SUSLIN'S CONJECTURE ON THE REDUCED WHITEHEAD GROUP OF A SIMPLE ALGEBRA

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Abstract. In 1991, A. Suslin conjectured that if the index of a central simple algebra A is not square-free, then the reduced Whitehead group of A is nontrivial generically. We prove this conjecture in the present paper.

1. Introduction

Let A be a central simple algebra over a field F. The *reduced norm* homomorphism $A^\times \to F^\times$ yields a homomorphism

$$\mathrm{Nrd} : K_1(A) \to F^\times = K_1(F).$$

The kernel $\mathrm{SK}_1(A)$ of Nrd is the *reduced Whitehead group* of A. Wang proved in [25] that if $\mathrm{ind}(A)$ is a square-free integer, then $\mathrm{SK}_1(A) = 0$. He also proved that the reduced Whitehead group is always trivial if F is a number field. Platonov found examples of A with nontrivial $\mathrm{SK}_1(A)$ (see [17]).

In 1991, Suslin conjectured in [23] that if $\mathrm{ind}(A)$ is not square-free, then the reduced Whitehead group $\mathrm{SK}_1(A)$ of A is *generically nontrivial*, i.e., there is a field extension L/F such that $\mathrm{SK}_1(A \otimes_F L) \neq 0$.

Suslin's Conjecture was proved in the case when $\mathrm{ind}(A)$ is divisible by 4 (see [13] and [15]).

In this paper we prove Suslin's Conjecture (Theorem 8.1):

Theorem. Let A be a central simple F-algebra. If $\mathrm{ind}(A)$ is not square-free, then there is a field extension L/F such that $\mathrm{SK}_1(A \otimes_F L) \neq 0$.

Note that the group $\mathrm{SK}_1(A)$ coincides with the group of R-equivalence classes in the special linear group $\mathbf{SL}_1(A)$. In particular, generic non-triviality of the reduced Whitehead group of A implies that $\mathbf{SL}_1(A)$ is not a retract rational variety (Corollary 8.2).

2. Cycle modules and spectral sequences

Let Z be a variety over a field F and let M_* be a cycle module over Z (see [20, §2]). This is a collection of group $M_n(z)$ for $n \in \mathbb{Z}$ and a point $z : L \to Z$

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over F having certain compatibility properties. We write K_* for the cycle module over $\operatorname{Spec} F$ given by (Quillen's) K-groups (see [20, Remark 2.5]).

For every integer $r \geq 0$, denote by $Z^{(r)}$ the set of points of Z of codimension r. We write $A^r(Z, M_n)$ for the homology group of the complex [20, §5]

$$\coprod_{z\in Z^{(r-1)}} M_{n-r+1}F(z) \xrightarrow{\partial} \coprod_{z\in Z^{(r)}} M_{n-r}F(z) \xrightarrow{\partial} \coprod_{z\in Z^{(r+1)}} M_{n-r-1}F(z).$$

For example, $A^r(Z, K_r)$ is the Chow group $\mathrm{CH}^r(Z)$ of classes of codimension r algebraic cycles on Z. If Z is smooth, $A^*(Z, K_*)$ is a bi-graded commutative ring.

If $f : Y \to Z$ is a flat morphism of equidimensional varieties and M a cycle module over Y, for every $n \in \mathbb{Z}$, there is a spectral sequence [20, Corollary 8.2]

$$E_1^{r,s}(f, n) = \coprod_{z\in Z^{(r)}} A^s(f^{-1}(z), M_{n-r}) \Rightarrow A^{r+s}(Y, M_n),$$

where $f^{-1}(z)$ is the fiber of f over $z \in Z$.

Very often we will be considering the projections $f : W \times Z \to Z$ with Z and W smooth varieties. In this case $f^{-1}(z) = W_{F(z)}$. The associated spectral sequences have the following functorial properties. A morphism $h : W \to W'$ of smooth varieties yields a pull-back morphism of spectral sequences

$$h^* : E_*^{*,*}(f', n) \to E_*^{*,*}(f, n)$$

for every n (here $f' : W' \times Z \to Z$ is the projection). If h is a closed embedding of codimension c, we have a push-forward morphism of spectral sequences

$$h_* : E_*^{*,*}(f, n) \to E_*^{*,*+c}(f', n + c).$$

More generally, every correspondence λ between W and W' of degree d (see [3, §63]) yields a morphism

$$\lambda^* : h^* : E_*^{*,*}(f', n) \to E_*^{*,*-d}(f, n - d).$$

This is because the four basic maps of complexes of $W \times Z$ and $W' \times Z$ respect the filtration when projected to Z (see [20, §3]).

3. Chern classes

Let X be a smooth variety. There are Chern classes (see [7]):

$$c_{i,n} : K_n(X) \to A^{i-n}(X, K_i)$$

for $i \geq n \geq 0$. We will only need the classes

$$c_i := c_{i+1,1} : K_1(X) \to A^i(X, K_{i+1}).$$

There is the following product formula (see [21]):

Proposition 3.1. *If $x \in K_0(X)$ is the class of a line bundle L and $y \in K_1(X)$, we have*

$$c_i(xy) = \sum_{j=0}^{i} (-1)^j \binom{i}{j} h^j \cup c_{i-j}(y),$$

where h is the first (classical) Chern class of L in $A^1(X, K_1) = \mathrm{CH}^1(X)$.

Let $E \to X$ be a vector bundle of rank n and $\mathbf{SL}(E)$ the group scheme over Z of determinant 1 automorphisms of E. We will be using the following result due to Suslin [22, Th. 4.2].

Proposition 3.2. *If X is a smooth variety, the ring $A^*(\mathbf{SL}(E), K_*)$ is almost exterior algebra over $A^*(X, K_*)$ with generators $c_1(\beta), c_2(\beta), \ldots, c_{n-1}(\beta)$, where $\beta \in K_1(\mathbf{SL}(E))$ is the generic element. In particular,*

$$\mathrm{CH}(\mathbf{SL}(E)) \simeq \mathrm{CH}(X).$$

4. Severi-Brauer varieties

Let A be a central simple algebra of degree n over F, $X = \mathrm{SB}(A)$ the Severi-Brauer variety of rank n right ideals of A. If A is split, i.e., $A = \mathrm{End}(V)$ for a vector space of dimension n, the variety X is isomorphic to the projective space $\mathbb{P}(V)$.

The variety X has a point over a field extension L/F if and only if A is split over L, i.e., $A_L := A \otimes_F L \simeq M_n(L)$.

Write h for the class of a hyperplane section in $\mathrm{CH}^1(\mathbb{P}^{n-1})$. The Chow group $\mathrm{CH}^i(\mathbb{P}^{n-1})$ for $i = 0, 1, \ldots, n-1$ is infinite cyclic generated by h^i.

In the general case, the kernel of the *degree* homomorphism

$$\deg : \mathrm{CH}^i(X) \to \mathrm{CH}^i(X_{\mathrm{sep}}) = \mathbb{Z}h^i$$

coincides with the torsion part of $\mathrm{CH}^i(X)$. The group $\mathrm{CH}_0(X)$ is torsion free (see [16] or [1, Corollary 7.3]). Therefore, the classes in $\mathrm{CH}_0(X)$ of every two points of the same degree are equal.

If $A = M_m(B)$ for a central simple algebra B over F and $S = \mathrm{SB}(B)$, then S is a closed subvariety of $X = \mathrm{SB}(A)$. Moreover, the Chow motive $M(X)$ of X is isomorphic to the direct sum $M(S) \oplus M(S)\{k\} \oplus \cdots \oplus M(S)\{(m-1)k\}$, where $k = n/m$.

Let $I \to X$ be the *tautological* rank n vector bundle. The fiber of this bundle over a right ideal in A, a point of X, is the ideal itself. In the split case $A = \mathrm{End}(V)$, where V is a vector space of dimension n, a line $l \subset V$ as a point of $X = \mathbb{P}(V)$ corresponds to the right ideal $\mathrm{Hom}(V, l) = V^\vee \otimes l$. Therefore, $I = V^\vee \otimes L_t$, where L_t is the tautological line bundle over $\mathbb{P}(V)$. The *canonical* bundle J over X, the dual of I, is equal then to $V \otimes L_c$, where L_c is the canonical line bundle, dual of L_t. We have in the split case

$$X \times X = X \times \mathbb{P}(V) = \mathbb{P}_X(V) = \mathbb{P}_X(V \otimes L_c) = \mathbb{P}_X(J).$$

Note that the projective linear group $\mathbf{PGL}(V)$ acts on $\mathbb{P}(V)$ and the vector bundles I and J. In the general case, twisting by the $\mathbf{PGL}(V)$-torsor corresponding to the algebra A, we get an isomorphism

$$X \times X \simeq \mathbb{P}_X(J),$$

i.e., $X \times X$ is a projective vector bundle of J over X (with respect to the first of the two projections $q_1, q_2 : X \times X \to X$).

The tautological line bundle $\mathcal{L}_t$ over $X \times X = \mathbb{P}_X(J)$ is the sub-bundle $q_1^*(L_t) \otimes q_2^*(L_c)$ of the bundle $q_1^*(J) = V \otimes q_2^*(L_c)$ in the split case. Therefore,

$$\mathcal{L}_c = q_1^*(L_c) \otimes q_2^*(L_t), \tag{4.1}$$

where $\mathcal{L}_c$ is the canonical bundle over over $X \times X$.

Lemma 4.2. *Let $x \in X$ be a closed point. Then the push-forward homomorphism $\mathbb{Z} = \mathrm{CH}(X_{F(x)}) \to \mathrm{CH}(X \times X)$ for the closed embedding*

$$i : X_{F(x)} = X \times \operatorname{Spec} F(x) \hookrightarrow X \times X$$

depends only on the degree of x.

Proof. The canonical line bundle L over the projective space is the pull-back of the canonical bundle $\mathcal{L}$ on $X \times X$. Hence the class $h_1 = c_1(L)$ is equal to $i^*(h)$, where $h = c_1(\mathcal{L})$. By the projection formula,

$$i_*(h_1^i) = i_*(i^*(h^i)) = i_*(1) \cdot h^i = [X_{F(x)}] \cdot h^i.$$

The class of $X_{F(x)}$ in $\mathrm{CH}(X \times X)$ is the image of $[X] \times [x]$ under the exterior product map

$$\mathrm{CH}(X) \otimes \mathrm{CH}_0(X) \to \mathrm{CH}(X \times X).$$

Finally, the class of x in $\mathrm{CH}_0(X)$ depends only on the degree of x. $\square$

Choose a splitting field extension L/F of the smallest degree $\mathrm{ind}(A)$. We have $X_L \simeq \mathbb{P}_L^{n-1}$. Let $l_i \in \mathrm{CH}_i(X_L)$ be the class of a projective linear subspace of dimension i and $e_i = e_i(A)$ the image of l_i under the norm homomorphism

$$N_{L/F} : \mathrm{CH}_i(X_L) \to \mathrm{CH}_i(X).$$

Then e_i is independent of the choice of L. Indeed, choose a closed point $x \in X$ such that $F(x) \simeq L$. Then e_i is the image of l_i under the composition

$$\mathrm{CH}_i(X_L) = \mathrm{CH}_i(X_{F(x)}) \to \mathrm{CH}_i(X \times X) \to \mathrm{CH}_i(X),$$

where the last map is induced by the first projection. By Lemma 4.2, the composition does not depend on the choice of x.

The proof of Lemma 4.2 shows that for every closed point $x \in X$, we have

$$N_{F(x)/F}(l_i) = \frac{\deg(x)}{\mathrm{ind}(A)}\, e_i(A). \tag{4.3}$$

Lemma 4.4. *If K/F is a finite extension, then*

$$N_{K/F}(e_i(A_K)) = \frac{[K:F]\,\mathrm{ind}(A_K)}{\mathrm{ind}(A)}\, e_i(A).$$

Proof. Let L/K be a splitting field of A_K of degree $\mathrm{ind}(A_K)$. Choose an L-point $\mathrm{Spec}(L) \to X$. Let $\{x\}$ be the image of this morphism. We have by (4.3),

$$\begin{aligned} N_{K/F}(e_i(A_K)) &= N_{L/F}(e_i(A_L)) = [L : F(x)] \cdot N_{F(x)/F}(e_i(A_L)) \\ &= \frac{[L:F]}{\mathrm{ind}(A)}\, e_i(A) = \frac{[K:F]\,\mathrm{ind}(A_K)}{\mathrm{ind}(A)}\, e_i(A). \quad \square \end{aligned} \tag{4.5}$$

Proposition 4.6. *Let p be a prime integer and A a central simple F-algebra of p-primary degree, $X = \mathrm{SB}(A)$ the Severi-Brauer variety of A. Then $\mathrm{CH}_i(X) = \mathbb{Z}e_i$ for $i = 0, 1, \ldots, p-2$. In particular, these groups have no torsion.*

Proof. If D is a division algebra Brauer equivalent to A, the Severi-Brauer variety $Y = \mathrm{SB}(D)$ is a closed subscheme of X. The push-forward map $\mathrm{CH}_i(Y) \to \mathrm{CH}_i(X)$ is an isomorphism for $i \leq \dim(Y)$ taking $e_i(D)$ to $e_i(A)$. Thus, in the proof of the proposition it suffices to assume that A is a division algebra.

We prove the proposition by induction on $\mathrm{ind}(A)$. The case $\mathrm{ind}(A) = p$ was considered in [12, Corollary 8.7.2]. A standard restriction-corestriction argument reduces the proof to the case when F is a p-special field, i.e., the degree of every finite field extension of F is a power of p.

Let A be a central division algebra of p-primary degree n and $L \subset A$ a maximal subfield (of degree n over F). The torus $T = R_{L/F}(\mathbb{G}_m)/\mathbb{G}_m$ acts naturally on X making X a toric variety. Write U for the open T-invariant orbit and Z for $X \setminus U$. Thus, U is a T-torsor over $\mathrm{Spec}(F)$.

Conversely, let U be a T-torsor over $\mathrm{Spec}(F)$ and let A be a central simple algebra degree n over F with class in the relative Brauer group $\mathrm{Br}(L/F) = H^1(F, T)$ corresponding to the class of U. Then U is the open orbit of the T-action on $\mathrm{SB}(A)$.

In the split case, $X = \mathbb{P}^{n-1}$ and T is the torus of invertible diagonal matrices modulo the scalar matrices. Then U consists of all points in $\mathbb{P}^{n-1}$ with all coordinates $\neq 0$. The T-orbits are the subsets in $\mathbb{P}^{n-1}$ with zeros on the fixed set of coordinates.

Let Σ be the set of all n primitive idempotents of $L \otimes_F F_{\mathrm{sep}} = F_{\mathrm{sep}} \times \cdots \times F_{\mathrm{sep}}$. Every $\sigma \in \Sigma$ yields a co-character $\chi_\sigma : \mathbb{G}_{m,F_{\mathrm{sep}}} \to T_{\mathrm{sep}}$ which belongs to an edge (1-dimensional cone) in the fan of the toric variety X_{sep}. Moreover, the correspondence $\sigma \mapsto \chi_\sigma$ yields a bijection between the set of nonempty subsets in Σ and the set of cones in the fan (or the set of T-orbits in X_{sep}). The absolute Galois group $\Gamma = \mathrm{Gal}(F_{\mathrm{sep}}/F)$ of F acts transitively on the set Σ.

Lemma 4.7. *We have $\mathrm{CH}_i(U) = 0$ for $i = 0, 1, \ldots, p-2$.*

Proof. If $\mathrm{ind}(A) = p$, every cycle c in $\mathrm{CH}_i(U)$ comes by restriction from $\mathrm{CH}_i(X) = p\mathbb{Z}$ and therefore, by the norm, comes from $\mathrm{CH}_i(X_L)$. Hence c comes by the norm from $\mathrm{CH}_i(U_L)$. But $U_L \simeq T_L$, hence $\mathrm{CH}_i(U_L) = 0$.

In the general case, since F is a p-special field, there is a subfield $K \subset L$ of degree p over F. Consider the subtorus $S := R_{K/F}(\mathbb{G}_m)/\mathbb{G}_m$ of T, the S-torsor

$$f : U \to X := U/S$$

and Rost's spectral sequence for f converging to $\mathrm{CH}_i(U)$. On the zero diagonal, we have the groups $\coprod_{x \in X_{(j)}} \mathrm{CH}_k(f^{-1}(x))$ with $j + k = i$. Note that $f^{-1}(x)$ is an S-torsor over $\mathrm{Spec}\, F(x)$. Since $k \leq i \leq p-2$, by the first part of the proof, $\mathrm{CH}_k(f^{-1}(x)) = 0$. $\square$

The T-orbits in Z_{sep} correspond to proper subsets of the set of Σ. No such subset is fixed by Γ, hence no orbit in Z_{sep} is fixed by Γ.

We have a sequence of closed T-invariant subsets

$$Z = Z_0 \supset Z_1 \supset \cdots \supset Z_m \supset Z_{m+1} = \emptyset \tag{4.8}$$

such that every variety $(Z_j \setminus Z_{j+1})_{\mathrm{sep}}$ is the disjoint union of T-orbits of the same dimension which are permuted by Γ. It follows that each $Z_j \setminus Z_{j+1}$ is a disjoint union of varieties defined over finite separable field extensions K/F corresponding to the stabilizers $\Gamma' \subset \Gamma$ of T-orbits. The group Γ' does not act transitively on the set Σ, hence $L \otimes_F K$ is not a field and therefore, A_K is not a division algebra, i.e., $\mathrm{ind}(A_K) < \mathrm{ind}(A)$.

If W is a scheme over a finite separable field extension K/F, the norm map $\mathrm{CH}(W \otimes_F K) \to \mathrm{CH}(W)$ is surjective, since K is a direct factor of $K \otimes_F K$.

Fix an integer $i = 0, 1, \dots, p-2$. We say that a variety W over F satisfies the condition $(*)$ if $\mathrm{CH}_i(W)$ is generated by the images of the norm maps $\mathrm{CH}_i(W \otimes_F K) \to \mathrm{CH}_i(W)$ over finite field extensions K/F with $\mathrm{ind}(A_K) < \mathrm{ind}(A)$. We have proved that all the differences $Z_j \setminus Z_{j+1}$ satisfy $(*)$.

Let W' be a closed subvariety of W. The exactness of the localization sequence

$$\mathrm{CH}_i(W') \to \mathrm{CH}_i(W) \to \mathrm{CH}_i(W \setminus W') \to 0$$

shows that if W' and $W \setminus W'$ satisfy $(*)$, then so does W. It follows from (4.8) that Z satisfies $(*)$. By Lemma 4.7, U satisfies $(*)$, hence so does X.

By the induction hypothesis, $\mathrm{CH}_i(X_K)$ for K as above, is generated by $e_i(A_K)$. By Lemma 4.4, $\mathrm{CH}_i(X)$ is generated by $e_i(A)$. □

Corollary 4.9. *The degree map* $\mathrm{CH}_i(X) \to \mathrm{CH}_i(X_{\mathrm{sep}}) = \mathbb{Z}l_i$ *is injective, it takes* e_i *to* $\mathrm{ind}(A)l_i$. *Thus,* $\mathrm{CH}_i(X)$ *is identified with the subgroup* $\mathrm{ind}(A)\mathbb{Z}l_i$ *in* $\mathbb{Z}l_i$.

By the Projective Bundle Theorem, for every $j \geq 0$, we have

$$\mathrm{CH}^{d-j}(X \times X) \simeq \mathrm{CH}^{d-j}(X) \oplus \mathrm{CH}^{d-j-1}(X)h \oplus \cdots \oplus \mathrm{CH}^0(X)h^{d-j},$$

where $h \in \mathrm{CH}^1(X \times X)$ is the first Chern class of the canonical line bundle $\mathcal{L}_c$ over $X \times X$. The element $\lambda_j := h^{d-j}$ can be viewed as a degree j correspondence from X to itself and hence λ_j yields the homomorphism (see Section 2):

$$\lambda_j^* : \mathrm{CH}_0(X) \to \mathrm{CH}_j(X).$$

Lemma 4.10. *The maps* λ_j^* *are isomorphisms for* $j = 0, 1, \dots, p-2$, *taking* $e_0(A)$ *to* $e_j(A)$.

Proof. By (4.1), in the split case, $h = h_2 - h_1$, where h_i are the pull-backs to $X \times X$ of the classes of the hyperplanes in X, hence $\lambda_j = (h_2 - h_1)^{d-j}$. Therefore, λ_j^* takes the generator l_0 of the infinite cyclic group $\mathrm{CH}_0(X)$ to the generator l_j of $\mathrm{CH}_j(X)$.

By Proposition 4.6, in the general case, the degree map $\mathrm{CH}_j(X) \to \mathrm{CH}(X_{\mathrm{sep}}) = \mathbb{Z}l_j$ identifies the group $\mathrm{CH}_j(X)$ with $\mathrm{ind}(A)l_j$ by Corollary 4.9. The result follows. □

5. Two cycle modules

Let A be a central simple algebra over F. The first cycle module K_*^{QA} is defined by

$$K_n^{QA}(L) = K_n(A_L)$$

for a field extension L/F. The reduced norm map $\mathrm{Nrd} : K_n(A_L) \to K_n(L)$ is defined for $n = 0, 1, 2$ (see [12, §6]).

Let $G = \mathbf{SL}_1(A)$ be the algebraic group of reduced norm 1 elements in A. There is a canonical isomorphism (see [6, Proposition 7.3])

$$A^1(G, K_2) \simeq \mathbb{Z}.$$

The group $A^1(G, K_2)$ does not change under field extensions.

In particular, we have a homomorphism

$$\mathrm{Nrd}^{QA} : A^1(G, K_2^{QA}) \to A^1(G, K_2) = \mathbb{Z}.$$

Let X be the Severi-Brauer variety of A of dimension d. We will be using another cycle module K_*^A over F defined by

$$K_n^A(L) = A^d(X_L, K_{d+n}).$$

The push-forward homomorphism for the morphism $X_L \to \mathrm{Spec}(L)$ yields a map $A^d(X_L, K_{d+n}) \to K_n(L)$ and therefore, a morphism of cycle modules $K_*^A \to K_*$. In particular, we have a homomorphism

$$\mathrm{Nrd}^A : A^1(G, K_2^A) \to A^1(G, K_2) = \mathbb{Z}.$$

There is a natural homomorphism $A^d(X_L, K_{d+n}) \to K_n(A_L)$ which is an isomorphism for $n = 0$ and 1 (see [14]). Thus, we have a morphism of cycle modules $K_*^A \to K_*^{QA}$ that is isomorphism in degree 0 and 1. It follows that the images of the maps Nrd^{QA} and Nrd^A coincide.

If A is split, $K_*^{QA} = K_*^A = K_*$.

6. A reduction

Recall that $G = \mathbf{SL}_1(A)$ for a central simple algebra A of degree n over F. For every commutative F-algebra R there is a natural composition

$$G(R) \hookrightarrow A_R^\times \to K_1(A_R),$$

where $A_R = A \otimes_F R$.

Consider the generic point $\xi \in G(F[G])$ and its image $\xi_{F(G)}$ in $G(F(G))$. Let α be the image of ξ under the map

$$G(F[G]) \to K_1(A_{F[G]}),$$

and let $\alpha_{F(G)}$ be the image of $\xi_{F(G)}$ under the map

$$G(F(G)) \to K_1(A_{F(G)}).$$

We will prove that $\alpha_{F(G)}$ is nontrivial in $K_1(A_{F(G)})$ when A is a central simple algebra with $\mathrm{ind}(A)$ not square-free.

Filtering the category of coherent $A\otimes_F \mathcal{O}_G$-modules by codimension of support as in [18, §7.5], we get the Brown-Gersten-Quillen spectral sequence (see [18, §7])

$$E_1^{r,s}=\coprod_{g\in G^{(r)}} K_{-r-s}(A_{F(g)})\Rightarrow K_{-r-s}(A_{F[G]}),$$

where the limit is the K-group of the category of coherent $A\otimes_F \mathcal{O}_G$-modules equipped with the topological filtration (by codimension of support). In particular,

$$E_2^{r,s}=A^r(G,K_{-s}^{QA})$$

and the first term of the topological filtration on $K_1(A_{F[G]})$ is equal to

$$K_1(A_{F[G]})^{(1)}=\operatorname{Ker}\bigl(K_1(A_{F[G]})\to K_1(A_{F(G)})\bigr).$$

The spectral sequence gives then a homomorphism

$$\varepsilon : K_1(A_{F[G]})^{(1)}\to A^1(G,K_2^{QA}).$$

If $\alpha_{F(G)}$ is trivial in $K_1(A_{F(G)})$, then $\alpha\in K_1(A_{F[G]})^{(1)}$. Therefore, we have an element $\varepsilon(\alpha)\in A^1(G,K_2^{QA})$.

We compute $\varepsilon(\alpha)$ in the split case. We have $G=\mathbf{SL}_n$ and

$$\alpha\in K_1(A_{F[G]})=K_1(F[G])=K_1(G).$$

By [22, Th. 2.7], the first Chern class $c_1(\alpha)$ of α generates the group $A^1(G,K_2^{QA})=A^1(G,K_2)=\mathbb{Z}$.

Lemma 6.1. *In the split case, $\varepsilon(\alpha)=c_1(\alpha)$.*

Proof. Let $H:=\mathbf{GL}_n$ and $\beta\in K_1(H)$ be the element given by the generic matrix. By [22, Th. 3.10], $\gamma_{i+1}(\beta)\in K_1(H)^{(i)}$ for all $i\geq 0$, where γ is the gamma operation, and the image of $-\gamma_2(\beta)$ under the canonical homomorphism

$$K_1(H)^{(1)}\to A^1(H,K_2)$$

is equal to $c_1(\beta)$. On the other hand, the sum of $\gamma_i(\beta)$ for all $i\geq 1$ coincides with $\Lambda^n(\beta)=\det(\beta)$ by [22, p. 65]. Hence $-\gamma_2(\beta)\equiv \beta-\det(\beta)$ modulo $K_1(H)^{(2)}$.

Pulling back with respect to the embedding of G into H we have $-\gamma_2(\alpha)\equiv\alpha$ modulo $K_1(G)^{(2)}$ since $\det(\alpha)$ is trivial and therefore, the image of α under the homomorphism $K_1(G)^{(1)}\to A^1(G,K_2)$ is equal to $c_1(\alpha)$. □

Let L/F be a splitting field of A. We have a commutative diagram

$$\begin{tikzcd}
K_1(A_{F[G]})^{(1)} \arrow[r, "\varepsilon"] \arrow[d] & A^1(G,K_2^{QA}) \arrow[d] & \\
K_1(A_{L[G]})^{(1)} \arrow[r, "\varepsilon"] & A^1(G_L,K_2^{QA}) \arrow[r, equal] & \mathbb{Z}.
\end{tikzcd}$$

The right vertical homomorphism factors as follows:

$$A^1(G,K_2^{QA})\xrightarrow{\mathrm{Nrd}^{QA}} A^1(G,K_2)\xrightarrow{\sim} A^1(G_L,K_2)=A^1(G_L,K_2^{QA}).$$

Assume that $\alpha_{F(G)}$ is trivial in $K_1(A_{F(G)})$, hence $\alpha \in K_1(A_{F[G]})^{(1)}$. By Lemma 6.1, $\varepsilon(\alpha)_L$ in $A^1(G_L, K_2^{QA}) = \mathbb{Z}$ is a generator. It follows that the image of $\varepsilon(\alpha)$ under the map $\mathrm{Nrd}^{QA} : A^1(G, K_2^{QA}) \to A^1(G, K_2) = \mathbb{Z}$ is equal to ± 1, hence Nrd^{QA} is surjective.

We have proved:

Proposition 6.2. *Suppose that the map* $\mathrm{Nrd}^{QA} : A^1(G, K_2^{QA}) \to A^1(G, K_2) = \mathbb{Z}$ *is not surjective. Then Suslin's Conjecture holds for* A.

Let A be a central simple F-algebra such that $\mathrm{ind}(A)$ is not square-free, i.e., $\mathrm{ind}(A)$ is divisible by p^2 for a prime integer p. We want to prove that $\mathrm{SK}_1(A)$ is nontrivial generically. Replacing F by a field extension over which A has index exactly p^2 and replacing A by a Brauer equivalent division algebra, we may assume that A is a division algebra of degree p^2. Moreover, an application of the index reduction formula shows that we may assume that A is decomposable, i.e., A is a tensor product of two algebras of degree p (see [19, Theorem 1.20]).

We will prove that if A is a decomposable division algebra of degree p^2, then the map Nrd^{QA} is not surjective. Recall that the maps Nrd^{QA} and Nrd^{A} have the same images. Therefore, it suffices to prove that the map Nrd^{A} : $A^1(G, K_2^{A}) \to A^1(G, K_2) = \mathbb{Z}$ is not surjective.

7. A spectral sequence

Let A be a central simple F-algebra of degree p^2 and X the Severi-Brauer variety of A with $\dim(X) = d = p^2 - 1$. We would like to find a reasonable description the group $A^1(G, K_2^A)$ via algebraic cycles on $G \times X$.

Consider the spectral sequence associated with the projection $q : G \times X \to G$ (see Section 2):

$$(7.1) \quad E_1^{r,s} = E_1^{r,s}(q, d+2) = \coprod_{g \in G^{(r)}} A^s(X_{F(g)}, K_{d+2-r}) \Rightarrow A^{r+s}(G \times X, K_{d+2}).$$

We have $E_1^{r,s} = 0$ if $s > d$ and

$$E_2^{r,d} = A^r(G, K_2^A).$$

There are no nontrivial differentials arriving at $E_*^{r,d}$.

Proposition 7.2. *We have* $E_2^{i,d+2-i} = 0$ *for* $i = 2, 3, \dots, p$. *In particular,*

$$A^1(G, K_2^A) = E_2^{1,d} = E_3^{1,d} = \cdots = E_p^{1,d}.$$

Proof. Let $j = i - 2$ and λ_j be the correspondence on $X \times X$ of degree j considered in Section 4. By Lemma 4.10, the maps

$$\lambda_j^* : \mathrm{CH}_0(X_L) \to \mathrm{CH}_j(X_L).$$

are isomorphisms for $j = 0, 1, \dots, p-2$ and every field extension L/F.

Consider the spectral sequence

$$\widehat{E}_1^{r,s} := E_1^{r,s}(q, d+i) = \coprod_{g \in G^{(r)}} A^s(X_{F(g)}, K_{d+i-r}) \Rightarrow A^{r+s}(G \times X, K_{d+i}). \tag{7.3}$$

The edge homomorphism

$$\mathrm{CH}^{d+i}(G \times X) = A^{d+i}(G \times X, K_{d+i}) \to \widehat{E}_2^{i,d}$$

is surjective. By Proposition 3.2,

$$\mathrm{CH}^{d+i}(G \times X) = \mathrm{CH}^{d+i}(X) = 0$$

since $d + i > \dim(X)$. It follows that $\widehat{E}_2^{i,d} = 0$.

The correspondence λ_j yields a morphism between the spectral sequences (7.1) and (7.3). In particular, we have a homomorphism

$$\tilde{\lambda}_j : \widehat{E}_2^{i,d} \to E_2^{i,d-i+2}.$$

Since λ_j^* is an isomorphism for X_L for every field extension L/F, the map $\tilde{\lambda}_j$ is surjective. As $\widehat{E}_2^{i,d} = 0$, we have $E_2^{i,d+2-i} = 0$. $\square$

By Proposition 7.2, we have a differential

$$A^1(G, K_2^A) = E_p^{1,d} \xrightarrow{\delta} E_p^{p+1,d+1-p}.$$

Proposition 7.4. *If* $\mathrm{ind}(A) = p$*, the image of* $\mathrm{Ker}(\delta)$ *under the homomorphism*

$$\mathrm{Nrd}^A : A^1(G, K_2^A) \to A^1(G, K_2) = \mathbb{Z}$$

is equal to $p\mathbb{Z}$*.*

We will prove this proposition in Section 10.

Let A be a division algebra of degree p^2 over F. Choose a field extension K/F such that $\mathrm{ind}(A_K) = p$ and set $\widetilde{A} = A_K$, $\widetilde{X} = X_K$, $\widetilde{G} = G_K$ and write $\widetilde{E}_*^{r,s}$ for the terms of the spectral sequence associated with the projection $\widetilde{G} \times \widetilde{X} \to \widetilde{G}$. We have the following commutative diagram

$$\begin{array}{ccccc} A^1(G, K_2^A) & = & E_p^{1,d} & \xrightarrow{\delta} & E_p^{p+1,d+1-p} \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle\kappa} \\ A^1(\widetilde{G}, K_2^{\widetilde{A}}) & = & \widetilde{E}_p^{1,d} & \xrightarrow{\tilde{\delta}} & \widetilde{E}_p^{p+1,d+1-p}, \end{array}$$

where δ and $\tilde{\delta}$ are the differentials in the p-th pages of the spectral sequences.

Proposition 7.5. *If* A *is decomposable degree* p^2 *division algebra, then* $\kappa : E_p^{p+1,d+1-p} \to \widetilde{E}_p^{p+1,d+1-p}$ *is the zero map.*

We will prove this proposition in Section 11.

8. Main theorem

We deduce the following theorem from Propositions 7.4 and 7.5.

Theorem 8.1. *Let A be a central simple F-algebra. If* $\operatorname{ind}(A)$ *is not square-free, then there is a field extension L/F such that* $\mathrm{SK}_1(A_L) \neq 0$.

Proof. We may assume that A is a decomposable division algebra of degree p^2 for a prime integer p. Note that $\operatorname{ind}(\widetilde{A}) = p$.

By Propositions 7.4 (applied to the algebra $\widetilde{A}$) and 7.5, the image of the composition

$$E_p^{1,d} \to \widetilde{E}_p^{1,d} = A^1(\widetilde{G}, K_2^{\widetilde{A}}) \xrightarrow{\mathrm{Nrd}^{\widetilde{A}}} A^1(\widetilde{G}, K_2) = \mathbb{Z}$$

is contained in $p\mathbb{Z}$. On the other hand, this composition coincides with

$$E_p^{1,d} = A^1(G, K_2^A) \xrightarrow{\mathrm{Nrd}^A} A^1(G, K_2) = \mathbb{Z}.$$

Therefore, the norm homomorphism $\mathrm{Nrd}^A : A^1(G, K_2^A) \to A^1(G, K_2)$ is not surjective and this finishes the proof by Proposition 6.2 since $\mathrm{Im}(\mathrm{Nrd}^{QA}) = \mathrm{Im}(\mathrm{Nrd}^A)$. $\square$

An irreducible variety Z over F is called a *retract rational* variety if there exist rational morphisms $\alpha : Z \dashrightarrow \mathbb{P}^m$ and $\beta : \mathbb{P}^m \dashrightarrow Z$ for some m such that the composition $\beta \circ \alpha$ is defined and equal to the identity of Z.

Corollary 8.2. *Let A be a central simple algebra over F. Then the following are equivalent:*

(1) *The group* $\mathbf{SL}_1(A)$ *is a retract rational variety;*
(2) $\mathrm{SK}_1(A_L) = 0$ *for every field extension L/F;*
(3) *The index* $\operatorname{ind}(A)$ *is square-free.*

Proof. (1) $\Rightarrow$ (2): If $G := \mathbf{SL}_1(A)$ is a retract rational variety, then G_L is so for every field extension L/F. By [2, Proposition 11], the group of R-equivalence classes $G(L)/R$ is trivial. But $G(L)/R$ is isomorphic to $\mathrm{SK}_1(A_L)$ by [24, §18.2].

(2) $\Rightarrow$ (1): This is proved in [5, Proposition 2.4] and [10, Proposition 5.1].

(2) $\Leftrightarrow$ (3): This is Theorem 8.1. $\square$

9. Chow ring of G

Let $G = \mathbf{SL}_1(A)$ for a central simple algebra A of p-primary degree.

Lemma 9.1. *The Chow groups* $\mathrm{CH}^i(G)$ *are trivial for $i = 1, 2, \dots, p$ and* $p \cdot \mathrm{CH}^{p+1}(G) = 0$.

Proof. Since $\mathrm{CH}(G_{\mathrm{sep}}) = \mathbb{Z}$ by [22, Theorem 2.7], the groups $\mathrm{CH}^i(G)$ are p-primary torsion if $i > 0$. As $K_0(G) = \mathbb{Z}$ (see [22, Theorem 4.1]), by [4, Example 15.3.6], we have $(i-1)!\,\mathrm{CH}^i(G) = 0$ for $i > 0$. The result follows. $\square$

Consider the Brown-Gersten-Quillen spectral sequence

$$E_2^{r,s} = A^r(G, K_{-s}) \Rightarrow K_{-r-s}(G).$$

It follows from Lemma 9.1 that

$$A^1(G, K_2) = E_2^{1,-2} = E_3^{1,-2} = \cdots = E_p^{1,-2}.$$

Moreover, by [8, §3],

$$\mathrm{CH}^{p+1}(G) = E_2^{p+1,-p-1} = E_3^{p+1,-p-1} = \cdots = E_p^{p+1,-p-1}.$$

We have then a differential

$$\delta : A^1(G, K_2) = E_p^{1,-2} \to E_p^{p+1,-p-1} = \mathrm{CH}^{p+1}(G).$$

Write $h \in \mathrm{CH}^{p+1}(G)$ for the image under δ of the canonical generator of the group $A^1(G, K_2) = \mathbb{Z}$.

Proposition 9.2. *Suppose that* $\mathrm{ind}(A) = p$*, i.e.,* $A = M_n(B)$ *and* $G = \mathbf{SL}_n(B)$ *for some* n *and a central division algebra* B *of degree* p*. Then*

$$\mathrm{CH}^*(G) = \mathbb{Z} \oplus (\mathbb{Z}/p\mathbb{Z})h \oplus (\mathbb{Z}/p\mathbb{Z})h^2 \oplus \cdots \oplus (\mathbb{Z}/p\mathbb{Z})h^{p-1}.$$

Proof. Induction on n. The case $n = 1$ is done in [8, Theorem 9.7].

Let $H = \mathbf{SL}_{n-1}(B)$. We view H as a subgroup of G with respect to the embedding $x \mapsto diag(1, x)$. Consider the closed subvariety V of the affine space B^{2n} consisting of tuples $(b_1, \dots, b_n, c_1, \dots, c_n)$ such that $\sum b_i c_i = 1$. Define the morphism

$$f : G \to V, \qquad a = (a_{ij}) \mapsto (a_{11}, \dots, a_{1n}, a'_{11}, \dots, a'_{n1}),$$

where $(a'_{ij}) = a^{-1}$. Clearly, f is an H-torsor over V. For any field extension L/F, in the exact sequence of Galois cohomology

$$G(L) \xrightarrow{f(L)} V(L) \to H^1(L, H) \xrightarrow{r} H^1(L, G)$$

the map r is a bijection (both sets are identified with $L^\times/\operatorname{Nrd}(B_L^\times)$ and r is the identity map by [11, Cor. 2.9.4]). Hence f is surjective on L-points.

Let W be the open subset of the affine space B^n consisting of all tuples $(b_1, \dots, b_n)$ such that $\sum b_i B = B$. We have $\mathrm{CH}^i(W) = 0$ for $i > 0$. The obvious projection $V \to W$ is an affine bundle, hence by the homotopy invariance property,

$$\mathrm{CH}^i(V) \simeq \mathrm{CH}^i(W) = 0 \tag{9.3}$$

for every $i > 0$.

For every m, consider the spectral sequence associated with the morphism f:

$$E_1^{r,s} = E_1^{r,s}(f, m) = \coprod_{v \in V^{(r)}} A^s(f^{-1}(v), K_{m-r}) \Rightarrow A^{r+s}(G, K_m).$$

Since f is surjective on L-points, $f^{-1}(v) \simeq H_{F(v)}$.

We claim that $E_2^{r,s} = 0$ if $r + s = m$ and $r > 0$. By induction, the group $A^s(G_v, K_s) = \mathrm{CH}^s(G_v)$ is trivial unless $s = (p+1)i$ for $i = 0, 1, \dots, p-1$. In the latter case the map $\mathrm{CH}^r(V) \to E_2^{r,s}$ of multiplication by h^i is surjective by the induction hypothesis. The claim follows from the triviality of $\mathrm{CH}^r(V)$ for $r > 0$.

By the claim, $\mathrm{CH}(G) \simeq \mathrm{CH}(H_{F(V)})$. The statement of the proposition follows by induction. □

Corollary 9.4. *Let A be a central simple algebra of degree p^2. Then for every field extension L/F such that* $\mathrm{ind}(A_L) \le p$, *the map*

$$\mathrm{CH}(G) \to \mathrm{CH}(G_L)$$

is surjective.

Proof. The element h belongs to $\mathrm{CH}^{l+1}(G)$. As $\mathrm{ind}(A_L) \le p$, by Proposition 9.2, the element h_L generates the ring $\mathrm{CH}(G_L)$, whence the result. □

10. Proof of Proposition 7.4

In this section, A is a central simple algebra of degree p^2 and index p, so that $A = M_p(B)$, where B is a division algebra of degree p. We write S for the Severi-Brauer variety $\mathrm{SB}(B)$ of dimension $p-1$. Recall that the variety S can be viewed as a closed subvariety of X. Moreover, the Chow motive $M(X)$ of X is isomorphic to $M(S) \oplus M(S)\{p\} \oplus \cdots \oplus M(S)\{(p-1)p\}$.

Consider the spectral sequence associated with the projection $t : G \times S \to G$:

$$\hat{E}_1^{r,s} := E_1^{r,s}(t, p+1) = \coprod_{g \in G^{(r)}} A^s(S_{F(g)}, K_{p+1-r}) \Rightarrow A^{p+q}(G \times S, K_{p+1}). \tag{10.1}$$

The embedding of S into X induces the push-forward morphisms between the spectral sequences (10.1) and (7.1). Moreover, (10.1) is a direct summand of (7.1). More precisely, the maps

$$\hat{E}_*^{r,s} \to E_*^{r,s+d+1-p}$$

are isomorphisms for $s = 0, 1, \ldots, p-1$.

By Proposition 7.2, we have $E_2^{i,d+2-i} = 0$ for $i = 2, 3, \ldots, p$. It follows that $\hat{E}_2^{i,p+1-i} = 0$ for $i = 2, 3, \ldots, p$, i.e., all the terms but $\hat{E}_2^{p+1,0}$ on the diagonal $r + s = p + 1$ on page $\hat{E}_2^{*,*}$ are zero. Moreover,

$$E_2^{p+1,d+1-p} = \hat{E}_2^{p+1,0} = \mathrm{CH}^{p+1}(G). \tag{10.2}$$

It follows that

$$A^1(G, K_2^A) = \hat{E}_2^{1,p-1} = \hat{E}_3^{1,p-1} = \cdots = \hat{E}_p^{1,p-1}$$

and the only potentially nonzero differential starting in $\hat{E}_{\ge 2}^{1,p-1}$ appears on page p:

$$A^1(G, K_2^A) = \hat{E}_p^{1,p-1} \xrightarrow{\hat{\delta}} \hat{E}_p^{p+1,0}.$$

The spectral sequence (10.1) yields then an exact sequence

$$A^l(G \times S, K_{p+1}) \to \hat{E}_p^{1,p-1} \xrightarrow{\hat{\delta}} \hat{E}_p^{p+1,0}. \tag{10.3}$$

The differential $\delta : E_p^{1,d} \to E_p^{p+1,d+1-p}$ in (10.1) is identified with the differential $\hat{\delta} : \hat{E}_p^{1,p-1} \to \hat{E}_p^{p+1,0}$ in (7.1). Thus, to prove the proposition, it suffices to show that the image of the composition

$$A^p(G \times S, K_{p+1}) \to \hat{E}_p^{1,p-1} = A^1(G, K_2^A) \xrightarrow{\mathrm{Nrd}^A} A^1(G, K_2) = \mathbb{Z}$$

is equal to $p\mathbb{Z}$.

This composition is the push-forward homomorphism

$$t_* : A^p(G \times S, K_{p+1}) \to A^1(G, K_2) = \mathbb{Z}$$

with respect to the projection $t : G \times S \to G$.

Over S, the algebra A is isomorphic to $\mathcal{E}nd_{\mathcal{O}_S}(J^p)$, where J is the canonical vector bundle over S of rank p. By Proposition 3.2,

$$A^p(G \times S, K_{p+1}) = \mathrm{CH}^{p-1}(S) \cdot c_1(\beta) \oplus \mathrm{CH}^{p-2}(S) \cdot c_2(\beta) \oplus \cdots \oplus \mathrm{CH}^0(S) \cdot c_p(\beta),$$

where $\beta \in K_1(G \times S)$ is the generic element.

Since the group $A^1(G, K_2)$ does not change under field extensions, it is sufficient to compute the image over a field extension L/F splitting A. Over such a field extension the group G_L is isomorphic to $\mathbf{SL}_{p^2}$. Let $\beta' \in K_1(G_L)$ be the class of the generic matrix, so that $\beta = [L_c] \cdot t^*(\beta')$, where L_c is the canonical line bundle over X. By Proposition 3.1,

$$c_i(\beta) = \sum_{j=0}^{i} (-1)^j \binom{i}{j} \; h^j \; c_{i-j}(t^* \beta')$$

for every $i = 1, 2, \ldots, p$, where $h \in \mathrm{CH}^1(S_L)$ is the first Chern class of L_c. Note that $c_1(\beta')$ is the canonical generator of $A^1(G_L, K_2)$. By the projection formula, the image of t_* is the sum of the subgroups

$$\binom{i}{j} \cdot t_* \left[\mathrm{CH}^{p-i}(S) \cdot h^j\right] \cdot c_{i-j}(\beta')$$

over all $i = 1, 2, \ldots, p$ and $j = 0, 1, \ldots, i$. By dimension consideration, the subgroup is trivial if $j \neq i - 1$. Consider the case $j = i - 1$. If $p - i > 0$, then the image of $\mathrm{CH}^{p-i}(S)$ in $\mathbb{Z}$ (when splitting S) is equal to $p\mathbb{Z}$. Finally, if $i = p$, the multiple $\binom{i}{j}$ is equal to p. The proposition is proved.

11. Proof of Proposition 7.5

In this section we assume that A is a decomposable division algebra of degree p^2.

Since for every i and j with $i + j = d + 2$ the natural homomorphism $E_2^{i,j} \to E_p^{i,j}$ is surjective, it is sufficient to prove that the homomorphism

$$E_2^{p+1,d+1-p} \to \widetilde{E}_2^{p+1,d+1-p}$$

is trivial.

Let L/F be a field extension. Considering X over a separable closure of L we get the homomorphisms

$$A^i(X_L, K_{i+n}) \to A^i(X_{L_{\mathrm{sep}}}, K_{i+n}) = K_n(L)$$

for $i = d+1-p$ and $n = 0, 1$. These homomorphisms induce the vertical maps in the following commutative diagram

$$\begin{tikzcd} E_2^{p+1,d+1-p} \arrow[r] \arrow[d, "\varphi"'] & \widetilde{E}_2^{p+1,d+1-p} \arrow[d, "\widetilde{\varphi}"] \\ \mathrm{CH}^{p+1}(G) \arrow[r] & \mathrm{CH}^{p+1}(\widetilde{G}). \end{tikzcd}$$

Since $\mathrm{ind}(\widetilde{A}) = p$, it follows from (10.2) (applied to $\widetilde{A}$) that $\widetilde{\varphi}$ is an isomorphism. Thus, it is sufficient to prove that $\varphi = 0$.

Recall that A is a decomposable algebra. By a theorem of Karpenko [9, Th. 1],

$$\mathrm{Im}\big(\mathrm{CH}^{d+1-p}(X_{F(g)}) \xrightarrow{\deg} \mathbb{Z}\big) = \begin{cases} p\mathbb{Z}, & \text{if } \mathrm{ind}\, A_{F(g)} = p^2; \\ \mathbb{Z}, & \text{if } \mathrm{ind}\, A_{F(g)} \leq p. \end{cases} \tag{11.1}$$

Let $Y = \mathrm{SB}(p, A)$ be the generalized Severi-Brauer variety and set $m := \dim(Y) = p^3 - p^2$. Consider the cycle module M_* over F defined by

$$M_n(L) = A^m(Y_L, K_{n+m}).$$

There is the norm morphism $N : M_* \to K_*$ well defined. The variety Y has a point over a field extension L/F if and only if $\mathrm{ind}(A_L) \leq p$. It follows that

$$\mathrm{Im}\big(A^m(Y_L, K_m) \xrightarrow{N} \mathbb{Z}\big) = \begin{cases} p\mathbb{Z}, & \text{if } \mathrm{ind}\, A_L = p^2; \\ \mathbb{Z}, & \text{if } \mathrm{ind}\, A_L \leq p. \end{cases}$$

Therefore the image of φ coincide with the image of the map

$$\psi : A^{p+1}(G, M_{p+1}) \to A^{p+1}(G, K_{p+1}) = \mathrm{CH}^{p+1}(G)$$

induced by the norm map N. It is sufficient to prove that $\psi = 0$.

The spectral sequence for the projection $G \times Y \to G$,

$$E_1^{r,s} = \coprod_{g \in G^{(r)}} A^s(Y_{F(g)}, K_{m+p+1-r}) \Rightarrow A^{r+s}(G \times Y, K_{m+p+1})$$

yields a surjective homomorphism $\mathrm{CH}^{m+p+1}(G \times Y) \to A^{p+1}(G, M_{p+1})$. The composition

$$\mathrm{CH}^{m+p+1}(G \times Y) \to A^{p+1}(G, M_{p+1}) \xrightarrow{\psi} A^{p+1}(G, K_{p+1}) = \mathrm{CH}^{p+1}(G)$$

is the push-forward homomorphism with respect to the projection $G \times Y \to G$. Thus, it is sufficient to show that the push-forward homomorphism

$$\mathrm{CH}^{m+p+1}(G \times Y) \to \mathrm{CH}^{p+1}(G)$$

is zero.

Since $\operatorname{ind}(A_{F(y)}) \le p$ for every $y \in Y$, it follows from Corollary 9.4 that the map $\mathrm{CH}(G) \to \mathrm{CH}(G_{F(y)})$ is surjective. Then the proof of [3, Lemma 88.5] yields the following lemma.

Lemma 11.2. *The product homomorphism*

$$\mathrm{CH}(G) \otimes \mathrm{CH}(Y) \to \mathrm{CH}(G \times Y)$$

is surjective.

Lemma 11.3. *For every closed point $y \in Y$, the norm homomorphism*

$$N_{F(y)} : \mathrm{CH}^{p+1}(G_{F(y)}) \to \mathrm{CH}^{p+1}(G)$$

is trivial.

Proof. The first map in the composition

$$\mathrm{CH}^{p+1}(G) \to \mathrm{CH}^{p+1}(G_{F(y)}) \xrightarrow{N_{F(y)/F}} \mathrm{CH}^{p+1}(G)$$

is surjective by Corollary 9.4 since $\operatorname{ind}(A_{F(y)}) \le p$. The composition is multiplication by $\deg(y)$. Note that $\deg(y)$ is divisible by p since $\operatorname{ind}(A) = p^2$. The result follows from Lemma 9.1. □

Proposition 11.4. *If A is a division algebra, the push-forward homomorphism*

$$\mathrm{CH}^{m+p+1}(G \times Y) \to \mathrm{CH}^{p+1}(G)$$

is trivial.

Proof. By Lemma 11.2, it is sufficient to show that for every closed point $y \in Y$ the norm homomorphism $\mathrm{CH}^{p+1}(G_{F(y)}) \to \mathrm{CH}^{p+1}(G)$ is trivial. This is proved in Lemma 11.3. □

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