On decomposable biquaternion algebras with involution of orthogonal type

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Abstract

We investigate the pfaffians of decomposable biquaternion algebras with involution of orthogonal type. In characteristic two, a classification of these algebras in terms of their pfaffians is studied. Also, in arbitrary characteristic, a criterion for an orthogonal involution on a biquaternion algebra to be metabolic is obtained.
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1 Introduction

A biquaternion algebra is a tensor product of two quaternion algebras. Every biquaternion algebra is a central simple algebra of degree 4 and exponent 2 or 1. A result proved by A. A. Albert shows that the converse is also true (see [9, (16.1)]). An *Albert form* of a biquaternion algebra A is a 6-dimensional quadratic form with trivial discriminant whose Clifford algebra is isomorphic to $M_2(A)$. It is known that two biquaternion algebras over a field F are isomorphic as F-algebras if and only if their Albert forms are similar (see [9, (16.3)]).

The Albert form of a biquaternion algebra with involution arises naturally as the quadratic form induced by a *pfaffian* (see [13, (3.3)]). The classical pfaffian is a polynomial map Pf defined on alternating matrices under the transpose involution, which satisfies $\mathrm{Pf}(X)^2 = \det X$ for every alternating matrix X (see [2, (3.27)]). In [10], a pfaffian of certain modules over Azumaya algebras was defined and used to classify 6-dimensional quadratic spaces over commutative rings. This construction was used in [13] to find a criterion for involutions on an Azumaya algebra of rank 16, which contains 2 as a unit, to admit an invariant rank 4 Azumaya subalgebra. A similar decomposition criterion for involutions on a biquaternion algebra in arbitrary characteristic was also obtained in [11].

It is known that involutions of symplectic type on a biquaternion algebra can be classified, up to conjugation, by their *pfaffian norms* (see [9, (16.19)]). For orthogonal involutions the situation is a little more complicated. In characteristic $\neq 2$, [13, (5.3)] yields a classification of decomposable orthogonal involutions on a biquaternion algebra A in terms of the pfaffian and the *pfaffian adjoint* (introduced in [10] and [13]). This classification was originally stated in [13] for the more general case where A is an Azumaya algebra which contains 2 as a unit.

In this work, the pfaffians of decomposable biquaternion algebras with orthogonal involution are investigated. In §3, we recall the notions of pfaffian and pfaffian adjoint of a biquaternion algebra with involution (A,σ). For a decomposable orthogonal involution σ, let q_σ be a pfaffian satisfying $q_\sigma(x)^2=\mathrm{Nrd}_A(x)$ for every alternating element x. Set $\mathrm{Alt}(A,\sigma)^+=\{x+p_\sigma(x)\mid x\in\mathrm{Alt}(A,\sigma)\}$ and $\mathrm{Alt}(A,\sigma)^-=\{x-p_\sigma(x)\mid x\in\mathrm{Alt}(A,\sigma)\}$, where p_σ is the linear endomorphism of $\mathrm{Alt}(A,\sigma)$ satisfying $xp_\sigma(x)=p_\sigma(x)x=q_\sigma(x)$ and $p_\sigma^2(x)=x$ for $x\in\mathrm{Alt}(A,\sigma)$. We shall see in (3.9) that the union of the sets $\mathrm{Alt}(A,\sigma)^+$ and $\mathrm{Alt}(A,\sigma)^-$ coincides with the set of all square-central elements in $\mathrm{Alt}(A,\sigma)$. At the end of §3, we study in more details the classification of orthogonal involutions on biquaternion algebras in characteristic $\neq 2$, obtained in [13]. Although this result is already presented in [13], it is useful to restate it to enable comparison with the corresponding result in characteristic 2 (see (3.17) and (4.12)).

In §4, we study the characterization of decomposable biquaternion algebras with involution in characteristic 2. We also investigate the relation between the restriction of the form q_σ to $\mathrm{Alt}(A,\sigma)^+$, denoted by q_σ^+ and the *Pfister invariant* $\mathfrak{Pf}(A,\sigma)$ introduced in [4]. The key result is (4.11), which states that if σ and σ' are two decomposable orthogonal involutions on A, then $q_\sigma^+\simeq q_{\sigma'}^+$ if and only if $\mathfrak{Pf}(A,\sigma)\simeq\mathfrak{Pf}(A,\sigma')$. Using this and [15, (6.5)], it can be shown that σ and σ' are conjugate if and only if $q_\sigma^+\simeq q_{\sigma'}^+$ (see (4.12)). Finally, we obtain in (4.16) and (4.19) some criteria for an orthogonal involution on a biquaternion algebra in arbitrary characteristic to be metabolic.

2 Preliminaries

Let V be a finite dimensional vector space over a field F. A *quadratic form* over F is a map $q:V\to F$ such that (i) $q(av)=a^2q(v)$ for every $a\in F$ and $v\in V$; (ii) the map $\mathfrak{b}_q:V\times V\to F$ defined by $\mathfrak{b}_q(u,v)=q(u+v)-q(u)-q(v)$ is a bilinear form. Note that for every $v\in V$ we have $\mathfrak{b}_q(v,v)=2q(v)$. In particular, if $\mathrm{char}\,F=2$, then $\mathfrak{b}_q(v,v)=0$ for $v\in V$, i.e., $\mathfrak{b}_q$ is an *alternating* form. The *orthogonal complement* of a subspace $W\subseteq V$ is defined as $W^\perp=\{x\in V\mid b_q(x,y)=0 \text{ for all } y\in W\}$. If $V=U\oplus W$ is the direct sum of two subspaces U and W with $W\subseteq U^\perp$, we write $(V,q)=(U\perp W,q|_U\perp q|_W)$.

A quadratic form q (resp. a bilinear form $\mathfrak{b}$) on V is called *isotropic* if there exists a nonzero vector $v\in V$ such that $q(v)=0$ (resp. $\mathfrak{b}(v,v)=0$). For $\alpha\in F$, we say that q (resp. $\mathfrak{b}$) *represents* α if there exists a nonzero vector $v\in V$ such that $q(v)=\alpha$ (resp. $\mathfrak{b}(v,v)=\alpha$). The sets of all elements of F represented by q and $\mathfrak{b}$ are denoted by $D_F(q)$ and $D_F(\mathfrak{b})$ respectively. For $\alpha\in F^\times$, the *scaled* quadratic form $\alpha\cdot q$ is defined as $\alpha\cdot q(v)=\alpha q(v)$ for every $v\in V$.

If $\mathrm{char}\,F=2$, for $a\in F$, we denote by $[a]$ (the isometry class of) the quadratic form $q(x)=ax^2$. If $\mathrm{char}\,F\neq 2$ and $a_1,\cdots,a_n\in F$, the quadratic form $q(x_1,\cdots,x_n)=\sum_{i=1}^n a_ix_i^2$ is denoted by $\langle a_1,\cdots,a_n\rangle_q$. Also, in arbitrary characteristic, the bilinear form defined by $\mathfrak{b}((x_1,\cdots,x_n),(y_1,\cdots,y_n))=\sum_{i=1}^n a_ix_iy_i$ is denoted by $\langle a_1,\cdots,a_n\rangle$. Finally, the bilinear form $\langle 1,a_1\rangle\otimes\cdots\otimes\langle 1,a_n\rangle$ is called a *bilinear n-fold Pfister form* and is denoted by $\langle\langle a_1,\cdots,a_n\rangle\rangle$.

An *involution* on a central simple F-algebra A is an antiautomorphism σ of A of order 2. We say that σ is *of the first kind* if $\sigma|_F=\mathrm{id}$. An involution σ of the first kind is said to be of *symplectic type* (or *symplectic*) if over a splitting field of A, it becomes adjoint to an alternating bilinear form. Otherwise σ is said to be of *orthogonal type* (or *orthogonal*). The discriminant of an orthogonal involution

σ is denoted by $\operatorname{disc}\sigma$. The set of *alternating elements* of A is defined as

$$\mathrm{Alt}(A,\sigma) = \{a - \sigma(a) \mid a \in A\}.$$

A *quaternion algebra* over a field F is a central simple algebra Q of degree 2. The *canonical* involution γ on Q is defined by $\gamma(x) = \mathrm{Trd}_Q(x) - x$ for $x \in Q$, where $\mathrm{Trd}_Q(x)$ is the reduced trace of x. It is known that the canonical involution on Q is the unique involution of symplectic type on Q and it satisfies $\gamma(x)x \in F$ for every $x \in Q$ (see [9, Ch. 2]). The map $N_Q : Q \to F$ defined by $N_Q(x) = \gamma(x)x$ is called the *norm form* of Q. An element $x \in Q$ is called a *pure quaternion* if $\mathrm{Trd}_Q(x) = 0$. The set of all pure quaternions of Q is a 3-dimensional subspace of Q denoted by Q_0. Note that an element $x \in Q$ lies in Q_0 if and only if $\gamma(x) = -x$, or equivalently, $N_Q(x) = -x^2$.

A central simple algebra with involution (A,σ) over a field F is called *totally decomposable* if it decomposes as tensor products of σ-invariant quaternion F-algebras. If A is a biquaternion algebra, we will use the term *decomposable* instead of "totally decomposable". It is known that a biquaternion algebra with orthogonal involution (A,σ) is decomposable if and only if $\operatorname{disc}\sigma$ is trivial (see [11, (3.7)]).

Let (A,σ) be an algebra with involution over a field F. An idempotent $e \in A$ is called a *metabolic* (resp. *hyperbolic*) idempotent with respect to σ if $\sigma(e)e = 0$ and $(1-e)(1-\sigma(e)) = 0$ (resp. $\sigma(e) = 1-e$). The pair (A,σ) is called *metabolic* (resp. *hyperbolic*) if A contains a metabolic (resp. hyperbolic) idempotent with respect to σ. Every hyperbolic involution σ is metabolic but the converse is not always true. If σ is symplectic or $\operatorname{char} F \neq 2$, the involution σ is metabolic if and only if it is hyperbolic, (see [5, (4.10)] and [3, (A.3)]).

3 The pfaffian and the pfaffian adjoint

We begin our discussion by looking at special cases of [12, (2.1)] and [12, (3.1)]:

Theorem 3.1. ([12]) *Let (A,σ) be a biquaternion algebra with orthogonal involution over a field F and let $d_\sigma \in F^\times$ be a representative of the class $\operatorname{disc}\sigma \in F^\times/F^{\times 2}$, i.e., $d_\sigma F^{\times 2} = \operatorname{disc}\sigma$. There exists a map $pf_\sigma : \mathrm{Alt}(A,\sigma) \to F$ such that $pf_\sigma(x)^2 = d_\sigma \mathrm{Nrd}_A(x)$ for every $x \in \mathrm{Alt}(A,\sigma)$. The map pf_σ is uniquely determined up to a sign. Moreover, there exists an F-linear map $\pi_\sigma : \mathrm{Alt}(A,\sigma) \to \mathrm{Alt}(A,\sigma)$ such that $x\pi_\sigma(x) = \pi_\sigma(x)x = pf_\sigma(x)$ and $\pi_\sigma^2(x) = d_\sigma x$ for every $x \in \mathrm{Alt}(A,\sigma)$.*

Remark 3.2. The map π_σ in (3.1) is uniquely determined by pf_σ. In fact it is easily seen by scalar extension to a splitting field that $\mathrm{Alt}(A,\sigma)$ has a basis $\mathcal{B}$ consisting of invertible elements. For every $x \in \mathcal{B}$, we must have $\pi_\sigma(x) = x^{-1}pf_\sigma(x)$. As π_σ is F-linear, it is uniquely defined on $\mathrm{Alt}(A,\sigma)$.

Definition 3.3. The map pf_σ in (3.1) is called a *pfaffian of* (A,σ). We also call the map π_σ, the *pfaffian adjoint* of pf_σ.

Note that by [13, (3.3)], every pfaffian of (A,σ) is an Albert form of A.

Definition 3.4. Let F be a field. The *pfaffian* of an alternating matrix $X = (x_{ij}) \in M_4(F)$ (under transpose involution) is defined as

$$\mathrm{Pf}(X) = x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}.$$

Notations 3.5. Let (A, σ) be a decomposable biquaternion algebra with orthogonal involution over a field F. Since $\operatorname{disc}\sigma$ is trivial, using (3.1) one can find a pfaffian pf_σ of $\operatorname{Alt}(A, \sigma)$ satisfying $pf_\sigma(x)^2 = \operatorname{Nrd}_A(x)$ for every $x \in \operatorname{Alt}(A, \sigma)$. If pf'_σ is another pfaffian with this property, then $pf'_\sigma = \pm pf_\sigma$. After scalar extension to an algebraic closure of F, exactly one of these pfaffians corresponds to the pfaffian Pf. We denote this pfaffian by q_σ. Moreover, we denote by p_σ the pfaffian adjoint of q_σ, hence

$$q_\sigma(x)^2 = \operatorname{Nrd}_A(x), \quad xp_\sigma(x) = p_\sigma(x)x = q_\sigma(x) \quad \text{and} \quad p_\sigma^2(x) = x,$$

for every $x \in \operatorname{Alt}(A, \sigma)$. We also use the following notations:

$$\operatorname{Alt}(A, \sigma)^+ := \{x + p_\sigma(x) \mid x \in \operatorname{Alt}(A, \sigma)\},$$
$$\operatorname{Alt}(A, \sigma)^- := \{x - p_\sigma(x) \mid x \in \operatorname{Alt}(A, \sigma)\}.$$

Note that if $\operatorname{char} F = 2$, then $\operatorname{Alt}(A, \sigma)^+ = \operatorname{Alt}(A, \sigma)^-$. As proved in [13, p. 597] and [11, (3.5)], $\operatorname{Alt}(A, \sigma)^+$ and $\operatorname{Alt}(A, \sigma)^-$ are 3-dimensional subspaces of $\operatorname{Alt}(A, \sigma)$. Since $p_\sigma^2 = \operatorname{id}$, we have $p_\sigma(x) = x$ for every $x \in \operatorname{Alt}(A, \sigma)^+$ and $p_\sigma(x) = -x$ for every $x \in \operatorname{Alt}(A, \sigma)^-$. The converse is also true, i.e.,

$$\operatorname{Alt}(A, \sigma)^+ = \{x \in \operatorname{Alt}(A, \sigma) \mid p_\sigma(x) = x\}, \tag{1}$$
$$\operatorname{Alt}(A, \sigma)^- = \{x \in \operatorname{Alt}(A, \sigma) \mid p_\sigma(x) = -x\}. \tag{2}$$

In fact if $\operatorname{char} F \neq 2$, then for every $x \in \operatorname{Alt}(A, \sigma)$ with $p_\sigma(x) = x$ we have $x = \frac{1}{2}(x + p_\sigma(x)) \in \operatorname{Alt}(A, \sigma)^+$. Similarly if $p_\sigma(x) = -x$, then $x = \frac{1}{2}(x - p_\sigma(x)) \in \operatorname{Alt}(A, \sigma)^-$. If $\operatorname{char} F = 2$, then the relation (1) follows from the dimension formula for the image and the kernel of the linear map $p_\sigma + \operatorname{id}$.

The following result is implicitly contained in [9, pp. 249-250] over a field of characteristic different form 2:

Lemma 3.6. ([9]) *Let (A, σ) be a decomposable biquaternion algebra with orthogonal involution over a field F. Then p_σ is an isometry of $(\operatorname{Alt}(A, \sigma), q_\sigma)$. Furthermore $\mathfrak{b}_{q_\sigma}(x, y) = xp_\sigma(y) + yp_\sigma(x)$, for $x, y \in \operatorname{Alt}(A, \sigma)$.*

Proof. For every $x \in \operatorname{Alt}(A, \sigma)$ we have $q_\sigma(p_\sigma(x)) = p_\sigma(p_\sigma(x))p_\sigma(x) = xp_\sigma(x) = q_\sigma(x)$. Thus, p_σ is an isometry. The second assertion is easily obtained from the relations $q_\sigma(x) = xp_\sigma(x)$ and $\mathfrak{b}_{q_\sigma}(x, y) = q_\sigma(x + y) - q_\sigma(x) - q_\sigma(y)$. □

Lemma 3.7. *Let (A, σ) be a decomposable biquaternion algebra with orthogonal involution over a field F. Then $\operatorname{Alt}(A, \sigma)^+ = (\operatorname{Alt}(A, \sigma)^-)^\perp \subseteq C_A(\operatorname{Alt}(A, \sigma)^-)$.*

Proof. Let $\mathfrak{b} = \mathfrak{b}_{q_\sigma}$ and let $x \in \operatorname{Alt}(A, \sigma)^+$. Since $p_\sigma \in O(\operatorname{Alt}(A, \sigma), q_\sigma)$, we have $\mathfrak{b}(x, y) = \mathfrak{b}(p_\sigma(x), p_\sigma(y)) = \mathfrak{b}(x, p_\sigma(y))$ for every $y \in \operatorname{Alt}(A, \sigma)$. Thus, $\mathfrak{b}(x, y - p_\sigma(y)) = 0$, i.e., $\operatorname{Alt}(A, \sigma)^+ \subseteq (\operatorname{Alt}(A, \sigma)^-)^\perp$. By dimension count we obtain $\operatorname{Alt}(A, \sigma)^+ = (\operatorname{Alt}(A, \sigma)^-)^\perp$. Now let $z \in \operatorname{Alt}(A, \sigma)^-$. By (3.6) we have $0 = \mathfrak{b}(x, z) = -xz + zx$. Thus, $xz = zx$, which implies that $\operatorname{Alt}(A, \sigma)^+$ commutes with $\operatorname{Alt}(A, \sigma)^-$, i.e., $\operatorname{Alt}(A, \sigma)^+ \subseteq C_A(\operatorname{Alt}(A, \sigma)^-)$. □

Lemma 3.8. *Let (A, σ) be a decomposable biquaternion algebra with orthogonal involution over a field F and let $x \in \operatorname{Alt}(A, \sigma)$. If $x^2 \in F$, then $p_\sigma(x) = \pm x$.*

Proof. Set $\alpha = x^2 \in F$ and $\beta = q_\sigma(x) \in F$. Then $\beta^2 = q_\sigma(x)^2 = \mathrm{Nrd}_A(x) = \pm\alpha^2$. Thus, $\beta = \lambda\alpha$ for some $\lambda \in F$ with $\lambda^4 = 1$, i.e., $q_\sigma(x) = \lambda x^2$. If $\alpha \neq 0$, then multiplying $xp_\sigma(x) = q_\sigma(x) = \lambda x^2$ on the left by x^{-1} we obtain $p_\sigma(x) = \lambda x$. The relation $p_\sigma^2 = \mathrm{id}$ then implies that $\lambda = \pm 1$ and we are done. So suppose that $\alpha = 0$, i.e., $x^2 = 0$. By (3.6) we have $\mathfrak{b}_{q_\sigma}(p_\sigma(x), x) = p_\sigma(x)^2 + x^2 = p_\sigma(x)^2$, hence $p_\sigma(x)^2 \in F$. On the other hand, the relations $xp_\sigma(x) = q_\sigma(x) = \lambda x^2 = 0$ show that $p_\sigma(x)$ is not invertible. Thus,

$$p_\sigma(x)^2 = 0. \tag{3}$$

Suppose that $p_\sigma(x) \neq x$, hence $x \notin \mathrm{Alt}(A,\sigma)^+$. In view of (3.7) one can find $w \in \mathrm{Alt}(A,\sigma)^-$ such that $\mathfrak{b}_{q_\sigma}(x, w) = 1$. By (3.6) we have

$$-xw + wp_\sigma(x) = 1. \tag{4}$$

Multiplying (4) on the left by x we get $xwp_\sigma(x) = x$. Using (4), it follows that $(wp_\sigma(x) - 1)p_\sigma(x) = x$, which yields $p_\sigma(x) = -x$ by (3). This completes the proof (note that if $\mathrm{char}\, F = 2$, this argument shows that the assumption $p_\sigma(x) \neq x$ leads to the contradiction $p_\sigma(x) = -x$, hence $p_\sigma(x) = x$). □

Proposition 3.9. *Let (A,σ) be a decomposable biquaternion algebra with orthogonal involution over a field F and let $\mathrm{Alt}(A,\sigma)^0 = \mathrm{Alt}(A,\sigma)^+ \cup \mathrm{Alt}(A,\sigma)^-$. Then $\mathrm{Alt}(A,\sigma)^0 = \{x \in \mathrm{Alt}(A,\sigma) \mid p_\sigma(x) = \pm x\} = \{x \in \mathrm{Alt}(A,\sigma) \mid x^2 \in F\}$.*

Proof. The relations (1) and (2) below (3.5) yield the first equality. The second equality follows from (3.8). □

Notation 3.10. For a decomposable biquaternion algebra with involution of orthogonal type (A,σ) over a field F, we use the notations $Q(A,\sigma)^+ = F + \mathrm{Alt}(A,\sigma)^+$ and $Q(A,\sigma)^- = F + \mathrm{Alt}(A,\sigma)^-$. We will simply denote $Q(A,\sigma)^+$ by Q^+ and $Q(A,\sigma)^-$ by Q^-, if the pair (A,σ) is clear from the context.

Lemma 3.11. ([11]) *Let (A,σ) be a decomposable biquaternion algebra with orthogonal involution over a field F.*

(1) *If $\mathrm{char}\, F \neq 2$, then Q^+ and Q^- are two σ-invariant quaternion subalgebras of A with $Q_0^+ = \mathrm{Alt}(A,\sigma)^+$ and $Q_0^- = \mathrm{Alt}(A,\sigma)^-$. Furthermore we have $(A,\sigma) \simeq (Q^+, \sigma|_{Q^+}) \otimes (Q^-, \sigma|_{Q^-})$, where $\sigma|_{Q^+}$ and $\sigma|_{Q^-}$ are the canonical involutions of Q^+ and Q^- respectively.*

(2) *If $\mathrm{char}\, F = 2$, then $Q^+ = Q^-$ is a maximal commutative subalgebra of F satisfying $x^2 \in F$ for every $x \in Q^+$.*

Proof. As observed in [11, (3.5)], Q^+ is a σ-invariant quaternion subalgebra of A and $\sigma|_{Q^+}$ is of symplectic type. By dimension count and (3.7) we obtain $Q^- = C_A(Q^+)$, hence $A \simeq Q^+ \otimes_F Q^-$. By [9, (2.23 (1))], $\sigma|_{Q^-}$ is of symplectic type. Finally, since $\mathrm{Trd}_{Q^+}(x) = 0$ for every $x \in \mathrm{Alt}(A,\sigma)^+$, we have $Q_0^+ = \mathrm{Alt}(A,\sigma)^+$. Similarly $Q_0^- = \mathrm{Alt}(A,\sigma)^-$. This proves the first part. The second part follows from [11, (3.6)]. □

Notation 3.12. Let (A,σ) be a decomposable biquaternion algebra with orthogonal involution over a field F. We denote by q_σ^+ and q_σ^- the restrictions of q_σ to $\mathrm{Alt}(A,\sigma)^+$ and $\mathrm{Alt}(A,\sigma)^-$ respectively.

Lemma 3.13. *Let (A,σ) be a decomposable biquaternion algebra with orthogonal involution over a field F.*

(1) *Every unit $u \in \mathrm{Alt}(A,\sigma)^+$ (resp. $u \in \mathrm{Alt}(A,\sigma)^-$) can be extended to a basis $\{u,v,w\}$ of $\mathrm{Alt}(A,\sigma)^+$ (resp. $\mathrm{Alt}(A,\sigma)^-$) such that $w = uv$.*

(2) *Every basis $\{u,v,w\}$ of $\mathrm{Alt}(A,\sigma)^+$ (resp. $\mathrm{Alt}(A,\sigma)^-$) with $w = uv$ is orthogonal with respect to q_σ^+ (resp. q_σ^-).*

(3) *If $\operatorname{char} F \neq 2$, then $N_{Q^+} \simeq \langle 1\rangle_q \perp (-1)\cdot q_\sigma^+$ and $N_{Q^-} \simeq \langle 1\rangle_q \perp q_\sigma^-$.*

(4) *If $\operatorname{char} F = 2$ and $(A,\sigma) \simeq (Q_1,\sigma_1)\otimes(Q_2,\sigma_2)$ is a decomposition of (A,σ), then $q_\sigma^+ \simeq [\alpha] \perp [\beta] \perp [\alpha\beta]$, where $\alpha \in F^\times$ and $\beta \in F^\times$ are representatives of the classes $\operatorname{disc}\sigma_1 \in F^\times/F^{\times 2}$ and $\operatorname{disc}\sigma_2 \in F^\times/F^{\times 2}$ respectively.*

Proof. We just prove the result for q_σ^+. The proof for q_σ^- is similar.

(1) Choose an element $u' \in \mathrm{Alt}(A,\sigma)^+ \setminus Fu$ and set $\alpha = u^2 \in F^\times$. By (3.11), $uu' \in Q^+ = F + \mathrm{Alt}(A,\sigma)^+$. Thus, there exist $\lambda \in F$ and $w \in \mathrm{Alt}(A,\sigma)^+$ such that $uu' = \lambda + w$. Set $v = u' - \lambda\alpha^{-1}u \in \mathrm{Alt}(A,\sigma)^+$. Then $uv = w \in \mathrm{Alt}(A,\sigma)^+$. Thus, $\{u,v,w\}$ is the desired basis.

(2) Let $\mathcal{B} = \{u,v,w\}$ be a basis of $\mathrm{Alt}(A,\sigma)^+$ with $w = uv$. Then $vu = \sigma(uv) = -uv$. Using (3.6) we obtain $\mathfrak{b}(u,v) = uv + vu = 0$. Similarly, $\mathfrak{b}(u,w) = \mathfrak{b}(v,w) = 0$.

(3) Let $\{u,v,w\}$ be a basis of $\mathrm{Alt}(A,\sigma)^+$ with $w = uv$. By (2), $q_\sigma^+ \simeq \langle \alpha, \beta, -\alpha\beta\rangle_q$, where $\alpha = u^2 \in F$ and $\beta = v^2 \in F$. On the other hand since $vu = -uv$, $\{1,u,v,w\}$ is a quaternion basis of Q^+. Thus, $N_{Q^+} \simeq \langle 1, -\alpha, -\beta, \alpha\beta\rangle_q$ by [6, (9.6)].

(4) Let $u \in \mathrm{Alt}(Q_1,\sigma_1)$ and $v \in \mathrm{Alt}(Q_2,\sigma_2)$ be two units and set $\alpha = u^2$, $\beta = v^2$ and $w = uv$. Then $\operatorname{disc}\sigma_1 = \alpha F^{\times 2} \in F^\times/F^{\times 2}$ and $\operatorname{disc}\sigma_2 = \beta F^{\times 2} \in F^\times/F^{\times 2}$. Also, since $w \in \mathrm{Alt}(A,\sigma)$ and $w^2 \in F$, by (3.9) we obtain $w \in \mathrm{Alt}(A,\sigma)^+$. So $\{u,v,w\}$ is a basis of $\mathrm{Alt}(A,\sigma)^+$ and $q_\sigma^+ \simeq [\alpha] \perp [\beta] \perp [\alpha\beta]$. ☐

Proposition 3.14. ([13, (5.3)]) *Let (A,σ) and (A',σ') be two decomposable biquaternion algebras with orthogonal involution over a field F. If $(A,\sigma) \simeq (A',\sigma')$, then there exists an isometry $f : (\mathrm{Alt}(A,\sigma), q_\sigma) \to (\mathrm{Alt}(A',\sigma'), q_{\sigma'})$ such that $f(\mathrm{Alt}(A,\sigma)^+) = \mathrm{Alt}(A',\sigma')^+$.*

Proof. See the implication (1) ⇒ (2) in [13, (5.3)] (note that the proof given there also works in characteristic 2). ☐

Lemma 3.15. ([13]) *Let (A,σ) be a decomposable biquaternion algebra with orthogonal involution over a field F of characteristic different from 2. Then $(\mathrm{Alt}(A,\sigma), q_\sigma) \simeq (\mathrm{Alt}(A,\sigma)^+, q_\sigma^+) \perp (\mathrm{Alt}(A,\sigma)^-, q_\sigma^-)$.*

Proof. See [13, p. 597]. ☐

Remark 3.16. Let (A,σ) be a decomposable biquaternion algebra with orthogonal involution over a field F of characteristic different from 2. By (3.15) every $x \in \mathrm{Alt}(A,\sigma)$ can be written uniquely as $x = x^+ + x^-$, where $x^+ \in \mathrm{Alt}(A,\sigma)^+ = Q_0^+$ and $x^- \in \mathrm{Alt}(A,\sigma)^- = Q_0^-$. In view of [9, (16.24)] and (3.11 (1)), the maps p_σ and q_σ can be defined explicitly as $p_\sigma(x^+ + x^-) = x^+ - x^-$ and $q_\sigma(x^+ + x^-) = (x^+)^2 - (x^-)^2$. Using this fact, one can find a shorter proof of (3.8) in characteristic different from 2 (see also the decomposition of q_σ^+ and q_σ^- in the proof of (3.13 (3))).

The next result complements [13, (5.3)] for biquaternion algebras:

Theorem 3.17. ([13]) *Let (A,σ) and (A',σ') be two decomposable biquaternion algebras with orthogonal involution over a field F of characteristic different from 2. Let $Q^+ = Q(A,\sigma)^+$, $Q^- = Q(A,\sigma)^-$, $Q'^+ = Q(A',\sigma')^+$ and $Q'^- = Q(A',\sigma')^-$. The following statements are equivalent:*

(1) $(A,\sigma) \simeq (A',\sigma')$.

(2) $q_\sigma \simeq q_{\sigma'}$ *and* $q^+_\sigma \simeq q^+_{\sigma'}$.

(3) $A \simeq A'$ *and* $q^+_\sigma \simeq q^+_{\sigma'}$.

(4) $A \simeq A'$ *and* $Q^+ \simeq Q'^+$.

Furthermore, the above statements are equivalent to those obtained by changing "+" to "−".

Proof. The implication (1) ⇒ (2) follows from (3.14). Since q_σ and $q_{\sigma'}$ are Albert forms of (A,σ) and (A',σ') respectively, $q_\sigma \simeq q_{\sigma'}$ implies that $A \simeq A'$ by [9, (16.3)]. This proves (2) ⇒ (3). The implication (3) ⇒ (4) follows from (3.13 (3)) and [14, Ch. III, (2.5)]. To prove (4) ⇒ (1) observe that by (3.11 (1)) we have $C_A(Q^+) = Q^-$ and $C_{A'}(Q'^+) = Q'^-$. Thus, the isomorphisms $Q^+ \simeq_F Q'^+$ and $A \simeq_F A'$ imply that $Q^- \simeq_F Q'^-$. Since the restrictions of σ to Q^+ and Q^- and the restrictions of σ' to Q'^+ and Q'^- are all symplectic, we obtain

$$\begin{aligned}(A,\sigma) &\simeq_F (Q^+,\sigma|_{Q^+}) \otimes_F (Q^-,\sigma|_{Q^-})\\ &\simeq_F (Q'^+,\sigma'|_{Q'^+}) \otimes_F (Q'^-,\sigma'|_{Q'^-}) \simeq_F (A',\sigma').\end{aligned}$$

To prove the last statement of the result, observe that by (3.14), (3.15) and the Witt's cancellation theorem [14, Ch. I, (4.2)], the statement (1) implies that $q_\sigma \simeq q_{\sigma'}$ and $q^-_\sigma \simeq q^-_{\sigma'}$. Thus, the implication (1) ⇒ (2) again follows from (3.14). The rest implications can be verified easily. □

4 Relation with the Pfister invariant in characteristic two

Throughout this section, unless stated otherwise, F is a field of characteristic 2.

Definition 4.1. Let A be a finite-dimensional associative F-algebra. The minimum number r such that A can be generated as an F-algebra by r elements is called the *minimum rank* of A and is denoted by $r_F(A)$.

Theorem 4.2. ([15]) *Let (A,σ) be a totally decomposable algebra with involution of orthogonal type over F. There exists a symmetric and self-centralizing subalgebra $S \subseteq A$ such that $x^2 \in F$ for every $x \in S$ and $\dim_F S = 2^n$, where $n = r_F(S)$. Furthermore, for every subalgebra S with these properties, we have $S = F + S_0$, where $S_0 = S \cap \mathrm{Alt}(A,\sigma)$. In particular, $S \subseteq F + \mathrm{Alt}(A,\sigma)$. Finally, the subalgebra S is uniquely determined up to isomorphism.*

Proof. See [15, pp. 10-11]. □

Notation 4.3. The isomorphism class of S in (4.2) is denoted by $\Phi(A,\sigma)$.

The next result shows that for biquaternion algebras with orthogonal involution, the subalgebra $\Phi(A,\sigma)$ is unique as a set:

Corollary 4.4. *Let (A,σ) be a decomposable biquaternion algebra with involution of orthogonal type over F. Then $\Phi(A,\sigma)=Q^+$.*

Proof. Write $\Phi(A,\sigma)=F+S_0$, where $S_0=\Phi(A,\sigma)\cap \mathrm{Alt}(A,\sigma)$. Since every element of $\Phi(A,\sigma)$ is square-central, using (3.9) we have $S_0\subseteq \mathrm{Alt}(A,\sigma)^+$. Then $S_0=\mathrm{Alt}(A,\sigma)^+$ by dimension count, hence $\Phi(A,\sigma)=F+\mathrm{Alt}(A,\sigma)^+=Q^+$. $\square$

The following result is implicitly contained in [15]:

Lemma 4.5. *Let (A,σ) be a totally decomposable algebra of degree 2^n with involution of orthogonal type over F. Suppose that there exists a set $\{u_1,\cdots,u_n\}\subseteq \mathrm{Alt}(A,\sigma)\cap A^*$ consisting of pairwise commutative square-central elements such that $u_{i_1}\cdots u_{i_l}\in \mathrm{Alt}(A,\sigma)$ for every $1\leq l\leq n$ and $1\leq i_1<\cdots<i_l\leq n$. Then $\Phi(A,\sigma)\simeq F[u_1,\cdots,u_n]$.*

Proof. By [8, (2.2.3)], $S:=F[u_1,\cdots,u_n]$ is self-centralizing. The other required properties of S, stated in (4.2), are easily verified. $\square$

Definition 4.6. The set $\{u_1,\cdots,u_n\}\subseteq \mathrm{Alt}(A,\sigma)\cap A^*$ in (4.5) is called a *set of alternating generators* of $\Phi(A,\sigma)$.

We recall the following definition from [4]:

Definition 4.7. Let $(A,\sigma)=(Q_1,\sigma_1)\otimes\cdots\otimes(Q_n,\sigma_n)$ be a totally decomposable algebra with orthogonal involution over F. Let $\alpha_i\in F^\times$, $i=1,\cdots,n$, be a representative of the class $\operatorname{disc}\sigma_i\in F^\times/F^{\times2}$. The bilinear n-fold Pfister form $\langle\langle\alpha_1,\cdots,\alpha_n\rangle\rangle$ is called the *Pfister invariant* of (A,σ) and is denoted by $\mathfrak{Pf}(A,\sigma)$.

Note that by [4, (7.5)], $\mathfrak{Pf}(A,\sigma)$ is independent of the decomposition of (A,σ).

Theorem 4.8. ([15]) *Let (A,σ) be a totally decomposable algebra of degree 2^n with involution of orthogonal type over F. Then $\Phi(A,\sigma)$ can be considered as an underlying vector space of the bilinear form $\mathfrak{Pf}(A,\sigma)$ in such a way that $\mathfrak{Pf}(A,\sigma)(x,x)=x^2$ for every $x\in\Phi(A,\sigma)$. Also, $\mathfrak{Pf}(A,\sigma)\simeq\langle\langle\alpha_1,\cdots,\alpha_n\rangle\rangle$ if and only if there exists a set of alternating generators $\{u_1,\cdots,u_n\}$ of $\Phi(A,\sigma)$ such that $u_i^2=\alpha_i\in F^\times$, $i=1,\cdots,n$.*

Proof. See [15, pp. 11-12]. $\square$

Lemma 4.9. *Let $\langle\langle\alpha,\beta\rangle\rangle$ be an isotropic bilinear Pfister form over F. If $\alpha\beta\neq0$, then $\langle\langle\alpha,\beta\rangle\rangle\simeq\langle\langle\alpha,\beta+\alpha^{-1}\lambda^2\rangle\rangle$ for every $\lambda\in F$.*

Proof. Since $\langle\langle\alpha,\beta\rangle\rangle$ is isotropic, by [6, (4.14)] either $\alpha\in F^{\times2}$ or $\beta\in D_F\langle1,\alpha\rangle$. If $\alpha\in F^{\times2}$, using [6, (4.15 (2))] and [6, (4.15 (1))] we obtain

$$\begin{aligned}\langle\langle\alpha,\beta\rangle\rangle&\simeq\langle\langle\beta+\alpha^{-1}\lambda^2,\alpha\beta\rangle\rangle\simeq\langle\langle\beta+\alpha^{-1}\lambda^2,\alpha\beta(\alpha^{-1}\lambda^2-(\beta+\alpha^{-1}\lambda^2))\rangle\rangle\\&\simeq\langle\langle\beta+\alpha^{-1}\lambda^2,\alpha\beta^2\rangle\rangle\simeq\langle\langle\alpha,\beta+\alpha^{-1}\lambda^2\rangle\rangle.\end{aligned}$$

If $\beta\in D_F\langle1,\alpha\rangle$, then there exist $b,c\in F$ such that $\beta=b^2+c^2\alpha$. Let $s=\alpha^{-1}\beta^{-1}\lambda\in F$. Using [6, (14.15 (1))] we obtain

$$\begin{aligned}\langle\langle\alpha,\beta\rangle\rangle&\simeq\langle\langle\alpha,\beta((1+cs\alpha)^2-(bs)^2\alpha)\rangle\rangle\simeq\langle\langle\alpha,\beta(1+c^2s^2\alpha^2+b^2s^2\alpha)\rangle\rangle\\&\simeq\langle\langle\alpha,\beta+s^2\alpha\beta(c^2\alpha+b^2)\rangle\rangle\simeq\langle\langle\alpha,\beta+s^2\alpha\beta^2\rangle\rangle\simeq\langle\langle\alpha,\beta+\alpha^{-1}\lambda^2\rangle\rangle.\end{aligned}\qquad\square$$

Lemma 4.10. *Let (A,σ) be a decomposable biquaternion algebra with involution of orthogonal type over F and let $\alpha,\beta\in F^\times$. Then $\mathfrak{Pf}(A,\sigma)\simeq\langle\langle\alpha,\beta\rangle\rangle$ if and only if $q_\sigma^+\simeq[\alpha]\perp[\beta]\perp[\alpha\beta]$.*

Proof. If $\mathfrak{Pf}(A,\sigma)\simeq\langle\langle\alpha,\beta\rangle\rangle$, by (4.8) there exists a set of alternating generators $\{u,v\}$ of $\Phi(A,\sigma)$ such that $u^2=\alpha$ and $v^2=\beta$. By (4.4) and (3.13 (2)), $\{u,v,uv\}$ is an orthogonal basis of $\mathrm{Alt}(A,\sigma)^+$, hence $q_\sigma^+\simeq[\alpha]\perp[\beta]\perp[\alpha\beta]$.

To prove the converse, choose a basis $\{x,y,z\}$ of $\mathrm{Alt}(A,\sigma)^+$ with $x^2=\alpha$, $y^2=\beta$ and $z^2=\alpha\beta$. Consider the element $xy\in\Phi(A,\sigma)$. By (4.4), $\Phi(A,\sigma)=F+\mathrm{Alt}(A,\sigma)^+$. Thus, there exists $a,b,c,d\in F$ such that

$$xy=a+bx+cy+dz. \tag{5}$$

If $a=0$ then $xy=bx+cy+dz\in\mathrm{Alt}(A,\sigma)^+$, which implies that $\{x,y\}$ is a set of alternating generators of $\Phi(A,\sigma)$. As $x^2=\alpha$ and $y^2=\beta$ we obtain $\mathfrak{Pf}(A,\sigma)\simeq\langle\langle\alpha,\beta\rangle\rangle$ by (4.8).

So suppose that $a\neq 0$. By squaring both sides of (5), we obtain $\alpha\beta=a^2+b^2\alpha+c^2\beta+d^2\alpha\beta$, which yields

$$1+(ba^{-1})^2\alpha+(ca^{-1})^2\beta+((d+1)a^{-1})^2\alpha\beta=0.$$

Therefore, the form $\langle\langle\alpha,\beta\rangle\rangle$ is isotropic. Set $y'=y+\alpha^{-1}ax\in\mathrm{Alt}(A,\sigma)^+$. By (5) we have $xy'=xy+a=bx+cy+dz\in\mathrm{Alt}(A,\sigma)^+$, hence $\{x,y'\}$ is a set of alternating generators of $\Phi(A,\sigma)$. As $x^2=\alpha$ and $y'^2=\beta+\alpha^{-1}a^2$, using (4.8) we obtain $\mathfrak{Pf}(A,\sigma)\simeq\langle\langle\alpha,\beta+\alpha^{-1}a^2\rangle\rangle$. Thus, $\mathfrak{Pf}(A,\sigma)\simeq\langle\langle\alpha,\beta\rangle\rangle$ by (4.9). □

Using (4.10) and (3.13 (4)), we obtain the following relation between the Pfister invariant and the quadratic form q_σ^+:

Proposition 4.11. *Let (A,σ) and (A',σ') be two decomposable biquaternion algebras with orthogonal involution over F. Then $q_\sigma^+\simeq q_{\sigma'}^+$ if and only if $\mathfrak{Pf}(A,\sigma)\simeq\mathfrak{Pf}(A',\sigma')$.*

The following result is analogous to (3.17):

Theorem 4.12. *Let (A,σ) and (A',σ') be two decomposable biquaternion algebras with involution of orthogonal type over F. The following statements are equivalent:*

(1) $(A,\sigma)\simeq(A',\sigma')$.

(2) $q_\sigma\simeq q_{\sigma'}$ *and* $q_\sigma^+\simeq q_{\sigma'}^+$.

(3) $A\simeq A'$ *and* $q_\sigma^+\simeq q_{\sigma'}^+$.

(4) $A\simeq A'$ *and* $\mathfrak{Pf}(A,\sigma)\simeq\mathfrak{Pf}(A',\sigma')$.

Proof. The implication (1) ⇒ (2) follows from (3.14).

(2) ⇒ (3): Since q_σ and $q_{\sigma'}$ are Albert forms of (A,σ) and (A',σ') respectively, $q_\sigma\simeq q_{\sigma'}$ implies that $A\simeq A'$ by [9, (16.3)].

The implications (3) ⇒ (4) and (4) ⇒ (1) follow from (4.11) and [15, (6.5)] respectively. □

Lemma 4.13. *If $\langle\langle\alpha,\beta\rangle\rangle$ be an anisotropic bilinear Pfister form over F, then $\langle\langle\alpha,\beta\rangle\rangle\not\simeq\langle\langle\alpha+1,\beta\rangle\rangle$.*

Proof. As proved in [1, p. 16], two bilinear Pfister forms are isometric if and only if their pure subforms are isometric. Thus, it is enough to show that the pure subform of $\langle\langle\alpha,\beta\rangle\rangle$ does not represents $\alpha+1$. If $\alpha+1\in D_F(\langle\alpha,\beta,\alpha\beta\rangle)$, then there exists $a,b,c\in F$ such that $a^2\alpha+b^2\beta+c^2\alpha\beta=\alpha+1$. Thus, $1+(a+1)^2\alpha+b^2\beta+c^2\alpha\beta=0$, i.e., $\langle\langle\alpha,\beta\rangle\rangle$ is isotropic which contradicts the assumption. □

Definition 4.14. For $\alpha\in F^\times$, define an involution $T_\alpha: M_2(F)\to M_2(F)$ via

$$T_\alpha\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} a & c\alpha^{-1}\\ b\alpha & d\end{pmatrix}.$$

Note that T_α is an involution of orthogonal type on $M_2(F)$ and $\operatorname{disc} T_\alpha=\alpha F^{\times 2}\in F^\times/F^{\times 2}$.

The following example shows that if $\operatorname{char} F=2$, the conditions $A\simeq_F A'$ and $Q^+\simeq_F Q'^+$ don't necessarily imply that $(A,\sigma)\simeq(A',\sigma')$ (compare (3.17)):

Example 4.15. Let $\langle\langle\alpha,\beta\rangle\rangle$ be an anisotropic Pfister form over a field F of characteristic 2 and let $A=M_4(F)$. Consider the involutions $\sigma=T_\alpha\otimes T_\beta$ and $\sigma'=T_{\alpha+1}\otimes T_\beta$ on A. Then $\mathfrak{Pf}(A,\sigma)\simeq\langle\langle\alpha,\beta\rangle\rangle$ and $\mathfrak{Pf}(A,\sigma')\simeq\langle\langle\alpha+1,\beta\rangle\rangle$, hence $\mathfrak{Pf}(A,\sigma)\not\simeq\mathfrak{Pf}(A,\sigma')$ by (4.13). Using (4.12), we obtain $(A,\sigma)\not\simeq(A,\sigma')$.

On the other hand by (4.8) there exists a set of alternating generators $\{u,v\}$ (resp. $\{u',v'\}$) of $\Phi(A,\sigma)$ (resp. $\Phi(A,\sigma')$) such that $u^2=\alpha$ and $v^2=\beta$ (resp. $u'^2=\alpha+1$ and $v'^2=\beta$). Then $\Phi(A,\sigma)\simeq F[u,v]$ and $\Phi(A,\sigma')\simeq F[u',v']$. The linear map $f:F[u,v]\to F[u',v']$ induced by $f(1)=1$, $f(u)=u'+1$, $f(v)=v'$ and $f(uv)=(u'+1)v'$ is an F-algebra isomorphism. Thus, $\Phi(A,\sigma)\simeq\Phi(A,\sigma')$, which implies that $Q(A,\sigma)^+\simeq Q(A,\sigma')^+$ by (4.4).

We conclude with some results on metabolic involutions. We recall that if $\operatorname{char} F\neq 2$, then an involution on a central simple F-algebra is metabolic if and only if it is hyperbolic (see [5, (4.10)]).

Proposition 4.16. *Let (A,σ) be a decomposable biquaternion algebra with orthogonal involution over a field F of arbitrary characteristic. The following statements are equivalent:*

(1) *(A,σ) is metabolic.*

(2) *Q^+ or Q^- is not a division ring.*

(3) *$1\in D_F(q_\sigma^+)$ or $-1\in D_F(q_\sigma^-)$.*

(4) *q_σ^+ or q_σ^- is isotropic.*

Proof. If $\operatorname{char} F\neq 2$, by (3.11 (1)) we have $(A,\sigma)\simeq(Q^+,\sigma|_{Q^+})\otimes(Q^-,\sigma|_{Q^-})$, where $\sigma|_{Q^+}$ and $\sigma|_{Q^-}$ are the canonical involutions of Q^+ and Q^- respectively. Thus, the equivalence (1) ⇔ (2) follows from [7, (3.1)]. The equivalences (2) ⇔ (3) and (2) ⇔ (4) both follow from (3.13 (3)) and [14, Ch. III, (2.7)].

Now, suppose that $\operatorname{char} F=2$. Then the equivalence (1) ⇔ (2) follows from [15, (6.6)].

(1) ⇒ (3): Let e be a metabolic idempotent with respect to σ and let $x=e+\sigma(e)$. By (4.17), we have $x^2=1$. Since $x\in\operatorname{Alt}(A,\sigma)$, (3.9) implies that $x\in\operatorname{Alt}(A,\sigma)^+$, hence $q_\sigma^+(x)=1$.

(3) $\Rightarrow$ (4): Suppose that $q_\sigma^+(u) = 1$ for some $u \in \mathrm{Alt}(A,\sigma)^+$. By (3.13 (1)) and (3.13 (2)), $\{u\}$ extends to an orthogonal basis $\{u, v, w\}$ of $\mathrm{Alt}(A,\sigma)^+$ with $w = uv$. Since Q^+ is commutative (3.11 (2)), we obtain $q_\sigma^+(v+w) = (v+w)^2 = v^2 + (uv)^2 = 0$, i.e., q_σ^+ is isotropic.

(4) $\Rightarrow$ (2): If q_σ^+ is isotropic, then there exists a nonzero $x \in \mathrm{Alt}(A,\sigma)^+ \subseteq Q^+$ such that $x^2 = 0$. Thus, Q^+ is not a division ring. □

Lemma 4.17. *Let (A,σ) be a central simple algebra with orthogonal involution over F and let $e \in A$ be a metabolic idempotent. Then $(e + \sigma(e))^2 = 1$.*

Proof. As $(1-e)(1-\sigma(e)) = 0$, we obtain $1-\sigma(e)-e+e\sigma(e) = 0$, which implies that $e + \sigma(e) = 1 + e\sigma(e)$. Thus,

$$(e + \sigma(e))^2 = (1 + e\sigma(e))^2 = 1 + e\sigma(e)e\sigma(e) = 1.$$ □

Corollary 4.18. *Let (A,σ) be a central simple algebra with involution over a field F of arbitrary characteristic. If σ is metabolic, then* $\mathrm{disc}\,\sigma$ *is trivial.*

Proof. The result follows from (4.17) and [3, (2.3)]. □

Proposition 4.19. *Let (A,σ) be a biquaternion algebra with involution of orthogonal type over a field F of arbitrary characteristic. Then σ is metabolic if and only if there exists $u \in \mathrm{Alt}(A,\sigma)$ such that $u^2 = 1$.*

Proof. If σ is metabolic, then by (4.18), $\mathrm{disc}\,\sigma$ is trivial. Thus, σ is decomposable and the result follows from (4.16). Conversely, suppose that there exists $u \in \mathrm{Alt}(A,\sigma)$ such that $u^2 = 1$. Then $\mathrm{disc}\,\sigma = \mathrm{Nrd}_A(u)F^{\times 2}$ is trivial, so (A,σ) is decomposable by [11, (3.7)]. Since $u^2 = 1 \in F$ and $u \in \mathrm{Alt}(A,\sigma)$, by (3.9) we have $u \in \mathrm{Alt}(A,\sigma)^+ \cup \mathrm{Alt}(A,\sigma)^-$. Therefore, either $u \in \mathrm{Alt}(A,\sigma)^+$ (i.e., $q_\sigma^+(u) = 1$) or $u \in \mathrm{Alt}(A,\sigma)^-$ (i.e., $q_\sigma^-(u) = -1$). By (4.16), σ is metabolic. □

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