrm{ed}(\rho) \leqslant n^2/4$.

(b) If ρ is arbitrary, then $\mathrm{ed}(\rho) \leqslant |G|^2/4$.

(c) $\mathrm{ed}(\mathbf{Rep}_{G,k}) \leq |G|^2/4$ *for any base field k. Here $\mathbf{Rep}_{G,k}$ is the functor defined at the beginning of Section 6.*

Proof. (a) By Lemma 6.3 we may assume that the character χ of ρ is k-valued. By Lemma 7.3, χ is irreducible over k. Write $\chi = m(\chi_1 + \dots + \chi_r)$, where $m \geq 1$ divides $m_k(\chi_1) = \dots = m_k(\chi_r)$. By Corollary 9.1

$$\mathrm{ed}(\rho) \leqslant rm(m_k(\chi_1) - m) \leqslant r\frac{m_k(\chi_1)^2}{4}\,. \tag{9.3}$$

Now recall that by Theorem 2.4(d), $\mathrm{Env}_k(\rho)$ is a central simple algebra of index $m_k(\chi_1)$ over a field Z such that $[Z : k] = r$. Thus

$$rm_k(\chi_1)^2 \leqslant r\dim_Z(\mathrm{Env}_k(\rho)) = \dim_k(\mathrm{Env}_k(\rho)) = \dim_K(\mathrm{Env}_K(\rho)) \leqslant n^2\,. \tag{9.4}$$

Here the equality $\dim_k(\mathrm{Env}_k(\rho)) = \dim_K(\mathrm{Env}_K(\rho))$ follows from Lemma 2.3, and the inequality $\dim_K(\mathrm{Env}_K(\rho)) \leqslant n^2$ follows from the fact that $\mathrm{Env}_K(\rho)$ is a K-subalgebra of $\mathrm{M}_n(K)$. Combining (9.3) and (9.4), we obtain $\mathrm{ed}(\rho) \leqslant n^2/4$.

(b) Decompose $\rho = a_1\rho_1 \oplus \dots \oplus a_s\rho_s$ as a direct sum of distinct K-irreducibles. By Lemma 6.2, $\mathrm{ed}(\rho) \leqslant \mathrm{ed}(\rho_1) + \dots + \mathrm{ed}(\rho_s)$. Thus in view of part (b) we only need to show that

$$\sum_{i=1}^{s} \dim(\rho_i)^2 \leq |G|^2\,. \tag{9.5}$$

Over $\overline{K}$, $\rho_i \simeq m_i(\rho_{i1} \oplus \dots \oplus \rho_{ir_i})$, where the ρ_{ij} are distinct $\overline{K}$-irreducibles, and m_i is the common Schur index $m_K(\rho_{i1}) = \dots = m_K(\rho_{ir_i})$, as in Theorem 2.4(a). The representations $\rho_{i1}, \dots, \rho_{ir_i}$ are conjugate over K and thus have the same dimension. By Theorem 2.4(c), the $\overline{K}$-irreducible representations ρ_{ij} are pairwise non-isomorphic, and by Theorem 2.4(f), $m_i \leqslant \dim(\rho_{i1}) = \dots = \dim(\rho_{ir_i})$. Thus

$$\dim(\rho_i) = m_i r_i \dim(\rho_{i1}) \leqslant r_i \dim(\rho_{i1})^2$$

and

$$\sum_{i=1}^{s} \dim(\rho_i)^2 \leqslant \sum_{i=1}^{s} r_i^2 \dim(\rho_{i1})^4 \leqslant (\sum_{i=1}^{s} r_i \dim(\rho_{i1})^2)^2 = (\sum_{i=1}^{s}\sum_{j=1}^{r_i} \dim(\rho_{ij})^2)^2 \leqslant |G|^2.$$

Here the last inequality follows from the fact that the sum of the squares of the dimensions of $\overline{K}$-irreducible representations is $|G|$; see, e.g., [30, Corollary 2.4.2(a)].

(c) is just a restatement of (b); see Section 3. $\square$

10. A variant of a theorem of Brauer

A theorem of R. Brauer [6] asserts for every integer $l \geqslant 1$ there exists a number field k, a finite group G and an absolutely irreducible character χ_1 such that the Schur index $m_k(\chi_1) = l$. For alternative proofs of Brauer's theorem, see [3] or [32].

In this section we will prove an analogous statement with the Schur index replaced by the essential dimension. Note however, that the analogy is not perfect, because the representation we construct will not be irreducible for any $l \geqslant 2$.

Proposition 10.1. *For every integer $l \geqslant 0$ there exists a number field k, a finite group G, and a character $\chi\colon G \to k$ such that* $\mathrm{ed}(\chi) = l$.

Proof. The proposition is obvious for $l = 0$. Indeed, the trivial representation $\rho\colon G \to \mathrm{GL}_1(k)$ has essential dimension 0 for any group G and any number field k. We may thus assume that $l \geqslant 1$.

The strategy of the proof is as follows. We will construct finite groups $G_1, \dots, G_l$ and 2-dimensional absolutely irreducible characters $\chi_i\colon G_i \to k$, for a suitable number field k, such that the Brauer classes of the quaternion algebras $A_i := \mathrm{Env}_k(\chi_i)$ are linearly independent over $\mathbb{Z}/2\mathbb{Z}$ in $\mathrm{Br}(k)$. (Proving linear independence for these Brauer classes will be the most delicate part of the argument. We will defer it to Lemma 10.3.) We will view each χ_i as a character of $G = G_1 \times \cdots \times G_l$ via the natural projection $G \to G_i$ and set $\chi := \chi_1 + \cdots + \chi_r\colon G \to k$. By Theorem 8.4

$$\mathrm{ed}(\chi) = \mathrm{cd}(X_\chi)\,,$$

where $X_\chi := X_{\chi_1} \times_k \cdots \times_k X_{\chi_l}$, and X_{χ_i} is the 1-dimensional Severi-Brauer variety $\mathrm{SB}(A_i)$. Since the Brauer classes of $A_1, \dots, A_l$ are linearly independent over $\mathbb{Z}/2\mathbb{Z}$ in $\mathrm{Br}(k)$, [22, Theorem 2.1] tells us that $\mathrm{cd}(X_\chi) = l$, as desired.

We now proceed with the construction of k, $G_1, \dots, G_l$ and $\chi_1, \dots, \chi_l$. Choose l distinct prime integers $p_1, \dots, p_l \equiv 3 \pmod 4$ and let $F = \mathbb{Q}(\zeta_{p_1}, \dots, \zeta_{p_l})$, where as usual, ζ_p denotes a primitive pth root of unity. The extension $F/\mathbb{Q}$ is Galois with

$$\mathrm{Gal}(F/\mathbb{Q}) = \Gamma_1 \times \cdots \times \Gamma_l\,,$$

where $\Gamma_i \simeq \mathbb{Z}/(p-1)\mathbb{Z}$. Since $p_i \equiv 3 \pmod 4$, Γ_i has a unique subgroup of order 2, which we will denote by $\Gamma_i[2]$. The non-trivial element of $\Gamma_i[2]$ takes ζ_{p_i} to $\zeta_{p_i}^{-1}$ and preserves ζ_{p_j} for every $j \neq i$. The elementary 2-group $\Gamma_1[2] \times \cdots \times \Gamma_l[2]$ is the Sylow 2-subgroup of Γ; we will denote it by $\Gamma[2]$. We now set

$$k := F^{\Gamma[2]} = \mathbb{Q}(\zeta_{p_1} + \zeta_{p_1}^{-1}, \dots, \zeta_{p_l} + \zeta_{p_l}^{-1})\,.$$

Let A_i be the quaternion algebra $((\zeta_{p_i} - \zeta_{p_i}^{-1})^2, -1)$ over k. That is, A_i is the 4-dimensional k-algebra generated by two elements, x and y, subject to the relations

$$x^2 = (\zeta_{p_i} - \zeta_{p_i}^{-1})^2,\ y^2 = -1 \text{ and } xy = -yx.$$

Note that $k(\zeta_{p_i} - \zeta_{p_i}^{-1}) = k(\zeta_{p_i})$ is a maximal subfield of A_i, and

$$G_i := <\zeta_{p_i}> \rtimes <y> \simeq \mathbb{Z}/p_i\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$$

is a multiplicative subgroup of A_i^*. Since $k(\zeta_{p_i})$ splits A_i and $k(\zeta_{p_i}) \subset F$, the inclusion $G_i \hookrightarrow A_i$ gives rise to a 2-dimensional representation

$$\rho_i \colon G_i \hookrightarrow A_i \hookrightarrow \mathrm{M}_2(F)\,.$$

By our construction $A_i = \mathrm{Env}_k(\rho_i)$ and the character χ_i of ρ_i is k-valued. Since ρ_i is faithful and G_i is a non-abelian group, this representation is absolutely irreducible. (Otherwise ρ_i would embed G as a subgroup of the abelian group $\mathrm{GL}_1(\overline{\mathbb{Q}}) \times \mathrm{GL}_1(\overline{\mathbb{Q}})$, a contradiction.) Proposition 10.1 is now a consequence of Lemma 10.3 below. □

Remark 10.2. Proposition 10.1 implies that there exists a field extension K/k and a linear representation $\rho\colon G \to \mathrm{GL}_n(K)$ such that $\mathrm{ed}(\rho) = l$. Note however, that ρ is not the same as $\rho_1 \times \cdots \times \rho_l \colon G \to \mathrm{GL}_n(F)$, even though they have the same character. Indeed, since F/k is a finite extension, $\mathrm{ed}(\rho_1 \times \cdots \times \rho_l) = 0$. Under the isomorphism of functors $\mathbf{Rep}_\chi \simeq \mathcal{D}_{X_\chi}$ from Theorem 8.4, $\rho_1 \times \cdots \times \rho_l$ corresponds to an F-point of X_χ, while ρ corresponds to the generic point.

Lemma 10.3. *Let $p_1, \dots, p_l$ be distinct prime integers, each $\equiv 3 \pmod 4$, and A_i be the quaternion algebra $((\zeta_{p_i} - \zeta_{p_i}^{-1})^2, -1)$ over $k = \mathbb{Q}(\zeta_{p_1} + \zeta_{p_1}^{-1}, \dots, \zeta_{p_l} + \zeta_{p_l}^{-1})$. Then the classes of $A_1, \dots, A_l$ are linearly independent over $\mathbb{Z}/2\mathbb{Z}$ in $\mathrm{Br}(k)$.*

Proof of Lemma 10.3. After renumbering $p_1, \dots, p_l$, it suffices to show that $A_1 \otimes_k \cdots \otimes_k A_s$ is not split for any $s = 1, \dots, l$. Since $[(a,c)] \otimes [(b,c)] = [(ab,c)]$ in $\mathrm{Br}(k)$, we see that

$$[A_1 \otimes_k \cdots \otimes_k A_s] = [(T, -1)]\,.$$

Here $[A]$ denote the Brauer class of a central simple algebra A/k, and

$$T := \prod_{i=1}^{s} (\zeta_{p_i} - \zeta_{p_i}^{-1})^2 \in k\,.$$

Now recall that the quaternion algebra $(T, -1)$ splits over k if and only if T is a norm in $k(\sqrt{-1})/k$; see, e.g., [24, Theorem 2.7]. Thus it suffices to show that T is not a norm in $k(\sqrt{-1})/k$. Assume the contrary: $T = N_{k(\sqrt{-1})/k}(x)$ for some $x \in k(\sqrt{-1})$. Taking the norm in $k/\mathbb{Q}$ on both sides, we see that

$$N_{k/\mathbb{Q}}(T) = N_{k/\mathbb{Q}}(N_{k(\sqrt{-1})/k}(x)) = N_{k(\sqrt{-1})/\mathbb{Q}}(x) = N_{\mathbb{Q}(\sqrt{-1})/\mathbb{Q}}(N_{k(\sqrt{-1})/\mathbb{Q}(\sqrt{-1})}(x))$$

is a norm in $\mathbb{Q}(\sqrt{-1})/\mathbb{Q}$. Thus it suffices to prove the following

Claim: $N_{k/\mathbb{Q}}(T)$ is not a norm in $\mathbb{Q}(\sqrt{-1})/\mathbb{Q}$, i.e., is not the sum of two rational squares.

Our proof of this claim will be facilitated by the following diagram.

$$\begin{tikzcd}[arrows=dash]
 & & F \arrow[d, "2^l"] \arrow[ddl] & = & \mathbb{Q}(\zeta_{p_1}, \dots, \zeta_{p_l}) \\
 & & k \arrow[dd, "\text{odd}"] & = & \mathbb{Q}(\zeta_{p_1} + \zeta_{p_1}^{-1}, \dots, \zeta_{p_l} + \zeta_{p_l}^{-1}) \\
 & \mathbb{Q}(\zeta_{p_i}) \arrow[dr, "2"] & & & \\
(\zeta_{p_i} - \zeta_{p_i}^{-1})^2 & \in & k_i \arrow[d, "\frac{p_i-1}{2}\text{, odd}"] & = & \mathbb{Q}(\zeta_{p_i} + \zeta_{p_i}^{-1}) \\
 & & \mathbb{Q} & &
\end{tikzcd}$$

We now proceed to compute $N_{k/\mathbb{Q}}(T)$:

$$N_{k/\mathbb{Q}}(T) = \prod_{i=1}^{s} N_{k/\mathbb{Q}}((\zeta_{p_i} - \zeta_{p_i}^{-1})^2) = \prod_{i=1}^{s} N_{k_i/\mathbb{Q}}((\zeta_{p_i} - \zeta_{p_i}^{-1})^2)^{[k:k_i]},$$

and since $(\zeta_{p_i} - \zeta_{p_i}^{-1})^2 = -N_{\mathbb{Q}(\zeta_{p_i})/k_i}(\zeta_{p_i} - \zeta_{p_i}^{-1})$, for each i,

$$N_{k_i/\mathbb{Q}}((\zeta_{p_i} - \zeta_{p_i}^{-1})^2) = N_{k_i/\mathbb{Q}}(-N_{\mathbb{Q}(\zeta_{p_i})/k_i}(\zeta_{p_i} - \zeta_{p_i}^{-1})) =$$
$$(-1)^{[k_i:\mathbb{Q}]} N_{\mathbb{Q}(\zeta_{p_i})/\mathbb{Q}}(\zeta_{p_i} - \zeta_{p_i}^{-1}) = -p_i.$$

The last equality follows from the identities $\prod_{j=1}^{p_i-1} \zeta_{p_i}^j = 1$ and $\prod_{j=1}^{p_i-1}(1 - \zeta_{p_i}^j) = p_i$, where the latter is obtained by substituting $t = 1$ into $\prod_{j=1}^{p_i-1}(t - \zeta_{p_i}^j) = t^{p_i-1} + \cdots + t + 1$. In summary,

$$N_{k_i/\mathbb{Q}}(T) = (-1)^s \prod_{i=1}^{s} p_i^{[k:k_i]} \in \mathbb{Z}.$$

If s is odd then $N_{k_i/\mathbb{Q}}(T) < 0$, and the claim is obvious. In the case where s is even, recall that by a classical theorem of Fermat, a positive integer n can be written as a sum of two rational squares if and only if it can be written as a sum of two integer squares if and only if every prime p which is $\equiv 3 \pmod 4$ occurs to an even power in the prime decomposition of n. Since each p_i is $\equiv 3 \pmod 4$ and each $[k_i : k]$ is odd, we conclude that $N_{k_i/\mathbb{Q}}(T)$ cannot be written as a sum of two rational squares for any $s = 1, \dots, l$. This completes the proof of the claim and thus of Lemma 10.3 and Proposition 10.1. $\square$

11. Computation of canonical p-dimension

This section aims to determine canonical p-dimension of a broad class of Weil transfers of generalized Severi-Brauer varieties. Here p is a fixed prime integer. The base field k is allowed to be of arbitrary characteristic.

Let Z/k be a finite Galois field extension (not necessarily abelian). We will work with Chow motives with coefficients in a finite field of p elements; see [13, §64]. For a motive

M over Z, $R_{Z/k}M$ is the motive over k given by the Weil transfer of M introduced in [15]. Although the coefficient ring is assumed to be $\mathbb{Z}$ in [15], and the results obtained there over $\mathbb{Z}$ do not formally imply similar results for other coefficients, the proofs go through for an arbitrary coefficient ring.

For any finite separable field extension K/k and a motive M over K, the *corestriction* of M is a well-defined motive over k; see [17].

Lemma 11.1. *Let Z/k be an arbitrary finite Galois field extension and let $M_1, \ldots, M_m$ be $m \geqslant 1$ motives over Z. Then the motive $R_{Z/k}(M_1 \oplus \cdots \oplus M_m)$ decomposes in a direct sum*

$$R_{Z/k}(M_1 \oplus \cdots \oplus M_m) \simeq R_{Z/k}M_1 \oplus \cdots \oplus R_{Z/k}M_m \oplus N,$$

where N is a direct sum of corestrictions to k of motives over fields K with $k \subsetneq K \subset Z$.

Proof. For $m = 1$ the statement is void. For $m = 2$ use the same argument as in [18, Proof of Lemma 2.1]. For $m \geqslant 3$ argue by induction. □

Now recall from Section 3b that a k-variety X is called incompressible if $\operatorname{cd}(X) = \dim(X)$ and p-incompressible if $\operatorname{cd}_p(X) = \dim(X)$.

Theorem 11.2. *Let p be a prime number, Z/k a finite Galois field extension of degree p^r for some $r \geqslant 0$, D a balanced central division Z-algebra of degree p^n for some $n \geqslant 0$, and X the generalized Severi-Brauer variety $\mathrm{SB}(D, p^i)$ of D for some $i = 0, 1, \ldots, n$. Then the k-variety $R_{Z/k}X$, given by the Weil transfer of X, is p-incompressible.*

Note that in the case, where Z/k is a quadratic Galois extension, D is balanced if the k-algebra given by the norm of D is Brauer-trivial; ${}^{\alpha}D$ for $\alpha \neq 1$ is then opposite to D. In this special case Theorem 11.2 was proved in [18, Theorem 1.1].

Proof of Theorem 11.2. In the proof we will use Chow motives with coefficients in a finite field of p elements. Therefore the Krull-Schmidt principle holds for direct summands of motives of projective homogeneous varieties by [10] (see also [20]).

We will prove Theorem 11.2 by induction on $r + n$. The base case, where $r + n = 0$, is trivial. Moreover, in the case where $r = 0$ (and n is arbitrary), we have $Z = k$ and thus $R_{Z/k}X = X$ is p-incompressible by [20, Theorem 4.3]. Thus we may assume that $r \geqslant 1$ from now on.

If $i = n$, then $X = \operatorname{Spec} Z$, $R_{Z/k}X = \operatorname{Spec} k$, and the statement of Theorem 11.2 is trivial. We will thus assume that $i \leqslant n - 1$ and, in particular, that $n \geqslant 1$.

Let k' be the function field of the variety $R_{Z/k}\,\mathrm{SB}(D, p^{n-1})$. Set $Z' := k' \otimes_k Z$. By Proposition 5.1(b), the index of the central simple Z'-algebra $D_{Z'} = D \otimes_Z Z' = D \otimes_k k'$ is p^{n-1}. Thus there exists a central division Z'-algebra D' such that the algebra of $(p \times p)$-matrices over D' is isomorphic to $D_{Z'}$. Let $X' = \mathrm{SB}(D', p^i)$. By [14, Theorem 10.9 and Corollary 10.19] (see also [9]) and [20, Theorems 3.8 and 4.3], the motive of the variety $X_{Z'}$ decomposes in a direct sum

$$M(X_{Z'}) \simeq U(X') \oplus U(X')(p^{i+n-1}) \oplus U(X')(2p^{i+n-1}) \oplus \cdots \oplus U(X')((p-1)p^{i+n-1}) \oplus N,$$

where $U(X')$ is the upper motive of X' and N is a direct sum of shifts of upper motives of the varieties $\mathrm{SB}(D', p^j)$ with $j < i$. Therefore, by Lemma 11.1 and [15, Theorem 5.4], the motive of the variety $(R_{Z/k}X)_{k'} \simeq R_{Z'/k'}(X_{Z'})$ decomposes in a direct sum

$$
\begin{aligned}
(11.3)\quad M(R_{Z/k}X)_{k'} \simeq\ & R_{Z'/k'}U(X') \oplus R_{Z'/k'}U(X')(p^{r+i+n-1})\oplus \\
& R_{Z'/k'}U(X')(2p^{r+i+n-1}) \oplus \cdots \oplus R_{Z'/k'}U(X')((p-1)p^{r+i+n-1}) \oplus N \oplus N',
\end{aligned}
$$

where now N is a direct sum of shifts of $R_{Z'/k'}U(\mathrm{SB}(D',p^j))$ with $j < i$, and N' is a direct sum of corestrictions of motives over fields K with $k' \subsetneq K \subset Z'$. By the induction hypothesis, the variety $R_{Z'/k'}X'$ is p-incompressible. By [16, Theorem 5.1], this means that no positive shift of the motive $U(R_{Z'/k'}X')$ is a direct summand of the motive of $R_{Z'/k'}X'$. It follows by [17] that $R_{Z'/k'}U(X')$ is a direct sum of $U(R_{Z'/k'}X')$, of $U(R_{Z'/k'}\,\mathrm{SB}(D',p^j))$ with $j < i$, and of corestrictions of motives over fields K with $k' \subsetneq K \subset Z'$. Therefore we may exchange $R_{Z'/k'}$ with U in (11.3) and get a decomposition of the form

$$
\begin{aligned}
(11.4)\quad M(R_{Z/k}X)_{k'} \simeq\ & U(R_{Z'/k'}X') \oplus U(R_{Z'/k'}X')(p^{r+i+n-1})\oplus \\
& U(R_{Z'/k'}X')(2p^{r+i+n-1}) \oplus \cdots \oplus U(R_{Z'/k'}X')((p-1)p^{r+i+n-1}) \oplus N \oplus N',
\end{aligned}
$$

where N is now a direct sum of shifts of some $U(R_{Z'/k'}\,\mathrm{SB}(D',p^j))$ with $j < i$, and N' is a direct sum of corestrictions of motives over fields K with $k' \subsetneq K \subset Z'$. Note that the first p summands of decomposition (11.4) (that is, all but the last two) are shifts of an indecomposable motive; moreover, no shift of this motive is isomorphic to a summand of N or of N'. Since the variety $R_{Z'/k'}X'$ is p-incompressible, we have

$$\dim U(R_{Z'/k'}X') = \dim R_{Z'/k'}X' = [Z' : k'] \cdot \dim X' = p^r \cdot p^i(p^{n-1} - p^i)\,.$$

(We refer the reader to [16, Theorem 5.1] for the definition of the dimension of the upper motive, as well as its relationship to the dimension and p-incompressibility of the corresponding variety). Note that the shifting number of the p-th summand in (11.4) plus $\dim R_{Z'/k}X'$ equals $\dim R_{Z/k}X$:

$$(p-1)p^{r+i+n-1} + p^r p^i(p^{n-1} - p^i) = p^r p^i(p^n - p^i).$$

We want to show that the variety $R_{Z/k}X$ is p-incompressible. In other words, we want to show that $\dim U(R_{Z/k}X) = \dim R_{Z/k}X$. Let l be the number of shifts of $U(R_{Z'/k'}X')$ contained in the complete decomposition of the motive $U(R_{Z/k}X)_{k'}$. Clearly, $1 \leqslant l \leqslant p$ and it suffices to show that $l = p$ because in this case the p-th summand of (11.4) is contained in the complete decomposition of $U(R_{Z/k}X)_{k'}$.

The complete motivic decomposition of $R_{Z/k}X$ contains several shifts of $U(R_{Z/k}X)$. Let N be any of the remaining (indecomposable) summands. Then, by [17], N is either a shift of the upper motive $U(R_{Z/k}\,\mathrm{SB}(D,p^j))$ with some $j < i$ or a corestriction to k of a motive over a field K with $k \subsetneq K \subset Z$. It follows that the complete decomposition of $N_{k'}$ does not contain any shift of $U(R_{Z'/k'}X')$. Therefore l divides p, that is, $l = 1$ or $l = p$, and we only need to show that $l \neq 1$.

We claim that $l > 1$ provided that $\dim U(R_{Z/k}X) > \dim U(R_{Z'/k'}X')$. Indeed, by [19, Proposition 2.4], the complete decomposition of $U(R_{Z/k}X)_{k'}$ contains as a summand the motive $U(R_{Z'/k'}X')$ shifted by the difference $\dim U(R_{Z/k}X) - \dim U(R_{Z'/k'}X')$. Therefore, in order to show that $l \neq 1$ it is enough to show that

$$\dim U(R_{Z/k}X) > \dim U(R_{Z'/k'}X').$$

We already know the precise value of the dimension on the right, so we only need to find a good enough lower bound on the dimension on the left. This will be given by $\dim U((R_{Z/k}X)_{\tilde{k}})$, where $\tilde{k}/k$ is a degree p Galois field subextension of Z/k. We can determine the latter dimension using the induction hypothesis.

Indeed, since $R_{Z/k}X \simeq R_{\tilde{k}/k}R_{Z/\tilde{k}}X$, the variety $(R_{Z/k}X)_{\tilde{k}}$ is isomorphic to

$$(R_{Z/k}X)_{\tilde{k}} \simeq \prod_{\tilde{\alpha}\in\tilde{\Gamma}} {}^{\tilde{\alpha}}R_{Z/\tilde{k}}X \simeq R_{Z/\tilde{k}} \prod_{\tilde{\alpha}\in\tilde{\Gamma}} {}^{\alpha}X,$$

where Γ is the Galois group of Z/k, $\tilde{\Gamma}$ is the Galois group of $\tilde{k}/k$, and $\alpha \in \Gamma$ is a representative of $\tilde{\alpha} \in \tilde{\Gamma}$ (see [5, §2.8]). Since D is balanced, the product $\prod_{\tilde{\alpha}\in\tilde{\Gamma}} {}^{\alpha}X$ is equivalent to X. Hence, by Lemma 3.3 these varieties have the same canonical p-dimension (i.e., the dimensions of their upper motives coincide). The latter variety is p-incompressible by the induction hypothesis. Consequently,

$$\dim U(R_{Z/k}X) \geqslant \dim U((R_{Z/k}X)_{\tilde{k}}) = \dim R_{Z/\tilde{k}}X = p^{r-1}\cdot p^i(p^n - p^i).$$

The lower bound $p^{r-1}\cdot p^i(p^n-p^i)$ on $\dim U(R_{Z/k}X)$ thus obtained is good enough for our purposes, because

$$p^{r-1}\cdot p^i(p^n - p^i) > p^r \cdot p^i(p^{n-1} - p^i) = \dim U(R_{Z'/k'}X').$$

This completes the proof of Theorem 11.2. □

The following example, due to A. Merkurjev, shows that Theorem 11.2 fails if D is not assumed to be balanced.

Example 11.5. Let L be a field containing a primitive 4-th root of unity. Let Z be the field $Z := L(x, y, x', y')$ of rational functions over L in four variables x, y, x', y'. Consider the degree 4 cyclic central division Z-algebras $C := (x, y)_4$ and $C' := (x', y')_4$. Let $k \subset Z$ be the subfield Z^α of the elements in Z fixed under the L-automorphism α of Z exchanging x with x' and y with y'. The field extension Z/k is then Galois of degree 2, and the algebra C' is conjugate to C.

The index of the tensor product of Z-algebras $C \otimes C'^{\otimes 2}$ is 8. Let D/Z be the underlying (unbalanced!) division algebra of degree 8. The subgroup of the Brauer group $\mathrm{Br}(Z)$ generated by the classes of D and ${}^{\alpha}D = C' \otimes C^{\otimes 2}$ coincides with the subgroup generated by the classes of C and ${}^{\alpha}C = C'$. Therefore the varieties $X_1 := R_{Z/k}\,\mathrm{SB}(D)$ and $X_2 := R_{Z/k}\,\mathrm{SB}(C)$ are equivalent. Thus, by Lemma 3.3,

$$\mathrm{cd}(X_1) = \mathrm{cd}(X_2) \leqslant \dim(X_2) < \dim(X_1)$$

and consequently, X_1 is compressible (and in particular, 2-compressible).

12. Some consequences of Theorem 11.2

Theorem 11.2 makes it possible to determine the canonical p-dimension of the Weil transfer in the situation, where the degrees of Z/k and of D are not necessarily p-powers.

Corollary 12.1. *Let Z/k be a finite Galois field extension and D a balanced central division Z-algebra. For any positive integer m dividing* $\deg(D)$, *one has*

$$\mathrm{cd}_p\, R_{Z/k}\,\mathrm{SB}(D, m) = \dim R_{Z/k'}\,\mathrm{SB}(D', m') = [Z : k']\cdot m'(\deg D' - m'),$$

where m' is the p-primary part of m (i.e., the highest power of p dividing m), D' is the p-primary component of D, and $k' = Z^{\Gamma_p}$, where Γ_p is a Sylow p-subgroup of $\Gamma := \mathrm{Gal}(Z/k)$ (so that $[Z : k']$ is the p-primary part of $[Z : k]$).

Proof. Since the degree $[k' : k]$ is prime to p, we have

$$\mathrm{cd}_p\, R_{Z/k}\, \mathrm{SB}(D, m) = \mathrm{cd}_p(R_{Z/k}\, \mathrm{SB}(D, m))_{k'}\,;$$

see [26, Proposition 1.5(2)]. The k'-variety $R_{Z/k}\,\mathrm{SB}(D, m)_{k'}$ is isomorphic to a product of $R_{Z/k'}\,\mathrm{SB}(D, m)$ with several varieties of the form $R_{Z/k'}\,\mathrm{SB}(\tilde{D}, m)$ where $\tilde{D}$ ranges over a set of conjugates of D. Since D is balanced, these algebras $\tilde{D}$ are Brauer-equivalent to powers of D. Thus the product is equivalent to the k'-variety $R_{Z/k'}\,\mathrm{SB}(D, m)$. We conclude by Lemma 3.3 that $\mathrm{cd}_p\, R_{Z/k}\,\mathrm{SB}(D, m) = \mathrm{cd}_p\, R_{Z/k'}\,\mathrm{SB}(D, m)$. In the sequel we will replace k by k', so that the degree $[Z : k]$ becomes a power of p.

We may now replace k by its p-special closure; see [13, Proposition 101.16]. This will not change the value of $\mathrm{cd}_p(X)$. In other words, we may assume that k is p-special. Under this assumption the algebras D and D' become Brauer-equivalent and consequently, the k-varieties $R_{Z/k}\,\mathrm{SB}(D, m)$ and $R_{Z/k}\,\mathrm{SB}(D', m')$ become equivalent. By Lemma 3.3,

$$\mathrm{cd}_p\, R_{Z/k}\,\mathrm{SB}(D, m) = \mathrm{cd}_p\, R_{Z/k}\,\mathrm{SB}(D', m').$$

Since the Z-algebra D' is balanced over k, Theorem 11.2 tells us that $R_{Z/k}\,\mathrm{SB}(D', m')$ is p-incompressible. That is,

$$\mathrm{cd}_p\, R_{Z/k}\,\mathrm{SB}(D', m') = \dim(R_{Z/k}\,\mathrm{SB}(D', m')) = [Z : k] \cdot m'(\deg D' - m')\,,$$

and the corollary follows. □

Remark 12.2. Corollary 12.1 can be used to compute the p-canonical dimension of $R_{Z/k}\,\mathrm{SB}(D, j)$ for any $j = 1, \ldots, \deg(D)$, even if j does not divide $\deg(D)$. Indeed, let m is the greatest common divisor of j and $\deg(D)$. Proposition 5.1(a) tells us that for any field extension K/k, $R_{Z/k}\,\mathrm{SB}(D, j)$ has a K-point if and only if $R_{Z/k}\,\mathrm{SB}(D, m)$ has a K-point. In other words, the detection functors for these two varieties are isomorphic. Consequently,

$$\mathrm{cd}(R_{Z/k}\,\mathrm{SB}(D, j)) = \mathrm{cd}(R_{Z/k}\,\mathrm{SB}(D, m)) \text{ and } \mathrm{cd}_p(R_{Z/k}\,\mathrm{SB}(D, j)) = \mathrm{cd}_p(R_{Z/k}\,\mathrm{SB}(D, m)),$$

and the value of $\mathrm{cd}_p(R_{Z/k}\,\mathrm{SB}(D, m))$ is given by Corollary 12.1.

We now return to the setting of Sections 7–9. In particular, G is a finite group, and the base field k is of characteristic 0.

Corollary 12.3. *Let $\chi = m(\chi_1 + \cdots + \chi_r)\colon G \to k$ be an irreducible k-valued character, where $\chi_1, \ldots, \chi_r$ are absolutely irreducible and conjugate over k, and m divides $m_k(\chi_1) = \cdots = m_k(\chi_r)$, as in Section 7.*

(a) $\mathrm{ed}_p(\chi) = r'm'(m_k(\chi_1)' - m')$. Here x' denotes the p-primary part of x (i.e., the highest power of p dividing x) for any integer $x \geq 1$.

(b) If r and $m_k(\chi_1)$ are powers of p, then $\mathrm{ed}_p(\chi) = \mathrm{ed}(\chi) = \dim(X_\chi) = rm(m_k(\chi_1) - m)$. Here X_χ is as in Definition 8.3.

Proof. (a) Let D be the underlying division algebra and Z/k be the center of $\mathrm{Env}_k(\chi)$. By Theorem 8.4, $\mathrm{ed}_p(\chi) = \mathrm{cd}_p(X_\chi)$. By Proposition 4.1, D is balanced. The desired conclusion now follows from Corollary 12.1.

(b) Here $r' = r$, $m_k(\chi_1)' = m_k(\chi)$ and thus $m' = m$. By part (a),

$$\dim(X_\chi) = rm(m_k(\chi_1) - m) = \mathrm{ed}_p(\chi) \leqslant \mathrm{ed}(\chi)\,.$$

On the other hand, by Corollary 9.1, $\mathrm{ed}(\chi) \leqslant rm(m_k(\chi_1) - m)$, and part (b) follows. □

Remark 12.4. While a priori $\mathrm{ed}_p(\chi)$ depends on k, G, and χ, Corollary 12.3(a) shows that, in fact, $\mathrm{ed}_p(\chi)$ depends only on the integers r, m, and $m_k(\chi_1)$. (Here we are assuming that χ is irreducible.) We do not know if the same is true of $\mathrm{ed}(\chi)$.

Remark 12.5. We do not know a formula for the canonical p-dimension of a product of Weil transfers of generalized Severi-Brauer varieties similar to Corollary 12.1, except for [22, Theorem 2.1], in the case, where $Z = k$ and $m = 1$. (Note that this special case played a key role in the proof of Proposition 10.1.) For this reason we do not know how to generalize Corollary 12.3 to the case, where the character χ is not k-irreducible.

13. A variant of a theorem of Schilling

Let G be a p-group and χ_1 be an absolutely irreducible character of G. It is well known that for any field k of characteristic 0, $m_k(\chi_1) = 1$ if p is odd, and $m_k(\chi_1) = 1$ or 2 if $p = 2$. Following C. Curtis and I. Reiner, we will attribute this theorem to O. Schilling; see [12, Theorem 74.15]. For further bibliographical references, see [33, Corollary 9.8].

In this section we will use Corollary 12.3 to prove the following analogous statement, with the Schur index replaced by the essential dimension.

Proposition 13.1. *Let k be a field of characteristic* 0*, G be a p-group, and $\chi\colon G \to k$ be an irreducible character over k.*

(a) If p is odd then $\mathrm{ed}(\chi) = 0$.

(b) If $p = 2$ then $\mathrm{ed}_2(\chi) = \mathrm{ed}(\chi) = 0$ *or* 2^l *for some integer* $l \geqslant 0$.

(c) Moreover, every $l \geqslant 0$ in part (b) can occur with $k = \mathbb{Q}$, for suitable choices of G and χ.

Proof. Write $\chi = m(\chi_1 + \cdots + \chi_r)$, where $\chi_i\colon G \to \overline{k}$ are absolutely irreducible characters and m divides $m_k(\chi_1)$. If $m = m_k(\chi_1)$ then $\mathrm{ed}(\chi) = 0$ by Corollary 9.1.

(a) In particular, this will always be the case if p is odd. Indeed, by Schilling's theorem, $m_k(\chi_1) = 1$ and thus $m = 1$. (Also cf. Lemma 6.4.)

(b) By Schilling's theorem, $m_k(\chi) = 1$ or 2, and by the above argument, we may assume that $m < m_k(\chi_1)$. Thus the only case we need to consider is $m_k(\chi_1) = 2$ and $m = 1$. By Theorem 2.4(b), $r = [k(\chi_1) : k]$. Since $k(\chi_1) \subset k(\zeta_e)$, where the exponent e of G is a power of 2, we see that r divides $[k(\zeta_e) : k]$, which is, once again, a power of 2. Thus we conclude that r is a power of 2. Corollary 12.3(b) now tells us that

$$\mathrm{ed}_2(\chi) = \mathrm{ed}(\chi) = rm(m_k(\chi) - m) = r \cdot 1 \cdot (2 - 1) = r \qquad (13.2)$$

is a power of 2, as claimed.

(c) Let $s = 2^{l+2}$, and $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_s)/\mathbb{Q})$ be complex conjugation, and

$$F := \mathbb{Q}(\zeta_s)^\sigma = \mathbb{Q}(\zeta_s) \cap \mathbb{R} = \mathbb{Q}(\zeta_s + \zeta_s^{-1})\,.$$

Consider the quaternion algebra $A = ((\zeta_s - \zeta_s^{-1})^2, -1)$ over F, i.e., the F-algebra generated by elements x and y, subject to the relations

$$x^2 = (\zeta_s - \zeta_s^{-1})^2,\ y^2 = -1 \text{ and } xy = -yx.$$

Arguing as in the proof of Proposition 10.1, we see that $F(\zeta_s - \zeta_s^{-1}) = \mathbb{Q}(\zeta_s)$ is a maximal subfield of A, ζ_s and y generate a a multiplicative subgroup of A of order $2s$, which spans A as an F-vector space, and the inclusion $G \hookrightarrow A$ gives rise to an absolutely irreducible 2-dimensional representation

$$\rho_1 \colon G \hookrightarrow A \hookrightarrow \mathrm{GL}_2(F)\,.$$

Denote the character of ρ_1 by $\chi_1 \colon G \to F$. Since F is generated by $\chi_1(\zeta_s) = \zeta_s + \zeta_s^{-1}$ as a field extension of $\mathbb{Q}$, we have $\mathbb{Q}(\chi_1) = F$. Thus χ_1 has exactly

$$r = [F : \mathbb{Q}] = \frac{1}{2}[\mathbb{Q}(\zeta_s) : \mathbb{Q}] = 2^l$$

conjugates over $\mathbb{Q}$, and $\chi = \chi_1 + \cdots + \chi_r$ is an irreducible character over $\mathbb{Q}$.

Note that since $s = 2^{l+2} \geqslant 4$, $(\zeta_s - \zeta_s^{-1})^2 < 0$, $A \otimes_F \mathbb{R}$ is $\mathbb{R}$-isomorphic to the Hamiltonian quaternion algebra $\mathbb{H} = (-1, -1)$ and hence, is non-split. Thus $\mathrm{ind}(A) = 2$. Since $A = \mathrm{Env}_{\mathbb{Q}}(\rho)$, Theorem 2.4(d) tells us that $m_{\mathbb{Q}}(\chi_1) = 2$. Applying Corollary 12.3(b), as in (13.2), we conclude that $\mathrm{ed}_2(\chi) = \mathrm{ed}(\chi) = r = 2^l$, as desired. □

14. Essential dimension of modular representations

Let G be a finite group and $\mathbf{Rep}_{G,k}$ be the functor of representations defined at the beginning of Section 6. In the non-modular setting (where $\mathrm{char}(k)$ does not divide $|G|$), we know that

$$\mathrm{ed}(\mathbf{Rep}_{G,k}) \text{ is } \begin{cases} 0, \text{ if } \mathrm{char}(k) > 0, \text{ by Remark 6.5, and} \\ \leqslant |G|^2/4, \text{ if } \mathrm{char}(k) = 0, \text{ by Proposition 9.2.} \end{cases}$$

We shall now see that essential dimension of representations behaves very differently in the modular case.

Proposition 14.1. *Let k be a field of characteristic p. Suppose a finite group G contains an elementary abelian subgroup $E = \langle g_1, g_2 \rangle \simeq (\mathbb{Z}/p\mathbb{Z})^2$ of rank 2. Then $\mathrm{ed}(\mathbf{Rep}_{G,k}) = \infty$.*

Proof. Let $\mathbf{Sub}_{\mathbb{P}^1,k} \colon \mathbf{Fields}_k \to \mathbf{Sets}$ be a covariant functor, given by

$$\mathbf{Sub}_{\mathbb{P}^1,k}(K) := \{\text{closed subvarieties of } \mathbb{P}^1_K\}.$$

Here subvarieties of $\mathbb{P}^1_K$ are required to be reduced but not necessarily irreducible. Closed subvarieties $X, Y \subset \mathbb{P}^1_K$ represent the same element in $\mathbf{Sub}_{\mathbb{P}^1,k}(K)$ if $X(\overline{K}) = Y(\overline{K})$ in $\mathbb{P}^1(\overline{K})$. We will now consider the morphism of functors

$$\mathbf{V}_{E,k} \colon \mathbf{Rep}_{G,k} \to \mathbf{Sub}_{,\mathbb{P}^1,k}$$

which associates to a representation $\rho\colon G \to \mathrm{GL}_n(K)$, the rank variety $V_E(\rho)$, as defined by J. Carlson [8]. Recall that in projective coordinates $(x_1 : x_2)$ on $\mathbb{P}^1_K$, the rank variety $V_E(\rho)$ is given by

$$\text{rank}(A_{x_1,x_2}) < \frac{(p-1)n}{p}\,, \tag{14.2}$$

where $A = x_1(\rho(g_1) - I_n) + x_2(\rho(g_2) - I_n) \in \mathrm{M}_n(K[x_1, x_2])$, and I_n is the $n \times n$ identity matrix. In other words, condition (14.2) is equivalent to the vanishing of certain minors of A_{x_1,x_2}. These minors are homogeneous polynomials in $K[x_1, x_2]$, and $V_E(\rho)$ is the reduced subvariety of $\mathbb{P}^1_K$ they cut out. Note that the generators g_1, g_2 of E are assumed to be fixed throughout. For details on this construction, see [8, Section 4] or [1, Section 5.8].

To make the rest of the proof more transparent, we will first consider the case, where $G = E$. In this case it follows from the work of Carlson that the functor $\mathbf{V}_{E,k}$ is surjective. That is, for any given field K/k, every reduced subvariety $X \subset \mathbb{P}^1_K$ can be realized as the rank variety of a suitable representation $\rho\colon E \to \mathrm{GL}_n(K)$; see [1, Corollary 5.9.2]. Thus $\mathrm{ed}(\mathbf{Rep}_{G,k}) \geqslant \mathrm{ed}(\mathbf{Sub}_{\mathbb{P}^1,k})$; see [2, Lemma 1.9]. It now suffices to show that $\mathrm{ed}(\mathbf{Sub}_{\mathbb{P}^1,k}) = \infty$. Let L/k be a field, $a_1, \dots, a_n \in L$, and $X[n]$ be the union of the points

$$X_1 = (1 : a_1), \dots, X_n = (1 : a_n) \tag{14.3}$$

in $\mathbb{P}^1$. We view $X[n]$ as an element of $\mathbf{Sub}_{\mathbb{P}^1,k}(L)$.

Claim: Suppose $X[n]$ descends to a subfield $K \subset L$. Then a_i is algebraic over K for every $i = 1, \dots, n$.

By the definition of the functor $\mathbf{Sub}_{\mathbb{P}^1,k}$, $X[n]$ descends to K if $X[n]$ can be cut out (set-theoretically) by homogeneous polynomials $f_1, \dots, f_s \in K[x_1, x_2]$. In other words, the points $X_1, \dots, X_n$ in (14.3) are the only non-zero solutions, in the algebraic closure $\overline{L}$, of a system of homogeneous equations

$$f_1(x_1, x_2) = \cdots = f_s(x_1, x_2) = 0$$

with coefficients in K. Since every solution of such a system can be found over $\overline{K}$, we have $a_1, \dots, a_n \in \overline{K}$. This proves the claim.

Taking $a_1, \dots, a_n$ to be independent variables and $L := k(a_1, \dots, a_n)$, we see that $\mathrm{trdeg}_k(K) = \mathrm{trdeg}_k(L) = n$ and thus in this case $\mathrm{ed}(X[n]) = n$. Therefore,

$$\mathrm{ed}(\mathbf{Sub}_{\mathbb{P}^1,k}) \geqslant \sup_{n \geqslant 1} \mathrm{ed}(X[n]) = \infty\,.$$

This completes the proof of the proposition in the case where $G = E$.

We now proceed with the proof of Proposition 14.1 for a general group G. Denote the centralizer and the normalizer of E in G by $C_G(E)$ and $N_G(E)$, respectively. Then $W_G(E) := N_G(E)/C_G(E)$ acts on $\mathbb{P}^1_k$. By the Quillen Stratification Theorem [1, Theorem 5.6.3] and [1, Corollary 5.9.2], there exists a closed $W_G(E)$-invariant k-subvariety $B \subsetneq \mathbb{P}^1_k$ with the following property. Every $X \in \mathbf{Sub}_{\mathbb{P}^1,k}(L)$ satisfying conditions

(i) X is $W_G(E)$-invariant, and

(ii) no irreducible component of X is contained in B_K, lies in the image of $\mathbf{V}_{E,k}(L)$. (The subvariety $B \subset \mathbb{P}^1_k$ comes from the cohomology of elementary abelian subgroups of G strictly contained in E; see the bottom of [1, p. 178].) Consequently, for any $X \in \mathbf{Sub}_{\mathbb{P}^1,k}(L)$ satisfying (i) and (ii), $\mathrm{ed}(\mathbf{Rep}_{G,k}) \geqslant \mathrm{ed}(X)$.

In particular, let $L := k(a_1, \ldots, a_n)$, where $a_1, \ldots, a_n$ are independent variables over k, $X_1, \ldots, X_n \subset \mathbb{P}^1_L$ be as in (14.3), and

$$Z[n] := \bigcup w(X_i)\,,$$

where the union is taken over all $i = 1, \ldots, n$ and all $w \in W_G(E)$. Note that every point of the form $w(X_i)$ corresponds to a dominant k-morphism $\mathrm{Spec}(L) \to \mathbb{P}^1_k$ and hence, cannot lie in B_K for any proper subvariety $B \subset \mathbb{P}^1_k$. Thus $Z[n]$ satisfies conditions (i) and (ii). We conclude that $Z[n]$ lies in the image of $\mathbf{V}_{E,k}$, and thus

$$\mathrm{ed}(\mathbf{Rep}_{G,k}) \geqslant \mathrm{ed}(Z[n]) \tag{14.4}$$

for every $n \geqslant 1$. On the other hand, the Claim above shows that if $Z[n]$ descends to an intermediate field $k \subset K \subset L$ then every a_i is algebraic over K. Hence, $\mathrm{ed}(Z[n]) = \mathrm{trdeg}_k(L) = n$, and (14.4) tells us that $\mathrm{ed}(\mathbf{Rep}_{G,k}) = \infty$, as desired. □

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References

[1] Benson, D. J. *Representations and cohomology. II*, second ed., vol. 31 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1998. Cohomology of groups and modules.

[2] Berhuy, G., and Favi, G. Essential dimension: a functorial point of view (after A. Merkurjev). *Doc. Math. 8* (2003), 279–330 (electronic).

[3] Berman, S. D. On Schur's index. *Uspehi Mat. Nauk 16*, 2 (98) (1961), 95–99.

[4] Blanchet, A. Function fields of generalized Brauer-Severi varieties. *Comm. Algebra 19*, 1 (1991), 97–118.

[5] Borel, A., and Serre, J.-P. Théorèmes de finitude en cohomologie galoisienne. *Comment. Math. Helv. 39* (1964), 111–164.

[6] Brauer, R. Untersuchungen über die arithmetischen Eigenschaften von Gruppen linearer Substitutionen. *Math. Z. 31*, 1 (1930), 733–747.

[7] Brosnan, P., Reichstein, Z., and Vistoli, A. Essential dimension of moduli of curves and other algebraic stacks. *J. Eur. Math. Soc. (JEMS) 13*, 4 (2011), 1079–1112. With an appendix by Najmuddin Fakhruddin.

[8] Carlson, J. F. The varieties and the cohomology ring of a module. *J. Algebra 85*, 1 (1983), 104–143.

[9] Chernousov, V., Gille, S., and Merkurjev, A. Motivic decomposition of isotropic projective homogeneous varieties. *Duke Math. J. 126*, 1 (2005), 137–159.

[10] Chernousov, V., and Merkurjev, A. Motivic decomposition of projective homogeneous varieties and the Krull-Schmidt theorem. *Transform. Groups 11*, 3 (2006), 371–386.

[11] Curtis, C. W., and Reiner, I. *Methods of representation theory. Vol. I.* John Wiley & Sons, Inc., New York, 1981. With applications to finite groups and orders, Pure and Applied Mathematics, A Wiley-Interscience Publication.

[12] Curtis, C. W., and Reiner, I. *Methods of representation theory. Vol. II.* Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, 1987. With applications to finite groups and orders, A Wiley-Interscience Publication.

[13] Elman, R., Karpenko, N., and Merkurjev, A. *The algebraic and geometric theory of quadratic forms*, vol. 56 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2008.

[14] Karpenko, N. A. Cohomology of relative cellular spaces and of isotropic flag varieties. *Algebra i Analiz 12*, 1 (2000), 3–69.

[15] Karpenko, N. A. Weil transfer of algebraic cycles. *Indag. Math. (N.S.) 11*, 1 (2000), 73–86.

[16] Karpenko, N. A. Canonical dimension. In *Proceedings of the International Congress of Mathematicians. Volume II* (New Delhi, 2010), Hindustan Book Agency, pp. 146–161.

[17] Karpenko, N. A. Upper motives of outer algebraic groups. In *Quadratic forms, linear algebraic groups, and cohomology*, vol. 18 of *Dev. Math.* Springer, New York, 2010, pp. 249–258.

[18] Karpenko, N. A. Incompressibility of quadratic Weil transfer of generalized Severi-Brauer varieties. *J. Inst. Math. Jussieu 11*, 1 (2012), 119–131.

[19] Karpenko, N. A. Sufficiently generic orthogonal Grassmannians. *J. Algebra 372* (2012), 365–375.

[20] Karpenko, N. A. Upper motives of algebraic groups and incompressibility of Severi-Brauer varieties. *J. Reine Angew. Math. 677* (2013), 179–198.

[21] Karpenko, N. A., and Merkurjev, A. S. Canonical p-dimension of algebraic groups. *Adv. Math. 205*, 2 (2006), 410–433.

[22] Karpenko, N. A., and Merkurjev, A. S. Essential dimension of finite p-groups. *Invent. Math. 172*, 3 (2008), 491–508.

[23] Karpenko, N. A., and Merkurjev, A. S. On standard norm varieties. *Ann. Sci. Éc. Norm. Supér. (4) 46*, 1 (2013), 175–214.

[24] Lam, T. Y. *Introduction to quadratic forms over fields*, vol. 67 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2005.

[25] Lang, S. *Algebra*, third ed., vol. 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002.

[26] Merkurjev, A. S. Essential dimension. In *Quadratic Forms – Algebra, Arithmetic, and Geometry*, vol. 493 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 2009, pp. 299–326.

[27] Merkurjev, A. S. Essential dimension: a survey. *Transform. Groups 18*, 2 (2013), 415–481.

[28] Reichstein, Z. Essential dimension. In *Proceedings of the International Congress of Mathematicians. Volume II* (2010), Hindustan Book Agency, New Delhi, pp. 162–188.

[29] Reichstein, Z., and Youssin, B. Essential dimensions of algebraic groups and a resolution theorem for G-varieties. *Canad. J. Math. 52*, 5 (2000), 1018–1056. With an appendix by János Kollár and Endre Szabó.

[30] Serre, J.-P. *Linear representations of finite groups*. Springer-Verlag, New York-Heidelberg, 1977. Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42.

[31] Wadsworth, A. R. The index reduction formula for generic partial splitting varieties. *Comm. Algebra 21*, 4 (1993), 1063–1070.

[32] Yamada, T. On the group algebras of metabelian groups over algebraic number fields. I. *Osaka J. Math. 6* (1969), 211–228.

[33] Yamada, T. *The Schur subgroup of the Brauer group*. Lecture Notes in Mathematics, Vol. 397. Springer-Verlag, Berlin-New York, 1974.

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