UNRAMIFIED DEGREE THREE INVARIANTS OF REDUCTIVE GROUPS

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Abstract. We prove that if $G$ is a reductive group over an algebraically closed field $F$, then for a prime integer $p \neq \text{char}(F)$, the group of unramified Galois cohomology $H^3_{nr}(F(BG), \mathbb{Q}_p/\mathbb{Z}_p(2))$ is trivial for the classifying space $BG$ of $G$ if $p$ is odd or the commutator subgroup of $G$ is simple.

1. Introduction

The notion of a cohomological invariant of an algebraic group was introduced by J-P. Serre in [6]. Let $G$ be an algebraic group over a field $F$ and $M$ a Galois module over $F$. A degree $d$ invariant of $G$ assigns to every $G$-torsor over a field extension $K$ over $F$ an element in the Galois cohomology group $H^d(K, M)$, functorially in $K$. In this paper we consider the cohomology groups $H^d(K) = H^d(K, \mathbb{Q}/\mathbb{Z}(d-1))$, where $\mathbb{Q}/\mathbb{Z}(d-1)$ is defined as the Galois module of $(d-1)$-twisted roots of unity. The $p$-part of this module requires special care if $p = \text{char}(F) > 0$. All degree $d$ invariants of $G$ form an abelian group $\text{Inv}^d(G)$. An invariant is normalized if it takes a trivial torsor to the trivial cohomology class. The group $\text{Inv}^d(G)$ is the direct sum of the subgroup $\text{Inv}^d(G)_{\text{norm}}$ of normalized invariants and the subgroup of constant invariants isomorphic to $H^d(F)$.

The group $\text{Inv}^d(G)_{\text{norm}}$ for small values of $d$ is well understood. The group $\text{Inv}^1(G)_{\text{norm}}$ is trivial if $G$ is connected. There is a canonical isomorphism $\text{Inv}^2(G)_{\text{norm}} \simeq \text{Pic}(G)$ for every reductive group $G$ (see [2, Theorem 2.4]). M. Rost proved (see [6, Part 2]) that if $G$ is simple simply connected then the group $\text{Inv}^3(G)_{\text{norm}}$ is cyclic of finite order with a canonical generator called the Rost invariant. The group $\text{Inv}^3(G)_{\text{norm}}$ for an arbitrary semisimple group $G$ was studied in [10].

For a prime integer $p$, write $H^d(K, p)$ and $\text{Inv}^d(G, p)$ for the $p$-primary components of $H^d(K)$ and $\text{Inv}^d(G)$ respectively. If $v$ is a discrete valuation of a field extension $K/F$ trivial on $F$ with residue field $F(v)$, then there is defined the residue homomorphism $\partial_v : H^d(K, p) \rightarrow H^{d-1}(F(v), p)$ for every $p \neq \text{char}(F)$. An element $a \in H^d(K, p)$ is unramified with respect to $v$ if $\partial_v(a) = 0$. We write $H^d_{\text{nr}}(K, p)$ for the subgroup of all elements unramified

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with respect to every discrete valuation of $K$ over $F$. An invariant in $\text{Inv}^d(G, p)$ is called unramified if all values of the invariant over every $K/F$ belongs to $H^d_{nr}(K, p)$. We write $\text{Inv}^d_{nr}(G, p)$ for the group of all unramified invariants.

Let $V$ be a generically free representation of $G$. There is a nonempty $G$-invariant open subscheme $U \subset V$ and a versal $G$-torsor $U \to X$ for a variety $X$ over $F$. We think of $X$ as an approximation of the classifying space $BG$ of $G$. The larger the codimension of $V \setminus U$ in $V$ the better $X$ approximates $BG$. Abusing notation, we will write $BG$ for $X$. Note that the stable birational type of $BG$ is well defined.

The generic fiber of the versal $G$-torsor is the generic $G$-torsor over the function field $F(BG)$ of the classifying space. A theorem of Rost and Totaro asserts that the evaluation at the generic $G$-torsor yields an isomorphism between $\text{Inv}^d(G, p)$ and the subgroup of $H^d_{nr}(F(BG), p)$ of all elements unramified with respect to the discrete valuations associated with all irreducible divisors of $BG$. This isomorphism restricts to an isomorphism

$$\text{Inv}^d_{nr}(G, p) \xrightarrow{\sim} H^d_{nr}(F(BG), p).$$

A classical question is whether the classifying space $BG$ of an algebraic group $G$ is stably rational. To disprove stable rationality of $BG$ it suffices to show that the map $H^d(F, p) \to H^d_{nr}(F(BG), p)$ is not surjective for some $d$ and $p$ or, equivalently, to find a non-constant unramified invariant of $G$. For example, D. Saltman disproved in [14] the Noether Conjecture (that $V/G$ is stably rational for a faithful representation $V$ of a finite group $G$ over an algebraically closed field) by proving that $H^2_{nr}(F(BG), p) \neq H^2(F, p)$ for some $G$ and $p$, i.e., by establishing a non-constant degree 2 invariant of $G$. E. Peyre found new examples of finite groups with non-constant unramified degree 3 invariants in [12]. Degree 3 unramified invariants of simply connected groups (over arbitrary fields) were studied in [11] (classical groups) and [7] (exceptional groups).

It is still a wide open problem whether there exists a connected algebraic group $G$ over an algebraically closed field $F$ with the classifying space $BG$ that is not stably rational. Connected groups have no non-trivial degree 1 invariants. F. Bogomolov proved in [3, Lemma 5.7] (see also [2, Theorem 5.10]) that connected groups have no non-trivial degree 2 unramified invariants. In [15] and [16], D. Saltman proved that the projective linear group $\text{PGL}_n$ has no non-trivial degree 3 unramified invariants.

In the present paper, we study unramified degree 3 invariants of an arbitrary (connected) reductive group $G$ over an algebraically closed field, or equivalently, the unramified elements in $H^3(F(BG))$. The language of invariants seems easier to work with. The main result is the following theorem (see Theorems 8.4 and 11.3):

**Theorem.** Let $G$ be a split reductive group over an algebraically closed field $F$ and $p$ a prime integer different from char($F$). Then

$$\text{Inv}^3_{nr}(G, p) = H^3_{nr}(F(BG), p) = 0$$
if $p$ is odd or the commutator subgroup of $G$ is (almost) simple.

Let $H$ be the commutator subgroup of a split reductive group $G$. We have $\text{Inv}_{\text{nr}}^3(G, p) = \text{Inv}_{\text{nr}}^3(H, p)$ (see Proposition 6.1). If $H$ is a simple group, we compare the group $\text{Inv}^3(H)$ with the group $\text{Inv}^3(\tilde{H}^{\text{gen}})$, where $\tilde{H}^{\text{gen}}$ is the simply connected cover of $H$ twisted by a generic $H$-torsor, and use our knowledge of the unramified degree 3 invariants in the simply connected case. The key statement is the injectivity of the homomorphism $\text{Inv}^3(H) \rightarrow \text{Inv}^3(\tilde{H}^{\text{gen}})$ (see Section 8).

In general, when $H$ is semisimple but not necessarily simple, we consider an embedding of $H$ into a reductive group $G'$ as the commutator subgroup. Then $\text{Inv}^3(G')$ is identified with a subgroup of $\text{Inv}^3(H)$. If $G'$ is strict, i.e., the center of $G'$ is a torus, this subgroup is the smallest possible and is independent of the choice of $G'$. We write $\text{Inv}^3_{\text{red}}(H)$ for this subgroup. It satisfies

$$\text{Inv}^3_{\text{nr}}(H, p) \subset \text{Inv}^3_{\text{red}}(H, p) \subset \text{Inv}^3(H, p)$$

for every prime $p \neq \text{char}(F)$. The group $\text{Inv}^3_{\text{red}}(H, p)$ is easier to control than $\text{Inv}^3_{\text{nr}}(H, p)$. We show that $\text{Inv}^3_{\text{red}}(H, p) = 0$ which implies that $\text{Inv}^3_{\text{nr}}(H, p)$ is also trivial.

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# 2. Basic definitions and facts

Let $F$ be a field. If $d \geq 1$, we write $H^d(F)$ for the Galois cohomology group $H^d(F, \mathbb{Q}/\mathbb{Z}(d-1))$, with $\mathbb{Q}/\mathbb{Z}(d-1)$ the direct sum of colim $\mu_n^{\otimes (d-1)}$, where $\mu_n$ is the group of roots of unity of degree $n$, and the $p$-component if $p = \text{char}(F) > 0$ (see [6, Part 2, Appendix A]). In particular, $H^1(F)$ is the group of (continuous) characters of the absolute Galois group $\text{Gal}(F_{\text{sep}}/F)$ of $F$ and $H^2(F)$ is the Brauer group $\text{Br}(F)$. We view $H^d$ as a functor from the category $\text{Fields}_F$ of field extensions of $F$ to the category of abelian groups (or the category $\text{Sets}$ of sets).

Let $G$ be a (linear) algebraic group over a field $F$. The notion of an **invariant** of $G$ was defined in [6] as follows. Consider the functor

$$\text{Tors}_G : \text{Fields}_F \rightarrow \text{Sets}$$

taking a field $K$ to the set $\text{Tors}_G(K) := H^1(K, G)$ of isomorphism classes of (right) $G$-torsors over $\text{Spec} K$. A **degree $d$ cohomological invariant** of $G$ is then a morphism of functors

$$\text{Tors}_G \rightarrow H^d,$$

i.e., a functorial in $K$ collection of maps of sets $\text{Tors}_G(K) \rightarrow H^d(K)$ for all field extensions $K/F$. We denote the group of such invariants by $\text{Inv}^d(G)$.

An invariant $I \in \text{Inv}^d(G)$ is called **normalized** if $I(E) = 0$ for a trivial $G$-torsor $E$. The normalized invariants form a subgroup $\text{Inv}^d(G)_{\text{norm}}$ of $\text{Inv}^d(G)$.
and there is a natural isomorphism
\[ \text{Inv}^d(G) \simeq H^d(F) \oplus \text{Inv}^d(G)_{\text{norm}}. \]

**Example 2.1.** Let \( G \) be a (connected) reductive group over \( F \). It is shown in [2, Theorem 2.4] that there is an isomorphism
\[ \beta_G : \text{Pic}(G) \simrightarrow \text{Inv}^2(G)_{\text{norm}}. \]

Let \( G \) be a split reductive group and \( H \) the commutator subgroup of \( G \). Let \( \pi : \tilde{H} \rightarrow H \) be a simply connected cover with kernel \( \tilde{C} \). There are canonical isomorphisms (see [17, §6])
\[ (2.2) \quad \text{Pic}(G) \simrightarrow \text{Pic}(H) \simeq \tilde{C}^* := \text{Hom}(\tilde{C}, \mathbb{G}_m). \]

Take any character \( \chi \in \tilde{C}^* \) and consider the push-out diagram
\[
\begin{array}{cccccc}
1 & \longrightarrow & \tilde{C} & \longrightarrow & \tilde{H} & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow \pi & & \\
1 & \longrightarrow & \mathbb{G}_m & & H' & \longrightarrow & H & \longrightarrow & 1.
\end{array}
\]

The isomorphism \( \tilde{C}^* \simeq \text{Pic}(H) \) takes a character \( \chi \) to the class the line bundle \( L_\chi \) on \( H \) given by the \( \mathbb{G}_m \)-torsor \( H' \rightarrow H \) in the bottom row of the diagram. For a field extension \( K/F \) and an \( H \)-torsor \( E \) over \( K \), the value of the invariant \( \beta_H(L_\chi) \) is equal to \( \delta([E]) \in H^2(K, \mathbb{G}_m) = \text{Br}(K) \), where \([E]\) is the class of \( E \) in \( H^1(K, H) \) and \( \delta : H^1(K, H) \longrightarrow H^2(K, \mathbb{G}_m) \) is the connecting map for the bottom exact sequence in the diagram.

If \( f : G_1 \longrightarrow G_2 \) is a homomorphism of algebraic groups over \( F \) and \( E_1 \) is a \( G_1 \)-torsor over a field extension \( K/F \), then \( E_2 := (E_1 \times G_2)/G_1 \) is the \( G_2 \)-torsor over \( K \) which we denote by \( f_*(E_1) \). If \( I \) is a degree \( d \) invariant of \( G_2 \), we define an invariant \( f^*(I) \) of \( G_1 \) by \( f^*(I)(E_1) := I(f_*(E_1)) \). Thus, we have a group homomorphism
\[ (2.3) \quad f^* : \text{Inv}^d(G_2) \longrightarrow \text{Inv}^d(G_1) \]
taking normalized invariants to the normalized ones.

Let \( G \) be an algebraic group over a field \( F \) and let \( V \) be a generically free representation of \( G \). There is a nonempty \( G \)-invariant open subscheme \( U \subseteq V \) such that \( U \) is a \( G \)-torsor over a variety which we denote by \( U/G \) (see [18, Remark 1.4]). We think of \( U/G \) as an approximation of the “classifying space” \( BG \) of \( G \) and abusing notation write \( U/G = BG \). The space \( BG \) is better approximated by \( U/G \) if the codimension of \( V \setminus U \) in \( V \) is large. For our purposes it suffices to assume that this codimension is at least 3 (see [2]).

Note that by the No-name Lemma, the stable rationality type of \( BG \) is uniquely determined by \( G \).

The generic fiber \( E^\text{gen} \longrightarrow \text{Spec}(F(BG)) \) of the projection \( U \longrightarrow U/G \) is called the generic \( G \)-torsor. The value of an invariant of \( G \) at the generic
torsor $E^\text{gen}$ yields a homomorphism

$$\text{Inv}^d(G) \rightarrow H^d(F(BG)).$$

Rost proved (see [6, Part 2, Th. 3.3] or [2, Theorem 2.2]) that this map is injective, i.e., every invariant is determined by its value at the generic torsor.

We decompose the group of invariants into a direct sum of primary components:

$$\text{Inv}^d(G) = \bigoplus_{p \text{ prime}} \text{Inv}^d(G, p).$$

Let $K$ be a field extension of $F$. For a prime integer $p$, write $H^d(K, p)$ for the $p$-primary component of $H^d(K)$. Let $v$ be a discrete valuation of $K$ over $F$ with residue field $F(v)$. If $\text{char}(F) \neq p$, there is the residue map (see [6, Chapter 2])

$$\partial_v : H^d(K, p) \rightarrow H^{d-1}(F(v), p).$$

An element $a \in H^d(K, p)$ is unramified with respect to $v$ if $\partial_v(a) = 0$.

A point $x$ of codimension 1 in $BG$ for an algebraic group $G$ yields a discrete valuation $v_x$ on the function field $F(BG)$ over $F$. Write $A^0(BG, H^d, p)$ for the group of all elements in $H^d(F(BG), p)$ that are unramified with respect to $v_x$ for all points $x$ of codimension 1 in $BG$. It is proved in [6, Part 1, Theorem 11.7] that the value of every invariant from $\text{Inv}^d(G, p)$ at the generic $G$-torsor $E^\text{gen}$ belongs to $A^0(BG, H^d, p)$. Moreover, we have the following theorem (see [6, Part 1, Appendix C]):

**Theorem 2.4.** Let $G$ be an algebraic group over $F$ and $p$ a prime different from $\text{char}(F)$. Then the evaluation of an invariant at the generic $G$-torsor yields an isomorphism

$$\text{Inv}^d(G, p) \sim \rightarrow A^0(BG, H^d, p).$$

The inverse isomorphism is defined as follows. Let $E$ be a $G$-torsor over a field extension $K/F$ and $BG = U/G$. We have the following canonical morphisms:

$$\text{Spec } K = E/G \xleftarrow{f} (E \times U)/G \xrightarrow{h} U/G = BG.$$ 

Note that the groups $H^d(K, p)$ for all $d$ and all field extensions $K/F$ form a cycle module in the sense of Rost (see [13]). In particular, we have flat pull-back homomorphisms

$$H^d(K, p) = A^0(\text{Spec } K, H^d, p) \xrightarrow{f^*} A^0((E \times U)/G, H^d, p) \xleftarrow{h^*} A^0(BG, H^d, p).$$

The variety $(E \times U)/G$ is an open subscheme of the vector bundle $(E \times V)/G$ over $\text{Spec } K$. By the homotopy invariance property, the pull-back homomorphism

$$H^d(K, p) = A^0(\text{Spec } K, H^d, p) \rightarrow A^0((E \times V)/G, H^d, p)$$
is an isomorphism. Since the inclusion of \((E \times U)/G\) into \((E \times V)/G\) is a bijection on points of codimension 1 (by our assumption on the codimension of \(V \setminus U\) in \(V\)), the restriction homomorphism
\[
A^0((E \times V)/G, H^d, p) \longrightarrow A^0((E \times U)/G, H^d, p)
\]
is an isomorphism. It follows that \(f^*\) is an isomorphism.

Let \(a \in A^0(BG, H^d, p)\). The invariant \(I \in \text{Inv}^d(G, p)\) defined by \(I(E) = (f^*)^{-1}h^*(a)\) is the inverse image of \(a\) under the isomorphism in Theorem 2.4.

3. Decomposable invariants

The group of decomposable degree 3 invariants of a semisimple group was defined in [10, §1]. We extend this definition to the class of split reductive groups.

Let \(G\) be a split reductive group over \(F\). The \(\cup\)-product \(H^2(K) \otimes K^\times \longrightarrow H^3(K)\) for any field extension \(K/F\) yields a pairing
\[
\text{Inv}^2(G)_{\text{norm}} \otimes F^\times \longrightarrow \text{Inv}^3(G)_{\text{norm}}.
\]
The subgroup of decomposable invariants \(\text{Inv}^3(G)_{\text{dec}}\) is the image of the pairing.

**Proposition 3.1.** Let \(G\) be a split reductive group over \(F\). Then the composition
\[
\text{Pic}(G) \otimes F^\times \longrightarrow \text{Inv}^2(G)_{\text{norm}} \otimes F^\times \longrightarrow \text{Inv}^3(G)_{\text{dec}}
\]
is an isomorphism.

**Proof.** The surjectivity of the composition follows from the definition. Let \(H\) be the commutator subgroup of \(G\). By [10, Theorem 4.2]), the composition is an isomorphism when \(G\) is replaced by \(H\). The injectivity of the composition for \(G\) follows then from the fact that the map \(\text{Pic}(G) \longrightarrow \text{Pic}(H)\) in (2.2) is an isomorphism.

It follows from the proposition that \(\text{Inv}^3(G)_{\text{dec}} = 0\) if \(\text{Pic}(G) = 0\) (for example, \(G\) is semisimple simply connected) or \(F\) is algebraically closed.

We write
\[
\text{Inv}^3(G)_{\text{ind}} := \text{Inv}^3(G)_{\text{norm}}/\text{Inv}^3(G)_{\text{dec}}.
\]

4. Unramified invariants

Let \(K/F\) be a field extension and \(p\) a prime integer different from \(\text{char}(F)\). We write \(H^d_{\text{nr}}(K/F, p)\) for the subgroup of all elements in \(H^d(K, p)\) that are unramified with respect to all discrete valuations of \(K\) over \(F\). A field extension \(L/K\) yields a natural homomorphism \(H^d(K) \longrightarrow H^d(L)\) that takes \(H^d_{\text{nr}}(K/F, p)\) into \(H^d_{\text{nr}}(L/F, p)\) by [6, Part 1, Proposition 8.2].

Let \(G\) be an algebraic group over \(F\). An invariant \(I \in \text{Inv}^d(G, p)\) is called unramified if for every field extension \(K/F\) and every \(E \in \text{Tors}_G(K)\), we have \(I(E) \in H^d_{\text{nr}}(K/F, p)\). Note that the constant invariants are always unramified. We will write \(\text{Inv}^d_{\text{nr}}(G, p)\) for the subgroup of all unramified invariants in \(\text{Inv}^d(G, p)\).
If \( f : G_1 \to G_2 \) is a group homomorphism, then the map \( f^* \) in (2.3) takes \( \text{Inv}^d_{\text{nr}}(G_2, p) \) into \( \text{Inv}^d_{\text{nr}}(G_1, p) \).

**Proposition 4.1.** Let \( G \) be an algebraic group over \( F \). An invariant \( I \in \text{Inv}^d(G, p) \) is unramified if and only if the value of \( I \) at the generic \( G \)-torsor in \( H^d(FBG, p) \) is unramified. In particular, \( \text{Inv}^d_{\text{nr}}(G, p) \cong H^d_{\text{nr}}(FBG, p) \).

**Proof.** It suffices to show that the inverse of the isomorphism in Theorem 2.4 takes unramified elements to unramified invariants. Let \( a \in H^d_{\text{nr}}(FBG, p) \subset A^0(BG, H^d, p) \). The corresponding invariant \( I \in \text{Inv}^d(G, p) \) is defined by \( I(E) = (f^* h)^{-1} h(a) \) (see Section 2). Note that \( h^* \) takes unramified elements to unramified ones and \( f^* \) yields an isomorphism on the unramified elements as the function field of \( (E \times U)/G \) is a purely transcendental extension of \( K \). It follows that \( I(E) \) is unramified for all \( E \), hence the invariant \( I \) is unramified. \( \square \)

Unramified invariants are constant along rational families of torsors. Precisely, if \( K/F \) is a purely transcendental field extension and \( E \) is a \( G \)-torsor over \( K \), then for every invariant \( I \in \text{Inv}^d_{\text{nr}}(G, p) \) we have

\[
I(E) \in \text{Im}\left( H^d(F, p) \to H^d(K, p) \right).
\]

Indeed, \( I(E) \in H^d_{\text{nr}}(K, p) \) which is the image of \( H^d(F, p) \) in \( H^d(K, p) \).

### 5. Abstract Chern classes

Let \( A \) be a lattice (written additively). Consider the symmetric ring \( S^*(A) \) over \( \mathbb{Z} \) and the group ring \( \mathbb{Z}[A] \) of \( A \). We use the exponential notation for \( \mathbb{Z}[A] \): every element can be written as a finite sum \( \sum_{a \in A} n_a e^a \) with \( n_a \in \mathbb{Z} \). There are the abstract Chern classes (see [10, 3c])

\[
c_i : \mathbb{Z}[A] \to S^i(A), \quad i \geq 0
\]

satisfying in particular,

\[
c_1 \left( \sum_i e^{a_i} \right) = \sum_i a_i \in A \quad \text{and} \quad c_2 \left( \sum_i e^{a_i} \right) = \sum_{i<j} a_i a_j \in S^2(A).
\]

The map \( c_1 \) is a homomorphism and

\[
c_2(x + y) = c_2(x) + c_2(y) + c_1(x) c_1(y)
\]

for all \( x, y \in \mathbb{Z}[A] \).

If \( A \) is a \( W \)-lattice for a group \( W \) acting on \( A \), then all the \( c_i \)'s are \( W \)-equivariant. It follows that \( c_2 \) yields a map (not a homomorphism in general) of groups of \( W \)-invariant elements:

\[
c_2^W : \mathbb{Z}[A]^W \to S^2(A)^W.
\]

The group \( \mathbb{Z}[A]^W \) is generated by the elements \( \sum e^{a_i} \), where the \( a_i \)'s form a \( W \)-orbit in \( A \). It follows that the subgroup of \( S^2(A)^W \) generated by the image of \( c_2^W \) is generated by \( \sum_{i<j} a_i a_j \) with the \( a_i \)'s forming a \( W \)-orbit in \( A \) and \( a a' \).
for $a, a' \in A^W$. The elements of these two types can be viewed as “obvious” elements in $S^2(A)^W$ which we call decomposable.

Write $S^2(A)_{\text{dec}}^W$ for the subgroup of $S^2(A)^W$ generated by the decomposable elements, or equivalently, by the image of $c_2^W$. Set

$$S^2(A)_{\text{ind}}^W := S^2(A)^W / S^2(A)_{\text{dec}}^W.$$  

Note that if $A^W = 0$, the map $c_2^W$ is a homomorphism and $S^2(A)_{\text{ind}}^W$ is the cokernel of $c_2^W$.

**Lemma 5.1.** Let $A_1$ and $A_2$ be $W_1$- and $W_2$-lattices respectively. Then there is a canonical isomorphism

$$S^2(A_1 \oplus A_2)_{\text{ind}}^{W_1 \times W_2} \simeq S^2(A_1)_{\text{ind}}^{W_1} \oplus S^2(A_2)_{\text{ind}}^{W_2}.$$  

**Proof.** We have

$$S^2(A_1 \oplus A_2)^{W_1 \times W_2} \simeq S^2(A_1)^{W_1} \oplus S^2(A_2)^{W_2} \oplus (A_1^{W_1} \otimes A_2^{W_2})$$

and

$$Z[A_1 \oplus A_2]^{W_1 \times W_2} \simeq Z[A_1]^{W_1} \otimes Z[A_2]^{W_2}.$$  

The standard formulas on the Chern classes show that $c_1(Z[A_1]^{W_i}) = A_i^{W_i}$ and

$$S^2(A_1 \oplus A_2)^{W_1 \times W_2}_{\text{dec}} \simeq S^2(A_1)^{W_1}_{\text{dec}} \oplus S^2(A_2)^{W_2}_{\text{dec}} \oplus (A_1^{W_1} \otimes A_2^{W_2}),$$

whence the result. \hfill \Box

**Lemma 5.2.** Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an exact sequence of $W$-lattices. Suppose that $W$ acts trivially on $A$ and $C^W = 0$. Then

1. The sequence

$$0 \rightarrow S^2(A) \rightarrow S^2(B)^W \rightarrow S^2(C)^W$$

is exact.

2. The natural homomorphism $S^2(B)^W_{\text{ind}} \rightarrow S^2(C)^W_{\text{ind}}$ is injective.

**Proof.** The first statement is proved in [5, Lemma 4.9]. Since $W$ acts trivially on $A$, for every subgroup $W' \subset W$, we have $H^1(W', A) = 0$, hence the map $B^{W'} \rightarrow C^{W'}$ is surjective. The group $Z[C]^W$ is generated by elements of the form $\sum_i e^{c_i}$, where the $c_i$'s form a $W$-orbit in $C$. By the surjectivity above, applied to the stabilizer $W' \subset W$, this orbit can be lifted to a $W$-orbit in $B$. Therefore, the map $Z[B]^W \rightarrow Z[C]^W$ is surjective. The second statement follows from this, the first statement of the lemma and the fact that $S^2(A) = S^2(A)^{W}_{\text{dec}} \subset S^2(B)^{W}_{\text{dec}}$. \hfill \Box

6. **Degree 3 invariants of split reductive groups**

Let $G$ be a split reductive group over $F$ and let $H$ be the commutator subgroup of $G$. Thus, $H$ is a split semisimple group and the factor group $Q := G/H$ is a split torus.
**Proposition 6.1.** 1. The restriction maps $\text{Inv}^d(G) \to \text{Inv}^d(H)$ and $\text{Inv}^d(G)_{\text{ind}} \to \text{Inv}^d(H)_{\text{ind}}$ are injective.

2. For every prime $p \neq \text{char}(F)$, the restriction map $\text{Inv}^d_{\text{nr}}(G, p) \to \text{Inv}^d_{\text{nr}}(H, p)$ is an isomorphism.

**Proof.** For a field extension $K/F$, the map $j : H^1(K, H) \to H^1(K, G)$ is surjective as $H^1(K, Q) = 1$ and the group $Q(K)$ acts transitively on the fibers of $j$. It follows that the restriction map $\text{Inv}^d(G) \to \text{Inv}^d(H)$ is injective. The injectivity of $\text{Inv}^d(G)_{\text{ind}} \to \text{Inv}^d(H)_{\text{ind}}$ follows then from Proposition 3.1.

As $Q$ is a rational variety, the fibers of $j$ are rational families of $H$-torsors. Since an unramified invariant of $H$ must be constant on the fibers, it defines an invariant of $G$. This proves the second statement. □

Let $G$ be a split reductive group, $T \subset G$ a split maximal torus. By [10, 3d], there is a commutative diagram

\[
\begin{array}{cccccc}
0 & \to & \text{CH}^2(BG) & \to & \overline{H}_{\text{ét}}^{1,2}(BG) & \to & \text{Inv}^3(G)_{\text{norm}} & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \text{CH}^2(BT) & \to & \overline{H}_{\text{ét}}^{1,2}(BT) & \to & \text{Inv}^3(T)_{\text{norm}} & \to & 0
\end{array}
\]

with the exact rows, where $\overline{H}_{\text{ét}}^{1,2}(BH) = \overline{H}^1(BH, \mathbb{Z}(2))$ for an algebraic group $H$ is the reduced weight two étale motivic cohomology group (see [9, §5]). The group $\text{Inv}^3(T)_{\text{norm}}$ is trivial as $T$ has no nontrivial torsors and $\text{CH}^2(BT) = S^2(T^*)$ by [2, Example A.5], hence the middle term in the bottom row is isomorphic to $S^2(T^*)$.

Let $N$ be the normalizer of $T$ in $G$ and $W = N/T$ the Weyl group. The group $W$ acts naturally on $BT$. Moreover, if $w \in W$, the composition

\[BT \to_{w} BT \to_{s} BG,\]

where $s$ is the natural morphism, coincides with $s$. Therefore, the image of the middle vertical homomorphism in the diagram

\[\overline{H}_{\text{ét}}^{1,2}(BG) \to \overline{H}_{\text{ét}}^{1,2}(BT) = S^2(T^*)\]

is contained in the subgroup $S^2(T^*)^W$ of $W$-invariant elements. By [10, Lemma 3.8], the image of $\text{CH}^2(BG)$ under this homomorphism is equal to $S^2(T^*)_{\text{dec}}$. Therefore, by diagram chase, we have a homomorphism $\text{Inv}^3(G)_{\text{norm}} \to S^2(T^*)_{\text{ind}}$. The group of decomposable invariants $\text{Inv}^3(G)_{\text{dec}}$ is in the kernel of this map since $\text{Inv}^3(G)_{\text{dec}}$ vanishes over an algebraic closure of $F$ and the group $S^2(T^*)_{\text{ind}}$ does not change. Therefore, we have a well-defined homomorphism

\[\alpha_G : \text{Inv}^3(G)_{\text{ind}} \to S^2(T^*)_{\text{ind}}.\]

**Theorem 6.2.** Let $G$ be a split reductive group over $F$. Then the map $\alpha_G$ is injective. If $G$ is semisimple, then $\alpha_G$ is an isomorphism.
Proof. The second statement is proved in [10, Theorem 3.9]. The first statement follows from Proposition 6.1(1), the commutativity of the diagram

\[
\begin{array}{c}
\text{Inv}^3(G)_{\text{ind}} \xrightarrow{\alpha_G} S^2(T^*)^W_{\text{ind}} \\
\downarrow \quad \downarrow \\
\text{Inv}^3(H)_{\text{ind}} \xrightarrow{\alpha_H} S^2(S^*)^W_{\text{ind}},
\end{array}
\]

where \( H \) is the commutator subgroup of \( G \) and \( S \) is a maximal torus of \( H \), and the second statement applied to \( H \). □

Proposition 3.1 and Lemma 5.1 yield the following additivity property.

**Corollary 6.3.** Let \( H_1 \) and \( H_2 \) be two split semisimple groups. Then there is a canonical isomorphism

\[
\text{Inv}^3(H_1 \times H_2) \simeq \text{Inv}^3(H_1) \oplus \text{Inv}^3(H_2).
\]

Let \( H \) be a split semisimple group over a field \( F \), \( \pi : \tilde{H} \to H \) a simply connected cover, \( \tilde{S} \) the pre-image of a split maximal torus \( S \) of \( H \), so \( \tilde{S} \) is a split maximal torus of \( \tilde{H} \). Then \( S^2(S^*) \) can be viewed with respect to \( \pi \) as a sublattice of \( S^2(\tilde{S}^*) \) of finite index and we have the following commutative diagram

\[
\begin{array}{c}
\text{Inv}^3(H)_{\text{ind}} \xrightarrow{\alpha_H} S^2(S^*)^W_{\text{ind}} \\
\downarrow \quad \downarrow \\
\text{Inv}^3(\tilde{H})_{\text{norm}} \xrightarrow{\alpha_{\tilde{H}}} S^2(\tilde{S}^*)^W_{\text{ind}}.
\end{array}
\]

If \( H \) is simple, the group \( S^2(\tilde{S}^*)^W \) is infinite cyclic with a canonical generator \( q \) (see [6, Part 2, §7]). It follows that \( S^2(S^*)^W \) is also infinite cyclic with \( kq \) a generator for a unique integer \( k > 0 \). The invariant \( R \in \text{Inv}^3(\tilde{H})_{\text{norm}} \) corresponding to the generator \( q \) is called the **Rost invariant** of \( \tilde{H} \). It is a generator of the cyclic group \( \text{Inv}^3(\tilde{H}) \).

7. **Change of groups**

In this section we prove the following useful property.

**Proposition 7.1.** Let \( p \) be a prime integer different from \( \text{char}(F) \), \( G \) an algebraic group over \( F \), \( C \subset G \) a finite central diagonalizable subgroup of order not divisible by \( p \), \( H = G/C \). Then the natural maps \( \text{Inv}^d(H,p) \to \text{Inv}^d(G,p) \) and \( \text{Inv}^d_{\text{nr}}(H,p) \to \text{Inv}^d_{\text{nr}}(G,p) \) are isomorphisms.

Proof. Both functors in the definition of an invariant can be naturally extended to the category \( C \) of \( F \)-algebras that are finite product of fields, and every invariant extends uniquely to a morphism of extended functors. If \( K \to L \) is a morphism in \( C \) and \( M \) is an étale \( K \)-algebra, then \( L \otimes_K M \) is also an object of the category \( C \).
For any $K$ in $\mathcal{C}$ we have an exact sequence

$$H^1_{\text{et}}(K, G) \rightarrow H^1_{\text{et}}(K, H) \xrightarrow{\delta_K} H^2_{\text{et}}(K, C)$$

and the group $H^1_{\text{et}}(K, C)$ acts transitively on the fibers of the first map in the sequence.

**Proof of injectivity.** Let $I \in \text{Inv}^d(H, p)$ be such that $f^*(I) = 0$, where $f : G \rightarrow H$ is the canonical homomorphism. We prove that $I = 0$. Take any $K$ in $\mathcal{C}$ and $E \in \text{Tors}_H(K)$. As an element of the group $H^2_{\text{et}}(K, C)$ is a tuple of elements in $\text{Br}(K)$ of order prime to $p$, there is an étale $K$-algebra $L$ of (constant) finite rank $[L : K]$ prime to $p$ such that $\delta_L(E_L) = 0$. It follows that $E_L = f_*(E')$ for some $E' \in \text{Tors}_G(L)$. We have

$$I(E)_L = I(E) = I(f_*(E')) = f^*(I)(E') = 0.$$ 

Since $[L : K]$ is prime to $p$, we have $I(E) = 0$, i.e., $I = 0$.

**Proof of surjectivity.** Let $J \in \text{Inv}^d(G, p)$. We construct an invariant $I \in \text{Inv}^d(H, p)$ such that $J = f^*(I)$. Take any $K$ in $\mathcal{C}$ and $E \in \text{Tors}_H(K)$. As above, choose an étale $K$-algebra $L$ of finite rank prime to $p$ such that $\delta_L(E_L) = 0$ and an element $E' \in \text{Tors}_G(L)$ with $E_L = f_*(E')$. We set

$$I(E) = \frac{1}{[L : K]} \text{cor}_{L/F}(J(E')).$$

This is independent of the choice of $E'$. Indeed, if $E_L = f_*(E'')$ for $E'' \in \text{Tors}_G(L)$, then there exists $\nu \in H^1_{\text{et}}(L, C)$ with $E'' = \nu(E')$. Choose an $L$-algebra $P$ of constant rank $[P : L]$ prime to $p$ such that $\nu_P = 1$. It follows that $E''_P = E'_P$ and therefore,

$$[P : L] \text{cor}_{L/F}(J(E'')) = \text{cor}_{P/F}(J(E''_P)) = \text{cor}_{P/F}(J(E'_P)) = [P : L] \text{cor}_{L/F}(J(E')).$$

Since $[P : L]$ is prime to $p$, we have $\text{cor}_{L/F}(J(E'')) = \text{cor}_{L/F}(J(E'))$.

In order to show that the value $I(E)$ is independent of the choice of $L$, for the two choices $L$ and $L'$, it suffices to compare the formulas for $L$ and $LL' := L \otimes_F L'$:

$$\frac{1}{[L : K]} \text{cor}_{L/F}(J(E')) = \frac{[L' : K]}{[LL' : K]} \text{cor}_{L/F}(J(E')) = \frac{1}{[LL' : K]} \text{cor}_{LL'/F}(J(E'_{LL'})).$$

We have constructed the invariant $I \in \text{Inv}^d(H, p)$. For any $K$ in $\mathcal{C}$ and $E' \in \text{Tors}_G(K)$, by the definition of $I$, we have $f^*(I)(E') = I(f_*(E)) = J(E')$, hence $f^*(J) = J$. Note that if $J$ is an unramified invariant, $I$ is also unramified since the corestriction map preserves unramified elements by [6, Part 1, Proposition 8.6].

8. Degree 3 unramified invariants of simple groups

The following statement was proved in [11] (classical groups) and [7] (exceptional groups).
Proposition 8.1. Let $H$ be an absolutely simple simply connected group over $F$ and $p$ a prime different from $\text{char}(F)$.

1. If the Dynkin diagram of $H$ is different from $2A_n$, $n$ odd, and $3D_4$, then $\text{Inv}_m^3(H,p)_{\text{norm}} = 0$.

2. If $H$ is split, then $\text{Inv}_m^3(H,p)_{\text{norm}} = 0$.

Let $H$ be a semisimple group over $F$, $E$ an $H$-torsor over $\text{Spec}(K)$ for a field extension $K/F$. The twist $H^E := \text{Aut}_H(E)$ of $H$ by $E$ is a semisimple group over $K$. The twisting argument shows that $BH^E = BH_K$ and there is a canonical isomorphism $\text{Inv}^d(H^E) \simeq \text{Inv}^d(H_K)$. If $E_{\text{gen}}$ is a generic $H$-torsor, we write $H^\text{gen}$ for $H^{E_{\text{gen}}}$. Let $H^\text{gen} \to H^\text{gen}$ be a simply connected cover.

Proposition 8.2. Let $H$ be a split simple group. Then the composition

$$\text{Inv}^3(H)_{\text{ind}} \to \text{Inv}^3(H^\text{gen})_{\text{ind}} \to \text{Inv}^3(\tilde{H}^\text{gen})_{\text{ind}} = \text{Inv}^3(\tilde{H}^\text{gen})$$

is injective.

Proof. The statement is clear if $H$ is a simply connected group. The case of an adjoint group $H$ was considered in [10, Theorem 4.10]. Consider the other split semisimple groups type-by-type. It suffices to restrict to the $p$-component of $\text{Inv}^3(H)$ for a prime $p$.

Type $A_{n-1}$, $n \geq 2$. We have $H = S_{\mu_m}$ for an integer $m$ dividing $n$. By Proposition 7.1, we may assume that $m = p^r$ for some $r$. It is shown in [1, Theorem 4.1] and Theorem 6.2 that

$$\text{Inv}^3(H)_{\text{ind}} \xrightarrow{\sim} S^2(S^*)^W_{\text{ind}} \hookrightarrow (\mathbb{Z}/m\mathbb{Z})q.$$

On the other hand, an $H$-torsor yields a central simple algebra of degree $n$ and exponent dividing $m$. A generic torsor gives an algebra with the exponent exactly $m$, hence $\text{Inv}^3(\tilde{H}^\text{gen}) = (\mathbb{Z}/m\mathbb{Z})R$ by [6, Part 2, Theorem 11.5].

Type $D_n$, $n \geq 4$. We have $H = O^{\pm}_{2n}$, the special orthogonal group or $H = \text{HSpin}_{2n}$, the half-spin group if $n$ is even. It is shown in [6, Part 1, Chapter VI] in the case $\text{char}(F) \neq 2$ that $\text{Inv}^3(O^+_{2n})_{\text{ind}} = 0$. In general, recall that the character group of a maximal split torus $S$ is a free group of rank $n$. Let $x_1, x_2, \ldots, x_n$ be a basis for $S^*$ such that the Weyl group $W$ acts on the $x_i$’s by permutations and change of signs. The generator of $S^2(S^*)^W$ is the quadratic form $q = x_1^2 + x_2^2 + \cdots + x_n^2$. It is in $S^2(S^*)_{\text{dec}}^W$ since $c_2(\sum_i e^{x_i} + e^{-x_i}) = -q$. By [10, Theorem 3.9], $\text{Inv}^3(O^+_{2n})_{\text{ind}} = 0$.

Finally, assume that $n$ is even and $H = \text{HSpin}_{2n}$, the half-spin group. It follows from [1, Theorem 5.1] and Theorem 6.2 that

$$\text{Inv}^3(H)_{\text{ind}} \xrightarrow{\sim} S^2(S^*)^W_{\text{ind}} \hookrightarrow (\mathbb{Z}/4\mathbb{Z})q$$

and $\text{Inv}^3(H)_{\text{ind}} = 0$ if $n = 4$. On the other hand, an $H$-torsor yields a central simple algebra of degree $2n$. A generic torsor gives a nonsplit algebra. By [6, Part 2, Theorem 15.4], $\text{Inv}^3(\tilde{H}^\text{gen}) = (\mathbb{Z}/4\mathbb{Z})R$ if $n > 4$. □

Remark 8.3. The statement fails for semisimple groups that are not simple, see Example 11.2.
Theorem 8.4. Let $H$ be a split simple group over an algebraically closed field $F$ and $p$ a prime integer different from $\text{char}(F)$. Then $\text{Inv}^3_{nr}(H, p) = 0$.

Proof. Let $I \in \text{Inv}^3_{nr}(H, p)$. Note that since $F$ is algebraically closed, every decomposable invariant is trivial.

The pull-back $\tilde{I}$ of $I$ under the composition in Proposition 8.2 is an unramified invariant. As $\tilde{H} \text{gen}$ is an inner form of $\tilde{H}$, by Proposition 8.1, $\tilde{I} = 0$ and hence $I = 0$ by Proposition 8.2 unless the Dynkin diagram of $H$ is $D_4$.

If $H$ is a simply connected group of type $D_4$, then $I = 0$ by [1, Theorem 5.1]. Finally assume that $H$ is an adjoint group of type $D_4$. By [10, Theorem 4.7], the group $\text{Inv}^3(H)$ is cyclic of order 2.

Assume that $I \neq 0$. The group $\tilde{H} \text{gen}$ is the spinor group of a central simple algebra $A$ of degree 8 with orthogonal involution $\sigma$ of trivial discriminant. Consider the corresponding special orthogonal group $\hat{H} \text{gen} := O^+(A, \sigma)$ of $(A, \sigma)$. An $\hat{H} \text{gen}$-torsor over a field $K$ is given by a pair $(a, x)$, where $a$ is an invertible $\sigma$-symmetric element in $A$ and $x \in K^\times$ such that $\text{Nrd}(a) = x^2$ and $\text{Nrd}$ is the reduced norm map (see [8, 29.27]).

The canonical homomorphism $\text{Inv}^3(H \text{gen}) \rightarrow \text{Inv}^3(\hat{H} \text{gen})$ factors through $\text{Inv}^3(\tilde{H} \text{gen})$. By [10, §4, type $D_n$], the pull-back of $I$ in $\text{Inv}^3(\hat{H} \text{gen})$ is the class of the invariant taking a pair $(a, x)$ to the cup-product $(x) \cup [A] \in H^3(K)$. This invariant is ramified as it is non-constant when $a$ runs over a subfield of $A$ of dimension $n$ fixed by $\sigma$ element-wise, a contradiction. □

9. Structure of reductive groups

Let $H$ be a split semisimple group over a field $F$, $S \subset H$ a split maximal torus. Write $\Lambda_r \subset S^*$ for the root lattice of $H$. Let $\tilde{H} \rightarrow H$ be a simply connected cover and let $\tilde{S}$ for the inverse image of $S$, a maximal torus in $\tilde{H}$. Write $\Lambda_w$ for the character group of $\tilde{S}$. This is the weight lattice freely generated by the fundamental weights. We have

$$\Lambda_r \subset S^* \subset \Lambda_w.$$

The center $C$ of $H$ is a finite diagonalizable group with $C^* = S^*/\Lambda_r$.

Let $G$ be a split reductive group over a field $F$ with the commutator subgroup $H$. Choose a split maximal $T \subset G$ such that $T \cap H = S$. The roots of $H$ can be uniquely lifted to $T^*$ (to the roots of $G$), so the inclusion of $\Lambda_r$ into $S^*$ is lifted to the inclusion of $\Lambda_r$ into $T^*$. The composition $\tilde{S} \rightarrow S \rightarrow T$ yields a homomorphism $T^* \rightarrow \Lambda_w$ of lattices. Thus, we have the two homomorphisms

\begin{equation} (9.1) \Lambda_r \rightarrow T^* \rightarrow \Lambda_w \end{equation}

with the composition the canonical embedding of $\Lambda_r$ into $\Lambda_w$. The image of $f$ in (9.1) is equal to $S^*$. The center $Z$ of $G$ is a diagonalizable group with $Z^* = T^*/\Lambda_r$. The factor group $G/H = T/S$ is a torus $Q$ with the character lattice $Q^* = \text{Ker}(f)$. 

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We would like to study all split reductive groups with the fixed commutator subgroup \( H \).

Let \( H \) be a split semisimple group over \( F \). Fix a split maximal torus \( S \subset H \) and consider the root system of \( H \) relative to \( S \) with the root and weight lattices \( \Lambda_r \subset \Lambda_w \) respectively.

Consider a category \( \text{Red}(H) \) with objects split reductive groups \( G \) over \( F \) with the commutator subgroup \( H \). A morphism between \( G_1 \) and \( G_2 \) in this category is a group homomorphism \( G_1 \to G_2 \) over \( F \) that is the identity on \( H \).

Consider another category \( \text{Lat}(H) \) with objects the diagrams of the form

\[
\begin{align*}
\Lambda_r & \to A \\
A & \to \Lambda_w,
\end{align*}
\]

where \( A \) is a lattice, \( \text{Im}(f) = S^* \) and the composition is the embedding of \( \Lambda_r \) into \( \Lambda_w \). A morphism in \( \text{Lat}(R) \) is a morphism between the diagrams which is identity on \( \Lambda_r \) and \( \Lambda_w \).

Let \( G \) be an object in \( \text{Red}(H) \). Write \( Z \) for the center of \( G \). Then \( T := S \cdot Z \) is a split maximal torus of \( G \). The diagram (9.1) yields then a contravariant functor

\[
\rho : \text{Red}(H) \to \text{Lat}(H).
\]

Proposition 9.3. For every split semisimple group \( H \), the functor \( \rho \) is an equivalence of categories \( \text{Red}(H) \) and \( \text{Lat}(H)^{op} \).

Proof. We construct a functor \( \varepsilon : \text{Lat}(H) \to \text{Red}(H) \) as follows. Given the diagram (9.2), let \( T \) be a split torus with \( T^* = A \) and \( Z \) a diagonalizable subgroup of \( T \) with \( Z^* = A/\Lambda_r \). We view the torus \( S \) as a subgroup of \( T \) via the dual surjective homomorphism \( A \to \text{Im}(f) = S^* \).

We embed the center \( C \) of \( H \) into \( Z \) via a homomorphism dual to the surjective composition

\[
Z^* = A/\Lambda_r \to \text{Im}(f)/\Lambda_r = S^*/\Lambda_r = C^*.
\]

The sequence

\[
0 \to A \xrightarrow{g} S^* \oplus (A/\Lambda_r) \xrightarrow{h} S^*/\Lambda_r \to 0,
\]

where \( g(a) = (f(a), a + \Lambda_r) \) and \( h(x, a + \Lambda_r) = (x - f(a)) + \Lambda_r \) is exact. It follows that the product homomorphism \( S \times Z \to T \) is surjective with the kernel \( C \) embedded into \( S \times Z \) via \( c \mapsto (c, c^{-1}) \), i.e., \( T \simeq (S \times Z)/C \).

We set \( G = (H \times Z)/C \). The group \( Z \) is naturally a subgroup of \( G \) which coincides with the center of \( G \). The torus \( T \) is a subgroup of \( G \) generated by \( S \) and \( Z \), hence \( T \) is a split maximal torus of \( G \). The natural sequence

\[
0 \to \ker(f) \to A/\Lambda_r \to \text{Im}(f)/\Lambda_r \to 0
\]

is exact. It follows that \( Z/C \) is a torus dual to \( \ker(f) \). Since \( G/H \simeq Z/C \), \( G \) is a (smooth connected) reductive group with \( H \) the commutator subgroup. The functor \( \varepsilon \), by definition, takes the diagram (9.2) to the group \( G \). By construction, both compositions of \( \rho \) and \( \varepsilon \) are isomorphic to the identity functors. \( \square \)
Let $H$ be a split semisimple group as above. We consider another category $\text{Mor}(H)$ with objects homomorphisms $h : B \rightarrow \Lambda_w/\Lambda_r$ with $B$ a finitely generated abelian group, $\text{Im}(h) = S^*/\Lambda_r$, and torsion free $\text{Ker}(h)$. Morphisms are defined in the obvious way. Consider a contravariant functor $\nu : \text{Red}(H) \rightarrow \text{Mor}(H)$ taking a reductive group $G$ to the composition $Z^* \rightarrow C^* \hookrightarrow \Lambda_w/\Lambda_r$, where $Z$ is the center of $G$. The kernel of this homomorphism is the character lattice of the torus $Z/C = G/H$ and hence has no torsion.

**Proposition 9.4.** For every split semisimple group $H$, the functor $\nu$ is an equivalence of categories $\text{Red}(H)$ and $\text{Mor}(H)^{\text{op}}$.

**Proof.** We construct a functor $\lambda : \text{Mor}(H) \rightarrow \text{Red}(H)$ as follows. Let $h : B \rightarrow \Lambda_w/\Lambda_r$ be an object in $\text{Mor}(H)$ and $Z$ a diagonalizable group with $Z^* = B$. The map $h$ yields an embedding of $C$ into $Z$ and the factor group $Z/C$ is a torus. Set $G = (H \times Z)/C$ as in the proof of Proposition 9.3. The factor group $G/H$ is isomorphic to the torus $Z/C$, hence $G$ is a reductive group with the commutator subgroup $H$, i.e., $G$ is an object of $\text{Red}(H)$. Then $Z$ is the center of $G$ as the group $G/Z \cong H/C$ is adjoint. We set $\lambda(h) = G$. By construction, both compositions of $\rho$ and $\lambda$ are isomorphic to the identity functors. □

**Remark 9.5.** It follows from Propositions 9.3 and 9.4 that the categories $\text{Lat}(H)$ and $\text{Mor}(H)$ are equivalent. An equivalence between the categories can be described directly as follows. If $\Lambda_r \rightarrow A \xrightarrow{f} \Lambda_w$ is an object in $\text{Lat}(H)$, then the induced morphism $A/\Lambda_r \rightarrow \Lambda_w/\Lambda_r$ is the corresponding object in $\text{Mor}(H)$. Conversely, let $\mu : B \rightarrow \Lambda_w/\Lambda_r$ be an object in $\text{Mor}(H)$. Write $A$ for the kernel of the homomorphism

$$h : S^* \oplus B \rightarrow S^*/\Lambda_r$$

defined by $h(x, b) = (x + \Lambda_r) - \mu(b)$. The corresponding object

$$\Lambda_r \rightarrow A \xrightarrow{f} \Lambda_w$$

in $\text{Lat}(H)$ is defined as follows. The map $f$ is given by the first projection followed by the inclusion of $S^*$ into $\Lambda_w$ and the inclusion $\Lambda_r \rightarrow A$ takes $x$ to $(x, 0)$. Note that $W$ acts on $S^*$ naturally on $S^*$ and trivially on $B$.

A split reductive group $G$ is called *strict* if the center $Z$ of $G$ is a torus, i.e., $Z^*$ is a lattice. An object $G$ of $\text{Red}(H)$ is *strict* if $G$ is strict. If $B \rightarrow \Lambda_w/\Lambda_r$ is the object $\nu(G)$ of $\text{Mor}(H)$, then $G$ is strict if and only if $B$ is torsion-free.

A semisimple group is strict if and only if it is adjoint. A *strict envelope* of a split semisimple group $H$ is a strict object in $\text{Red}(H)$.

**Example 9.6.** The group $\text{GL}_n$ is a strict envelope of $\text{SL}_n$.

**Example 9.7.** The object $G$ in $\text{Red}(H)$ corresponding to the composition $S^* \rightarrow S^*/\Lambda_r \hookrightarrow \Lambda_w/\Lambda_r$, viewed as an object of the category $\text{Mor}(H)$, is
strict. We call such $G$ the *standard* strict envelope of $H$. By Remark 9.5, the lattice $T^*$ is the subgroup in $S^* \oplus S^*$ consisting of all pairs $(x, y)$ such that $x - y \in \Lambda_r$. Note that the Weyl group acts naturally on the first component of $S^* \oplus S^*$ and trivially on the second.

A strict envelope of $H$ behaves like an “injective resolution” of $H$.

**Lemma 9.8.** Let $G_1$ and $G_2$ be two objects in $\text{Red}(H)$. If $G_2$ is strict, then there is a morphism $G_1 \to G_2$ in $\text{Red}(H)$.

**Proof.** Let $h_i : B_i \to \Lambda_w/\Lambda_r$ be the object $\nu(G_i)$ in $\text{Mor}(H)$ for $i = 1, 2$. By assumption, $B_2$ is a free $\mathbb{Z}$-module. Therefore, there is a group homomorphism $g : B_2 \to B_1$ such that $g \circ h_1 = h_2$, i.e., $g$ is a morphism in $\text{Mor}(H)$. By Proposition 9.4, there is a morphism $G_1 \to G_2$ in $\text{Red}(H)$ corresponding to $g$. □

10. Reductive invariants

Let $H$ be a split semisimple group and $G$ is a reductive group with the commutator subgroup $H$, i.e., $G$ is an object in $\text{Red}(H)$. By Proposition 6.1, the map $\text{Inv}^d(G) \to \text{Inv}^d(H)$ is injective. We view $\text{Inv}^d(G)$ as a subgroup of $\text{Inv}^d(H)$. If $G'$ is a strict envelope of $H$, then it follows from Lemma 9.8 that $\text{Inv}^d(G') \subset \text{Inv}^d(G)$. Therefore, the subgroup $\text{Inv}^d(G')$ is independent of the choice of the strict resolution $G'$ of $G$. We write $\text{Inv}^d_{\text{red}}(H)$ for this subgroup and call the invariants in this group the reductive invariants. By Proposition 6.1, for any prime $p \neq \text{char}(F)$ we have

$$\text{Inv}^d_{\text{red}}(H, p) \subset \text{Inv}^d_{\text{red}}(H, p) \subset \text{Inv}^d(H, p).$$

Let $A$ be a lattice and $q \in S^2(A)$. We can view $q$ as an integral quadratic form on the lattice $\hat{A}$ dual to $A$. The polar bilinear form $h$ of $q$ is the image of $q$ under the polar map $\text{pol} : S^2(A) \to A \otimes A$, $aa' \mapsto a \otimes a' + a' \otimes a$. The polar form $h$ is symmetric and *even*, i.e., $h(x, x) \in 2\mathbb{Z}$ for all $x \in \hat{A}$. Conversely, if $h \in A \otimes A$ is a symmetric even bilinear form, then $q(x) = \frac{1}{2}h(x, x)$ is an integral quadratic form with the polar form $h$.

Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a set of simple roots of an irreducible root system, $\{w_1, w_2, \ldots, w_n\}$ the corresponding fundamental weights generating the weight lattice $\Lambda_w$ and $W$ the Weyl group. Let $d_i$ be the square of the length of the co-root $\alpha_i^\vee$ (we assume that the length of the shortest co-root is 1). Consider the bilinear form

$$h = \sum_{i=1}^{n} w_i \otimes d_i \alpha_i = \sum_{i,j} w_i \otimes d_i c_{ij} w_j \in \Lambda_w \otimes \Lambda_w,$$

where $(c_{ij})$ is the Cartan matrix (see [4, Chapitre VI]). The matrix $(d_i c_{ij})$ is symmetric with even diagonal terms, hence $h$ is a symmetric even bilinear form.
form. The corresponding quadratic form
\[ q = \frac{1}{2} \sum_{i=1}^{n} d_i w_i \alpha_i \in S^2(\Lambda_w) \]
is \(W\)-invariant by [10, Lemma 3.2]. It follows that the polar form \(h\) of \(q\) is also \(W\)-invariant.

Consider the three embeddings \(i_1, i_2, j = i_1 + i_2 : \Lambda_w \to \Lambda_w^2 := \Lambda_w \oplus \Lambda_w\) given by \(x \mapsto (x, 0), (0, x), (x, x)\) respectively, and the two quadratic forms \(q^{(1)}, q^{(2)}\) that are the images of \(q\) under the maps \(S^2(i_1), S^2(i_2) : S^2(\Lambda_w) \to S^2(\Lambda_w^2)\) respectively. We let \(W\) act on \(\Lambda_w^2\) naturally on the first summand and trivially on the second.

Let \(A\) be the sublattice of \(\Lambda_w^2\) of all pairs \((x, y)\) such that \(x - y \in \Lambda_r\). Note that \(\text{Im}(j) \subseteq A\). In particular, \(S^2(j)(q) \in S^2(A)\). Moreover, since \(h \in (\Lambda_r \otimes \Lambda_w) \cap (\Lambda_w \otimes \Lambda_r)\) by [9, Lemma 2.1], we have \((i_k \otimes j)(h) \in A \otimes A\) and \((j \otimes i_k)(h) \in A \otimes A\) for \(k = 1, 2\).

Write \(m : \Lambda_w^2 \otimes \Lambda_w^2 \to S^2(\Lambda_w^2)\) for the canonical homomorphism. We have \(m(i_k \otimes j)(h) \in S^2(A)\) and \(m(j \otimes i_k)(h) \in S^2(A)\) for \(k = 1, 2\).

**Proposition 10.2.** We have \(q^{(1)} - q^{(2)} \in S^2(A)^W\) with the polar form \(h^{(1)} - h^{(2)} = (j \otimes i_1)(h) - (i_2 \otimes j)(h) \in A \otimes A\).

**Proof.** By construction, \(q^{(1)} - q^{(2)}\) is \(W\)-invariant. We have
\[
q^{(1)} - q^{(2)} = (q^{(1)} + q^{(2)}) - 2q^{(2)} = q^{(1)} + q^{(2)} - m(i_2 \otimes i_2)(h) = q^{(1)} + q^{(2)} + m(i_1 \otimes i_2)(h) - m(j \otimes i_2)(h)
= S^2(j)(q) - m(j \otimes i_2)(h) \in S^2(A).
\]
The second statement follows from the equality \(j = i_1 + i_2\).

**Corollary 10.3.** The image of \(h^{(1)} - h^{(2)}\) under the map
\[
A \otimes A \overset{p_1 \otimes 1}{\longrightarrow} \Lambda_w \otimes A,
\]
where \(p_1\) is the first projection, coincides with the image of \(h\) under the natural map
\[
\Lambda_w \otimes \Lambda_r \overset{1 \otimes i_1}{\longrightarrow} \Lambda_w \otimes A.
\]

**Proof.** The statement follows from Proposition 10.2 and the equalities \(p_1 \circ j = p_1 \circ i_1 = 1\) and \(p_1 \circ i_2 = 0\).

Let \(\tilde{H}\) be a split simply connected cover of \(H\) with a split maximal torus \(\tilde{S}\), thus \(\tilde{S}^* = \Lambda_w\). Consider the standard strict envelope \(\tilde{G}\) of \(\tilde{H}\) (see Example 9.7). The character group \(\tilde{T}^*\) of the maximal torus \(\tilde{T}\) of \(\tilde{G}\) coincides with the group \(A\) as above. If \(\tilde{H}\) is simple, by Proposition 9.7, \(\tilde{q} := q^{(1)} - q^{(2)} \in S^2(\tilde{T}^*)^W\). The form \(\tilde{q}\) maps to \(q\) under the natural map \(S^2(\tilde{T}^*)^W \to S^2(\tilde{S}^*)^W = S^2(\Lambda_w^*)^W\).
In the general case,

\[ \tilde{H} = \tilde{H}_1 \times \tilde{H}_2 \times \cdots \times \tilde{H}_s, \]

with \( \tilde{H}_j \) the simple simply connected components of \( \tilde{H} \). The components define a basis \( q_1, q_2, \ldots, q_s \) of \( S^2(H^*)^W \). Every \( q_j \) has a lift \( \tilde{q}_j \in S^2(\tilde{T}^*)^W \) as above. Lemma 5.2 then yields the following statement.

**Corollary 10.4.** The map \( S^2(\tilde{T}^*)^W \to S^2(\tilde{S}^*)^W \) is surjective and \( S^2(\tilde{T}^*)_{\text{ind}}^W \to S^2(\tilde{S}^*)_{\text{ind}}^W \) is an isomorphism. In particular, \( S^2(\tilde{T}^*)_{\text{ind}}^W \) is generated by the classes of the forms \( \tilde{q}_j \).

We will write \( \alpha_{ij} \) for the simple roots of the \( j \)-th component and \( w_{ij} \) for the corresponding fundamental weights, etc.

Let \( \tilde{C} \subset \tilde{H} \) be a central subgroup and set \( G := \tilde{H}/\tilde{C} \) and \( T := \tilde{T}/\tilde{C} \). The character group \( \tilde{C}^* \) is a factor group of \( \Lambda_w/\Lambda_r \). Consider the composition

\[ S^2(\tilde{T}^*) \xrightarrow{\text{pol}} \tilde{T}^* \otimes \tilde{T}^* \xrightarrow{p_1} \Lambda_w \otimes \tilde{T}^* \to \tilde{C}^* \otimes \tilde{T}^*. \]

where \( \text{pol} \) denotes the image of an \( x \in \tilde{T}^* \) in \( \tilde{C}^* \).

Let \( \tilde{T}_j \subset \tilde{T} \) be a maximal torus of of the \( j \)-th simple component of \( \tilde{G} \), so that \( \tilde{T} = \tilde{T}_1 \times \cdots \times \tilde{T}_s \). Let \( \tilde{C}_j \) be the image of the projection \( \tilde{C} \to \tilde{T}_j \). Then \( \tilde{C}_j^* \) can be viewed as a subgroup of \( \tilde{C}^* \) and \( \overline{w}_{ij} \in \tilde{C}_j^* \).

**Proposition 10.6.** Let \( q := \sum_{j=1}^s k_j \tilde{q}_j \in S^2(\tilde{T}) \) be a linear combination with integer coefficients \( k_j \). If \( q \) has trivial image under the composition (10.5) (for example, if \( q \in S^2(T^*) \)), then the order of \( \overline{w}_{ij} \) in \( \tilde{C}_j^* \) divides \( k_j d_{ij} \) for all \( i \) and \( j \).

**Proof.** We have \( \sum_{i,j} k_j d_{ij} \overline{w}_{ij} \otimes (\alpha_{ij}, 0) = 0 \) in \( \tilde{C}_j^* \otimes \tilde{T}^* \). Note that the elements \( (\alpha_{ij}, 0) \) form part of a basis of \( \tilde{T}^* \) (with the complement \( (w_{ij}, w_{ij}) \)). It follows that \( k_j d_{ij} \overline{w}_{ij} = 0 \) in \( \tilde{C}_j^* \) for all \( i \) and \( j \), whence the result. \( \square \)

11. **Degree 3 unramified invariants of reductive groups**

We assume that the base field \( F \) is algebraically closed.

**Proposition 11.1.** Let \( H \) be a (split) semisimple group over \( F \) with the components of the Dynkin diagram of types \( A_m \) for some \( m \) or \( E_6 \). Suppose that \( E_6^{\text{sc}} \) does not split off \( H \) as a direct factor. Then \( \text{Inv}_{\text{red}}^3(H, p) = \text{Inv}_{\text{nr}}^3(H, p) = 0 \) for all odd primes \( p \neq \text{char}(F) \).
Proof. Let \( \tilde{H} \rightarrow H \) be a simply connected cover with kernel \( \tilde{C} \) and \( \tilde{G} \) be the standard strict envelope of \( \tilde{H} \). By Proposition 7.1, replacing \( \tilde{C} \) if necessary, we may assume that \( \tilde{C}^* \) is a \( p \)-group. Set \( G := \tilde{G}/\tilde{C} \). We choose split maximal tori \( S \subset H, \tilde{S} \subset \tilde{H}, T \subset G, \tilde{T} \subset \tilde{G} \) as in Section 10. The group \( Q := G/H = \tilde{G}/\tilde{H} \) is a torus.

By Proposition 6.1, it suffices to prove that \( \text{Inv}^3(G, p) = 0 \). By Theorem 6.2, we are reduced to proving that \( S^2(T^*)^W_{\text{ind}}(p) = 0 \).

By Lemma 5.2(1) and Corollary 10.4, the rows of the diagram

\[
\begin{array}{cccc}
0 & \longrightarrow & S^2(Q^*) & \longrightarrow & S^2(T^*)^W & \longrightarrow & S^2(S^*)^W \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & S^2(Q^*) & \longrightarrow & S^2(\tilde{T}^*)^W & \longrightarrow & S^2(\tilde{S}^*)^W & \longrightarrow & 0
\end{array}
\]

are exact.

Let \( \alpha \in S^2(T^*)^W_{\text{ind}}(p) \). Since \( p \) is odd, it sufficient to show that \( 2\alpha = 0 \). The element \( \alpha \) lifts to a form \( q \in S^2(T^*)^W \). Recall that \( S^2(\tilde{S}^*)^W \) is a free abelian group with basis \( \{ \tilde{q}_j \} \). Hence the image of \( q \) in \( S^2(\tilde{S}^*)^W \) is equal to \( \sum_{j=1}^{s} k_j \tilde{q}_j \) for some \( k_j \in \mathbb{Z} \). Write \( \tilde{q} \) for \( \sum_{j=1}^{s} k_j \tilde{q}_j \in S^2(\tilde{T}^*)^W \). Therefore, in \( S^2(T^*)^W \) we have \( q = \tilde{q} + t \) for some \( t \in S^2(Q^*) \).

Note that since the Dynkin diagram of \( H \) is simply laced all the integer \( d_{ij} \) are equal to 1 for all \( i \) and \( j \). The images of \( q \) and \( t \) are trivial under (10.5), hence so is \( \tilde{q} \). By Proposition 10.6, the order of \( \overline{w}_{ij} \) in \( \tilde{C}^*_j \) divides \( k_j \) for all \( i \) and \( j \).

We claim that the class of \( 2k_jq_j \) is contained in \( S^2(S^*)^W_{\text{dec}} \) for all \( j \).

Case 1: The \( j \)-th simple component \( \tilde{G}_j \) is of type \( A_m \) for some \( m \), i.e., \( \tilde{H}_j = \text{SL}_{m+1} \). The center of \( \tilde{H}_j \) is \( \mu_{m+1} \), hence \( \tilde{C}_j = \mu_{p^r} \) for some \( r \). The element \( \overline{w}_{ij} \) is a generator of \( \tilde{C}^* = \mathbb{Z}/p^r\mathbb{Z} \), hence the order of \( \overline{w}_{ij} \) is equal to \( p^r \). Therefore, \( k_j \) is divisible by \( p^r \). As \( p \) is odd, by [1, 4.2], the form \( p^r k_\tilde{q}_j \) and hence \( k_j q_j \) belongs to \( S^2(S^*)^W_{\text{dec}} \). Taking the image of \( k_j q_j \) under the homomorphism \( S^*_j \rightarrow S^* \), we see that \( k_j q_j \in S^2(S^*)^W_{\text{dec}} \).

Case 2: The \( j \)-th simple component \( \tilde{H}_j \) is of type \( E_6^{sc} \). The center of \( \tilde{H}_j \) is \( \mu_3 \), hence \( \tilde{C}_j \) is a subgroup of \( \mu_3 \). If \( \tilde{C}_j = 1 \), then \( \tilde{H}_j \) is a direct factor of \( H \) and hence \( \tilde{E}_6^{sc} \) is a direct factor of \( H \). This is impossible by the assumption. Therefore, \( \tilde{C}_j = \mu_3 \) (and hence \( p = 3 \)). The element \( \overline{w}_{ij} \) is a generator of \( \tilde{C}^* = \mathbb{Z}/3\mathbb{Z} \), hence \( k_j \) is divisible by 3. By [10, §4, type \( E_6 \)], the form \( 6q_j \) and hence \( 2k_j q_j \) belongs to \( S^2(S^*)^W_{\text{dec}} \). Taking the image of \( 2k_j q_j \) under the homomorphism \( S^*_j \rightarrow S^* \), we see that \( 2k_j q_j \in S^2(S^*)^W_{\text{dec}} \). The claim is proved.

It follows from the claim that \( 2\alpha \) belongs to the kernel of the map \( S^2(T^*)^W_{\text{ind}} \rightarrow S^2(S^*)^W_{\text{ind}} \). By Lemma 5.2, this map is injective, hence \( 2\alpha = 0 \). \qed
Example 11.2. The statement of the proposition is wrong if \( p = 2 \). Consider the group \( H := (\text{SL}_2)^n/\tilde{C} \), where \( \tilde{C} \subset (\mu_2)^n \) consists of all \( n \)-tuples with trivial product. Then the group \( G := (\text{GL}_2)^n/\tilde{C} \) is a strict envelope of \( H \). A \( G \)-torsor over a field \( K \) is a tuple \((Q_1, Q_2, \ldots, Q_n)\) of quaternion algebras over \( K \) such that \([Q_1]+[Q_2]+\cdots+[Q_n] = 0 \) in \( \text{Br}(K) \). Let \( \varphi_i \) be the reduced norm quadratic form of \( Q_i \). The sum \( \varphi \) of the forms \( \varphi_i \) in the Witt ring \( W(K) \) of \( K \) belongs to the cube of the fundamental ideal of \( W(K) \). The Arason invariant of \( \varphi \) in \( H^3(K) \) yields a degree 3 invariant \( I \) of \( G \) (see \cite[page 431]{bogomolov}). The restriction \( J \) of \( I \) to \( H \) belongs to \( \text{Inv}^3_{\text{red}}(H) = \text{Im}(\text{Inv}^3(G) \rightarrow \text{Inv}^3(H)) \), and \( I \) and \( J \) are nontrivial if \( n \geq 3 \). Note that the invariants \( I \) and \( J \) are ramified. Moreover, the map \( \text{Inv}^3(G) \rightarrow \text{Inv}^3(\tilde{H}^\text{gen}) \) factors through \( \text{Inv}^3(\tilde{G}^\text{gen}) \), where \( \tilde{G}^\text{gen} \) is the product of \( \text{GL}_1(Q_i^\text{gen}) \). The group \( \text{Inv}^3(\tilde{G}^\text{gen}) \) is trivial since \( \text{GL}_1(Q_i^\text{gen}) \) have only trivial torsors. It follows that \( J \) belong to the kernel of

\[
\text{Inv}^3(H) \rightarrow \text{Inv}^3(\tilde{H}^\text{gen}),
\]

hence the map in Proposition 8.2 is not injective.

Theorem 11.3. Let \( G \) be a (split) reductive group over an algebraically closed field \( F \). Then \( \text{Inv}^3_{\text{nr}}(G, p) = 0 \) for every odd prime \( p \neq \text{char}(F) \).

Proof. Let \( H \) be the commutator subgroup of \( G \). By Proposition 6.1(2), it suffices to prove that \( \text{Inv}^3_{\text{nr}}(H, p) = 0 \). Let \( \tilde{H} \rightarrow H \) be a simply connected cover with kernel \( \tilde{C} \). Let \( \tilde{C}' \subset \tilde{C} \) be a subgroup such that \((\tilde{C}/\tilde{C'})^* \) is the 2-component of \( \tilde{C}^* \). Since \( p \) is odd, by Proposition 7.1, \( \text{Inv}^3_{\text{nr}}(H, p) = \text{Inv}^3_{\text{nr}}(\tilde{H}/\tilde{C}', p) \). Replacing \( H \) by \( \tilde{H}/\tilde{C}' \), we may assume that \( \tilde{C}^* \) has odd order.

Write \( \tilde{H} \) as a product of simply connected groups \( \tilde{H}_j \) and let \( \tilde{C}_j \) be the center of \( \tilde{H}_j \). If the order of \( \tilde{C}_j^* \) is a power of 2, the projection \( \tilde{C} \rightarrow \tilde{C}_j \) is trivial and therefore, the simply connected group \( \tilde{H}_j \) splits off \( H \) as a direct factor. Thus, the simply connected simple groups of types \( B_n, C_n, D_n, E_7, E_8, F_4 \) and \( G_2 \) split off \( H \), i.e., \( H = H_1 \times H_2 \), where \( H_1 \) is simply connected and \( H_2 \) satisfies the conditions of Proposition 11.1. By the additivity property Corollary 6.3, Propositions 8.1(2) and 11.1, we have \( \text{Inv}^3_{\text{nr}}(H, p) = 0 \). \( \square \)

References


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