MULTIPLES OF PFISTER FORMS

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Abstract. The isotropy of multiples of Pfister forms is studied. Some classical results are recalled and their consequences presented, with certain of these statements being hitherto unpublished. In particular, a lower bound on the first Witt index of Pfister multiples is established and a number of its corollaries are outlined. The relationship between a form and its Pfister multiples is explored, with particular attention being devoted to the case where the Pfister form is generic. Isotropy statements are obtained in this context, with related results demonstrating a strong correspondence between the properties of a form of those of its generic Pfister multiples. These correspondence results are applied to discriminate between properties inherited by Pfister multiples, with some novel examples being provided in this regard.

1. Introduction

Given the centrality of Pfister forms to the theory of quadratic forms, the properties of their multiples has been a topic of long-standing interest. The classification problem can be viewed as one source of motivation for this study, as Pfister forms, and therefore their multiples, represent prominent examples of forms of low complexity. Thus, given a form q and a Pfister form π over a field of characteristic different from two, it is motivated to determine which properties of q are inherited by the Pfister multiple $\pi \otimes q$. In the same vein, one can look to clarify what relationships exist between the properties or invariants of q and those of $\pi \otimes q$. With respect to invariants defined on the level of the motives of the associated varieties, this comparison problem is not well understood, despite the efforts of leading specialists (see [28], [30], [17], [20]).

This article addresses these questions with respect to properties and invariants that are reflected in the splitting behaviour of the associated forms. Elman and Lam established a number of important results in this regard (see [3] and [4]), and these results form the basis of our study of the isotropy of Pfister multiples, the primary focus of the opening section of this article. In [3], it was proven that the Witt index of $\pi \otimes q$ is a multiple of the dimension of π. Thus, extending $\pi \otimes q$ to the generic extension over which it becomes isotropic, one obtains that its first Witt index is at least the dimension of π, a result that is regularly invoked in the literature. In Theorem 2.5, we establish that the first Witt index of $\pi \otimes q$ is at least the first Witt index of q times the dimension of π. This result lends support to the following conjectural description of the possible Witt indices of Pfister multiples:

Conjecture 1.1. *Let q be a form over F and π an n-fold Pfister form over F. Then $i(\pi \otimes q)$ is either a specialization of*

$$(2^n \cdot i(q), 2^n \cdot i_1(q), \ldots, 2^n \cdot i_h(q)) \text{ or } (2^n \cdot i(q), 2^n \cdot i_1(q), \ldots, 2^n \cdot i_h(q), 2^{n-1}),$$

depending on whether q is even or odd dimensional respectively.

Whereas the value of the first Witt index of a Pfister multiple can exceed the aforementioned bound (Example 2.11), we can establish conditions on the form q which ensure that this bound is met (Proposition 2.14 and Proposition 2.15). In particular, the maximal splitting property is shown to be inherited by Pfister multiples. These results have previously been applied and referenced (without proof) in [25]. An anonymous referee communicated Corollary 2.9, a general result on the isotropy of Pfister multiples. This result constitutes the strongest statement of its kind in Section 2, and we recommend it as being especially noteworthy.

Section 3 is devoted to the study of multiples of "generic" Pfister forms, those of shape $\langle 1, x_1\rangle \otimes \ldots \otimes \langle 1, x_n\rangle$ where $x_1, \ldots, x_n$ are variables. Knebusch's specialization results (established in [21]) motivate the connection of this topic to the general case. In particular, we highlight the following question concerning the splitting behaviour of such multiples, noting that an affirmative answer would confirm Conjecture 1.1.

Question 1.2. *Let q be an anisotropic form over F of dimension at least two with splitting pattern $(i_1(q), \ldots, i_h(q))$. Let $\pi \simeq \langle 1, x_1\rangle \otimes \ldots \otimes \langle 1, x_n\rangle$ over $F((x_1)) \ldots ((x_n))$. Is the splitting pattern of $\pi \otimes q$ given by*

$$((\dim \pi)i_1(q), \ldots, (\dim \pi)i_h(q)) \text{ or } ((\dim \pi)i_1(q), \ldots, (\dim \pi)i_h(q), 2^{n-1})$$

in accordance with whether q is even or odd dimensional?

In Theorem 3.3, we establish a general result on the isotropy of these generic Pfister multiples, invoking it to establish some partial results with respect to the above question (see Corollary 3.4 and Proposition 3.5). The main focus of our work in this section is to establish a correspondence between the properties of a form and those of its generic Pfister multiples. We obtain a number of results in this regard, showing that such a relationship holds with respect to the properties of being a Pfister neighbour, a Pfister multiple, an excellent form, a round form and a form with maximal splitting.

These correspondence results are employed in Section 4 to distinguish between certain properties inherited by Pfister multiples, as they provide a framework for extending phenomena known to hold in low dimensions. To this end, we construct examples of non-excellent Pfister neighbours (including a minimum-codimension example of a non-excellent "special Pfister neighbour"), generalise Hoffmann's construction of forms with maximal splitting that are not Pfister neighbours, and briefly discuss the open problem of determining the dimensions in which the maximal-splitting and Pfister-neighbour properties coincide. We apply a similar approach in the concluding section of this article. Here, we generalise some known phenomena with respect to forms of non-trivial first Witt index, before positing Question 5.3, which suggests a possible characterisation of these forms.

Henceforth, we will let F denote a field of characteristic different from two. The term "form" will refer to a regular quadratic form. Every form over F can be diagonalised. Given $a_1, \ldots, a_n \in F^\times$ for $n \in \mathbb{N}$, we denote by $\langle a_1, \ldots, a_n\rangle$ the n-dimensional quadratic form $a_1X_1^2 + \ldots + a_nX_n^2$. If p and q are forms over F, we denote by $p \perp q$ their orthogonal sum and by $p \otimes q$ their tensor product. For $n \in \mathbb{N}$, we will denote the orthogonal sum of n copies of q by $n \times q$. We use aq to denote $\langle a\rangle \otimes q$ for $a \in F^\times$. We write $p \simeq q$ to indicate that p and q are isometric, and say that p and q are *similar* (over F) if $p \simeq aq$ for some $a \in F^\times$. For q a form over F and K/F a field extension, we will employ the notation q_K when viewing q as a form over K via the canonical embedding. A form p is a *subform of* q if $q \simeq p \perp r$ for some form r, in which case we will write $p \subset q$. A form q *represents* $a \in F$ if there exists a vector v such that $q(v) = a$. We denote by $D_F(q)$ the set of values in $F^\times$ represented by q. A form over F is *isotropic* if it represents zero non-trivially, and *anisotropic* otherwise. Every form q has a decomposition $q \simeq q_{\text{an}} \perp i(q) \times \langle 1, -1\rangle$ where the anisotropic form q_{an} and the integer $i(q)$, the *Witt index of* q, are uniquely determined. A form q is *hyperbolic* if q_{an} is trivial, whereby $i(q) = \frac{1}{2}\dim q$. Two anisotropic forms p and q over F are *isotropy equivalent* if for every field extension K/F we have that p_K is isotropic if and only if q_K is isotropic. The following basic fact (see [24, Exercise I.16]) will be employed frequently.

Lemma 1.3. *If $p \subset q$ with $\dim p \geqslant \dim q - i(q) + 1$, then p is isotropic.*

We will let $c(q)$ denote the *Clifford invariant of a form q*: if q is even dimensional, then $c(q)$ is $[C(q)]$, the class of the Clifford algebra of q in the Brauer group; if q is odd dimensional, then $c(q)$ is $[C_0(q)]$, the Brauer class of the even Clifford algebra of q (the subalgebra of elements of even degree in $C(q)$). Formulae for the computation of the Clifford invariant can be found in [24, V.(3.13)]. The *(Schur) index of a central simple algebra* is the square root of the dimension of a Brauer-equivalent division algebra. An *ordering of F* is a set $P \subset F^\times$ such that $P \cup -P = F^\times$ and $x + y, xy \in P$ for all $x, y \in P$. We say that F is a *(formally) real field* if it has an ordering. Given a form q over F and an ordering P of F, the *signature of q at P*, denoted $\mathrm{sgn}_P(q)$, is the number of coefficients in a diagonalisation of q that are in P minus the number that are not in P. A form q over F is *indefinite at P* if $|\mathrm{sgn}_P(q)| < \dim q$.

For $n \in \mathbb{N}$, an *n-fold Pfister form* over F is a form isometric to $\langle 1, a_1\rangle \otimes \ldots \otimes \langle 1, a_n\rangle$ for some $a_1, \ldots, a_n \in F^\times$ (the form $\langle 1\rangle$ is the 0-fold Pfister form). Isotropic Pfister forms are hyperbolic [24, Theorem X.1.7]. A form τ over F is a *neighbour of a Pfister form π* if $\tau \subset a\pi$ for some $a \in F^\times$ and $\dim \tau > \frac{1}{2} \dim \pi$. For τ a neighbour of a Pfister form π with $\tau \perp \gamma \simeq a\pi$ for some $a \in F^\times$, the form γ is called the *complementary form* of τ. All forms of dimension not greater than one are said to be *excellent*; a form q of dimension $n \geqslant 2$ is *excellent* if q is a Pfister neighbour and the complementary form of q is excellent. A form q over F is *round* if $D_F(q) = G_F(q)$, where $G_F(q) = \{a \in F^\times \mid aq \simeq q\}$ is the group of similarity factors of q. Pfister forms are round (see [24, Theorem X.1.8]).

For a form q over F with $\dim q = n \geqslant 2$ and $q \not\simeq \langle 1, -1\rangle$, the *function field $F(q)$ of q* is the quotient field of the integral domain $F[X_1, \ldots, X_n]/(q(X_1, \ldots, X_n))$ (this is the function field of the affine quadric $q(X) = 0$ over F). To avoid case distinctions, we set $F(q) = F$ if $\dim q \leqslant 1$ or $q \simeq \langle 1, -1\rangle$. Letting $F_0 = F$, $i_0(q) = i(q)$ and $q_0 \simeq q_{\mathrm{an}}$, following Knebusch [21] we inductively define

$$F_{j+1} = F_j(q_j), \quad i_{j+1}(q) = i((q_j)_{F_{j+1}}) \quad \text{and} \quad q_{j+1} \simeq ((q_j)_{F_{j+1}})_{\mathrm{an}},$$

stopping when $\dim q_h \leqslant 1$. This integer h is the *height of q*, the tower of fields $F = F_0 \subset F_1 \subset \ldots \subset F_h$ is the *generic splitting tower of q*, the forms $q_1, \ldots, q_h$ are the *higher kernel forms of q* and the natural numbers $i_1(q), \ldots, i_h(q)$ are the *higher Witt indices of q*. The sequence $(i_1(q), \ldots, i_h(q))$ is called the *(incremental) splitting pattern of q*. For all forms p over F and all extensions K/F such that q_K is isotropic, we have that $i(p_{F(q)}) \leqslant i(p_K)$ (see [21, Proposition 3.1 and Theorem 3.3]). In particular, with respect to $i_1(q)$, the *first Witt index of q*, we have that $i_1(q) \leqslant i(q_K)$ for all extensions K/F such that q_K is isotropic. In light of Lemma 1.3, given an anisotropic form p over F, we define a form q over F to be a *neighbour of p* if $q \subset ap$ for some $a \in F^\times$ and $\dim q > \dim p - i_1(p)$. For q a neighbour of p with $q \perp r \simeq ap$ for some $a \in F^\times$, the form r is called the *complementary form of q (with respect to p)*. As per [24, Theorem X.4.1], $F(q)$ is a purely-transcendental extension of F if and only if q is isotropic over F. On account of this fact, one can see that two anisotropic forms p and q over F are isotropy equivalent if and only if $p_{F(q)}$ and $q_{F(p)}$ are isotropic. The behaviour of orderings with respect to function field extensions is governed by the following result due to Elman, Lam and Wadsworth [5, Theorem 3.5] and, independently, Knebusch [7, Lemma 10].

Theorem 1.4. *Let q be a form of dimension at least two over a real field F. An ordering P of F extends to $F(q)$ if and only if q is indefinite at P.*

[8, Theorem 1], [29, Corollary 3] and [19, Theorem 4.1] represent important isotropy criteria with respect to function fields of quadratic forms. We recall them below.

Theorem 1.5. (*Hoffmann*) *Let p and q be forms over F such that p is anisotropic. If $\dim p \leqslant 2^n < \dim q$ for some integer $n \geqslant 0$, then $p_{F(q)}$ is anisotropic.*

Theorem 1.6. (*Vishik*) *Let p and q be anisotropic forms over F that are isotropy equivalent. Then $\dim p - i_1(p) = \dim q - i_1(q)$.*

Theorem 1.7. (*Karpenko, Merkurjev*) *Let p and q be anisotropic forms over F such that $p_{F(q)}$ is isotropic. Then*

(*i*) $\dim p - i_1(p) \geqslant \dim q - i_1(q)$;

(*ii*) $\dim p - i_1(p) = \dim q - i_1(q)$ *if and only if $q_{F(p)}$ is isotropic.*

Applying Theorem 1.5 and Lemma 1.3 to an anisotropic form q of dimension $2^n + k$ for $1 \leqslant k \leqslant 2^n$, one sees that $i_1(q) \leqslant k$. An anisotropic form q is said to have *maximal splitting* if $\dim q - i_1(q)$ is a power of two.

Over $F(\!(x)\!)$, the Laurent series field in the variable x over F, we recall that every non-zero square class can be represented by a or ax for some $a \in F^\times$, whereby every form φ over $F(\!(x)\!)$ can be written as $p \perp xq$ for p and q forms over F. We recall the following folkloric result regarding forms over Laurent series fields.

Lemma 1.8. *Let p and q be forms over F. Considering $p \perp xq$ as a form over $F(\!(x)\!)$, we have that $i(p \perp xq) = i(p) + i(q)$.*

Proof. Applying Springer's Theorem for complete discretely valued fields [24, Theorem VI.1.4], one obtains that $p \perp xq$ is anisotropic over $F(\!(x)\!)$ if and only if p and q are anisotropic over F. The result follows by applying Witt decomposition to the forms p and q over F. □

2. The isotropy of multiples of Pfister forms

Since the isotropy of scalar multiples of Pfister forms is well understood (indeed, an anisotropic form q of dimension at least two is a scalar multiple of a Pfister form if and only if q is hyperbolic over $F(q)$, see [2, Corollary 23.4]), we will restrict our attention to multiples of Pfister forms with forms of dimension at least two.

Elman and Lam obtained a number of important results on the isotropy of multiples of Pfister forms in the early seventies. In particular, the following result is known.

Theorem 2.1. (*Elman, Lam*) *Let π be an anisotropic Pfister form over F and let q be a form over F of dimension at least two. If $\pi \otimes q$ is isotropic, then there exist forms q_1 and q_2 over F such that $\pi \otimes q_1$ is anisotropic, q_2 is hyperbolic, and $\pi \otimes q \simeq \pi \otimes q_1 \perp \pi \otimes q_2$. In particular, $i(\pi \otimes q) = (\dim \pi) i(q_2)$.*

Wadsworth and Shapiro [32, Theorem 2] established that the above result holds, more generally, for multiples of round forms. Theorem 2.1, as formulated above, is a consequence of Elman and Lam's representation theorem [3, Theorem 1.4]. A proof of this fact is contained in the proof of [9, Lemma 3.1]. We will provide another proof of Theorem 2.1 below. To do this, we will invoke another important result of Elman and Lam, [4, Proposition 2.2], which states the following:

Proposition 2.2. (*Elman, Lam*) *Let q be a form over F of dimension at least two and π a 1-fold Pfister form over F. If $\pi \otimes q$ is isotropic, then there exists a 2-dimensional form $\beta \subset q$ such that $\pi \otimes \beta$ is hyperbolic.*

Proof of Theorem 2.1. We begin by considering the case where π is a 1-fold Pfister form. As $\pi \otimes q$ is isotropic, we may repeatedly apply Proposition 2.2 to obtain the "beta decomposition" of q, whereby $\pi \otimes q \simeq \pi \otimes (\beta_1 \perp \ldots \perp \beta_r \perp \sigma)$ with $\pi \otimes \beta_j$ hyperbolic for $1 \leqslant j \leqslant r$ and $\pi \otimes \sigma$ anisotropic. Hence, $\pi \otimes q \simeq \pi \otimes (r \times \langle 1, -1\rangle) \perp \pi \otimes \sigma$, whereby we may let $q_1 \simeq \sigma$, a subform of q, and $q_2 \simeq r \times \langle 1, -1\rangle$. The general statement now follows by iterating this argument. □

Theorem 2.1 has a number of important consequences. The following statement, which is regularly applied in the literature, is one such result.

Corollary 2.3. *Let q a form of dimension at least two and π similar to a Pfister form be such that $\pi \otimes q$ is anisotropic over F. Then $i_1(\pi \otimes q) \geqslant \dim \pi$.*

Thus, we have that $i((\pi \otimes q)_K) \geqslant \dim \pi$ for K/F such that $\pi \otimes q$ is isotropic over K. Moreover, as $F(\pi \otimes q)$ is a generic zero field of $\pi \otimes q$, it is often the case that when $i_1(\pi \otimes q)$ can be precisely determined, its value actually equals $\dim \pi$. Roussey [26] and Totaro [27] defined a neighbour of a multiple of a Pfister form π to be (up to similarity) a subform of codimension less than $\dim \pi$. We will show that the statement of Corollary 2.3 may be refined, whereby our definition of a neighbour of a Pfister multiple diverges from that in [26] and [27]. To achieve this refinement, we will invoke [26, Théorème 6.4.2], stated below. In his thesis, Roussey offers a number of proofs of this result, which he introduces as being already known but hitherto unwritten.

Theorem 2.4. (*Roussey*) *Let p and q be forms over F of dimension at least two and let π be similar to a Pfister form over F. If p is isotropic over $F(q)$, then $\pi \otimes p$ is isotropic over $F(\pi \otimes q)$.*

With regard to Theorem 2.4, we note that the corresponding statement with respect to hyperbolicity also holds, having been established by Fitzgerald [6, Theorem 3.2].

Our opening result establishes the aforementioned refinement of Corollary 2.3. In Remark 2.10, we will see another, elementary way of recovering this result, which suggests that its statement was known to some experts in the theory.

Theorem 2.5. *Let q a form of dimension at least two and π similar to a Pfister form be such that $\pi \otimes q$ is anisotropic over F. Then $i_1(\pi \otimes q) \geqslant (\dim \pi) i_1(q)$.*

Proof. If $i_1(q) = 1$, then the statement is precisely Corollary 2.3. Hence, we may assume that $i_1(q) > 1$. Let $q' \subset q$ over F of dimension $\dim q - i_1(q) + 1$. Lemma 1.3 implies that q' is isotropic over $F(q)$. Hence, $\pi \otimes q'$ is isotropic over $F(\pi \otimes q)$ by Theorem 2.4. As $\pi \otimes q' \subset \pi \otimes q$, we have that $\pi \otimes q'$ is anisotropic over F by assumption and, furthermore, that $\pi \otimes q$ is isotropic over $F(\pi \otimes q')$, whereby $\pi \otimes q'$ and $\pi \otimes q$ are isotropy equivalent. Invoking Theorem 1.6, we have that $\dim(\pi \otimes q') - i_1(\pi \otimes q') = \dim(\pi \otimes q) - i_1(\pi \otimes q)$, whereby $i_1(\pi \otimes q) = i_1(\pi \otimes q') + \dim \pi(\dim q - \dim q') = i_1(\pi \otimes q') + \dim \pi(i_1(q) - 1)$. Since $i_1(\pi \otimes q') \geqslant \dim \pi$ by Corollary 2.3, we have that $i_1(\pi \otimes q) \geqslant (\dim \pi) i_1(q)$. □

Corollary 2.6. *Let q a form of dimension at least two and π similar to a Pfister form be such that $\pi \otimes q$ is anisotropic over F. If $p \subset \pi \otimes q$ over F of codimension less than $(\dim \pi) i_1(q)$, then p is isotropic over $F(\pi \otimes q)$.*

Proof. Theorem 2.5 implies that p is a subform of $\pi \otimes q$ of codimension less than $i_1(\pi \otimes q)$, whereby Lemma 1.3 implies that p is isotropic over $F(\pi \otimes q)$. □

Theorem 2.5 can be viewed as lending support to Conjecture 1.1. In accordance with Theorem 2.1, every higher Witt index of $\pi \otimes q$ is a multiple of $\dim \pi$, with the exception of $i_h(\pi \otimes q)$ in the case where q is an odd-dimensional form. At present, we do not have an analogue of Theorem 2.5 with respect to $i_r(\pi \otimes q)$ for $2 \leqslant r \leqslant h$. In certain situations, we can establish upper bounds on the values of some higher Witt indices.

Proposition 2.7. *Let q a form of dimension at least two and π similar to a Pfister form be such that $\pi \otimes q$ is anisotropic over F. Let $F = F_0 \subset F_1 \subset \ldots \subset F_h$ denote the generic splitting tower of q and $q \simeq q_0, q_1 \ldots, q_h$ the kernel forms of q.*

(*i*) *If $\pi \otimes q_1$ is anisotropic over F_1, then $i_1(\pi \otimes q) = (\dim \pi) i_1(q)$.*

(*ii*) *Suppose that q is not a Pfister neighbour of codimension at most one and that $\pi \otimes q$ is not similar to a Pfister form, whereby $i_2(q)$ and $i_2(\pi \otimes q)$ are defined. If $\pi \otimes q_1$ is anisotropic over F_1 and $\pi \otimes q_2$ is anisotropic over F_2, then $i_1(\pi \otimes q) = (\dim \pi) i_1(q)$ and $i_2(\pi \otimes q) \leqslant (\dim \pi) i_2(q)$.*

Proof. (*i*) Considering the extension of $\pi \otimes q$ to the field F_1, we have that

$$(\pi \otimes q)_{F_1} \simeq \pi_{F_1} \otimes (i_1(q) \times \langle 1, -1\rangle_{F_1} \perp q_1) \simeq ((\dim \pi) i_1(q)) \times \langle 1, -1\rangle_{F_1} \perp (\pi_{F_1} \otimes q_1),$$

with $\pi_{F_1} \otimes q_1$ being anisotropic by assumption. As $F(\pi \otimes q)$ is the generic zero field of $\pi \otimes q$, we thus have that $i_1(\pi \otimes q) \leqslant (\dim \pi) i_1(q)$. Invoking Theorem 2.5, we also have that $i_1(\pi \otimes q) \geqslant (\dim \pi) i_1(q)$, whereby the result follows.

(*ii*) As forms of height one are necessarily Pfister neighbours of codimension at most one, by [21, Theorem 5.8] (independently proved by Wadsworth [31]), we have that q and $\pi \otimes q$ are forms of height at least two, whereby $i_2(q)$ and $i_2(\pi \otimes q)$ are defined. Considering the extension of $\pi \otimes q$ to the field F_2, we have that

$$(\pi \otimes q)_{F_2} \simeq \pi_{F_2} \otimes ((i_1(q) + i_2(q)) \times \langle 1, -1\rangle_{F_2} \perp q_2),$$

with $\pi_{F_2} \otimes q_2$ being anisotropic by assumption. As $i_1(\pi \otimes q) = (\dim \pi) i_1(q)$ by (*i*), we thus have that $i_1(\pi \otimes q) < i((\pi \otimes q)_{F_2}) = \dim \pi (i_1(q) + i_2(q))$, whereby $i_2(\pi \otimes q) \leqslant (\dim \pi) i_2(q)$. □

Elman and Lam's Proposition 2.2 is an important ingredient in the proof of the following lemma, whose statement and proof were kindly communicated to me by an anonymous referee (we present a strengthened formulation of the statement of this result, which was implicit in the referee's proof).

Lemma 2.8. *Let p and q be forms over F of dimension at least two. Let π be a 1-fold Pfister form over F and K/F a field extension. If $\pi \otimes q$ is isotropic over K, then we have that*

$$i((\pi \otimes p)_K) \geqslant 2i(p_{F(\beta)}) \geqslant 2i(p_{F(q)}),$$

where β is some 2-dimensional subform of q.

Proof. Without loss of generality, we may assume that $K = F$ and that p is anisotropic. As the statement is trivial when q is isotropic over F, as $F(q)/F$ is purely transcendental in this case, we may assume that q is anisotropic. Now, since $\pi \otimes q$ is isotropic for π a 1-fold Pfister form, there exists a 2-dimensional form $\beta \subset q$ such that $\pi \otimes \beta$ is hyperbolic, by Proposition 2.2. As $i(p_{F(\beta)}) \geqslant i(p_{F(q)})$ by Knebusch's specialization results (see [2, Section 22.A]), it suffices to prove that $i(\pi \otimes p) \geqslant 2i(p_{F(\beta)})$. If p is anisotropic over $F(\beta)$, then there is nothing to prove. Otherwise, since p is anisotropic, Pfister's lemma on quadratic extensions [24, Theorem VII.3.1] implies that $p \simeq (\beta \otimes \tau) \perp \sigma$ for some forms σ and τ, with τ of dimension $i(p_{F(\beta)})$. Multiplying across by π, we have that $\pi \otimes p \simeq (\pi \otimes \beta \otimes \tau) \perp (\pi \otimes \sigma)$. As $\pi \otimes \beta$ is hyperbolic, it follows that $\pi \otimes p \simeq 2i(p_{F(\beta)}) \times \langle 1, -1\rangle \perp (\pi \otimes \sigma)$, establishing the result. □

The same referee also communicated the following result as being a corollary of Lemma 2.8. We offer a proof of this fact below. As highlighted in Remark 2.10, this powerful statement can be regarded as the most general result of its kind in this section.

Corollary 2.9. *Let p and q be forms over F of dimension at least two. Let π be an n-fold Pfister form over F and K/F a field extension. If $\pi \otimes q$ is isotropic over K, then we have that*

$$i\left((\pi \otimes p)_K\right) \geqslant i\left((\pi \otimes p)_{F(\pi\otimes q)}\right) \geqslant 2^n i\left(p_{F(q)}\right).$$

Proof. As $\pi \otimes q$ is isotropic over K, it follows that $i\left((\pi \otimes p)_K\right) \geqslant i\left((\pi \otimes p)_{F(\pi\otimes q)}\right)$ by Knebusch's specialization results (see [2, Section 22.A]). Hence, it suffices to prove that $i\left((\pi \otimes p)_{F(\pi\otimes q)}\right) \geqslant 2^n i\left(p_{F(q)}\right)$.

In the case where $\pi \simeq \langle 1, a_1\rangle$ for $a_1 \in F^\times$, an application of Lemma 2.8 in the case where $K = F(\langle 1, a_1\rangle \otimes q)$ establishes that

$$i\left((\langle 1, a_1\rangle \otimes p)_{F(\langle 1,a_1\rangle\otimes q)}\right) \geqslant 2i(p_{F(q)}).$$

Similarly, for $\pi \simeq \langle 1, a_1\rangle \otimes \langle 1, a_2\rangle$ for $a_1, a_2 \in F^\times$, an application of Lemma 2.8 in the case where $K = F(\langle 1, a_2\rangle \otimes (\langle 1, a_1\rangle \otimes q))$ establishes that

$$i\left((\langle 1, a_2\rangle \otimes (\langle 1, a_1\rangle \otimes p))_{F(\langle 1,a_2\rangle\otimes(\langle 1,a_1\rangle\otimes q))}\right) \geqslant 2i\left((\langle 1, a_1\rangle \otimes p)_{F(\langle 1,a_1\rangle\otimes q)}\right).$$

Hence, we have that $i\left((\langle 1, a_1\rangle \otimes \langle 1, a_2\rangle \otimes p)_{F(\langle 1,a_1\rangle\otimes\langle 1,a_2\rangle\otimes q)}\right) \geqslant 4i(p_{F(q)})$.

Thus, for $\pi \simeq \langle 1, a_1\rangle \otimes \ldots \otimes \langle 1, a_n\rangle$ for $a_1, \ldots, a_n \in F^\times$, we may iterate the above argument to establish that $i\left((\pi \otimes p)_{F(\pi\otimes q)}\right) \geqslant 2^n i\left(p_{F(q)}\right)$. □

Remark 2.10. We note that Theorem 2.4 and [6, Theorem 3.2] are inherent in the statement of Corollary 2.9. Moreover, as noted by the aforementioned referee, the statement of Theorem 2.5 can be recovered from Corollary 2.9 by letting $p = q$.

Returning to the statement of Theorem 2.5, we note the existence of forms q and π over F such that the value of $i_1(\pi \otimes q)$ is strictly greater than $(\dim \pi)i_1(q)$. The following example, communicated to me by Karim Becher, can be used to demonstrate this. We will also use this example to show that the converses of Theorem 2.4 and [6, Theorem 3.2] do not hold in general.

Example 2.11. Let $q \simeq \langle 1, 1, 1, 7\rangle$ and $\pi \simeq \langle 1, 1, 1, 1\rangle$ over $F = \mathbb{Q}$. Since $\det q \notin \mathbb{Q}^2$, the form q is not similar to a 2-fold Pfister form. Hence, by invoking the Cassels-Pfister Subform Theorem [L, Ch.X, Theorem 4.5], we may conclude that q is not hyperbolic over $\mathbb{Q}(\pi)$. Moreover, as a consequence of [2, Corollary 23.4], it follows that $i_1(q) = 1$. Hence, the form $\langle 1, 1, 1\rangle$ is anisotropic over $\mathbb{Q}(q)$ by Theorem 1.7 (i). As $7 \in D_{\mathbb{Q}}(\pi)$, we have that $7\langle 1, 1, 1, 1\rangle \simeq \langle 1, 1, 1, 1\rangle$, and thus that $q\otimes\pi \simeq 16 \times \langle 1\rangle$. Hence, we have that $i_1(\pi \otimes q) = 8 > (\dim \pi)i_1(q) = 4$. Moreover, the Pfister form $q\otimes\pi$ is hyperbolic over $\mathbb{Q}(\pi\otimes\pi)$. Furthermore, as $\langle 1, 1, 1\rangle\otimes\pi$ is a Pfister neighbour of $16 \times \langle 1\rangle$, we have that $\langle 1, 1, 1\rangle \otimes \pi$ is isotropic over $\mathbb{Q}(q \otimes \pi)$.

As above, the converse to Theorem 2.4 does not hold in general. The following result places a necessary condition on situations where this converse holds with respect to all forms over F.

Proposition 2.12. *Let q be a form of dimension at least two such that $\pi \otimes q$ is anisotropic over F, where π is similar to a Pfister form. For all forms p over F such that $\pi\otimes p$ is anisotropic over F, suppose that p is isotropic over $F(q)$ whenever $\pi \otimes p$ is isotropic over $F(\pi \otimes q)$. Then $i_1(\pi \otimes q) = \dim \pi(i_1(q))$.*

Proof. Invoking Theorem 2.5, we have that $i_1(\pi \otimes q) \geqslant \dim \pi(i_1(q))$. Suppose, for the sake of contradiction, that $i_1(\pi \otimes q) > \dim \pi(i_1(q))$. Let $p \subset q$ of codimension $i_1(q)$, whereby p is anisotropic over $F(q)$ by Theorem 1.7(i). However, as $\pi \otimes p \subset \pi \otimes q$ of codimension $\dim \pi(i_1(q))$, we have that $\pi \otimes p$ is isotropic over $F(\pi \otimes q)$ by Lemma 1.3, a contradiction. Hence $i_1(\pi \otimes q) = \dim \pi(i_1(q))$. □

Incorporating the above necessary condition, the next result establishes the converse of Theorem 2.4 with the aid of an additional assumption.

Proposition 2.13. *Let p and q be forms of dimension at least two such that $\pi \otimes p$ and $\pi \otimes q$ are anisotropic over F, where π is similar to a Pfister form. Suppose that $i_1(\pi \otimes q) = \dim \pi(i_1(q))$ and that q is isotropic over $F(p)$. If $\pi \otimes p$ is isotropic over $F(\pi \otimes q)$, then p is isotropic over $F(q)$.*

Proof. Suppose that $\pi \otimes p$ is isotropic over $F(\pi \otimes q)$. Invoking Theorem 1.7(i), we have that $\dim(\pi \otimes p) - i_1(\pi \otimes p) \geqslant \dim \pi(\dim q - i_1(q))$, whereby it follows that $i_1(\pi \otimes p) \leqslant \dim \pi(\dim p - \dim q + i_1(q))$. Invoking Theorem 2.5, we have that $\dim \pi(i_1(p)) \leqslant \dim \pi(\dim p - \dim q + i_1(q))$. As q is isotropic over $F(p)$ by assumption, we have that $\dim q - i_1(q) \geqslant \dim p - i_1(p)$ by Theorem 1.7(i). Hence, $\dim q - i_1(q) = \dim p - i_1(p)$, whereby p is isotropic over $F(q)$ by Theorem 1.7(ii). □

The preceding results provide some additional motivation for determining when the equality $i_1(\pi \otimes q) = \dim \pi(i_1(q))$ holds, a question which naturally arises in light of Theorem 2.5. Our next results establish conditions on the form q that ensure that this equality holds. Over real fields, we can establish the following statement.

Proposition 2.14. *Let q a form of dimension at least two and π similar to a Pfister form be such that $\pi \otimes q$ is anisotropic over a real field F. Let P be an ordering of F such that π is definite at P and q is indefinite at P, whereby $|\mathrm{sgn}_P(q)| \leqslant \dim q - 2i_1(q)$. If $|\mathrm{sgn}_P(q)| = \dim q - 2i_1(q)$, then $i_1(\pi \otimes q) = (\dim \pi)i_1(q)$.*

Proof. As q is indefinite at P, Theorem 1.4 implies that P extends to $F(q)$, whereby it follows that $|\mathrm{sgn}_P(q)| \leqslant \dim q - 2i_1(q)$. By assumption, we have that π is (positive) definite at P and $|\mathrm{sgn}_P(q)| = \dim q - 2i_1(q)$, whereby it follows that

$$|\mathrm{sgn}_P(\pi \otimes q)| = \dim \pi(\dim q - 2i_1(q)).$$

Hence, Theorem 1.4 implies that P extends to $K = F(\pi \otimes q)$. Over K, $(\pi \otimes q)_K \simeq ((\pi \otimes q)_K)_{\mathrm{an}} \perp i_1(\pi \otimes q) \times \langle 1, -1 \rangle_K$, whereby a comparison of signatures with respect to P yields that $i_1(\pi \otimes q) \leqslant (\dim \pi)i_1(q)$. Invoking Theorem 2.5, we also have that $i_1(\pi \otimes q) \geqslant (\dim \pi)i_1(q)$, whereby the result follows. □

In the preceding result, we established that the value of $i_1(\pi \otimes q)$ coincides with our lower bound when $i_1(q)$ is maximal with respect to the signature of q at an ordering. In a similar spirit, if $i_1(q)$ is maximal with respect to the dimension of q, we can also establish this equality.

Proposition 2.15. *Let q a form of dimension at least two and π similar to a Pfister form be such that $\pi \otimes q$ is anisotropic over F. If q has maximal splitting, then $\pi \otimes q$ has maximal splitting, whereby $i_1(\pi \otimes q) = (\dim \pi)i_1(q)$.*

Proof. Let $\dim q = 2^n + k$ for some integers n and k such that $1 \leqslant k \leqslant 2^n$, whereby $i_1(q) = k$. Theorem 2.5 implies that $i_1(\pi \otimes q) \geqslant k \dim \pi$, whereby the $(2^n \dim \pi + k \dim \pi)$-dimensional form $\pi \otimes q$ has maximal splitting. □

Proposition 2.15 was previously known to hold in the case where $\dim q = 2^n + 1$ for some $n \in \mathbb{N}$, where the statement follows through combining Theorem 1.5 with Corollary 2.3 (see [13, Corollary 8.9]).

Letting τ be a neighbour of a Pfister form π, we note that the statement of Proposition 2.15 holds with respect to the product $\tau \otimes q$ in the case where the codimension of $\tau \otimes q$ as a subform of $\pi \otimes q$ is less than $i_1(\pi \otimes q)$. As it can be difficult to determine the exact value of $i_1(\pi \otimes q)$ for a prescribed form q, we will invoke Theorem 2.5 to express this observation in terms of $i_1(q)$.

Corollary 2.16. *Let q a form of dimension at least two and π similar to a Pfister form be such that $\pi \otimes q$ is anisotropic over F. Let τ be a neighbour of π such that $\dim \tau > \dim \pi - \frac{(\dim \pi) i_1(q)}{\dim q}$. If q has maximal splitting, then $\tau \otimes q$ has maximal splitting, whereby $i_1(\tau \otimes q) = (\dim \pi) i_1(q) - (\dim q)(\dim \pi - \dim \tau)$.*

Proof. As per Corollary 2.6, since $\dim(\tau \otimes q) > \dim(\pi \otimes q) - (\dim \pi) i_1(q)$, it follows that $\tau \otimes q$ is isotropic over $F(\pi \otimes q)$. Thus, $\tau \otimes q$ is isotropy equivalent to $\pi \otimes q$, whereby Theorem 1.6 implies that $\dim(\tau \otimes q) - i_1(\tau \otimes q) = \dim(\pi \otimes q) - i_1(\pi \otimes q)$. Invoking Proposition 2.15, it follows that $\tau \otimes q$ has maximal splitting, whereby $i_1(\tau \otimes q) = (\dim \pi) i_1(q) - (\dim q)(\dim \pi - \dim \tau)$. □

The following example shows that the dimension condition in the preceding result can be sharp. This example furthermore demonstrates that the anisotropic product of two Pfister neighbours, both necessarily having maximal splitting, need not have maximal splitting.

Example 2.17. Let F be a field such that $\langle 1, 1, 1, d, d, d \rangle$ is anisotropic over F for some $d \in F^\times$. Let $K = F((x))((y))$ be the iterated Laurent series field in two variables over F. Consider the Pfister neighbours $\tau_1 \simeq \langle 1, 1, 1 \rangle$ and $\tau_2 \simeq \langle d \rangle \perp \langle 1, -x, -y, xy \rangle$ over K. Applying Springer's Theorem for complete discretely valued fields [24, Theorem VI.1.4], once with respect to the y-adic valuation and subsequently twice with respect to the x-adic valuation, one sees that the form $\tau_1 \otimes \tau_2$ is anisotropic over K. Suppose $\tau_1 \otimes \tau_2$ has maximal splitting. As $\dim(\tau_1 \otimes \tau_2) = 15$, it follows that $\tau_1 \otimes \tau_2$ is of height one, whereby [21, Theorem 5.8] (independently proved by Wadsworth [31]) implies that $\tau_1 \otimes \tau_2$ is a neighbour of some 4-fold Pfister form π over K. Comparing determinants, we have that $\tau_1 \otimes \tau_2 \perp \langle d \rangle \simeq a\pi$ for some $a \in K$. As $a\pi \in I^3 K$, it follows that $c(a\pi)$ is trivial (see [24, Corollary V.3.4]), whereby $c(\tau_1 \otimes \tau_2 \perp \langle d \rangle)$ is trivial. Applying [24, V.(3.13)], we thus obtain that $c(\tau_1 \otimes \tau_2)$ is trivial. However, by applying [24, V.(3.13)] to a decomposition of $\tau_1 \otimes \tau_2$, we see that $c(\tau_1 \otimes \tau_2)$ is Brauer equivalent to the biquaternion algebra $(-1, -1)_K \otimes_K (x, y)_K$. As above, iterated applications of Springer's Theorem [24, Theorem VI.1.4] with respect to the y-adic and x-adic valuations enable us to conclude that the form $\langle 1, 1, 1, x, y, -xy \rangle$ is anisotropic over K, whereby Albert's Theorem [24, Theorem III.4.8] implies that $(-1, -1)_K \otimes_K (x, y)_K$ is a division algebra over K. Hence, as $c(\tau_1 \otimes \tau_2)$ is non-trivial in the Brauer group of K, we may conclude that $\tau_1 \otimes \tau_2$ does not have maximal splitting.

3. Multiples of generic Pfister forms

A number of properties of a form are inherited by its Pfister multiples. This is clearly the case with respect to the properties of being a multiple or a neighbour of a Pfister form, whereby the property of being excellent is also inherited. Other properties inherited by Pfister multiples include roundness and, as established in Proposition 2.15, the property of having maximal splitting. While these properties of a form are reflected in those of its Pfister multiples, we cannot expect the converse to hold in general. Despite this, it seems reasonable to suggest that the behaviour of the generic Pfister multiples of a form might mirror the behaviour of the form itself. Thus, throughout this section, we will examine the relationship between a form q over F and its product with the generic Pfister form $\pi \simeq \langle 1, x_1\rangle \otimes \ldots \otimes \langle 1, x_n\rangle$ defined over the iterated Laurent series field $K = F(\!(x_1)\!) \ldots (\!(x_n)\!)$ (remarking that all the results we will mention also hold if π is considered as a form over the rational function field $F(x_1, \ldots, x_n)$). In this regard, we note that $i(\pi \otimes q_K) = (\dim \pi) i(q)$, as can be seen by iteratively invoking Lemma 1.8. In particular, q is anisotropic over F if and only if $\pi \otimes q$ is anisotropic over K.

An existing result which supports the above view is [26, Proposition 6.4.3]. We recall this result, which may be obtained by combining Roussey's Theorem 2.4 with applications of Izhboldin's [12, Lemma 5.4], in the proposition below.

Proposition 3.1. (*Roussey, Izhboldin*) *Let p and q be anisotropic forms over F of dimension at least two. Let $\pi \simeq \langle 1, x_1\rangle \otimes \ldots \otimes \langle 1, x_n\rangle$ over $K = F(\!(x_1)\!) \ldots (\!(x_n)\!)$. Then p is isotropic over $F(q)$ if and only if $\pi \otimes p$ is isotropic over $K(\pi \otimes q)$.*

In a similar spirit, by invoking Fitzgerald's result on the Witt kernels of Pfister multiples [6, Theorem 3.2] and adapting the proof of [12, Lemma 5.4], we can apply Lemma 1.8 to establish the hyperbolic analogue of Proposition 3.1:

Proposition 3.2. *Let p and q be anisotropic forms over F of dimension at least two. Let $\pi \simeq \langle 1, x_1\rangle \otimes \ldots \otimes \langle 1, x_n\rangle$ over $K = F(\!(x_1)\!) \ldots (\!(x_n)\!)$. Then p is hyperbolic over $F(q)$ if and only if $\pi \otimes p$ is hyperbolic over $K(\pi \otimes q)$.*

In preference to recording the aforementioned proof of Proposition 3.2, we instead note that the statements of Proposition 3.1 and Proposition 3.2 can be recovered as the extreme cases of the following general result.

Theorem 3.3. *Let p and q be anisotropic forms over F of dimension at least two. Over $K = F(\!(x_1)\!) \ldots (\!(x_n)\!)$, let $\pi \simeq \langle 1, x_1\rangle \otimes \ldots \otimes \langle 1, x_n\rangle$ and consider the forms $p \otimes \pi$ and $q \otimes \pi$. We have that $i\left((p \otimes \pi)_{K(q\otimes\pi)}\right) = (\dim \pi) i\left(p_{F(q)}\right)$.*

Proof. We recall that $p \otimes \pi$ is isotropic over $K(q \otimes \pi)$ if and only if p is isotropic over $F(q)$ by Proposition 3.1, whereby it suffices to prove the statement in the case where these forms are isotropic over the associated fields.

To do this, we begin by considering the forms $p \otimes \langle 1, x_1\rangle$ and $q \otimes \langle 1, x_1\rangle$ over the field $F(\!(x_1)\!)$, assuming that $p \otimes \langle 1, x_1\rangle$ is isotropic over $F(\!(x_1)\!)(q \otimes \langle 1, x_1\rangle)$. Invoking Knebusch's specialization results (see [2, Section 22.A]) together with the inclusion $F(\!(x_1)\!)(q) \subset F(q)(\!(x_1)\!)$, we have that

$$i\left(p \otimes \langle 1, x_1\rangle_{F(\!(x_1)\!)(q\otimes\langle 1,x_1\rangle)}\right) \leqslant i\left(p \otimes \langle 1, x_1\rangle_{F(\!(x_1)\!)(q)}\right) \leqslant i\left(p \otimes \langle 1, x_1\rangle_{F(q)(\!(x_1)\!)}\right).$$

Invoking Lemma 1.8, it follows that $i\left(p \otimes \langle 1, x_1\rangle_{F(\!(x_1)\!)(q\otimes\langle 1,x_1\rangle)}\right) \leqslant 2i\left(p_{F(q)}\right)$. An invocation of Lemma 2.8 shows that $i\left(p \otimes \langle 1, x_1\rangle_{F(\!(x_1)\!)(q\otimes\langle 1,x_1\rangle)}\right) \geqslant 2i\left(p_{F(\!(x_1)\!)(q)}\right)$. As $i\left(p_{F(q)}\right) \leqslant i\left(p_{F(\!(x_1)\!)(q)}\right) \leqslant i\left(p_{F(q)(\!(x_1)\!)}\right) = i\left(p_{F(q)}\right)$ by Lemma 1.8, it follows that $i\left(p_{F(\!(x_1)\!)(q)}\right) = i\left(p_{F(q)}\right)$. Thus, we have that

$$i\left(p \otimes \langle 1, x_1\rangle_{F(\!(x_1)\!)(q\otimes\langle 1,x_1\rangle)}\right) = 2i\left(p_{F(q)}\right).$$

The general statement now follows by iterating the above argument. □

Considering Theorem 3.3 in the case where $p \simeq q$, we obtain the following result, which can be viewed as a first step towards resolving Question 1.2.

Corollary 3.4. *Let q be an anisotropic form over F of dimension at least two. For $\pi \simeq \langle 1, x_1\rangle \otimes \ldots \otimes \langle 1, x_n\rangle$, the anisotropic form $q \otimes \pi$ over $F(\!(x_1)\!) \ldots (\!(x_n)\!)$ satisfies $i_1(\pi \otimes q) = (\dim \pi) i_1(q)$.*

In the spirit of Proposition 2.7, we can establish the following partial result with respect to Question 1.2.

Proposition 3.5. *Let q be an anisotropic form over F of dimension at least two. Suppose that q is not a Pfister neighbour of codimension at most one. For $\pi \simeq \langle 1, x_1\rangle \otimes \ldots \otimes \langle 1, x_n\rangle$, the anisotropic form $q \otimes \pi$ over $F(\!(x_1)\!)\ldots(\!(x_n)\!)$ satisfies $i_2(\pi \otimes q) \leqslant (\dim \pi) i_2(q)$.*

Proof. As forms of height one are necessarily Pfister neighbours of codimension at most one, by [21, Theorem 5.8] (independently proved by Wadsworth [31]), we have that $i_2(q)$ is defined. Moreover, $\pi \otimes q$ is not similar to a Pfister form in this case, as follows from combining parts (i) and (ii) of Proposition 3.7 (see Remark 3.8), whereby $i_2(\pi \otimes q)$ is defined.

Considering $q \simeq q_0$ as a form over F, let q_1 and q_2 denote its first two higher kernel forms. Extending $\pi \otimes q$ to the field $L = F(q)(q_1)(\!(x_1)\!)\ldots(\!(x_n)\!)$, we have that

$$(\pi \otimes q)_L \simeq \pi_L \otimes ((i_1(q) + i_2(q)) \times \langle 1, -1\rangle_L \perp (q_2)_L).$$

As q_2 is anisotropic over $F(q)(q_1)$ by definition, it follows that $\pi \otimes q_2$ is anisotropic over L by Lemma 1.8. As $i_1(\pi\otimes q) = (\dim \pi) i_1(q)$ by Corollary 3.4, we thus have that $i_1(\pi \otimes q) < i((\pi \otimes q)_L) = \dim \pi(i_1(q) + i_2(q))$, whereby $i_2(\pi \otimes q) \leqslant (\dim \pi) i_2(q)$. □

Proposition 3.6. *Let q be an anisotropic form over F of dimension at least two. For $\pi \simeq \langle 1, x_1\rangle \otimes \ldots \otimes \langle 1, x_n\rangle$, the form $q \otimes \pi$ over $F(\!(x_1)\!)\ldots(\!(x_n)\!)$ has maximal splitting if and only if q has maximal splitting.*

Proof. Assuming that q has maximal splitting, Proposition 2.15 implies that $q \otimes \pi$ has maximal splitting.

Conversely, let us assume that $q \otimes \pi$ has maximal splitting. Letting $\dim q = 2^m + k$, where $0 < k \leqslant 2^m$, we have that $\dim(q \otimes \pi) = 2^{m+n} + k(2^n)$. As $0 < k(2^n) \leqslant 2^{m+n}$, we have that $i_1(q \otimes \pi) = k(2^n)$. Invoking Corollary 3.4, it follows that $i_1(q) = k$, whereby q has maximal splitting. □

Proposition 3.7. *Let p and q be anisotropic forms over F. Let π denote the n-fold Pfister form $\langle 1, x_1\rangle \otimes \ldots \otimes \langle 1, x_n\rangle$ over $K = F(\!(x_1)\!)\ldots(\!(x_n)\!)$.*

- (i) *$q\otimes\pi$ is a neighbour of a Pfister form over K if and only if $q\otimes\pi$ is a neighbour of $\sigma \otimes \pi$ for σ some Pfister form over F.*
- (ii) *$q \otimes \pi$ is a neighbour of the form $p \otimes \pi$ over K if and only if q is a neighbour of the form p over F.*

Proof. (i) The "if" statement is clear. To prove the converse, we begin by considering the form $q \otimes \langle 1, x_1\rangle$ over $F(\!(x_1)\!)$. Suppose that $q \otimes \langle 1, x_1\rangle$ is a neighbour of an anisotropic Pfister form γ over $F(\!(x_1)\!)$. As the form $q \otimes \langle 1, x_1\rangle$ is isotropic (indeed, hyperbolic) over $F(\!(x_1)\!)(\langle 1, x_1\rangle)$, it follows that the Pfister form γ is hyperbolic over $F(\!(x_1)\!)(\langle 1, x_1\rangle)$, whereby we have that $\gamma \simeq \langle 1, x_1\rangle \otimes \vartheta$ for some Pfister form ϑ over $F(\!(x_1)\!)$ (see [24, Theorem X.4.11 and Corollary X.4.13]). As was observed in the proof of [8, Proposition 7], since every non-zero square class in $F(\!(x_1)\!)$ can be represented by a or ax_1 for some $a \in F^\times$, and since $\langle\!\langle -ax_1, -bx_1\rangle\!\rangle \simeq \langle\!\langle -ab, -ax_1\rangle\!\rangle$ for all $a, b \in F^\times$, we may assume that either ϑ is defined over F or that $\vartheta \simeq \delta \otimes \langle 1, ax_1\rangle$ for $a \in F^\times$ and δ a Pfister form over F. In the latter case, we have that

$$\gamma \simeq \langle 1, x_1\rangle \otimes \vartheta \simeq \langle 1, x_1\rangle \otimes \delta \otimes \langle 1, ax_1\rangle \simeq \langle 1, x_1\rangle \otimes \delta \otimes \langle 1, a\rangle.$$

Hence, in either case, we have that $\gamma \simeq \langle 1, x_1\rangle \otimes \sigma$ for σ a Pfister form over F. Statement (i) now follows by iterating the above argument.

(ii) To prove the "if" statement, we assume that q is a neighbour of p, whereby $\dim q > \dim p - i_1(p)$. Invoking Corollary 3.4, we have that $i_1(p\otimes\pi) = (\dim\pi)i_1(p)$, whereby it follows that $\dim(q\otimes\pi) > \dim(p\otimes\pi) - i_1(p\otimes\pi)$, and hence that $q\otimes\pi$ is a neighbour of $p\otimes\pi$.

To prove the converse, we begin by assuming that $q\otimes\langle 1, x_1\rangle$ is a neighbour of $p\otimes\langle 1, x_1\rangle$ over $F(\!(x_1)\!)$ for some form p over F. Letting γ denote the complementary form of $q\otimes\langle 1, x_1\rangle$, we have that $q\otimes\langle 1, x_1\rangle \perp \gamma \sim p\otimes\langle 1, x_1\rangle$. As every square class in $F(\!(x_1)\!)$ is represented by a or ax_1 for some $a\in F^\times$, and since $x_1 \in D(\langle 1, x_1\rangle) = G(\langle 1, x_1\rangle)$, we have that $q\otimes\langle 1, x_1\rangle \perp \gamma \simeq a(p\otimes\langle 1, x_1\rangle)$ for $a\in F^\times$. As $\gamma\simeq\gamma_1\perp x_1\gamma_2$ for some forms γ_1, γ_2 over F, we have that $(q\perp x_1q)\perp(\gamma_1\perp x_1\gamma_2)\simeq ap\perp x_1(ap)$. Invoking Lemma 1.8, it thus follows that

$$q\perp\gamma_1\simeq ap\simeq q\perp\gamma_2.$$

Thus, q is a neighbour of p with complementary form $\gamma_1\simeq\gamma_2$. Statement (ii) now follows by iterating the above argument. □

Remark 3.8. In [8, Proposition 7] it was established that a form q is a Pfister neighbour over F if and only if q is a Pfister neighbour over $F(x)$. Thus, in the case where q is an anisotropic and p is similar to a Pfister form, Proposition 3.7 (i) and (ii) may be combined to generalise this result. Moreover, by additionally adapting the isotropic part of the proof of [8, Proposition 7], one can invoke the preceding observation to establish that, for all forms q over F, we have that $q\otimes\langle 1, x_1\rangle\otimes\ldots\otimes\langle 1, x_n\rangle$ is a Pfister neighbour over $F(\!(x_1)\!)\ldots(\!(x_n)\!)$ if and only if q is a Pfister neighbour over F.

As was remarked in [22], if q is an excellent form and π is a Pfister form, then $q\otimes\pi$ is an excellent form.

Proposition 3.9. *Let q be an anisotropic form over F. For $\pi\simeq\langle 1, x_1\rangle\otimes\ldots\otimes\langle 1, x_n\rangle$, the form $q\otimes\pi$ over $F(\!(x_1)\!)\ldots(\!(x_n)\!)$ is excellent if and only if q is excellent.*

Proof. As above, it suffices to prove the "only if" statement. We will begin by assuming that $q\otimes\langle 1, x_1\rangle$ is excellent over $F(\!(x_1)\!)$. Hence, $q\otimes\langle 1, x_1\rangle$ is a Pfister neighbour over $F(\!(x_1)\!)$. Invoking Proposition 3.7 (i), we have that $q\otimes\langle 1, x_1\rangle$ is a Pfister neighbour of $\sigma\otimes\langle 1, x_1\rangle$ for some Pfister form σ over F. As per the proof of Proposition 3.7 (ii), the form $q\otimes\langle 1, x_1\rangle$ has complementary form $q_1\otimes\langle 1, x_1\rangle$ for some form q_1 over F, where q is a neighbour of σ with complementary form q_1. As $q\otimes\langle 1, x_1\rangle$ is excellent, we have that $q_1\otimes\langle 1, x_1\rangle$ is excellent. Arguing as above, we may establish that $q_1\otimes\langle 1, x_1\rangle$ has complementary form $q_2\otimes\langle 1, x_1\rangle$ for some form q_2 over F, and that q_1 is a Pfister neighbour with complementary form q_2. Iterating this argument until $\dim q_n\leqslant 1$, we thus obtain that q is an excellent form over F.

The general statement now follows by iterating the above argument. □

Proposition 3.10. *Let p and q be anisotropic forms over F. Let π denote the n-fold Pfister form $\langle 1, x_1\rangle\otimes\ldots\otimes\langle 1, x_n\rangle$ over $K = F(\!(x_1)\!)\ldots(\!(x_n)\!)$.*

- (i) *$q\otimes\pi$ is a multiple of an $(m+n)$-fold Pfister form ϑ over K if and only if $q\otimes\pi$ is a multiple of $\sigma\otimes\pi$ for some m-fold Pfister form σ over F.*
- (ii) *$q\otimes\pi$ is a multiple of $p\otimes\pi$ over K if and only if q is a multiple of p over F.*

Proof. (i) The "if" statement is clear. To prove the converse, we begin by considering the case where $q\otimes\langle 1, x_1\rangle$ is a multiple of $\vartheta\in P_{m+1}F(\!(x_1)\!)$. As per the proof of Proposition 3.7 (i), we may assume that either $\vartheta\simeq\langle\!\langle -a_1,\ldots,-a_{m+1}\rangle\!\rangle$ or $\vartheta\simeq\langle\!\langle -a_1,\ldots,-a_m,-a_{m+1}x_1\rangle\!\rangle$ for some $a_1,\ldots,a_{m+1}\in F^\times$.

In the case where $\vartheta \simeq \langle\langle -a_1, \ldots, -a_{m+1}\rangle\rangle$, let φ be a form over $F((x_1))$ such that $q \otimes \langle 1, x_1\rangle \simeq \vartheta \otimes \varphi$. As $\varphi \simeq \varphi_1 \perp x_1\varphi_2$ for some forms φ_1 and φ_2 over F, we have that $q \otimes \langle 1, x_1\rangle \simeq \vartheta \otimes \varphi_1 \perp x_1(\vartheta \otimes \varphi_2)$, whereby $\vartheta \otimes \varphi_1 \simeq q \simeq \vartheta \otimes \varphi_2$. Hence

$$q \otimes \langle 1, x_1\rangle \simeq (\vartheta \otimes \varphi_1) \otimes \langle 1, x_1\rangle \simeq (\vartheta \otimes \langle 1, x_1\rangle) \otimes \varphi_1,$$

whereby $q \otimes \langle 1, x_1\rangle$ is a multiple of $\vartheta \otimes \langle 1, x_1\rangle$ for $\vartheta \in P_{m+1}F$ in this case.

In the case where $\vartheta \simeq \langle\langle -a_1, \ldots, -a_m, -a_{m+1}x_1\rangle\rangle$, we will denote $\langle\langle -a_1, \ldots, -a_m\rangle\rangle$ by σ, whereby $\vartheta \simeq \sigma \otimes \langle 1, a_{m+1}x_1\rangle$. Let φ be a form over $F((x_1))$ such that $q \otimes \langle 1, x_1\rangle \simeq \vartheta \otimes \varphi$. As $\varphi \simeq \varphi_1 \perp x_1\varphi_2$ for some forms φ_1 and φ_2 over F, we have that

$$\begin{aligned} q \otimes \langle 1, x_1\rangle &\simeq \vartheta \otimes \varphi \simeq (\sigma \otimes \langle 1, a_{m+1}x_1\rangle) \otimes (\varphi_1 \perp x_1\varphi_2), \\ &\simeq \sigma \otimes \varphi_1 \perp a_{m+1}(\sigma \otimes \varphi_2) \perp x_1(a_{m+1}(\sigma \otimes \varphi_1) \perp \sigma \otimes \varphi_2), \\ &\simeq \sigma \otimes (\varphi_1 \perp a_{m+1}\varphi_2) \perp x_1(\sigma \otimes (\varphi_2 \perp a_{m+1}\varphi_1)). \end{aligned}$$

Hence, by taking the difference of isometric forms and invoking Lemma 1.8, it follows that

$$\sigma \otimes (\varphi_1 \perp a_{m+1}\varphi_2) \simeq q \simeq \sigma \otimes (\varphi_2 \perp a_{m+1}\varphi_1).$$

Thus, we have that

$$q \otimes \langle 1, x_1\rangle \simeq (\sigma \otimes (\varphi_1 \perp a_{m+1}\varphi_2)) \otimes \langle 1, x_1\rangle \simeq (\varphi_1 \perp a_{m+1}\varphi_2) \otimes (\sigma \otimes \langle 1, x_1\rangle),$$

whereby $q \otimes \langle 1, x_1\rangle$ is a multiple of $\sigma \otimes \langle 1, x_1\rangle$ for $\sigma \in P_mF$ in this case.

Statement (i) follows by iterating the above argument.

(ii) The "if" statement is clear. To prove the converse, we begin by considering the case where $q \otimes \langle 1, x_1\rangle$ is a multiple of $p \otimes \langle 1, x_1\rangle$ over $F((x_1))$. Let φ be a form over $F((x_1))$ such that $q \otimes \langle 1, x_1\rangle \simeq \varphi \otimes (p \otimes \langle 1, x_1\rangle)$. As $\varphi \simeq \varphi_1 \perp x_1\varphi_2$ for some forms φ_1 and φ_2 over F, we have that

$$q \otimes \langle 1, x_1\rangle \simeq (\varphi_1 \otimes p \perp \varphi_2 \otimes p) \perp x_1(\varphi_1 \otimes p \perp \varphi_2 \otimes p).$$

Hence, the difference of these forms is hyperbolic, whereby we may invoke Lemma 1.8 to establish that $q \simeq \varphi_1 \otimes p \perp \varphi_2 \otimes p \simeq (\varphi_1 \perp \varphi_2) \otimes p$, as desired. Statement (ii) now follows by iterating the above argument. $\square$

Witt's Round Form Theorem [24, Theorem X.1.14] states that the product of a Pfister form and a round form is round.

Proposition 3.11. *Let q be an anisotropic form over F. For $\pi \simeq \langle 1, x_1\rangle \otimes \ldots \otimes \langle 1, x_n\rangle$, the form $q \otimes \pi$ over $F((x_1))\ldots((x_n))$ is round if and only if q is round.*

Proof. As above, it suffices to prove the "only if" statement. We will begin by assuming that $q \otimes \langle 1, x_1\rangle$ is round over $F((x_1))$.

Hence, $1 \in D_{F((x_1))}(q \otimes \langle 1, x_1\rangle) = G_{F((x_1))}(q \otimes \langle 1, x_1\rangle)$, whereby $\langle -1\rangle \perp q \perp x_1q$ is isotropic over $F((x_1))$. Invoking Lemma 1.8, we obtain that $\langle -1\rangle \perp q$ is isotropic over F, whereby $1 \in D_F(q)$ and thus $G_F(q) \subset D_F(q)$.

Let $y \in D_F(q)$. As $y \in D_{F((x_1))}(q \perp x_1q) = G_{F((x_1))}(q \perp x_1q)$, it follows that $q \perp x_1q \simeq yq \perp x_1yq$ over $F((x_1))$. Thus, $q \perp -yq \perp x_1(q \perp -yq)$ is hyperbolic over $F((x_1))$. Invoking Lemma 1.8, it follows that $q \perp -yq$ is hyperbolic over F. Hence, we have that $y \in G_F(q)$, whereby $G_F(q) = D_F(q)$.

The general statement now follows by iterating the above argument. $\square$

4. Properties inherited by Pfister multiples

As discussed previously, the properties of being a Pfister multiple or neighbour, of being an excellent or round form, and of having maximal splitting are all inherited by the Pfister multiples of a form. Moreover, as per the results of the previous section, the absence of these properties is reflected in the generic Pfister multiples of a form. Thus, combining these observations, one can look to clarify how such properties relate to one another.

We begin by considering the relationship between excellence and the property of being a Pfister neighbour. In certain dimensions, these properties are equivalent:

Proposition 4.1. *Let φ be an anisotropic neighbour of a Pfister form π. If $\dim\varphi \geqslant \dim\pi - 3$, then φ is excellent.*

Proof. As the complementary form of φ has dimension at most three, it is a Pfister neighbour, whereby the result follows. $\square$

In [22], Knebusch characterised excellent forms of dimension at most 12. Invoking Proposition 4.1, one thus obtains characterisations of Pfister neighbours of dimension at most 8, formulated in terms of the classical quadratic-form invariants (we note that a geometric characterisation of 9-dimensional Pfister neighbours has been established by Karpenko [16]). The following example establishes the existence of Pfister neighbours of dimension 9, 10, 11 and 12 that are not excellent.

Example 4.2. Invoking Knebusch's characterisations [22, p.11], for q an excellent form of dimension 9, 10, 11 or 12, we note that the index of $c(q)$ is at most two. Over the field $K = \mathbb{C}(\!(x_1)\!)\dots(\!(x_4)\!)$, we will consider 9-, 10-, 11- and 12-dimensional forms q that are neighbours of $\langle\!\langle x_1,\dots,x_4\rangle\!\rangle$, with γ denoting their complementary forms. Invoking [24, V.(3.13)], we have that $c(q) = c(\gamma)$. In the case where $\dim\gamma = 4$, we can let $\gamma \simeq \langle x_1, x_2, x_3, x_4\rangle$ and invoke [24, V.(3.13)] to establish that $c(\gamma) = (x_1x_2, x_1x_3x_4) \otimes_K (x_3, x_4)$, which is a biquaternion division algebra by Albert's Theorem [24, Theorem III.4.8]. Similarly, in the case where $\dim\gamma = 5$ (respectively 6 and 7), we have that $c(\gamma)$ is the division algebra $(x_1, x_2) \otimes_K (x_3, x_4)$ for $\gamma \simeq \langle x_1, x_2, x_1x_2, x_3, x_4\rangle$ (respectively for $\gamma \simeq \langle x_1, x_2, x_1x_2, x_3, x_4, x_3x_4\rangle$ and $\gamma \simeq \langle 1, x_1, x_2, x_1x_2, x_3, x_4, x_3x_4\rangle$).

Remark 4.3. A similar approach to the above may be employed to construct non-excellent Pfister neighbours of dimension $2^n - k$ for $n \geqslant 5$ and $4 \leqslant k \leqslant 8$. Moreover, invoking Proposition 3.9, all generic Pfister multiples of non-excellent Pfister neighbours will also be non-excellent Pfister neighbours, whereby the above examples can be extended to higher dimensions. That these examples exist is unsurprising, as it seems reasonable to conjecture the existence of non-excellent Pfister neighbours of codimension k for all values of $k \geqslant 4$. The obstruction to establishing this conjecture is itself a more important problem in the theory, namely that of the lack of characterisations of the Pfister neighbour property (outside of the aforementioned characterisations that are known in low dimensions).

Remark 4.4. In [1], Ahmad and Ohm introduced the notion of a special Pfister neighbour; a special neighbour of an n-fold Pfister form is one that contains a subform similar to an $(n-1)$-fold Pfister form. We note that the non-excellent Pfister neighbours in Example 4.2 of dimension 9, 10 and 11 are not special by [1, Theorem 2.2]. The 12-dimensional non-excellent Pfister neighbour in Example 4.2 is special by [1, Corollary 2.5]. This form is an example of a codimension-4 special Pfister neighbour that is not excellent (Ahmad and Ohm gave an example of a codimension-5 special Pfister neighbour that is not excellent in [1, p. 663, Remark], while Pfister neighbours of codimension at most 3 are excellent by Proposition 4.1).

The second problem we will consider in this section is probably the most important one in this vein. This problem, first posed in [8], is to determine when the maximal splitting property implies that the Pfister neighbour property also holds. In particular, a condition that refers solely to the dimension of the form is sought. As above, characterisations of the Pfister neighbour property are known for forms of small dimension. In particular, an anisotropic 5-dimensional form is a Pfister neighbour if and only if its even Clifford algebra is of index two, as follows from [22, Corollary 8.2]. Thus, over the field $F = \mathbb{R}(w, x, y, z)$, the form $q \simeq \langle 1, w, x, y, z\rangle$ is not a Pfister neighbour: by applying [24, V.(3.13)], one sees that its even Clifford algebra is Brauer equivalent to $(-w, -x)_F \otimes (-yz, wxz)_F$, which is a biquaternion division algebra by Albert's Theorem [24, Theorem III.4.8]. Moreover, q trivially has maximal splitting. For all $n > 2$, Hoffmann considered the product $\pi_{n-2} \otimes q$, where π_{n-2} is the Pfister form $2^{n-2} \times \langle 1\rangle$ over F. In [8, Example 2], he established that $\pi_{n-2} \otimes q$ is a $(2^n + 2^{n-2})$-dimensional form with maximal splitting that is not a Pfister neighbour. The following proposition allows us to recover the existence of such $(2^n + 2^{n-2})$-dimensional forms. More generally, given any form q with maximal splitting that is not a Pfister neighbour, for all $n \in \mathbb{N}$ there exists an n-fold Pfister multiple of q with maximal splitting that is not a Pfister neighbour.

Proposition 4.5. *Let q be an anisotropic form over a field F that has maximal splitting but is not a Pfister neighbour. For $n \in \mathbb{N}$ and $\pi \simeq \langle 1, x_1\rangle \otimes \ldots \otimes \langle 1, x_n\rangle$ over $F(\!(x_1)\!) \ldots (\!(x_n)\!)$, the form $\pi \otimes q$ has maximal splitting but is not a Pfister neighbour.*

Proof. We note that $\pi \otimes q$ has maximal splitting by Proposition 2.15. The fact that $\pi \otimes q$ is not a Pfister neighbour follows from Proposition 3.7. □

For F, q and π as in Proposition 4.5, let $m \in \mathbb{N}$ be such that $2^m < 2^n(\dim q) \leqslant 2^{m+1}$. For $d \in \mathbb{N}$ such that $2^m < d \leqslant 2^n(\dim q)$, every d-dimensional subform of $\pi \otimes q$ has maximal splitting but is not a Pfister neighbour. In particular, as observed in [14, Proposition 1.5], for $n \geqslant 2$ and $d \in \mathbb{N}$ such that $2^n < d \leqslant 2^n + 2^{n-2}$, there exists a d-dimensional form with maximal splitting that is not a Pfister neighbour. The following conjecture, implicit in [8] (and stated in [14]), posits that the dimensions of such forms necessarily belong to these intervals.

Conjecture 4.6. *Let F be a field and q be an anisotropic form over F with maximal splitting. If $2^n + 2^{n-2} < \dim q \leqslant 2^{n+1}$ holds for $n \geqslant 2$, then q is a Pfister neighbour.*

Conjecture 4.6 appears to be very difficult to resolve. It is known to hold for $n \leqslant 3$ (see [10] or [11]). In order to establish the truth of this conjecture for a fixed value of $n \geqslant 4$, we remark that it suffices to prove the statement in the case where q is any form of dimension $2^n + 2^{n-2} + 1$. More generally, one can look to prove Conjecture 4.6 with respect to forms q of some prescribed dimension, an approach which has been successfully employed by a number of authors. For $n \geqslant 4$, the conjecture is known to hold when $2^{n+1} - 7 \leqslant \dim q \leqslant 2^{n+1}$ (see [14, Theorem 1.7]).

We remark that Proposition 4.5 is also of some relevance to these approaches towards resolving Conjecture 4.6. In particular, in order to establish the conjecture with respect to a form q, Proposition 4.5 implies that it is sufficient to prove the statement with respect to an m-fold generic Pfister multiple of q for any prescribed $m \in \mathbb{N}$. Thus, it suffices to prove the conjecture with respect to the forms belonging to any prescribed power of the fundamental ideal (generated by even-dimensional forms). Hence, when treating the general conjecture, there is no loss of generality in assuming that the first m cohomological invariants of q are trivial. The same considerations apply when seeking to establish the conjecture with respect to forms of prescribed dimension.

We conclude our discussion of Conjecture 4.6 by invoking descent results of Laghribi to establish the conjecture with respect to forms with specified properties.

Proposition 4.7. *For $n \geqslant 4$, let q be an anisotropic form over F of dimension at least $2^{n+1}-10$. Suppose that q contains a subform p of one of the following types:*

(*i*) $\dim p = 2^{n+1}-10$, $\det p = -1$ *and $c(p)$ has index at most two,*
(*ii*) $\dim p = 2^{n+1}-9$ *and $c(p)$ has index at most two,*
(*iii*) $\dim p = 2^{n+1}-8$ *and* $c(p) \otimes_F F(p)\left(\sqrt{\det p}\right)$ *has index at most two.*

If q has maximal splitting, then q is a Pfister neighbour.

Proof. Assuming that q has maximal splitting, we have that q and p are isotropy equivalent by Lemma 1.3, whereby p has maximal splitting by Theorem 1.6. We consider the extension of p to $F(p)$. Invoking [23, Théorème principal], we have that $(p_{F(p)})_{\mathrm{an}}$ is defined over F. Hence, we have that p is a Pfister neighbour, by [22, Theorem 7.13], whereby it follows that q is a Pfister neighbour. □

5. Forms with non-trivial first Witt index

Broadening the preceding discussion regarding forms with maximal splitting, one can consider the problem of characterising those forms whose first Witt indices are greater than one. By appealing to Corollary 2.3 or Theorem 2.5, one sees that Pfister multiples represent a prominent class of examples in this regard. Moreover, in accordance with Theorem 1.6, the class of forms with non-trivial first Witt index also includes those neighbours of Pfister multiples that are not of maximal codimension.

Clearly, in order to have non-trivial first Witt index, a form must be of dimension at least 4. We will begin by recalling what is known regarding forms of dimension less than 16.

Odd-dimensional forms q with $5 \leqslant \dim q \leqslant 13$ are either of trivial first Witt index or have maximal splitting (as can be seen by appealing to Karpenko's determination of the possible values of the first Witt index of a form, [18]). Hence, such forms with non-trivial first Witt index, and 15-dimensional forms with maximal splitting, are necessarily Pfister neighbours, by [10] or [11]. Furthermore, as determined by Vishik [30, pp. 80-81], a 15-dimensional form q satisfies $i_1(q) = 3$ if and only if the associated form $q \perp \langle \det q \rangle$ has splitting pattern $(4,4)$. By results of Izhboldin [13, Theorem 13.9] and Kahn [15, Theorem 2.12], such a form $q \perp \langle \det q \rangle$ is a multiple of a 2-fold Pfister form, whereby a 15-dimensional form q with $i_1(q) = 3$ is a neighbour of a Pfister multiple.

For q an even-dimensional form of dimension at least four and at most eight, it is known that $i_1(q)$ is divisible by two if and only if q is a multiple of a 1-fold Pfister form (see [10]). In [13, Proof of Conjecture 0.10], Izhboldin proved that a 10-dimensional form q satisfies $i_1(q) = 2$ if and only if q is a multiple of a 1-fold Pfister form or q is a Pfister neighbour. As per [30, pp. 94-95], Vishik established that a 12-dimensional form q satisfies $i_1(q) = 2$ if and only if its splitting pattern is of the form $(2,4)$ (in which case it is a multiple of a 1-fold Pfister form) or of the form $(2,2,2)$ (with Vishik hypothesising that q is a multiple of a 1-fold Pfister form in this case too). As per [10] or [11], a form q with $11 \leqslant \dim q \leqslant 16$ has maximal splitting if and only if it is a Pfister neighbour. Totaro classified 14-dimensional forms with first Witt index greater than one in [27, Theorem 4.2], determining that such a form q satisfies $i_1(q) = 2$ if and only if q is a multiple of a 1-fold Pfister form or q is a subform of a 16-dimensional multiple of a 2-fold Pfister form. As noted in [27], though these two classes of 14-dimensional forms q with $i_1(q) = 2$

overlap, neither of them may be omitted. The following examples show that similar phenomena occur in higher dimensions.

Example 5.1. As presented in [27, p. 263], over the field $K = F(x_1, \ldots, x_6)$, the 14-dimensional form

$$q \simeq (\langle\langle x_1, x_2\rangle\rangle \otimes \langle x_3, x_4, x_5, x_6\rangle \perp \langle -x_3, -x_4\rangle)_{\mathrm{an}}$$

satisfies $i_1(q) = 2$ but is not a multiple of a 1-fold Pfister form. Totaro demonstrates this by observing that $i\left(q_{K(\sqrt{-x_3x_4})}\right) = 3$, whereas the entries of the splitting pattern of a Pfister multiple are all even by Theorem 2.1. Now, over the field $L = K((y_1))\ldots((y_n))$ for $n \in \mathbb{N} \cup \{0\}$, let $\pi \simeq \langle 1, y_1\rangle \otimes \ldots \otimes \langle 1, y_n\rangle$ and consider $q \otimes \pi$, the generic Pfister multiples of q. As $i_1(q) = 2$, it follows that $i_1(q \otimes \pi) = 2^{n+1}$, in accordance with Corollary 3.4. Moreover, as q is not a multiple of a 1-fold Pfister form over K, we have that $q \otimes \pi$ is not a multiple of an $(n+1)$-fold Pfister form over L, as follows from n invocations of Proposition 3.10.

Example 5.2. Let p be a 12-dimensional subform of Totaro's 14-dimensional form

$$q \simeq (\langle\langle x_1, x_2\rangle\rangle \otimes \langle x_3, x_4, x_5, x_6\rangle \perp \langle -x_3, -x_4\rangle)_{\mathrm{an}}$$

over $K = F(x_1, \ldots, x_6)$. Consider the 26-dimensional form $\psi \simeq q \perp yp$ over $K((y))$. As $\psi \subset q \otimes \langle 1, y\rangle$, and $i_1(q \otimes \langle 1, y\rangle) = 4$ as in the preceding example, Theorem 1.6 implies that $i_1(\psi) = 2$. Suppose, for the sake of contradiction, that ψ is a multiple of a 1-fold Pfister form ρ over $K((y))$. Hence $\psi \simeq \rho \otimes \gamma$ for some form γ over $K((y))$, with $\gamma \simeq \gamma_1 \perp y\gamma_2$ for γ_1, γ_2 forms over K. As before, we have that $\rho \simeq \langle 1, a\rangle$ or $\rho \simeq \langle 1, ay\rangle$ for some $a \in K^{\times}$. For $\rho \simeq \langle 1, a\rangle$, it follows that $q \simeq \langle 1, a\rangle \otimes \gamma_1$, in contradiction to the fact that q is not a multiple of a 1-fold Pfister form. For $\rho \simeq \langle 1, ay\rangle$, it follows that $q \simeq \gamma_1 \perp a\gamma_2$ and $p \simeq \gamma_2 \perp a\gamma_1$, in contradiction to the fact that $\dim p \neq \dim q$. Hence, ψ is a 26-dimensional form over $K((y))$ satisfying $i_1(\psi) = 2$ that is not a multiple of a 1-fold Pfister form. By iterating this argument, it follows that, for any $n \in \mathbb{N}$, there exists forms φ of dimension $14(2^n) - 2^{n+1} + 2$ satisfying $i_1(\varphi) = 2$ that are not multiples of 1-fold Pfister forms.

Returning to the characterisations of forms of non-trivial first Witt index recalled above, if we assume that Vishik's hypothesis regarding 12-dimensional forms is true, then we have that a form q with $3 < \dim q < 16$ has non-trivial first Witt index if and only if it is a neighbour of a Pfister multiple. Vishik showed that this phenomenon does not extend further. As presented in [27, Lemma 7.1], Vishik established that, over the field $K = F(x_1, \ldots, x_5)$, the 16-dimensional form

$$q \simeq (\langle 1, x_1\rangle \otimes \langle 1, x_2\rangle \otimes \langle 1, x_3\rangle) \perp x_4\langle 1, x_1, x_2, x_3\rangle \perp x_5\langle 1, x_1, x_2, x_3\rangle$$

satisfies $i_1(q) = 2$ but is not a multiple of a 1-fold Pfister form. Since q is 16-dimensional, it cannot be a neighbour of a higher-dimensional Pfister multiple, in keeping with Theorem 1.5, whereby q represents the first known example of a form of non-trivial first Witt index that is not a neighbour of a Pfister multiple. Moreover, as noted in [27], q has splitting pattern $(2, 2, 2, 2)$, whereby it is the first example of a form that is not a Pfister multiple but whose higher Witt indices are all even.

It seems reasonable to conjecture the existence of forms of dimension greater than 16 which share these properties of Vishik's form. In this regard, it would be interesting to know whether there exist forms q of dimension 18 with $i_1(q) = 2$ that are not neighbours of Pfister multiples. In this vein, motivated by Conjecture 4.6, Izhboldin's result on 10-dimensional forms [13, Proof of Conjecture 0.10] and other related considerations, we conclude by posing a question. Considering the forms of dimension $2^n + 1$ for some $n \in \mathbb{N}$ to be those with trivial maximal splitting, we ask the following:

Question 5.3. *Let q be a form with non-trivial maximal splitting. Is q a neighbour of a Pfister multiple?*

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