MOTIVIC DECOMPOSITIONS OF TWISTED FLAG VARIETIES
AND REPRESENTATIONS OF HECKE-TYPE ALGEBRAS

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Abstract. Motivated by the motivic Galois group and the Kostant-Kumar results on equivariant cohomology of flag varieties, we provide a uniform description of motivic (direct sum) decompositions with integer coefficients of versal flag varieties in terms of integer representations of the associated affine nil-Hecke algebra $H$.

More generally, we establish an equivalence between the $h$-motivic subcategory generated by the motive of $E/B$ and the category of projective modules of the associated rational algebra $D$ of push-pull operators, where $E$ is a torsor for a split semisimple linear algebraic group $G$ over a field $k$, $B$ is a Borel subgroup of $G$, $h$ is an algebraic oriented cohomology theory in the sense of Levine-Morel (e.g. Chow ring $CH$ or an algebraic cobordism $\Omega$). The algebra $D$ can be think of as an integer-analogue of the 'Hopf-algebra of the $h$-motivic Galois group of $E/B$.

As an application, taking $h = CH$ and specializing the coefficients to the finite field $\mathbb{F}_p$ we obtain that $p$-modular projective representations of $D = H$ are generated by an irreducible $H$-module corresponding to the generalized Rost-Voevodsky motive for $(G, p)$.

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1. Introduction

Let $G$ be a split semisimple linear algebraic group over a field $k$, let $E$ be a $G$-torsor over $K/k$. Consider a twisted form $E/B$ over $K$ of the variety of Borel subgroups $G/B$ of $G$, e.g., a variety of (complete) flags of ideals in a central division algebra over $K$. In general, such a variety neither have a $K$-rational point nor any (relative) cellular filtration over $K$.

Consider the pseudo-abelian tensor category of Grothendieck-Chow motives of smooth projective varieties over $K$ with coefficients in a ring $R$. The main result of [25] says that the motive of $E/B$ with finite coefficients ($R = \mathbb{F}_p$) is always a direct sum of Tate twists of some indecomposable motive $R_{E,p}$, a generalization of the Rost-Voevodsky motive. In other words, the tensor subcategory $\langle E/B \rangle_{\mathbb{F}_p}$ generated by all direct summands of $E/B$ is, indeed, generated by $R_{E,p}$, i.e.,

$$\langle E/B \rangle_{\mathbb{F}_p} = \langle R_{E,p} \rangle.$$ 

The motive $R_{E,p}$ has several remarkable properties. For instance, it is closely related to various cohomological invariants of $G$-torsors. If $p$ is not a torsion prime of $G$ or if the coefficient ring $R$ has characteristic 0, then $R_{E,p}$ coincides with the motive of a point, so $(E/B)_{\mathbb{F}_p}$ gives the subcategory of Tate motives. While being indecomposable over $k$, the motive $R_{E,p}$ becomes isomorphic to a direct sum of Tate motives over the splitting field $\bar{k}$ of $E$ (as $\bar{k}$ one can always take the algebraic closure of $k$ or the function field of $E/B$). Moreover, the generating function of $R_{E,p}$ over $\bar{k}$ (counting the number of Tate motives in each dimension) is given by an explicit cyclotomic polynomial involving the $p$-exceptional degrees of V. Kac [19]. For example, if $E$ is a $G$-torsor, where $G$ is an exceptional group of type $F_4$ and $E$ splits by a cubic field extension, then $R_{E,p}$ corresponds to the Rost-Serre cohomological invariant and $R_{E,3}\bar{k} \cong \mathbb{F}_3 \oplus \mathbb{F}_3(4) \oplus \mathbb{F}_3(8)$.

As for integer coefficients ($R = \mathbb{Z}$) only very few facts are known concerning the category $\langle E/B \rangle_{\mathbb{Z}}$. An integer version of the motive $R_E$ was introduced and discussed in [26]; in [5], [10] it was shown that $\langle E/B \rangle_{\mathbb{Z}}$ is not Krull-Schmidt (the uniqueness of a direct sum decomposition fails).

In the present paper we cover all the mentioned cases ($R = \mathbb{Z}, \mathbb{F}_p$). More generally, we consider the category of $h$-correspondences with coefficients in $R$, where $h$ is any algebraic oriented Borel-Moore homology of $[23]$ (e.g. Chow groups, connective $K$-theory, elliptic cohomology, algebraic cobordism $\Omega$ of Levine-Morel) and $R = h(K)$ is its coefficient ring. Let $\langle E/B \rangle_h$ denote the respective tensor subcategory generated by indecomposable summands of the $h$-motive of $E/B$. Our main result (Theorem 8.1) establishes an equivalence between the motivic category $\langle E/B \rangle_h$ and certain category of finitely generated projective $D_F$-modules

$$(1) \quad \langle E/B \rangle_h \cong \text{Proj } \overline{D}_F,$$

where $\overline{D}_F$ is the $R$-algebra defined using the formal push-pull operators for the group $G$ and the theory $h$. So it provides a direct link between integer/modular $h$-motivic decompositions of twisted flag varieties and integer/modular representations of Hecke-type algebras $\overline{D}_F$.

If $E$ is a versal (generic) torsor, then $\overline{D}_F$ can be replaced by the formal affine Demazure algebra $D_F$. The theory of such algebras and formal push-pull operators
has been recently developed in [6], [18], [7], [8], [9] motivated by Bernstein-Gelfand- 
Gelfand [2], Demazure [11], [12], Bressler-Evens [3], [4] and Kostant-Kumar [21], 
[20] results. The key properties of $D_F$ are

- it is a free module over the $T$-equivariant oriented cohomology ring $S = h_T(K)$ of a point, where $T$ is a split maximal torus in $G$;
- its $S$-dual $D_F^* = Hom_S(D_F, S)$ is isomorphic to the $T$-equivariant oriented 
cohomology $h_T(G/B)$ of $G/B$ [9] and
- its structure (generators and relations) is very close to those of the affine 
Hecke algebra [18].

For example, if $h = CH$ (Chow groups) and $R = F_p$ as before, then $D_F = H_{nil,p}$ is the affine nil-Hecke algebra (in the notation of Ginzburg [16, §12]) over $F_p$ which is a free module of rank $[W]$ over the polynomial ring $S = F_p[x_1, \ldots, x_n]$, where $n$ is the rank of $G$ and $W$ is the Weyl group, and $D_F^* = CH_T(G/B; F_p)$ is the $T$-equivariant Chow groups. For a versal torsor $E$ the equivalence (1) then turns into

$$\langle R_{E,p} \rangle \simeq Proj H_{nil,p}$$

meaning that all indecomposable projective $H_{nil,p}$-modules are isomorphic to each other (up to a shift). Moreover, their ranks over $S$ equal to the $p$-part of the product of $p$-exceptional degrees of the group $G$.

Roughly speaking, the algebra $\overline{D}_F$ can be viewed as an integral analogue of the Hopf-algebra of the motivic Galois group of $E/B$ (see e.g. [1]). Indeed, if taken with $\mathbb{Q}$-coefficients (or if $E$ is split), the algebra $\overline{D}_F$ becomes isomorphic to $End_R h(G/B) \simeq M_{[W]}(R)$ and, hence, the category $Proj \overline{D}_F$ can be identified with the category of representations $Proj \mathbb{Q}[G_m] = Rep \mathbb{G}_m$ with $\mathbb{G}_m$ known to be the motivic Galois group of $(E/B)_\mathbb{Q}$. Observe that in general, $\overline{D}_F$ is not a matrix algebra over $R$.

In the paper we restrict ourselves to varieties $E/B$ of Borel subgroups only. However, by [5] we have $\langle E/B \rangle_h = \langle E/P \rangle_h$ for any special parabolic subgroup $P$. Hence, $B$ can be replaced by any such $P$ without affecting the equivalence (1). For instance, for $G = PGL_n$, $h = CH$, $R = \mathbb{Z}$ and $E$ corresponding to a generic central division algebra $A$ of degree $p^n$ we get

$$\langle SB(A) \rangle_\mathbb{Z} \simeq Proj H_{nil,\mathbb{Z}},$$

where $SB(A)$ is the Severi-Brauer variety of $A$ and $H_{nil,\mathbb{Z}}$ is the affine nil-Hecke algebra with integer coefficients.

The paper is organized as follows. In section 2 we recall definitions and basic facts concerning Borel-Moore homology $h$ and the respective category of $h$-motives. We state a version of the Künneth isomorphism for cellular spaces. In the next section we generalize it to the equivariant setting. In section 4 we introduce the convolution product on the equivariant cohomology of group powers and study its properties. In the next section we identify this equivariant cohomology with the endomorphism ring on equivariant cohomology of $G/B$ and then in section 6 with the formal affine Demazure algebra. In section 7 we introduce the notion of a rational algebra of push-pull operators $\overline{D}_F$ and identify it with the subring of rational endomorphisms. In the last section we prove the equivalence (1) and provide some applications and examples.
2. ORIENTED (CO-)HOMOLOGY

We recall definitions of an algebraic oriented Borel-Moore homology and of the respective category of correspondences. We also recall a version of the Künneth isomorphism for cellular spaces (Lemmas 2.4 and 2.5).

Fix a smooth scheme $S$ over a field $k$. Let $\text{Sch}_S$ denote the category of finite type quasi-projective separated $S$-schemes and let $\text{Sm}_S$ denote its full subcategory consisting of smooth quasi-projective $S$-schemes.

Following [23, Def. 5.1.3] consider an oriented graded Borel-Moore homology theory $h_\bullet$ defined on some admissible [23, (1.1)] subcategory $V$ of $\text{Sch}_S$. So that there are pull-backs $f^*: h_\bullet(X) \to h_{\bullet+d}(Y)$ for l.c.i. morphisms $f: Y \to X$ in $V$ of relative dimension $d$ and push-forwards $f_*: h_\bullet(Y) \to h_\bullet(X)$ for projective morphisms $f: X \to Y$ in $V$. According to [23, Prop. 5.2.1] the Borel-Moore homology $h_\bullet$ restricted to $\text{Sm}_S$ defines an algebraic oriented cohomology theory $h^\bullet$ (with values in the category of graded commutative rings with unit) in the sense of [23, Def. 1.1.2] by

$$h_{\dim_S X-\bullet}(X) := h_\bullet(X), \quad X \in \text{Sm}_S.$$ 

If the (co-)dimension is clear from the context we will write simply $h(X)$.

Following [26, §2] (see also [14, §63]) we define the category of $h$-correspondences $\text{h-Corr}_S$ over $S$. The objects are pairs $([X \to S], i)$, where $[X \to S]$ is an isomorphism class of a smooth projective map $X \to S$ and $i \in \mathbb{Z}$. The morphisms are defined by

$$\text{Hom}_{\text{h-Corr}_S}(([Y \to S], i), ([X \to S], j)) := \bigoplus_i \text{Hom}_{i-j}([Y \to S], [X \to S]),$$

taken over all connected components $Y_i$ of $Y$, where

$$\text{Hom}_i([Y \to S], [X \to S]) := h_{\dim_{Y_i} Y_i \times_S X}.$$ 

The composition of morphisms is given by the correspondence product. Namely, if $p_i: X_1 \times_S X_2 \times_S X_3 \to X_j \times_S X_j'$ denotes the projection obtained by removing the $i$-th coordinate, then given $\alpha \in h(X_1 \times_S X_2)$ and $\beta \in h(X_2 \times_S X_3)$ we set

$$\beta \circ \alpha := (p_2)_* (p_{i}^* (\beta) \cdot p_{3}^* (\alpha)) \in h(X_1 \times_S X_3).$$

Let $\text{h-Corr}_S^+$ denote the additive completion of $\text{h-Corr}_S$. We simply write $X$ for the respective class in $\text{h-Corr}_S^+$.

**Definition 2.1.** (cf. [23, (CD')]) Let $X$ be smooth projective over $S$. Suppose that there is a filtration by proper closed subschemes

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset X_n = X$$

such that

- each irreducible component $X_{ij}$ of $X_i \setminus X_{i-1}$ is a locally trivial affine fibration over $S$ of rank $d_{ij}$, and
- the closure of $X_{ij}$ in $X$ admits a resolution of singularities $\overline{X}_{ij} \to \overline{X}_{ij}$ over $S$; we set $g_{ij}: \overline{X}_{ij} \to \overline{X}_{ij} \hookrightarrow X$ and, therefore, $(g_{ij})_*(1_{\overline{X}_{ij}}) \in h_{d_{ij}}(X).

We call such $X$ (together with the filtration) a cellular space over $S$.

**Definition 2.2.** We say that the theory $h$ satisfies the cellular decomposition (CD) property if given a cellular space $X$ over $S$ the respective elements $(g_{ij})_*(1_{\overline{X}_{ij}})$ form a $h(S)$-basis of $h(X)$. 

Example 2.3. The property (CD) holds for any oriented Borel-Moore homology \( h \) over a field \( k \) of characteristic 0.

Indeed, the same reasoning as in [14, Thm. 66.2] shows that for every \( Z \in Sm_S \) there is an isomorphism

\[
\sum (g_{ij})_*(1) \times \text{id}_Z : \bigoplus_{ij} CH_{* - d_{ij}}(Z) \to CH_*(Z \times_S X).
\]

By the Yoneda lemma (cf. [14, Lemma 63.9]) the latter induces an isomorphism in the category \( CH-Corr_S^+ \) (cf. [14, Cor. 66.4]).

Following [29, §2] consider the specialization functor \( \Omega-Corr_S^+ \to CH-Corr_S^+ \), \([f: Y \to X] \mapsto f_*(1_Y)\). It is surjective on the classes of objects and morphisms. Moreover, for every \( X \) the kernel of

\[
\Omega_{\text{dim}_S} x(X \times_S X) \to CH_{\text{dim}_S} x(X \times_S X)
\]

is \( \Omega_{\geq 1}(k) \cdot \Omega(x(X \times_S X)) \) by [23, Rem 4.5.6]. Hence for every \( y \) in this kernel

\[
y^2(\text{dim}_S X + 1) \in \Omega_{\text{dim}_S} x(X \times_S X) \cap (\Omega_{\geq (\text{dim}_S X + 1)}(k) \cdot \Omega x(X \times_S X)).
\]

So \( y = 0 \) since \( \Omega_{<0}(Y) = 0 \). Therefore, the kernel of

\[
\text{End}_{\Omega-Corr_S^+}(X, i) \to \text{End}_{CH-Corr_S^+}(X, i)
\]

consists of nilpotents.

Finally, by [29, Lemma 2.1] the isomorphism \( \sum_{ij}(g_{ij})_*(1) \) in \( CH-Corr_S^+ \) can be lifted to an isomorphism in the category \( \Omega-Corr_S^+ \). Specializing it via \( \Omega \to h \) we obtain the desired isomorphism.

From this point on we assume that \( h \) satisfies the property (CD).

Lemma 2.4. Let \( X \) be a cellular space over \( S \). Then there is an isomorphism in \( h-Corr_S^+ \)

\[
\sum (g_{ij})_*(1_{X_{ij}}) : \bigoplus_{ij} h(S, d_{ij}) \to X,
\]

where \( (g_{ij})_*(1_{X_{ij}}) \in h_{d_{ij}}(X) = \text{Hom}_{h-Corr_S^+}((S, d_{ij}), X) \).

Proof. Transversal base change implies that there is an isomorphism

\[
\sum (g_{ij})_*(1) \times \text{id}_Z : \bigoplus_{ij} h_{* - d_{ij}}(Z \times_S S) \to h_*(Z \times_S X)
\]

for any \( Z \) smooth projective over \( S \). So by the Yoneda lemma (cf. [14, Lemma 63.9]) it induces an isomorphism in \( h-Corr_S^+ \) (cf. [14, Cor. 66.4]).

Lemma 2.5. The pairing \( \langle \cdot, \cdot \rangle : h(X) \otimes_{h(S)} h(X) \to h(S) \) given by \( (a, b) = p_*(ab) \) is non-degenerate and the map

\[
f : (h(X \times_S X), \circ) \to \text{End}_{h(S)} h(X) \text{ given by } a \mapsto f_a, \quad f_a(x) = (p_2)_*(p_1^*(x) \cdot a)
\]

is an \( h(S) \)-linear isomorphism of graded rings. In particular, it gives an \( h(S) \)-linear isomorphism

\[
(h_{\text{dim}_S} x(X \times_S X), \circ) \simeq \text{End}_{h-Corr_S^+}(X).
\]

Observe that the endomorphism ring of \( h(S) \)-linear operators \( \text{End}_{h(S)}(h(X)) \) is a graded ring. Its \( n \)-th graded component consists of operators increasing the codimension by \( n \). By definition the subring of degree-0 operators (preserving the codimension) coincides with \( \text{End}_{h-Corr_S^+}(X) \).
Proof. By the previous lemma there is an isomorphism
\[
\bigoplus_{ij} h(S) = \bigoplus_{k=-\infty}^\infty \text{Hom}((S, k), \oplus_{ij} (S, d_{ij})) \cong \bigoplus_{k=-\infty}^\infty \text{Hom}((S, k), X) = h(X),
\]
where each component is given by \( x \mapsto x \cdot (g_{ij})_*(1) \). Let \( \sum_{ij} a_{ij} : X \to \oplus_{ij} (S, d_{ij}) \) be the inverse isomorphism in \( h\text{-Corr}_G^1 \). Observe that
\[
a_{ij} \in \text{Hom}(X, (S, d_{ij})) = h_{\dim(X/S)-d_{ij}}(X).
\]
Since \( a_{ij} \circ (g_{ij})_*(1) = p_*(a_{ij} \circ (g_{ij})_*(1)) = \delta_{i,j} \), the pairing \( (\cdot, \cdot) \) is non-degenerate.

The pairing \( (\cdot, \cdot) \) gives an isomorphism \( h(X) \to \text{Hom}_{h(S)}(h(X), h(S)) \) and, hence, an isomorphism \( \text{End}_{h(S)} h(X) \cong h(X) \otimes_{h(S)} h(X) \). Consider the composition
\[
\rho : h(X \times_S X) \xrightarrow{\mathcal{L}} \text{End}_{h(S)} h(X) \xrightarrow{\cong} h(X) \otimes_{h(S)} h(X)
\]
and a map \( \pi : h(X) \otimes h(X) \to h(X \times_S X) \) given by \( \pi(a \otimes b) = p_1^*(a) \cdot p_2^*(b) \).

By definition, we have
\[
f_{p_1^*(a) p_2^*(b)}(x) = (p_2)_* (p_1^*(x) p_1^*(a) p_2^*(b)) = (x, a) b.
\]
Hence, \( \rho(\pi(a \otimes b)) = a \otimes b \) and the map \( \rho \) is surjective. By the property (CD) for \( X \times_S X \to X \), \( h(X \times_S X) \) is a free \( h(X) \)-module of rank \( \text{rk}_{h(S)} h(X) \). Thus, \( \rho \) is a surjective homomorphism between free modules of the same rank, hence, it is an isomorphism. 

\[\square\]

3. The equivariant Künneth isomorphism

In the present section we introduce an equivariant Borel-Moore homology following [7, §2] and [17]. We provide an equivariant analogue of the Künneth isomorphism (Lemma 3.7).

Let \( G \) be a smooth group scheme over \( S \). Consider an admissible subcategory \( \mathcal{V}^G \) of the category of \( G \)-varieties \( X \in \text{Sch}_S \) with \( G \)-equivariant morphisms. By a \( G \)-equivariant oriented (graded) Borel-Moore homology theory we will call an additive functor \( h^G_\bullet \) from \( \mathcal{V}^G \) to graded abelian groups such that

1. There are pull-backs for \( \text{l.c.i.} \) maps and push-forwards for projective maps that satisfy

\[\text{(TS) (l.c.i. base change)} \quad \text{For a Cartesian square } X' \xrightarrow{f'} Y' \xrightarrow{g'} Y \text{ where } f \text{ (hence } \text{f'} \text{) is l.c.i. and } g \text{ (hence } g' \text{) is projective, we have } f^*g_* = g'_{\ast}(f')^*\].

\[\text{(Loc) (localization)} \quad \text{If } U \subset X \text{ is an open } G\text{-equivariant embedding with } Z = X \setminus U, \text{ then there is a right exact sequence:}
\]
\[h^G_\bullet(Z) \to h^G_\bullet(X) \to h^G_\bullet(U) \to 0.\]
2. The functor $h_G^•$ restricted to $Sm_S$ defines a graded $G$-equivariant oriented cohomology theory $h_G^•$ in the sense of [9] (we refer to [9, §2, A1-9] for the precise definition) by

$$h_G^{\dim X−•}(X) := h_G^•(X), \quad X \in Sm_S.$$ 

In addition to the axioms of [9, §2] we require that $h_G$ satisfies the following stronger version of the homotopy invariance axiom:

(III) (extended homotopy invariance) Let $p: Y \to X$ be a $G$-equivariant torsor of a vector bundle of rank $r$ over $X$, then the pull-back induced by projection

$$p^*: h_G^•(X) \to h_G^•(Y)$$

is an isomorphism.

If a variety is smooth we will always use the cohomology notation.

Example 3.1. Given a linear algebraic group $G$ over a field $k$ of characteristic zero an example of such $G$-equivariant Borel-Moore homology theory $h_G^•$ was constructed in [17] as follows.

Consider a system of $G$-representations $V_i$ and its open subsets $U_i \subseteq V_i$ such that

- $G$ acts freely on $U_i$ and the quotient $U_i/G$ exists as a scheme over $k$,
- $V_{i+1} = V_i \oplus W_i$ for some representation $W_i$,
- $U_i \subseteq U_i \oplus W_i \subseteq U_{i+1}$, and $U_i \oplus W_i \to U_{i+1}$ is an open inclusion, and
- $\text{codim}(V_i \setminus U_i)$ strictly increases.

Such a system is called a good system of representations of $G$.

Let $X \in Sch_k$ be a $G$-variety. Following [17, §3 and §5] the inverse limit induced by pull-backs

$$\lim_{\leftarrow i} h_{−\dim G+\dim U_i}^•(X \times_G U_i), \quad X \times_G U_i = (X \times_k U_i)/G,$$

does not depend on the choice of the system $(V_i,U_i)$ and, hence, defines the $G$-equivariant oriented homology group $h_G^•(X)$.

In the present paper we will extensively use the following property (cf. [9, §2, A6]) of an equivariant theory

(Tor) Let $X \to X/G$ be a $G$-torsor over $S$ and a $G'$-equivariant map for some group scheme $G'$ over $S$. Then there is an isomorphism

$$h_{G \times G'}^•(X) \xrightarrow{\sim} h_{G'}^•(X/G).$$

that is natural with respect to the maps of pairs

$$(\phi, \gamma): (X, G \times G') \to (X_1, G_1 \times G'_1), \quad \phi(x \cdot (g,g')) = \phi(x) \cdot \gamma(g,g').$$

Observe that the theory of Example 3.1 satisfies this property by [17, Prop. 27].

We have the following equivariant analogues of Definitions 2.1 and 2.2

Definition 3.2. Let $X \in \mathcal{V}^G$. Suppose that there is a filtration by $G$-equivariant proper closed subschemes

$$\emptyset = X_{−1} \subset X_0 \subset X_1 \subset \ldots \subset X_n = X$$

such that

- each irreducible component $X_{ij}$ of $X_i \setminus X_{i−1}$ is a $G$-equivariant (locally trivial) affine fibration over $S$ of rank $d_{ij},$ and
• the closure of $X_{ij}$ in $X$ admits a $G$-equivariant resolution of singularities $g_{ij} : \bar{X}_{ij} \to \overline{X}_{ij}$ over $S$. We call such $X$ (together with the filtration) a $G$-equivariant cellular space over $S$.

**Definition 3.3.** We say that the equivariant theory $h^G$ satisfies the cellular decomposition (CD) property if given a $G$-equivariant cellular space $X$ over $S$ the respective elements $(g_{ij})_* (1_{\overline{X}_{ij}})$ form a $h^G(S)$-basis of $h^G(X)$.

**Lemma 3.4.** Suppose a morphism $f : X \to Y$ in $\text{Sm}_k$ factors as $f : X \xrightarrow{\delta} L \xrightarrow{\pi} Y$ where $\pi : L \to X$ is a vector bundle, $\delta : X \to L$ is a zero section and $\delta$ is an open embedding.

Then for every projective map $a : Y' \to Y$ and $X' = X \times_Y Y'$ the following diagram of pull-back and push-forward maps commutes (we omit the grading)

$$
\begin{array}{ccc}
X' & \xrightarrow{a'_*} & X \\
\downarrow f'^* & & \downarrow f^* \\
Y' & \xrightarrow{a_*} & Y
\end{array}
$$

**Proof.** Observe that the map $f' : X' \to Y'$ factors as $X' \xrightarrow{z'} L \times_Y Y' \xrightarrow{j'} Y'$ where $z'$ is the zero section of the vector bundle $p' : L' = L \times_Y Y' \to X'$ and $j'$ is an open embedding. Let $b$ denote the canonical map $L' \to L$. Since $j$ and $j'$ are flat, we have $j^*a_* = b_*j'^*$ by the l.c.i. base change for oriented theories. Note that by the homotopy invariance $z'^* = (p'^{-1})$ and $z'^* = (p'^{-1})$. Since $p$ and $p'$ are flat, $p'^*a_* = b_*p'^*$. Then $z'^*b_* = a'_*z'^* = a'_*b_*j'^*$. Since $z'$ and $j'$ are flat, $p'^*a_* = b_*p'^*$. Then $z'^*b_* = a'_*z'^* = a'_*b_*j'^* = a'_*j'^*$. \qed

**Remark 3.5.** If $(V_i, U_i)$ is a good system of representations of Example 3.1, then for any $G$-variety $X$ the connecting maps $X \times^G U_i \to X \times^G U_{i+1}$ factor as in Lemma 3.4, i.e., we have $X \times^G U_i \to X \times^G (U_i \oplus W_i) \to X \times^G U_{i+1}$.

**Example 3.6.** Let $h^G$ be the equivariant theory of Example 3.1. Then the property (CD) holds for $h^G$.

Indeed, consider a good system of representations $\{(V_j, U_j)\}_j$ for $X$. The sub-varieties $X_i \times^G U_{i,j}$, $i = 0 \ldots n$ form a cellular filtration on $X \times^G U_j$ over $S \times^G U_j$. Note that $\bar{X}_i \times^G U_j$ is a resolution of singularities of $X_i \times^G U_j$. By (CD) for $h$ the set $\{(f_i \times^G \text{id}_{U_j})_* (1)\}_i$ forms a basis of $h(X \times^G U_j)$ as a $h^{G}(S \times^G U_j)$-module. By Lemma 3.4 the following diagram commutes:

$$
\begin{array}{ccc}
h(\bar{X}_i \times^G U_{j+1}) & \xrightarrow{(g_{i,j+1})_*} & h(X \times^G U_{j+1}) \\
\downarrow g_{i,j} & & \downarrow g_{j} \\
h(\bar{X}_i \times^G U_{j}) & \xrightarrow{(g_{i,j})_*} & h(X \times^G U_{j})
\end{array}
$$

So $i_{\pi i}^* ((f_i \times^G \text{id}_{U_j})_* (1)) = (f_i \times^G \text{id}_{U_j})_* (1)$, which implies that the elements $f_* (1) = \lim_j ((f_i \times^G \text{id}_{U_j})_* (1))$ form a basis of $h^G(X)$ over $h^G(S)$.

From this point on we assume that $h^G$ satisfies the property (CD). As for usual oriented theories we then obtain
Lemma 3.7. The pairing $(\cdot, \cdot) : \mathfrak{h}^G(X) \otimes_{\mathfrak{h}G(S)} \mathfrak{h}^G(X) \to \mathfrak{h}^G(S)$ given by $(a, b) = p_1(ab)$ is non-degenerate and the map

$$f : (\mathfrak{h}^G(X \times_S X), \circ) \to \text{End}_{\mathfrak{h}G(S)} \mathfrak{h}^G(X)$$

given by $a \mapsto f_a$, $f_a(x) = (p_2)_*(p_1^*(x) \cdot a)$

is an $\mathfrak{h}^G(S)$-linear isomorphism. In particular, there is an $\mathfrak{h}^G(S)$-linear isomorphism

$$(\mathfrak{h}^G_{\dim_S X}(X \times_S X), \circ) \to \text{End}_{\mathfrak{h}G, \text{Corr}}^+(\mathfrak{h}^G(X)),$$

where $\mathfrak{h}^G, \text{Corr}^+$ is the respective category of $G$-equivariant correspondences.

4. The Convolution Product

In the present section we introduce the convolution product on the equivariant Borel-Moore homology (Definition 4.3) of group power. We relate this product to the usual correspondence product for the associated torsors (Lemma 4.6) and study its behaviour under the base change (diagram (6)).

Let $G$ be a smooth algebraic group over $k$ and let $E$ be a $G$-torsor over $k$ ($G$ acts on the right). By definition there is an isomorphism $\rho : E \times_k G \iso E \times_k E$ given on points by $(e, g) \mapsto (e, eg)$. For each $i \geq 0$ it induces an isomorphism

$$\rho_i : E \times_k G^i \to E^{i+1}, \quad (e, g_1, g_2, \ldots, g_i) \mapsto (e, eg_1, eg_2, \ldots, eg_i).$$

Consider the composition

$$\gamma_i : E^{i+1} \xrightarrow{\rho_i^{-1}} E \times_k G^i = E \times_k G^i \xrightarrow{pr} G^i.$$

The coordinate-wise right $G^{i+1}$-action on $E^{i+1}$ induces an action on $E \times_k G^i$ and, hence, on $G^i$. For instance, on points it is given by

$$(e, g_1, \ldots, g_i) \cdot (h_1, \ldots, h_{i+1}) = (eh_1, h_1^{-1}g_1h_2, \ldots, h_1^{-1}g_ih_{i+1}).$$

Consider projections $p_j : E^{i+1} \to E^j$ obtained by removing the $j$-th coordinate and the respective $G^j$-action on $E^j$. For each $i \geq 1$, $1 \leq j \leq i + 1$ there is a commutative diagram of $G^j$-equivariant maps

$$
\begin{array}{ccc}
E^{i+1} & \xrightarrow{\gamma_i} & G^i \\
p_j \downarrow & & \downarrow \pi_j \\
E^j & \xrightarrow{\gamma_{i-1}} & G^{i-1}
\end{array}
$$

where $\pi_j(g_1, \ldots, g_i) = (g_1^{-1}g_2, \ldots, g_{j-1}^{-1}g_j)$ and $\pi_j(g_1, \ldots, g_i) = (g_1, \ldots, g_{j-1}, \ldots, g_i)$ for $j > 1$.

Example 4.1. For $i = 1$ it gives a commutative diagram of $G$-equivariant maps

$$
\begin{array}{ccc}
E \times_k E & \xrightarrow{\gamma_0} & G \\
p_j \downarrow & & \downarrow \pi_j \\
E & \xrightarrow{\gamma_0} & \text{Spec } k
\end{array}
$$

where $\gamma_0, \pi_1, \pi_2$ are the structure maps, $p_1, p_2$ are the corresponding projections and $\gamma_1(e, eg) = g$. Moreover, if $E$ is trivial, then $\gamma_1 = \pi_1 : G \times_k G \to G, (g_1, g_2) \mapsto g_1^{-1}g_2$. 

Let $H$ be an algebraic subgroup of $G$ such that $G/H$ is a smooth variety over $k$. We can view $G^i$ as an $H$-torsor over $G^i/H$, where $H$ acts on $G^i$ via the $j$th coordinate of $G^{i+1}$. By definition, the $H^i$-equivariant map $\pi_j$ factors as

$$\pi_j : G^i \xrightarrow{\delta} G^i/H \xrightarrow{\pi_j} G^{i-1},$$

where the second map $\bar{\pi}_j$ is a fibration with a fibre $G/H$.

**Example 4.2.** The map $\pi_1$ factors through the quotient maps modulo the diagonal action

$$\pi_1 : G^i \xrightarrow{\delta} G^i/\Delta(H) \xrightarrow{\pi_1} G^i/\Delta(G) = G^{i-1},$$

which are equivariant with respect to the usual coordinate-wise $H^i$-action.

Consider an equivariant Borel-Moore homology theory $\mathfrak{h}$. For every $1 \leq j \leq i+1$ consider the action of the $j$-th copy of $H$ on $G^i$. The property (Tor) gives an isomorphism

$$\mathfrak{h}_{H^j}(G^i/H) \xrightarrow{\sim} \mathfrak{h}_{H^{j+1}}(G^i),$$

where $H^{j+1}$ acts on $G^i$ as in (3). Unless explicitly mentioned we will always identify these two rings.

Set $S = \mathfrak{h}_{H^0}(G^0) = \mathfrak{h}_{H^1}(k)$ and set the convolution product on $S$ to be the usual intersection product.

**Definition 4.3.** Assume that $G/H$ is a smooth projective variety over $k$. We define the $S$-linear convolution product $'\circ'$ on $\mathfrak{h}_{H^i}(G^{i-1})$, $i \geq 2$ to be the composite

$$\mathfrak{h}_{H^i}(G^{i-1}) \otimes \mathfrak{h}_{H^i}(G^{i-1}) \xrightarrow{\pi_{i-1} \otimes \pi_{i+1}} \mathfrak{h}_{H^{i+1}}(G^i) \otimes \mathfrak{h}_{H^{i+1}}(G^i) \xrightarrow{'}$$

$$\mathfrak{h}_{H^{i+1}}(G^i) \xrightarrow{(\pi_{i+1})^*} \mathfrak{h}_{H^i}(G^{i+1}),$$

where $\mathfrak{h}_{H^{i+1}}(G^i)$ is identified with $\mathfrak{h}_{H^i}(G^i/H)$ via (5) and $\bar{\pi}_i$ is projective because so is $G/H$.

The central object of the present paper is the convolution ring $(\mathfrak{h}_{H^2}(G), \circ)$, i.e., the case $i = 2$. In the next sections we will show that $(\mathfrak{h}_{B^2}(G), \circ)$ (where $B$ is a Borel subgroup of a semisimple split $G$) can be identified with the formal affine Demazure algebra.

**Example 4.4.** In the case $i = 3$ the convolution ring $(\mathfrak{h}_{H^3}(G^2), \circ)$ is isomorphic to $\mathfrak{h}_{\Delta(H)}((G/H)^2)$ with respect to the usual correspondence product. Indeed, the maps $\pi_i : G^3 \to G^2$, $i = 2, 3, 4$ induce $\Delta(H)$-equivariant projections $(G/H)^3 \to (G/H)^2$. The isomorphism then follows by (Tor).

Observe that if $G/H$ is an $H$-equivariant cellular space and $\mathfrak{h}_H$ satisfies (CD), then by Lemma 3.7 there is an $S$-linear ring isomorphism

$$(\mathfrak{h}_{H^i}(G^2), \circ) \simeq \text{End}_S \mathfrak{h}_H(G/H).$$

**Lemma 4.5.** For $i \geq 1$ the map $\pi_1$ induces an injective ring homomorphism with respect to the convolution products

$$\mathfrak{h}_{H^i}(G^{i-1}) \xrightarrow{\pi_1^*} \mathfrak{h}_{H^{i+1}}(G^i).$$
**Proof.** For \( i = 1 \) it follows from the fact that the convolution product on \( h_{H^2}(G) \) is \( S \)-linear.

For \( i \geq 2 \) for each \( i-1 \leq j \leq i+1 \) we have \( \pi_j \circ \pi_1 = \pi_1 \circ \pi_{j+1} \). Since push-forwards commute with flat pull-backs by (TS), there are commutative diagrams in equivariant cohomology

\[
\begin{array}{ccc}
\pi_i^* & \pi_i^* & \pi^* \\
\pi_i^* & \pi_i^* & \pi^* \\
\pi_i^* & \pi_i^* & \pi^* \\
\end{array}
\]

Finally, there is a \( H^i \)-equivariant section of the map \( \pi_1: G^i/\Delta(H) \to G^{i-1} \) given by \( (g_1, \ldots, g_{i-1}) \mapsto (1, g_1, \ldots, g_{i-1}) \), so \( \pi_1^* \) is injective.

**Lemma 4.6.** The map \( \gamma_1 \) induces a ring homomorphism

\[
(h_{H^2}(G), \circ) \xrightarrow{\gamma_1^*} (h_{H^2}(E^2), \circ) \xrightarrow{\sim} (h((E/H)^2), \circ),
\]

where the last ring is viewed with respect to the correspondence product (2).

**Proof.** By (TS) the diagram (4) gives rise to commutative diagrams in cohomology

\[
\begin{array}{ccc}
\gamma_1^* & \gamma_1^* & \gamma_1^* \\
\gamma_1^* & \gamma_1^* & \gamma_1^* \\
\gamma_1^* & \gamma_1^* & \gamma_1^* \\
\end{array}
\]

The last isomorphism follows by (Tor).

Let \( k \) denote the splitting field of a \( G \)-torsor \( E \) so that \( G_k = G_\bar{k} \). Since the base change preserves the convolution product, combining Lemmas 4.5 and 4.6 we obtain two commutative diagrams of convolution (correspondence) rings

\[
\begin{array}{ccc}
\gamma_1^*: h_{H^2}(G) \xrightarrow{pr^*} h_{H^2}(E \times_k G) & \xrightarrow{\pi^*_1} h_{H^2}(E^2) \\
\xrightarrow{res_{E/k}} & \xrightarrow{res_{E/k}} \\
\end{array}
\]

\[
\begin{array}{ccc}
\gamma_1^*: h_{H^2}(G_k) \xrightarrow{\pi_1^*} h_{H^2}(G_k^2/\Delta(H)) & \xrightarrow{q^*} h_{H^2}(G_k^2) \\
\end{array}
\]

and

\[
\begin{array}{ccc}
\gamma_0^*: h_{H}(k) \xrightarrow{\rho_0^*\circ pr^*} h_{H}(E) \\
\xrightarrow{res_{E/k}} & \xrightarrow{res_{E/k}} \\
\gamma_0^*: h_{H}(G_k) \xrightarrow{q^*\circ \pi_1^*} h_{H}(G_k) \\
\end{array}
\]

where \( res_{E/k} \) is the base change map. Combining these two diagrams we obtain a commutative diagram of convolution rings

\[
\begin{array}{ccc}
\gamma_0^*: h_{H}(E) \otimes h_{H^2}(G) & \xrightarrow{(pr^* \circ \gamma_1^*)} h_{H^2}(E^2) \\
\xrightarrow{res_{E/k}} & \xrightarrow{res_{E/k}} \\
\end{array}
\]

\[
\begin{array}{ccc}
h_{H}(G_k) \otimes h_{H^2}(G_k) & \xrightarrow{(pr^* \circ \gamma_1^*)} h_{H^2}(G_k^2) \\
\xrightarrow{res_{E/k}} & \xrightarrow{res_{E/k}} \\
\end{array}
\]
where the left convolution rings are $h_H(E)$- and $h_H(G_k)$-linear.

5. The Subring of Push-Pull Operators

In the present section we prove that if $H$ is the Borel subgroup of a split semisimple linear algebraic group, then the convolution ring $h_{H^2}(G)$ of Definition 4.3 can be identified with the subring of push-pull operators (Corollary 5.3). Our arguments are essentially based on the Bruhat decomposition of $G$ stated using the $G$-orbits on the product $G/H \times_k G/H$ and the resolution of singularities (8).

As before assume that $G/H$ is a smooth projective variety over $k$. In the notation of the previous section consider the $H^2$-equivariant maps of Example 4.2.

\[
\pi_1: G^2 \rightarrow G^2/\Delta(H) \rightarrow G^2/\Delta(G) = G, \quad (g_1, g_2) \mapsto g_1^{-1}g_2.
\]

Since $G^2$ is a $\Delta(G)$-torsor over $G$ ($\Delta(H)$-torsor over $G^2/\Delta(H)$), by the property (Tor) the induced $\Delta(G) \times H^2$-equivariant pull-backs on cohomology coincide with the forgetful maps

\[
\gamma^*_1: h_{H^2}(G) \cong h_{\Delta(G) \times H^2}(G^2) \rightarrow h_{H^2}(G^2) \rightarrow h_H((G/H)^2) \cong h((G/H)^2)
\]

Moreover, by Lemma 4.5 it is a commutative diagram of convolution rings.

Let $G$ be a split semisimple linear algebraic group over $k$ and let $h$ be an equivariant theory that satisfies property (CD). We fix a Borel subgroup $B$ of $G$ containing a split maximal torus $T$. By Bruhat decomposition (e.g. [27])

\[G = \coprod_{w \in W} B\dot{w}B, \quad \dot{w} \in N_T,\]

is the disjoint union of $B^2$-orbits of $G$, where $W = N_T/T$ is the Weyl group and $N_T$ is the normalizer of $T$ in $G$. Projecting this decomposition onto $X = G/B$ gives a $B$-equivariant cellular filtration on $X$ by closures $X_w$ of affine spaces $X_w = B\dot{w}B/B$ of dimension $l(w)$ (the length of $w$).

The preimage $\pi_1^{-1}(B\dot{w}B)$ is a $\Delta(G)$-orbit in $G^2$ (here $H = B$). Let $O_w$ denote its image via $G^2 \rightarrow X^2$ and let $\overline{O}_w$ denote its closure. Observe that both $O_w$ and $\overline{O}_w$ are $\Delta(G)$-invariant in $X^2$. By properties of the Bruhat decomposition (see [27, §1]) it follows that the projection $O_w \rightarrow X \rightarrow X$ is a torsor of a vector bundle over $X$ with fibre $X_w$. Indeed, the transition functions are affine since they are given by the action of $B$ on the left on $B\dot{w}B/B$ that is by $T$ acting on the product of the respective root subgroups $\prod_{\alpha \in \Phi^+ \cap \omega(w)\Phi^-} U_{\alpha}$ via the conjugation and, hence, by $T$ acting on the product of the respective $G_\alpha’s$ via the multiplication $t \cdot x = \alpha(t)x$, $t \in T$, $x \in G_\alpha$. So $X^2$ is a $G$-equivariant ($G$ acts diagonally) cellular space over $X$ with filtration given by the closures $\overline{O}_w$.

Assume that for each $w \in W$ we are given a $G$-equivariant resolution of singularities $\tilde{O}_w \rightarrow \overline{O}_w$. Let $[\tilde{O}_w|G]$ denote the respective class in $h_G^{\dim X - l(w)}(X^2)$. Then by the property (CD) the cohomology $h_G(X^2)$ (resp. $h_B(X^2)$ and $h(X^2)$) is a free module over $h_G(X)$ (resp. over $h_B(X)$ and $h(X)$) with basis $\{[\tilde{O}_w|G]w \in W$ (resp. $\{[\tilde{O}_w|^B]w \in W$ and $\{[\tilde{O}_w]w \in W$). Hence, the forgetful maps of (7) send $[\tilde{O}_w|G] \mapsto [\tilde{O}_w|^B \mapsto [\tilde{O}_w]$ and change the coefficients by $-\otimes_{h_G(X)} h_B(X)$ and
We now construct such $G$-equivariant resolutions as follows. For the $i$-th simple reflection $s_i$ we denote $X_{s_i}$ (resp. $O_{s_i}$) simply by $X_i$ (resp. by $O_i$). Let $P_i$ be the minimal parabolic subgroup corresponding to a simple root $\alpha_i$ and let $q_i : X \to G/P_i$ denote the respective quotient map.

**Lemma 5.1.** We have $\overline{O}_i = X \times G/P_i X$ and, in particular, $\overline{O}_i$ is smooth.

*Proof.* We have $(g_1 B, g_2 B) \in X \times G/P_i X$, $g_1, g_2 \in G$ if and only if $g_1 P_i = g_2 P_i$, so $g_2 = g_1 h$ for some $h \in P_i$. Since $P_i = B \cup B_{s_i} B$, it means that either $g_2 = g_1 B$ or $g_2 B = g_1 B_{s_i} B$, so $(g_1 B, g_2 B) \in O_{s_i} \cup \Delta_X = \overline{O}_i$. \(\square\)

For any $w \in W$ we choose a reduced decomposition $w = s_{i_1} s_{i_2} \ldots s_{i_t}$ and set $I_w = (i_1, i_2, \ldots, i_t)$. Consider a variety
\[
\tilde{O}_{I_w} = X \times_{G/P_{i_1}} X \times_{G/P_{i_2}} \ldots X_{G/P_{i_t}} X.
\]

The projection on the first and the last factor $pr: \tilde{O}_{I_w} \to X \times_k X$ gives a $G$-equivariant resolution of singularities of $\overline{O}_w$.

**Theorem 5.2.** For $H = B$ or $1$, the image of $[\tilde{O}_{I_w}]_H \subseteq h_H(X \times_k X)$ under the K"unneth isomorphism
\[
(h_H(X \times_k X), \circ) \xrightarrow{\sim} \text{End}_{h_H(k)}(h_H(X))
\]
is the composition of push-pull operators $q_{i_1}^* q_{i_t}^* \circ \ldots \circ q_{i_t}^* q_{i_1}^* \circ$.

*Proof.* By definition the image of $[\tilde{O}_{I_w}]_H$ is the $h_H(k)$-linear operator
\[
h^*_H(X) \xrightarrow{pr_{i_1}} h_H(X \times_k X) \xrightarrow{\tilde{O}_{I_w}} h_H(X \times_k X) \xrightarrow{pr_{i_t}} h^*_H(\tilde{O}_{I_w}(X)).
\]

By the projection formula and (TS) it can be also written as
\[
h^*_H(X) \xrightarrow{pr_{i_1}^{-1}} h_H(\tilde{O}_{I_w}) \xrightarrow{pr_{i_1}} h^*_H(\tilde{O}_{I_w}(X)),
\]
where $pr_j$ denotes the projection on the $j$-th coordinate (recall that $p_j$ denotes the projection obtained by removing the $j$-th coordinate).

By the property (TS) we obtain a commutative diagram
\[
\begin{array}{cccccccc}
h_H(X) & \xrightarrow{pr_{i_1}} & h_H(\tilde{O}_{i_1}) & \xrightarrow{pr_{i_1}^2} & h(H(\tilde{O}_{(i_1, i_1)})) & \xrightarrow{pr_{i_1}^3} & \ldots & \xrightarrow{pr_{i_1}^{t+1}} & h_H(\tilde{O}_{I_w}) \\
q_{i_1} & & q_{i_1} & & q_{i_1} & & & & q_{i_1}
\end{array}
\]

where $pr_{i_1 j k \ldots}$ denote the projection on the $i$-th, $j$-th, $k$-th, \ldots, coordinates. The result then follows since the top horizontal row gives $pr_{i_1}^{t+1}$ and the right vertical column gives $pr_{i_1}$. \(\square\)
Combining Diagram (7) and Theorem 5.2 we obtain

**Corollary 5.3.** There is a commutative diagram of convolution rings

\[
\begin{array}{ccc}
\h_B^2(G) & \xrightarrow{\pi_1} & \h_{\Delta(B) \times B^2}(G^2) \\
\downarrow{\phantom{\pi_1}} & & \downarrow{\pi_2} \\
\h_B^2(G^2) & \xrightarrow{\pi_3} & \h_B(X^2) & \xrightarrow{\sim} & \text{End}_S(\h_B(X)) \\
& & \downarrow{q^*} & & \downarrow{\sim} \\
& & \h(X^2) & \xrightarrow{\sim} & \text{End}_R(\h(X))
\end{array}
\]

where the image of \((\h_B^2(G), \alpha)\) in \(\text{End}_S(\h_B(X))\) is the subring generated by the push-pull operators \(q^*q_\alpha\) (of degree \((-1)\)) and the image of the forgetful map \(S = \h_B^\ast(X) \rightarrow \h_B^\ast(X)\) (of degrees \(\bullet\)) and the last vertical arrow is induced by the augmentation map \(S \rightarrow R = \h(k)\).

6. Self-duality of the algebra of push-pull operators

In the present section we identify the convolution ring \(\h_B^2(G)\) with the formal affine Demazure algebra \(D_F\) of [18] and show that it is self-dual with respect to the convolution product (Theorem 6.2). Our arguments are based on the results of [18], [7], [8] and, especially, [9]. We use the notation of [9].

Recall that algebraic oriented cohomology theories \(h\) correspond (up to universality) to one-dimensional commutative formal group laws \(F(u,v)\): the formal group law corresponds to \(h\) by means of the Quillen formula expressing the first characteristic classes

\[c_i^\ast(L_1 \otimes L_2) = F(c_1^\ast(L_1), c_1^\ast(L_2))\]

and the respective cohomology theory \(h\) is defined from \(F\) by tensoring with the algebraic cobordism

\[h(-) = \Omega(-) \otimes_{\Omega(k)} R,\]

where \(\Omega(k) \rightarrow R\) defines \(F\) by specializing the coefficients in the Lazard ring (see [9, §2] for details). For example, the additive formal group law correspond to \(h\) by means of the Quillen formula expressing the first characteristic classes

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where \(\Omega(k) \rightarrow R\) defines \(F\) by specializing the coefficients in the Lazard ring (see [9, §2] for details). For example, the additive formal group law correspond to \(h\) by means of the Quillen formula expressing the first characteristic classes

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and the respective cohomology theory \(h\) is defined from \(F\) by tensoring with the algebraic cobordism

\[h(-) = \Omega(-) \otimes_{\Omega(k)} R,\]

where \(\Omega(k) \rightarrow R\) defines \(F\) by specializing the coefficients in the Lazard ring (see [9, §2] for details). For example, the additive formal group law correspond to \(h\) by means of the Quillen formula expressing the first characteristic classes

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\[h(-) = \Omega(-) \otimes_{\Omega(k)} R,\]

where \(\Omega(k) \rightarrow R\) defines \(F\) by specializing the coefficients in the Lazard ring (see [9, §2] for details). For example, the additive formal group law correspond to \(h\) by means of the Quillen formula expressing the first characteristic classes

\[c_i^\ast(L_1 \otimes L_2) = F(c_1^\ast(L_1), c_1^\ast(L_2))\]

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\[h(-) = \Omega(-) \otimes_{\Omega(k)} R,\]

where \(\Omega(k) \rightarrow R\) defines \(F\) by specializing the coefficients in the Lazard ring (see [9, §2] for details). For example, the additive formal group law correspond to \(h\) by means of the Quillen formula expressing the first characteristic classes

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\[c_i^\ast(L_1 \otimes L_2) = F(c_1^\ast(L_1), c_1^\ast(L_2))\]

and the respective cohomology theory \(h\) is defined from \(F\) by tensoring with the algebraic cobordism

\[h(-) = \Omega(-) \otimes_{\Omega(k)} R,\]
By [7, Thm. 7.9] (see also [18, Thm. 5.14]) the \( R \)-algebra \( D_F \) satisfies the following (complete) set of relations: for \( i, j = 1 \ldots rk(G) \) and \( u \in S \)

- \( Y_i^2 = \kappa_i Y_i \), where \( \kappa_i = \frac{1}{x_i} + \frac{1}{x_{i}^\ast} \), and \( x_i = x_{\alpha_i} \),
- \( Y_i u = s_i(u) y_i + \Delta_{-i}(u) \), where \( \Delta_{-i}(u) = \frac{u - s_i(u)}{x_{i} - 1} \),
- \( (Y_i Y_j)^{m_{ij}} - (Y_j Y_i)^{m_{ij}} = \sum c_{I_w} Y_i Y_j \), where the sum is taken over all reduced expressions \( I_w \) of elements \( w \) of the subgroup \( \langle s_i, s_j \rangle \subseteq W \), and the coefficients \( c_{I_w} \) are given by the formulas of [18, Prop. 5.8]

**Example 6.1.** If \( F \) corresponds to Chow groups, then \( D_F = H_{nil} \) is the affine nil-Hecke algebra over \( \mathbb{Z} \) in the notation of [16]. If \( F \) corresponds to \( K \)-theory, then \( D_F \) is the 0-affine Hecke algebra over \( \mathbb{Z} \) (\( q \rightarrow 0 \) in the affine Hecke algebra). If \( F \) corresponds to the generic hyperbolic formal group law of [8, 9], then by [8, Prop. 9.2] the constant part of \( D_F \) is isomorphic to the localized classical Iwahori-Hecke algebra.

Let \( D_F^\ast = Hom_S(D_F, S) \) denote its dual. Observe that the main result of [9] (Thm. 8.2 loc. cit.) says that \( D_F^\ast \) is isomorphic to the \( R \)-algebra \( h_F(X) \).

**Theorem 6.2.** Let \( G \) be a split semisimple linear algebraic group over a field \( k \) and let \( h \) be an equivariant theory that satisfies property (CD).

Then the convolution algebra \( (h_{BZ}(G), \circ) \) is isomorphic (as an \( R \)-algebra) to the formal affine Demazure algebra \( D_F \). So there is an \( R \)-algebra isomorphism

\[
(D_F^\ast, \circ) \simeq (D_F, \circ)
\]

**Proof.** By Corollary 5.3 the ring \( (h_{BZ}(G), \circ) \simeq (h_B(X), \circ) \) is isomorphic to the subalgebra of \( \text{End}_S(h_B(X)) \) generated by the image of the forgetful map \( h_G(X) \rightarrow h_B(X) \) and push-pull operators \( q_i^\ast q_i \). Since the map \( B \rightarrow B/T \) is an affine fibration, the natural map \( h_B(X) \rightarrow h_T(X) \) is an isomorphism. Hence we may identify \( S \) with \( h_T(k) \) and \( \text{End}_S(h_B(X)) \) with \( \text{End}_S(h_T(X)) \). Observe that these identifications preserve push-pull operators. The inclusion of \( T \)-fixed point set \( W \rightarrow X \) gives an embedding \( h_T(X) \rightarrow h_T(W) = S_W \subseteq Q_W^\ast \). By [9, Corollary 8.7] there is the following commutative diagram

\[
\begin{array}{ccc}
h_T(X) & \overset{q_i^\ast q_i}{\longrightarrow} & S_W \subseteq Q_W^\ast \\
\downarrow h_T & & \downarrow A_i \\
h_T(X) & \overset{A_i}{\longrightarrow} & S_W \subseteq Q_W^\ast
\end{array}
\]

where the Hecke operator \( A_i \) is given by

\[
A_i(f)(x) = f(x \cdot Y_i) \quad \text{for } x \in Q_W, f \in Q_W^\ast.
\]

Moreover, the forgetful map

\[
S \cong h_G(X) \rightarrow h_T(X) = \oplus_{w \in W} S
\]

is given by the formula \( s \mapsto (w \cdot s)_{w \in W} \) for any \( s \in S \). Then the multiplication in \( h_T(X) = S_W \) by the image of any element in \( s \in h_G(X) \) induces a right multiplication by \( s \) in \( Q_W \). Since \( Q_W \) is a free \( Q \)-module of finite rank, the natural map \( \iota: Q_W \rightarrow \text{End}_Q(Q_W^\ast) \) given by \( \iota(x)(f)(y) = f(yx) \) is an inclusion. Note that every \( A_i \) lies in the image of \( \iota \). Then by diagram (9) the image of \( h_{BZ}(G) \) is isomorphic to a subalgebra of \( Q_W \) generated by \( S \) and \( Y_i \) which is \( D_F \). \( \square \)
7. The rational algebra of push-pull operators

In the present section we introduce the rational algebra of push-pull operators \( D_F \) and show that it can be identified with the subring of rational endomorphisms of \( G/B \) (Theorem 7.5).

The \( B^2 \)-equivariant isomorphism \( E \times_k G \rightarrow E \times_k E \), \((e, g)\rightarrow (e, eg)\) induces an isomorphism \( E \times B G/B \rightarrow E/B \times_k E/B \). For all \( w \in W \) fix a reduced decomposition \( I_w = (i_1, \ldots, i_l) \) and the corresponding Bott-Samelson resolution \( X_{I_w} \rightarrow G/B \) of the Schubert cell. This map is \( B \)-equivariant, so it descends to a map \( Y_{I_w} = E \times B X_{I_w} \rightarrow E \times B G/B \).

**Lemma 7.1.** The classes \([Y_{I_w}]\) form a basis of \( h(E/B \times_k E/B) \) over \( h(E/B) \), where the module structure is given by the pullback of the projection \( pr_i^*: h(E/B) \rightarrow h(E/B \times_k E/B) \).

**Proof.** Since \( B \) is special, \( G \)-torsor \( E \) splits over the function field of \( E/B \). Then by [25, Lemma 3.3] projection \( pr_i: E \times_k E/B \rightarrow E/B \) is a cellular fibration in the sense of [25, Definition 3.1] so that \( (E/B)^2 \) is a cellular space over \( E/B \). Let \( \xi \) be the generic point of \( E/B \). The pullback of an open embedding \( j^* : h(E/B \times_k E/B) \rightarrow h(\xi \times E/B) \approx h(G/B) \) is surjective and any preimage of \( R \)-basis of \( h(G/B) \) gives a basis of \( h(E/B \times_k E/B) \). Thus it is sufficient to check that \( j^* \) sends \([Y_{I_w}]\) to a basis of \( h(\xi \times E/B) \). Let \( p: E \rightarrow E/B \) be the projection. Note that

\[
E \times B X_{I_w} \times_{(E/B \times_k E/B)} (\xi \times E/B) = p^{-1}(\xi) \times B X_{I_w} = B X_{I_w} = \xi \times X_{I_w},
\]

since \( p^{-1}(\xi) \rightarrow \xi \) is a trivial \( B \)-torsor. Thus \( j^*([Y_{I_w}]) = [\xi \times X_{I_w}] \) that forms a basis of \( h(\xi \times E/B) \approx h(\xi \times G/B) \) over \( h(\xi) = R \).  

Consider a \( B \)-equivariant map

\[
f: E \times^B G \rightarrow B \backslash G, \quad (e, g)B \mapsto Bg.
\]

Let \( X_{I_w}' = (P_{i_1} \times \ldots \times P_{i_l})/B^l \) where \( B^l \)-action on \( P_{i_1} \times \ldots \times P_{i_l} \) is given by \((p_1, \ldots, p_l) \cdot (b_1, \ldots, b_l) = (b_1^{-1} p_1 b_2, \ldots, b_l^{-1} p_l)\). Then \( X_{I_w}' \) gives the Bott-Samelson class for \( B \backslash G \).

**Lemma 7.2.** The composition \( h_B(B \backslash G) \overset{I_{I_w}'}{\rightarrow} h_B(E \times^B G) \approx h(E/B \times_k E/B) \) maps \([X_{I_w}']_B\) to \([Y_{I_w}]\).

**Proof.** Consider the map \( P_{i_1} \times^B P_{i_2} \times^B \ldots \times^B P_{i_l} \rightarrow G \) given by \((p_1, \ldots, p_l) \mapsto p_1 \ldots p_l\). It is \( B \)-equivariant with respect to the left multiplication, so it descends to a map \( M_{I_w} = E \times^B P_{i_1} \times^B P_{i_2} \times^B \ldots P_{i_l} \rightarrow E \times^B G \). By construction we have an isomorphism

\[
M_{I_w} \approx [Y_{I_w}] \times_{E \times^B (G/B)} (E \times^B G). 
\]

Then \([M_{I_w}]_B \) is mapped to \([Y_{I_w}]\) via the isomorphism \( h(E \times^B G/B) \rightarrow h_B(E \times^B G) \). Thus it is sufficient to check that \( f^*[X_{I_w}']_B = [M_{I_w}]_B \), which follows from the fact that

\[
M_{I_w} = E \times^B (P_{i_1} \times \ldots \times P_{i_l}/B^{l-1}) \approx (E \times^B G) \times_{B \backslash B^l} X_{I_w}'.
\]

**Lemma 7.3.** (cf. [25, Corollary 3.4]) The composition

\[
(p_1^*, \gamma_1^*): h_B(E) \otimes_S h_{B^2}(G) \rightarrow h_{B^2}(E^2) \approx h((E/B)^2)
\]

of the diagram (6) (for \( H = B \)) is an isomorphism.
Consider the restriction map \( h(E/B) \to h(E_\ell/B) = h(X_\ell) \) on cohomology induced by the scalar extension \( k/k \) (here \( k \) is a splitting field of \( E \)). Let \( \overline{E}(X) \) denote its image.

**Corollary 7.4.** The image of the ring homomorphism
\[
\text{res}_{k/k}: (h(E/B \times_k E/B), \circ) \longrightarrow (h(X_\ell \times_k X_\ell), \circ).
\]
is the subalgebra generated by the multiplication by the elements of \( \overline{E}(X) \) and the push-pull operators \( q_i^* q_j*: h(X) \to h(G/P) \to h(X) \) for all simple roots \( \alpha_i \).

**Proof.** Follows by (6), Lemma 7.3 and Corollary 5.3. \( \square \)

There is a natural action of \( W \) on \( \overline{E}(X) \) that comes from the \( W \)-action on \( E/T \). So we can endow \( \overline{E}(X) \otimes_{R} \mathbb{Q}_W \) with a structure of an \( R \)-algebra. Let \( \overline{D}_F \) denote its subalgebra \( \overline{E}(X) \otimes_{R} \mathbb{D}_F \). We call it the rational algebra of push-pull operators.

**Theorem 7.5.** Consider the restriction
\[
\text{res}_{k/k}: \text{End}_{h, \text{Corr}^+_\ell}(E/B) \longrightarrow \text{End}_{h, \text{Corr}^+_\ell}(X_\ell)
\]
on endomorphism rings of the respective motives (i.e., preserving the grading of \( h(X) \)). Its image can be identified with \( \overline{D}_F^{(0)} \) via the injective forgetful map
\[
\phi: (\overline{E}(X) \otimes_{R} h_G(X^2_\ell))^{(N)}, \circ) \longrightarrow (h^N(X^2_\ell), \circ).
\]

**Proof.** By (7) both \( h_G(X^2) \) and \( h(X^2) \) are free modules over \( h_G(X) \) and \( h(X) \) with basis given by the classes \([\tilde{O}_{L_{\alpha}}]\) and \([\tilde{O}_{L_{\alpha}}]\) respectively. The map \( \phi \) sends \([\tilde{O}_{L_{\alpha}}]\) \( \to \) \([\tilde{O}_{L_{\alpha}}]\) and leaves the coefficients invariant. The result follows by Corollary 7.4, Corollary 5.3 and Theorem 6.2. \( \square \)

We say that a (co-)homology theory \( h \) satisfies the Dimension Axiom if

(1) For any smooth variety \( Y \) over \( k \) we have \( h^n(Y) \equiv 0 \) for all \( n > \dim Y \).

**Example 7.6.** Any theory \( h \) over a field \( k \) of characteristic 0 obtained by specialization of coefficients of the Lazard ring (e.g. Chow groups, connective \( K \)-theory, algebraic cobordism \( \Omega \)) satisfies (Dim).

The graded \( K \)-theory \( K_0(-)[\beta, \beta^{-1}] \) of \([23, \text{Example 1.1.5}] \) does not satisfy (Dim).

Observe that the image of the characteristic map \( c: S \to h(X) \) is contained in \( \overline{E}(X) \) (see \([15, \text{Thm. 4.5}] \)). Consider both the induced map \( c: \mathbb{D}_F^{(0)} \to \mathbb{D}_F^{(0)} \) and the restriction map \( \text{res}_{k/k}: (h_B(E) \otimes_{R} \mathbb{D}_F^{(0)} \to \mathbb{D}_F^{(0)}) \).

**Lemma 7.7.** Assume that the theory \( h \) satisfies (Dim), then the kernels of \( c \) and \( \text{res}_{k/k} \) are nilpotent.
In other words, there is a commutative diagram of maps of convolution rings

\[
\begin{array}{ccc}
D_F^{(0)} & \xrightarrow{\gamma_1} & h^N((E/B)^2) \\
\downarrow \epsilon & & \downarrow \text{res}_{\bar{k}/k} \\
\bar{D}_F^{(0)} & &
\end{array}
\]

with nilpotent kernels.

Proof. Let \( f = \epsilon \) or \( \text{res}_{\bar{k}/k} \). Then \( x \in \ker f \) means that \( x = \sum_w a_w [\tilde{O}_w]_G \) with \( f(a_w) = 0 \). By Theorem 5.2 each \([\tilde{O}_w]\) corresponds to the composite of push-pull elements \( Y_{I_w} \) in \( D_F \), so that \( x \) corresponds to \( \sum_w a_w Y_{I_w} \in \bar{D}_F^{(0)} \) and \( x^{2n} \) corresponds to

\[
(\sum_w a_w Y_{I_w})^n = \sum_w a_{w,n} Y_{I_w}, \quad a_{w,n} \in (\ker f)^n,
\]

Since \( \ker f \) is contained in the augmentation ideal, \( (\ker f)^n \subset S^{(\geq n)} \). Finally, observe that \( \deg Y_{I_w} \leq -N \), hence, for \( n > 2N \) we get \( x^{2n} = 0 \). □

Lemma 7.8. If \( E \) is a versal \( G \)-torsor, then the map \( \gamma_1^* \) and, hence, \( \epsilon \), of the lemma 7.7 is surjective.

Proof. Observe that if \( E \) is versal, then it admits an open \( G \)-equivariant embedding into \( A_{\bar{k}}^N \). So the projection \( E \times_k G^i \to G^i \) in the definition of \( \gamma_i \) factors through \( h_k^N \times_k G^i \). By (Loc) and (HI) the induced pullback \( \gamma_i^* \) is surjective. □

8. Applications to representation theory of Hecke rings

Let \( G \) be a split semisimple linear algebraic group over a field \( k \) and let \( E \) be a \( G \)-torsor over \( k \). Let \( \bar{k} \) be a splitting field of \( E \) and let \( E/B \) be the twisted form of \( G/B \) by means of \( E \). By definition, we have

\[
E/B \times_k \bar{k} \simeq G/B \times_k \bar{k}.
\]

Let \( h, h_B \) be an (\( B \)-equivariant) oriented theory over \( k \) that satisfies both (CD) and (Dim) axioms, e.g., Chow groups, connective \( K \)-theory or algebraic cobordism \( \Omega \). Let \( R = h(k) \) and \( S = h_B(k) \) be the respective coefficient rings.

Consider the endomorphism ring of the \( h \)-motive of \( E/B \)

\[
C_F = (\text{End}_{h_{\cdot \text{Corr}}^k}(E/B), \circ).
\]

By definition, any direct sum decomposition of the motive \([E/B]\) is given by a complete set of primitive pairwise orthogonal idempotents on \( C_F \) so that \( \langle E/B \rangle_h \simeq \text{Proj } C_F \). Our main result (Theorem 7.5) together with Lemma 7.7 says that the restriction map gives a surjective ring homomorphism with nilpotent kernel

\[
\text{res}_{\bar{k}/k}: C_F \to \bar{D}_F^{(0)}.
\]

By the standard idempotent lifting (see e.g. [25, §2 and Prop. 2.6]) we then obtain \( \text{Proj } C_F \simeq \text{Proj } \bar{D}_F^{(0)} \), so that we get the following
Theorem 8.1. There is a one-to-one correspondence between direct sum decompositions of the \( \mathbb{h} \)-motive \([E/B]\) and direct sum decompositions of \( \overline{\mathbb{D}}_F \)-module \( \overline{\mathbb{D}}_F \). This correspondence, induces an equivalence between the category \( \langle E/B \rangle_\mathbb{h} \) and the category \( \text{Proj} \overline{\mathbb{D}}_F \) of finitely generated projective \( \overline{\mathbb{D}}_F \)-modules.

If \( E \) is versal, then the algebra \( \overline{\mathbb{D}}_F \) can be replaced by the algebra \( \mathbb{D}_F \).

Observe that in general the ring \( \overline{\mathbb{D}}_F \) is not Krull-Schmidt (and not semi-simple).

Lemma 8.2. If the coefficient ring \( \mathbb{R} \) is Artinian, then both \( \mathbb{C}_F \) and \( \overline{\mathbb{D}}_F \) and, hence, the categories \( \langle E/B \rangle_\mathbb{h} \) and \( \text{Proj} \overline{\mathbb{D}}_F \) satisfy the Krull-Schmidt property (uniqueness of a direct sum decomposition).

Proof. If \( \mathbb{R} \) is Artinian, then both \( \mathbb{D}_F \) and \( \mathbb{C}_F \) are Artinian (as \( \mathbb{D}_F \) is finite dimensional over \( \mathbb{R} \)). So they are both Noetherian which implies that the respective tautological modules \( \mathbb{D}_F \) and \( \mathbb{C}_F \) have finite length and, hence, the Krull-Schmidt property holds for both \( \mathbb{D}_F \) and \( \mathbb{C}_F \). \( \square \)

As a direct application of the main result of [25] one obtains the following characterization of modular representations of the (affine) nil-Hecke algebra (\( F \) is additive and \( \mathbb{h} = CH \)).

Corollary 8.3. Let \( G \) be a split semisimple linear algebraic group over a field \( k \). Consider the affine nil-Hecke algebra \( \mathbb{H}_{nil} \) for \( G \) with coefficients in \( \mathbb{R} = \mathbb{F}_p \), \( p \) is a prime. Then

\[
\text{Proj} \mathbb{H}_{nil} \cong \langle \mathcal{R}_E, p \rangle,
\]

where \( E \) is a versal \( G \)-torsor.

In particular, all indecomposable submodules of \( \mathbb{H}_{nil} \) are free \( S \)-modules isomorphic to each other and their \( S \)-rank equals to the \( p \)-part of the product of \( p \)-exceptional degrees of \( G \).

Proof. The \( S \)-rank coincides with the number of Tate motives in the decomposition of \( \mathcal{R}_E, p \) over a splitting field of \( E \), that is \( \prod_{i=1}^r \frac{1-t_i^{q^{s_i}}}{1-t_i} \) (in the notation of [25]) which is equal to the \( p \)-part \( p^{\sum_{i=1}^r k_i} \) of \( p \)-exceptional degrees of [19, p.73]. \( \square \)

Example 8.4. Consider the root system of type \( A_1 \). In this case \( T^* = \mathbb{Z} \omega \) (\( G = SL_2 \)) or \( T^* = \mathbb{Z} \alpha \) (\( G = PGL_2 \)), \( \alpha = 2\omega \) is the simple root and \( \omega \) is the fundamental weight. The Weyl group \( W = \{ 1, s \} \) acts by \( s: \omega \mapsto -\omega \), where \( s \) is the simple reflection. By definition, \( S = \mathbb{R}[[x]] \) (where \( x = x_\omega \) or \( x = x_\alpha \)), \( \mathbb{Q} = S[1/2] \), \( \mathbb{Q}_W = \{ q(x)\delta_w \mid q(x) \in \mathbb{Q}, w \in W \} \) with
\[
q(-Fx)\delta_s = s(q(x))\delta_s = \delta_s q(x),
\]
where \( -Fx \) is the formal inverse of \( x \). Observe that \( x_\alpha = x_{\omega+\omega} = F(x_\omega, x_\omega) \) in \( S \).

The \( \mathbb{R} \)-algebra \( \mathbb{D}_F \) is a free left \( S \)-submodule of rank 2 in \( \mathbb{Q}_W \) with basis
\[
\{ 1, Y = \frac{1}{x_\omega} + \frac{1}{x_\alpha} \delta_s \}.
\]
It satisfies the relations
\[
Y^2 = \kappa Y \text{ and } Yq(x) = q(-Fx)Y + \Delta(q(x)),
\]
where \( \kappa = \frac{1}{x_\omega} + \frac{1}{x_\alpha} \) and \( \Delta(q(x)) = \frac{2(x)q(Fx)}{x_\omega} \).
Let \( p = a + bY \), \( a, b \in S \) be an idempotent in \( D_F \), i.e., \( p^2 = p \). Since \( \deg p = 0 \), we have \( \deg a = 0 \) and \( \deg b = 1 \) (the coefficient \( a_{ij} \in R \) at \( u^i v^j \) in \( F(u, v) \) has degree \( 1 - i - j \)). Then
\[
(a + bY)^2 = a^2 + abY + bY a + bY bY = a^2 + abY + b(s(a)Y + \Delta(a)) + b(s(b)Y + \Delta(b))Y = \\
\left( a^2 + b\Delta(a) \right) + \left( ab + bs(a) + bs(b)\kappa + b\Delta(b) \right)Y.
\]
So that
\[
a^2 + b\Delta(a) = a \quad \text{and} \quad ab + bs(a) + bs(b)\kappa + b\Delta(b) = b.
\]
Assume \( b \) is a non-zero divisor, then we obtain (in \( S \))
\[
(10) \quad a^2 + b\Delta(a) = a \quad \text{and} \quad a + s(a) + s(b)\kappa + \Delta(b) = 1.
\]
In the case \( h = CH \) and \( R = \mathbb{Z} \) (\( F(u, v) = u + v \)) we have \( \kappa = 0 \), \( -fx = -x \), \( x_\alpha = 2x_\omega, \ a \in \mathbb{Z}, \ b = cx, \ c \in \mathbb{Z} \) and, (10) becomes
\[
a = 0 \quad \text{and} \quad \Delta(b) = \frac{cx}{x_\omega} = 1
\]
or
\[
a = 1 \quad \text{and} \quad \Delta(b) = \frac{1}{x_\omega} = -1
\]
which have solutions only if \( x = x_\omega \) and \( c = \pm 1 \). Therefore, \( D_F \) has only two indecomposable submodules corresponding to the idempotents \( 1 - xY \) and \( xY \).

The latter translates into the following well-known fact concerning motivic decompositions:

The Chow motive of a conic \( E/B \) (for a versal \( G \)-torsor \( E \)) decomposes as a direct sum of two indecomposable summands if and only if \( G = SL_2 \), i.e. \( E \) is trivial and \( E/B = \mathbb{P}^1 \). In this case each summand correspond to the (shifted) Tate motive.

References


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