AROUND 16-DIMENSIONAL QUADRATIC FORMS IN $I^3_q$

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Abstract. We determine the indexes of all orthogonal Grassmannians of a generic 16-dimensional quadratic form in $I^3_q$. This is applied to show that the 3-Pfister number of the form is $\geq 4$. Other consequences are: a new and characteristic-free proof of a recent result by Chernousov–Merkurjev on proper subforms in $I^2_q$ (originally available in characteristic 0) as well as a new and characteristic-free proof of an old result by Hoffmann-Tignol and Izhboldin-Karpenko on 14-dimensional quadratic forms in $I^3_q$ (originally available in characteristic $\neq 2$). We also suggest an extension of the method, based on investigation of the topological filtration on the Grothendieck ring of a maximal orthogonal Grassmannian, which applies to quadratic forms of dimension higher than 16.

We work with non-degenerate quadratic forms over arbitrary fields. Recall that a quadratic form, similar to a Pfister form, is called a general Pfister form. We refer to [5] for general facts and terminology related to quadratic forms, especially for definition of a (quadratic) Pfister form in arbitrary characteristic. We write $I^d_q = I^d_q(F)$ for the Witt group of classes of even-dimensional quadratic forms over a field $F$. Recall that $I^d_q(F)$ is a module over the Witt ring $W(F)$ of classes of non-degenerate symmetric bilinear forms. There is a filtration by submodules $I^d_q = I^1_q \supset I^2_q \supset \ldots$ defined as follows: for any $d \geq 1$, $I^d_q := I^{d-1}(F) \cdot I_q(F)$, where $I(F) \subset W(F)$ is the fundamental ideal and $I^{d-1}(F)$ is its power.

Let $\varphi$ be an even-dimensional non-degenerate quadratic form over a field $F$ and let $d \geq 1$ be an integer such that the Witt class $[\varphi] \in I_q(F)$ is in $I^d_q(F)$. Then $[\varphi]$ can be written as a sum of classes of general $d$-fold Pfister forms. The minimal possible number of the summands is denoted $\text{Pf}_d(\varphi)$ and called the $d$-Pfister number of $\varphi$, cf. [14, §9c].

Given a base field $k$ and a positive even integer $m$, we are interested to determine

$$\text{Pf}_d(m) := \sup_{\varphi} \text{Pf}_d(\varphi),$$

where $\varphi$ runs over $m$-dimensional quadratic forms defined over some field $F \supset k$ and satisfying $[\varphi] \in I^d_q(F)$.

Trivially, $\text{Pf}_1(m) = m/2$ for any $m$. Also, it is known (and relatively easy to show) that $\text{Pf}_2(m) = (m - 2)/2$. In the present paper, we concentrate on the 3-Pfister number $\text{Pf}_3(m)$ which is known to be finite. Finiteness of $\text{Pf}_d(m)$ for $d \geq 4$ is an open question.

Classical results from the theory of quadratic forms provide us with values of $\text{Pf}_3(m)$ for $m$ up to 12: $\text{Pf}_3(m)$ is equal to 0 for $m < 8$, to 1 for $m \in \{8, 10\}$, and to 2 for $m = 12$.

Date: 29 November 2015.

Key words and phrases. Quadratic forms over fields; algebraic groups; projective homogeneous varieties; Chow groups. Mathematical Subject Classification (2010): 11E04; 20G15; 14C25.

This work has been accomplished during author’s stay at the Universität Duisburg-Essen; it has been supported by a Discovery Grant from the National Science and Engineering Board of Canada.
For \( k \) with \( \text{char} \ k \neq 2 \), the value \( \text{Pf}_3(14) = 3 \) has been determined independently in [6] and in [7]. More precisely, the lower bound \( \text{Pf}_3(14) \geq 3 \) has been established in the papers. The upper bound \( \text{Pf}_3(14) \leq 3 \) is a consequence of the classification of 14-dimensional quadratic forms in \( I_q^3 \) obtained by M. Rost in [18], see [6, Proposition 2.3].

The main result of the present paper is Theorem 0.1 providing a lower bound for \( \text{Pf}_3(16) \). (It is worthy to mention that no upper bound for \( \text{Pf}_3(16) \) is available.) The lower bound for \( \text{Pf}_3(14) \) mentioned above, can also be obtained (this time in arbitrary characteristic) by the same method, see Corollary 3.2. Unlike with the previous approaches, this new method applies directly to the generic quadratic form of given dimension in \( I_q^3 \) which a priori is expected to have the highest 3-Pfister number.

In view of recent [13, Conjecture 4.5] that the classifying spaces of spinor groups Spin\(_{16}\) and Spin\(_{15}\) are not retract rational (while the classifying space of Spin\(_m\) is retract rational for \( m \leq 14 \) in characteristic \( \neq 2 \), see [13, Theorem 4.4]), 16-dimensional forms in \( I_q^3 \) and any new piece of information on them become particularly intriguing.

**Theorem 0.1.** For any base field \( k \) (of arbitrary characteristic) one has \( \text{Pf}_3(16) \geq 4 \).

The proof is given in the beginning of §2.

**Remark 0.2.** A lower bound on \( \text{Pf}_3(m) \) for arbitrary \( m \) is obtained in [1] via essential dimension of spinor groups (in characteristic \( \neq 2 \)). Although this bound is very impressive for large \( m \), it does not provide any non-trivial information for \( m = 16 \). This situation is not changed even if the bound on ed Spin\(_{16}\), used in [1], is replaced by the precise value of ed Spin\(_{16}\), obtained later in [3] (in characteristic 0), or if the precise value of the essential dimension of the functor of 16-dimensional quadratic forms in \( I_q^3 \), obtained in [3] as well (still in characteristic 0), is inserted into the computations of [1].

Actually, Theorem 0.1 is an immediate consequence of Theorem 2.1 – a stronger result which also has another application: it allows one to recover – now in characteristic-free context – a recent result of [3] on absence of proper even-dimensional subforms of trivial discriminant inside of generic 16-dimensional quadratic forms in \( I_q^3 \). Basically, Theorem 2.1 determines the indexes of all orthogonal Grassmannians of a generic 16-dimensional quadratic form in \( I_q^3 \), or, equivalently, the maximal possible indexes of orthogonal Grassmannians of arbitrary 16-dimensional quadratic form in \( I_q^3 \), see Remark 2.2. Note that the index of the maximal orthogonal Grassmannian of a generic form in \( I_q^3 \) of arbitrary dimension \( m \) is the torsion index of the spinor group Spin\(_m\) and has been computed by B. Totaro in [19]. We show that our method does not apply directly to dimensions higher than 16 but we suggest its enhanced modification which does. This is based on the observation that the topological filtration on the Grothendieck group of the maximal orthogonal Grassmannian (as well as of any flag variety projecting to the maximal orthogonal Grassmannian) of a generic quadratic form in \( I_q^3 \) coincides with the gamma filtration, see Theorem 4.3 and Remark 4.5. In general, gamma filtration provides a computable lower bound for a much less accessible topological filtration. Usually, this lower bound is very far from being sharp, see, e.g., [10], where the computations for projective quadrics are performed. One class of projective homogeneous varieties for which this bound is sharp has been previously found in [12, Theorem 3.7] (these were the Severi-Brauer varieties of, in certain sense, generic central simple algebras).
The starting point here is matching of gamma and topological filtrations in the case of a generic quadratic form in $I_3^3$, or, equivalently, of a generic quadratic form of odd dimension, see Proposition 4.2 (the only condition satisfied by the generic form that matters is maximality of the index of the Clifford invariant). This is achieved due to the fact that the orders of cokernels of the change of field homomorphisms to an algebraic closure for the Grothendieck group and the Chow group of the variety, turn out to be equal to each other, which seems to be a lucky coincidence. The remaining step consisting in killing the Clifford invariant generically (thus getting a quadratic form in $I_q^3$ called generic here) is standard and similar to [12, Theorem 3.7].

1. Chow groups of maximal orthogonal Grassmannians

In this section, $\varphi$ is a non-degenerate quadratic of dimension $2n + 2$ (for some $n \geq 0$) and of trivial discriminant over a field $F$. The maximal orthogonal Grassmannian of $\varphi$ is a smooth projective variety consisting of two isomorphic connected components; let $X$ be one of them. We refer to [5, Chapter XVI] for general information on maximal orthogonal Grassmannians of quadratic forms. Note that $X$ can be identified with the maximal orthogonal Grassmannian of an arbitrary non-degenerate $2n + 1$-dimensional subform of $\varphi$, [5, Propositions 85.2 and 86.17]. We switch to odd-dimensional forms in §4, but we stay with even-dimensional $\varphi$ in the present section.

First we assume that the quadratic form $\varphi$ is hyperbolic. By [5, §86] (see [20] for original proofs), the Chow ring $\text{CH} X$ in this case is generated by the special Schubert classes (i.e., classes of the special Schubert varieties) $e_i \in \text{CH}^i X$ with $i = 1, \ldots, n$ subject to the relations

$$e_i^2 - 2e_{i-1}e_{i+1} + 2e_{i-2}e_{i+2} + \cdots + 2(-1)^{i-1}e_1e_{2i-1} + (-1)^i e_{2i} = 0,$$

where $e_i := 0$ for $i > n$. Let $Y$ be the projective quadric of $\varphi$ and let $f : Z \to X$ be the projective bundle given by the tautological vector bundle on $X$. Note that $Z$ is a closed subvariety of $Y \times X$. Let $g : Z \to Y$ be the first projection. For any $i = 1, \ldots, n$, the element $e_i$ satisfy (and can be defined by) the formula

$$e_i := f_* g^*(l_{n-i}),$$

where $l_{n-i} \in \text{CH}_{n-i} Y$ is the class of an $n - i$-dimensional linear subspace lying on $Y$ (given by an $n - i + 1$-dimensional totally isotropic subspace of $\varphi$). Note that the class $l_{n-i}$ does not depend on the choice of the linear subspace. By [5, Theorem 86.12], the additive group of the Chow ring $\text{CH} X$ is free with a basis given by the products $e_I := \prod_{i \in I} e_i$, where $I$ runs over the subsets of the set $\{1, \ldots, n\}$. The empty product $e_\emptyset$ is the unity $[X]$ of the ring, while $e_{\{1, \ldots, n\}}$ is the class of a rational point.

For arbitrary (i.e., not necessarily hyperbolic) $\varphi$, we fix a field extension $\bar{F}/F$ with hyperbolic $\varphi_{\bar{F}}$ and write $\bar{X}$ for $X_{\bar{F}}$. Let $\overline{\text{CH}} X$ be the image of the change of field homomorphism $\text{CH} X \to \text{CH} \bar{X}$. This is clearly a subring in $\text{CH} \bar{X}$. Moreover, $2e_i \in \overline{\text{CH}} X$ for all $i$ by [5, Proposition 86.13 and Remark 86.14]. Consequently, $2^{|I|} e_I \in \overline{\text{CH}} X$ for any $I$, where $|I|$ is the cardinality.

The even Clifford algebra $C_0(\varphi)$ is a direct product of two copies of a central simple $F$-algebra of degree $2^n$ which we, following [5, Remark 13.9], denote by $C^+(\varphi)$. We have
$C^+(\varphi) \simeq C_0(\varphi')$ for any non-degenerate subform $\varphi' \subset \varphi$ of dimension $2n + 1$. Besides, the Clifford algebra $C(\varphi)$ is isomorphic to the algebra of $2 \times 2$-matrices over $C^+(\varphi)$.

**Proposition 1.3.** If $C^+(\varphi)$ is a division algebra, then the subgroup $\overline{CH}_X \subset CH\bar{X}$ is generated by the elements $2|I|e_I$, $I \subset \{1, \ldots, n\}$. In particular, the subring $\overline{CH}_X \subset CH\bar{X}$ is generated by the elements $2e_i$, $i = 1, \ldots, n$.

**Proof.** Let us take an arbitrary element $x \in \overline{CH}_X$ and write it as a linear combination of the basic elements: $x = \sum_I a_I e_I$ with some $a_I \in \mathbb{Z}$. We want to show that for any $I$, the coefficient $a_I$ is divisible by $2^{|I|}$.

Assume that this is not the case. Multiplying $x$ by an appropriate power of 2, we come to the case with $\min_I (v_2(a_I) - |I|) = -1$, where $v_2$ is the 2-adic valuation. Let us choose a set $I$ with $v_2(a_I) - |I| = -1$ and minimal $|I|$. Let $J$ be the compliment of $I$. The product $y := x \cdot 2^{|I|} e_I$ is clearly in $\overline{CH}_X$.

Let us consider the degree homomorphism $deg : CH\bar{X} \rightarrow \mathbb{Z}$. By [5, Corollary 86.10], $deg(e_I \cdot e_J) = 1$. And $deg(e_{I'}, e_J) = 0 \pmod{2}$ for $I' \neq I$ by [5, Lemma 87.6 with Propositions 85.2 and 86.17]. It follows that $v_2(deg(y)) = n - 1$. Indeed,

$$v_2(deg(a_I e_I 2^{|J|} e_J)) = v_2(a_I) + |J| = |I| - 1 + |J| = n - 1.$$

At the same time, for any $I' \neq I$, we have $v_2(deg(a_{I'} e_J 2^{|J|} e_I)) \geq n$ because $v_2(a_{I'}) \geq |I'| - 1 \geq |I| - 1$ and $deg(e_{I'} \cdot e_J) = 0 \pmod{2}$.

On the other hand, since the residue field of any point on $X$ splits the algebra $C^+(\varphi)$, the degree of any element in $\overline{CH}_X$ is divisible by $\text{ind} C^+(\varphi) = 2^n$. In particular, $v_2(deg(y))$ cannot be $n - 1$. □

**Corollary 1.4.** For $\varphi$ and $X$ as in Proposition 1.3, the index $[CH\bar{X} : \overline{CH}_X]$ of the subgroup $\overline{CH}_X \subset CH\bar{X}$ is $[CH\bar{X} : \overline{CH}_X] = 2n^{2n-1}$.

Let now $S$ be the Severi-Brauer variety of the central simple $F$-algebra $C^+(\varphi)$. From now on, we write $\bar{X}$ for $X_{F(S)}$ and $\overline{CH}_X$ is the image of $CH_X$ in $\overline{CH}_X$. We also consider an intermediate ring $\overline{CH}_X \subset \overline{CH}_X \subset CH\bar{X}$ defined as the image of $CH_X$ in $CH\bar{X}$. The variety $X_{F(S)}$ is a component of the maximal orthogonal Grassmannian of the quadratic form $\varphi_{F(S)}$ whose Clifford invariant is trivial so that $[\varphi_{F(S)}] \in F^2_0(F(S))$, see [5, §16] for fields of characteristic 2 and [5, Chapter VIII] for field of characteristic $\neq 2$.

**Proposition 1.5.** The $\overline{CH}_X$-algebra $\overline{CH}_X F_{F(S)}$ is generated by $e_1$.

**Proof.** By [7, Proposition 4.3], the projection $X \times S \rightarrow X$ is a projective bundle. Therefore, by the Projective Bundle Theorem for Chow groups (see, e.g., [5, Theorem 57.14]), the Chow algebra $CH(X \times S)$ is generated by an element of $CH^1(X \times S)$. The epimorphism of Chow-algebras $CH(X \times S) \rightarrow CH X_{F(S)}$ given by pull-back with respect to the morphism of schemes $X_{F(S)} \rightarrow X \times S$ induced by the generic point of $S$ (for surjectivity of the pull-back see [5, Proposition 57.10]), shows that the Chow-algebra $CH X_{F(S)}$ is generated by an element of $CH^1 X_{F(S)}$. In particular, the $\overline{CH}_X$-algebra $\overline{CH}_X F_{F(S)}$ is generated by an element of $\overline{CH}^1 X_{F(S)}$.

The group $\overline{CH}^1 X$ is generated by $e_1$. Since $e_1 \in \overline{CH}_X F_{F(S)}$ by [5, Exercise 88.14(1)], the group $\overline{CH}^1 X_{F(S)}$ is also generated by $e_1$ and we are done.
Here is a solution of the part of [5, Exercise 88.14(1)] that has been used. Since the even Clifford algebra of $\varphi_{F(S)}$ is split, for any field extension $L/F(S)$ the Witt index of $\varphi_L$ differs from $n - 1$. It follows by [5, Proposition 88.8] that an odd multiple of $e_1$ is in the group $\overline{\text{CH}} X_{F(S)}$. Since $2e_1$ is there as well, we get that $e_1$ is there. □

**Corollary 1.6.** If $C^+(\varphi)$ is a division algebra, then the ring $\overline{\text{CH}} X_{F(S)}$ is generated by the elements $e_1$ and $2e_i$, $i = 2, \ldots, n$. □

The quadratic form $\varphi_{F(S)}$ as in Corollary 1.6 will be called a generic $2n + 2$-dimensional quadratic form in $I_q^3$. This is a slight abuse of terminology convenient for our purposes. For any given field $k$, such a form can be constructed over an appropriate field extension of $k$. An example is given by a generic (in a proper sense) form in $I_q^3$ which is obtained out of a generic form $\varphi$ in $I_q^2$ living over a purely transcendental extension $F/k$ of sufficiently large transcendence degree, by passing to the function field of the Severi-Brauer variety of the division algebra $C^+(\varphi)$.

### 2. Dimension 16

**Proof of Theorem 0.1.** Let $\varphi$ be a generic 16-dimensional quadratic form in $I_q^3$ and let $F$ be its field of definition. Assume that $[\varphi] = [\pi_1] + [\pi_2] + [\pi_3]$ for some general $3$-Pfister forms $\pi_1, \pi_2, \pi_3$ over $F$. Since non-zero Witt class of any general Pfister form vanishes over a quadratic field extension, there exists a finite field extension $L/F$ of degree dividing 4 such that $[\pi_2]_L = 0 = [\pi_3]_L$. It follows that $[\varphi_L] = [\pi_1]_L$ so that the Witt index $i_W(\varphi_L)$ of $\varphi_L$ is at least $4 = (\dim \varphi - \dim \pi)/2$. This contradicts Theorem 2.1 below. □

**Theorem 2.1.** Let $\varphi/F$ be a generic 16-dimensional quadratic form in $I_q^3$. Then the degree $[L:F]$ of any finite field extension $L/F$ with Witt index $i_W(\varphi_L) \geq 3$ is divisible by 8.

**Remark 2.2.** Theorem 2.1 completes computation of maximal possible indexes of the orthogonal Grassmannians of a 16-dimensional quadratic form in $I_q^3$. (The answer to the similar question in dimensions $\leq 14$ is known; for dimension 14 Theorem 3.1 below can be used.) By index $i(X)$ of a variety $X$ we mean the greatest common divisor of the degrees of closed points on $X$.

Namely, writing $i_n$ for the maximal index of the $n$-th Grassmannian (the numbering is such that the 1-st Grassmannian is the quadric), Proposition 2.1 says that $8 \mid i_3$. This implies that $i_3 = 8$. For the remaining Grassmannians one has: $i_1 = 2$ (trivial), $i_2 = 4$ (a consequence of [3, Theorem 4.2] as well as of [19, Theorem 0.1]), $i_3 | i_4 | i_5 = i_6 = i_7 = i_8$, and $i_8 = 8$ (again by [19, Theorem 0.1]).

**Proof of Theorem 2.1.** Let $X$ be a component of the maximal orthogonal Grassmannian of $\varphi$ and let $Y$ be the projective quadric of $\varphi$. We write $\bar{X}$ for $X_F$ and $\bar{Y}$ for $Y_F$, where $\bar{F}$ is an algebraic closure of $F$.

By Corollary 1.6, the image $\overline{\text{CH}} X$ of $\text{CH} X$ in $\overline{\text{CH}} \bar{X}$ coincides with the subring generated by $e_1, 2e_2, \ldots, 2e_7$.

Assume that there exists a finite field extension $L/F$ with $i_W(\varphi_L) \geq 3$ and with $[L:F]$ not divisible by 8. Let us fix an $F$-imbedding $L \hookrightarrow \bar{F}$. Then the elements $l_2, l_1, l_0 \in \text{CH} \bar{Y}$ are in the image of $\text{CH} Y_L$ and therefore, as follows from (1.2), $e_5, e_6, e_7$ belong to the ring...
Let \( R \) be the ring defined by generators and relations: the generators are \( e_1, \ldots, e_7 \) and relations are as in (1.1) with \( n = 7 \). Then the subring of \( R \) generated by \( e_1, 2e_2, \ldots, 2e_7 \) does not contain \( 4e_3e_6e_7 \).

Proof. Since relations (1.1) are homogeneous (with \( \deg e_i := i \)), the ring \( R \) is graded. By the shape of (1.1), the \( 2^7 \) products of distinct generators additively generate \( R \). Moreover, as we known, these products form a \( Z \)-basis of \( R \), cf. [5, Theorem 86.12 and Proposition 86.16].

It follows that the product \((e_2e_4) \cdot 4(e_3e_6e_7)\) is not divisible by \( 8 \) in \( R \). To show that \( 4e_3e_6e_7 \) is not a polynomial in \( e_1, 2e_2, \ldots, 2e_7 \), it suffices to show that every monomial \( M \) in \( e_1, 2e_2, \ldots, 2e_7 \) of degree \( 5 + 6 + 7 = 18 \), multiplied by \( e_2e_4 \), is divisible by \( 8 \) in \( R \).

If \( M \) contains at least \( 3 \) of \( 2e_2, \ldots, 2e_7 \) as factors, then already \( M \) itself is divisible by \( 8 \).

Note that \( e_2 = e_1^2 \) and \( e_4 = 2e_1e_3 - e_1^4 \) so that

\[
e_2e_4 = 2e_1^3e_3 - e_1^6.
\]

If \( M \) contains precisely \( 2 \) of \( 2e_2, \ldots, 2e_7 \) as factors, the remaining factors being copies of \( e_1 \), it contains \( 4e_1^4 \). Multiplying by \( e_2e_4 \) and taking into account formula (2.4) for \( e_2e_4 \), we get a sum of two monomials, one of which contains \( 8 \) as a factor and the other contains \( 4e_1^{10} \) as a factor.

However, relations (1.1) modulo \( 2 \) show that

\[
e_1^2 = e_2, \quad e_1^4 = e_2^2 = e_4, \quad \text{and} \quad e_1^8 = e_4^2 = e_8 = 0.
\]

In particular, \( e_1^8 \) is divisible by \( 2 \) in \( R \). Therefore \( 4e_1^{10} \) is divisible by \( 8 \).

Next we consider the case where \( M \) contains precisely \( 1 \) of \( 2e_2, \ldots, 2e_7 \). Then \( M \) contains \( 2e_1^{11} \). Multiplying \( M \) by \( e_2e_4 \) and taking into account formula (2.4), we get a sum of two monomials, one of which contains \( 4e_1^{14} \) as a factor and the other contains \( 2e_1^{17} \) as a factor. Since \( e_1^{14} \) is divisible by \( 2 \), the first monomial is divisible by \( 8 \). Since \( e_1^{17} = e_1^8 \cdot e_1^9 \cdot e_1 \) is divisible by \( 4 \), the second monomial is also divisible by \( 8 \).

Finally, when \( M \) is a power of \( e_1 \), then \( M \) is a multiple of \( e_1^{18} \). Multiplying \( M \) by \( e_2e_4 \), we also get divisibility by \( 8 \).

Using Theorem 2.1, we can recover

**Corollary 2.5** ([3]). Generic 16-dimensional quadratic form \( \varphi/F \) in \( I_9^3 \) does not contain \( (2- \) or \( 4- \) or \( 6- \) dimensional subforms of trivial discriminant. Moreover, the same property holds for \( \varphi_E \) with any finite field extension \( E/F \) of odd degree.

**Proof.** Assume that \( \varphi_E \) does contain a 6-dimensional subform \( \varphi' \) of trivial discriminant. The form \( \varphi'_L \) is hyperbolic for some finite extension \( L/E \) of degree dividing \( 4 \). We therefore have \( i_W(\varphi_L) \geq 3 \), contradicting Proposition 2.1.
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If \( \varphi_E \) contains 4- (or 2)-dimensional subform of trivial discriminant, we can construct a finite field extension \( L/E \) of degree dividing 4 with hyperbolic \( \varphi_L \). This contradicts not only Proposition 2.1, but also an earlier [19, Theorem 0.1].

Let us recall an open question (cf. [3, Theorem 4.2]): does any 16-dimensional quadratic form of trivial discriminant and Clifford invariant contain an 8-dimensional subform of trivial discriminant? The positive answer would provide an upper bound on \( \text{Pf}_3(16) \) (at least in characteristic \( \neq 2 \)):

**Proposition 2.6.** Let \( \varphi \) be a 16-dimensional quadratic form with trivial discriminant and Clifford invariant over a field \( F \) of characteristic \( \neq 2 \). Assume that \( \varphi \) contains a proper even-dimensional subform \( \varphi_1 \) of trivial discriminant. Then the Witt class of \( \varphi \) is a sum of classes of six general 3-Pfister forms.

**Proof.** We have \( \varphi = \varphi_1 \perp \varphi_2 \) for some \( \varphi_2 \). We may assume that \( \dim \varphi_1 \leq \dim \varphi_2 \) so that \( \dim \varphi_1 \leq 8 \). Choose \( a \in F^\times \) such that the form \( \varphi' := a \varphi_1 \perp \varphi_2 \) is isotropic. We have \( [\varphi] = [\varphi'] + [\langle a \rangle] \cdot [\varphi_1] \in I_q(F) \), where \( \langle a \rangle \) is the diagonal form \( (1, -a) \). As per [6, Proposition 2.3], \( [\varphi'] \) is a sum of classes of three general 3-Pfister forms. Since \( \dim \varphi_1 \leq 8 \), \( [\varphi_1] \) is a sum of classes of three general 2-Pfister forms, [5, Lemma 38.1].

Unfortunately, the way of proving Corollary 2.5 does not work for 8-dimensional subforms in \( \varphi \) of trivial discriminant. Absence of such subforms would follow from absence of finite field extensions \( L/F \) of degree dividing 8 with \( i_W(\varphi_L) \geq 4 \). However such a field extension does exist. More than that, \( \varphi \) becomes hyperbolic over certain field extension of degree 8.

### 3. Dimensions 14 and 18

As already mentioned, the following result has been obtained (in characteristic \( \neq 2 \)) independently in [6] and in [7]. The proof presented here is different; it is parallel to the proof of Theorem 2.1.

**Theorem 3.1.** Let \( \varphi \) be a generic 14-dimensional quadratic form in \( I_q^3 \) over a field \( F \) (of arbitrary characteristic). The degree of any finite field extension \( L/F \) with \( i_W(\varphi_L) \geq 2 \) is divisible by 4. Equivalently, for any finite extension \( E/F \) of odd degree, the quadratic form \( \varphi_L \) does not contain a 4-dimensional subform of trivial discriminant.

**Proof.** Assuming the contrary, we get that \( 2e_5e_6 \in \text{CH}X \), where \( X \) is a component of the maximal orthogonal Grassmannian of \( \varphi \). On the other hand, \( \text{CH}X \), being the subring of \( \text{CH}X \) generated by \( e_i \) and \( 2e_i \) with \( i = 2, \ldots, 6 \), does not contain \( 2e_5e_6 \).

Note that any 14-dimensional quadratic form in \( I_q^3 \) over a field of characteristic \( \neq 2 \) does contain a 6-dimensional subform of trivial discriminant as a consequence of M. Rost’s classification result.

**Corollary 3.2.** For any base field \( k \) (of arbitrary characteristic), one has \( \text{Pf}_3(14) \geq 3 \).

**Proof.** Let \( \varphi/F \supset k \) be a generic 14-dimensional quadratic form in \( I_q^3 \). If the Witt class \( [\varphi] \) is the sum of two classes of general Pfister forms, then \( i_W(\varphi_L) \geq 3 \) for some finite field extension \( L/F \) of degree dividing 2. This contradicts Theorem 3.1.
Now we look at dimension 18 and higher. It has been shown in [3, Theorem 4.2] that for any field \( k \) of characteristic 0 and any even \( m \geq 18 \), there exists a field \( F \supset k \) and an \( m \)-dimensional quadratic form \( \varphi \) over \( F \) with Witt class in \( I_q^3(F) \) such that for any finite field extension \( E/F \) of odd degree, \( \varphi_E \) does not contain any proper even-dimensional subform of trivial discriminant. We suggest the following conjecture implying this result:

**Conjecture 3.3.** For any even \( m \geq 18 \), the index of the \([m/4]\)-th orthogonal Grassmannian of any generic \( m \)-dimensional quadratic form \( \varphi/F \) in \( I_q^3 \) is equal to \( 2^{[m/4]} \).

Conjecture 3.3 actually implies that any generic \( m \)-dimensional quadratic form \( \varphi/F \) in \( I_q^3 \), for any even \( m \geq 18 \), has the property of [3, Theorem 4.2]: if this is not the case, i.e., if \( \varphi_E \) does contain a proper even-dimensional subform \( \psi \) of trivial discriminant for some finite field extension \( E/F \) of odd degree, then, possibly replacing \( \psi \) by its complement, we have \( d := (\dim \psi)/2 \leq [m/4] \) and \( \psi \) becomes hyperbolic over an extension of degree dividing \( 2^{d-1} \). Therefore \( \varphi_E \) acquires Witt index \( \geq [m/4] \) over an extension of \( L \) of degree dividing \( 2^{[m/4]-1} \), i.e., the index of the \([m/4]\)-th orthogonal Grassmannian of \( \varphi \) divides \( 2^{[m/4]-1} \).

Unfortunately, the method of proving Theorems 2.1 and 3.1 does not work for Conjecture 3.3. For instance, for \( m = 18 \) one needs to show that \( 2^3e_5e_6e_7e_8 \notin \overline{CH} X \) which is false – see next paragraph. This provides a motivation to develop a modification of the method; we do it in the next section.

Let us show that \( 2^3e_5e_6e_7e_8 \in \overline{CH} X \), where \( X \) is a component of the maximal orthogonal Grassmannian of a generic 18-dimensional quadratic form in \( I_q^3 \). We have \( e_2 = e_1^2 \in \overline{CH} X \), \( e_4 = e_1(2e_3) = e_3^2 \in \overline{CH} X \), so that \( e_8 = 2e_3e_5 = e_1(2e_7) = e_2(2e_6) = e_6^3 \in \overline{CH} X \). Multiplying by \( (2e_3)(2e_6)(2e_7) \in \overline{CH} X \), we get that \( 2^3e_5e_6e_7e_8 = 2^4e_3e_5^2e_6e_7 \in \overline{CH} X \). The second summand is in \( \overline{CH} X \) as well because \( e_5^3 = 2e_4e_6 - 2e_3e_7 + 2e_2e_8 \). Therefore the first summand \( 2^3e_5e_6e_7e_8 \) is in \( \overline{CH} X \).

### 4. Grothendieck ring of maximal orthogonal Grassmannian

For an integer \( n \geq 1 \), let \( \varphi : V \to F \) be a non-degenerate quadratic form of dimension \( 2n+1 \) over a field \( F \) (of arbitrary characteristic). The vector space \( V \) of definition of \( \varphi \) is a vector space over \( F \) of dimension \( 2n+1 \). Let \( X \) be the maximal orthogonal Grassmannian of \( \varphi \), i.e., \( X \) is the \( F \)-variety of \( n \)-dimensional totally isotropic subspaces in \( V \).

We are going to collect information on the Grothendieck group \( K(X) \). From Panin’s computation of \( K \)-theory of projective homogeneous varieties [16], we deduce

**Lemma 4.1.** There is a natural (with respect to field extensions of \( F \)) group isomorphism of \( K(X) \) onto the direct sum of \( 2^{n-1} \) copies of \( K(F) = \mathbb{Z} \) and \( 2^{n-1} \) copies of \( K(C_0(\varphi)) = \text{ind } C_0(\varphi) \cdot \mathbb{Z} \), where \( C_0(\varphi) \) is the even Clifford algebra of \( \varphi \).

**Proof.** Consider the category of \( K \)-correspondences as in [15, §1.8] (where it is called *motivic category*). Objects of this category are pairs \((Y, A)\), where \( Y \) is a smooth projective \( F \)-variety and \( A \) is a separable \( F \)-algebra. To simplify notation, one writes \( Y \) for the object \((Y, F)\) and \( A \) for \((\text{Spec } F, A)\).

As proved in [16], the object in the above category given by the maximal orthogonal Grassmannian \( X \) (as well as by any other projective homogeneous variety under the
orthogonal group $O^+(\varphi)$) is isomorphic to a separable algebra $A$ which is a finite direct product of $a$ copies of $F$ and of $b$ copies of $C_0(\varphi)$ for some integers $a, b \geq 0$. As a consequence, we get a natural group isomorphism $K(X) \simeq K(F)^{\oplus a} \oplus K(C_0(\varphi))^{\oplus b}$. The rank of the group $K(X)$ coincides with the rank of $\text{CH}(X)$ which, as we know, is equal to $2^n$. Therefore $a + b = 2^n$ and in order to finish the proof of Lemma 4.1 we only need to show that $a = b$ provided that the central simple algebra $C_0(\varphi)$ is not split (i.e., has index $> 1$).

Let $S$ be the Severi-Brauer variety of $C_0(\varphi)$. The projection $X \times S \to X$ is a projective bundle of rank $2^n - 1$. By Projective Bundle Theorem for $K$-theory [17, Proposition 4.3], there is a natural isomorphism $K(X \times S) \simeq K(X)^{\oplus 2^n}$. Therefore, $K(X)$ is naturally isomorphic to the direct sum of $2^n$ copies of $K(F)$ and $2^n$ copies of $K(C_0(\varphi))$.

On the other hand, $S$ as an object in the category of $K$-correspondences is isomorphic to the product of $2^{n-1}$ copies of $F$ and $2^{n-1}$ copies of $C_0(\varphi)$. (Since the algebra $C_0(\varphi)$ is of exponent 2, its higher tensor powers do not show up.) It follows that the object given by the direct product of varieties $X \times S$, which is the tensor product in the category of $K$-correspondences of the objects given by $X$ and by $S$, is isomorphic to the product of $2^{n-1}(a + b)$ copies of $F$ and $2^{n-1}(a + b)$ copies of $C_0(\varphi)$. This gives a natural isomorphism of $K(X \times S)$ with the direct sum of $2^{n-1}(a + b)$ copies of $K(F)$ and $2^{n-1}(a + b)$ copies of $K(C_0(\varphi))$. Comparing the resulting computation for the order of cokernel of the change of field homomorphism for $K(X \times S)$ to an algebraic closure of $F$, we get that

$$\text{ind} C_0(\varphi) 2^{n-1}(a + b).$$

Since $\text{ind} C_0(\varphi) \neq 1$, it follows that $2^n b = 2^{n-1}(a + b)$ giving $a = b$. \hfill \Box

**Proposition 4.2.** Assume that the index of the even Clifford algebra $C_0(\varphi)$ is maximal: $\text{ind} C_0(\varphi) = 2^n$. Then the topological filtration on $K(X)$ coincides with the gamma filtration.

**Proof.** Let us consider the filtration of the ring $K(X)$ generated by the $K$-theoretical Chern classes $c_i(\mathcal{T}), i \geq 0$ of the tautological vector bundle $\mathcal{T}$ on $X$. By definition, the $i$-th term $\mathcal{F}^i$ of this filtration is additively generated by the products $c_{i_1}(\mathcal{T}) \cdots c_{i_r}(\mathcal{T})$ with $r \geq 1$ and $i_1 + \cdots + i_r \geq i$.

Clearly, for any $i$, we have a chain of inclusions consisting of 4 terms: $\mathcal{F}^i$ is inside of the $i$-th term of the gamma filtration, which is inside of the $i$-term of the topological filtration on $K(X)$, which is inside of the intersection with $K(X)$ of the $i$-th term of the topological filtration on $K(X)$. We are going to show that the smallest term in this chain coincides with the largest term meaning that all four terms are the same. This will prove Proposition 4.2.

Let $\bar{F}$ be an algebraic closure of $F$ and $\bar{X} := X_{\bar{F}}$. By Lemma 4.1, the order of the cokernel of the change of field homomorphism $K(X) \to K(\bar{X})$ is equal to $2^{n2^n-1}$. On the other hand, the cokernel of the homomorphism of the associated graded rings $GK(X) \to GK(\bar{X})$ with respect to the filtration $\mathcal{F}$ on $K(X)$ and the topological filtration on $K(\bar{X})$, is identified with the cokernel of $\text{CH} X \to CH \bar{X}$ whose order also equals $2^{n2^n-1}$, see Corollary 1.4. The formula

$$|\text{Ker}(GK(X) \to GK(\bar{X}))| = \frac{|\text{Coker}(GK(X) \to GK(\bar{X}))|}{|\text{Coker}(K(X) \to K(\bar{X}))|}$$
(proved as [11, Proposition 2]) implies that the homomorphism $G_K(X) \to G_K(\bar{X})$ is injective, i.e., that the filtration $\mathcal{F}$ on $K(X)$ is induced by the topological filtration on $K(\bar{X})$. □

**Theorem 4.3.** For $\varphi$ as in Proposition 4.2, let $S$ be the Severi-Brauer variety of the division algebra $C_0(\varphi)$. Then the topological filtration on $K(X_{F(S)})$ coincides with the gamma filtration.

**Proof.** We proceed along the lines of [12, Theorem 3.7]. As we already observed in the proof of Proposition 1.5 and Lemma 4.1, the product $X \times S$ is a projective bundle over $X$. This implies that $\text{CH}(X \times S)$ and therefore $\text{CH}_X(S)$ is generated as $\text{CH}_X$-algebra by the first Chern class of certain linear vector bundle (well, this Chern class is equal to $\pm e_1$).

The epimorphism $\text{CH}_X(S) \to \text{GK}_X(S)$, where $\text{GK}_X$ is the associated graded ring of the topological filtration, shows that also $\text{GK}_X(S)$ is generated as $\text{GK}_X$-algebra by the first Chern class of a linear vector bundle. By Proposition 4.2, the ring $G_K(X)$ is generated by Chern classes of vector bundles. It follows that also the ring $G_K(X_{F(S)})$ is generated by Chern classes of vector bundles. This precisely means that the gamma filtration on $K(X_{F(S)})$ coincides with the topological one. □

**Remark 4.4.** The proofs of Theorem 4.3 and Proposition 4.2 actually provide a simpler than a priori description of gamma filtration on $K(X_{F(S)})$: it is generated by $e_1$ and the Chern classes of the tautological vector bundle. More precisely, for any $i \geq 0$, the $i$-th term of the filtration is additively generated by the products $e_1^{i_0}c_{i_1}(T)\ldots c_{i_r}(T)$ with $r \geq 0$ and $i_0 + i_1 + \cdots + i_r \geq i$.

**Remark 4.5.** Let $\bar{X}$ be any flag variety of $\varphi$ projecting to $X$, i.e., any variety of flags of totally isotropic subspaces in $\varphi$, where the flags under consideration include maximal totally isotropic subspaces. In particular, $\bar{X}$ can be the variety of complete flags. Since the projection $\bar{X} \to X$ is a flag bundle of a vector bundle on $X$ (namely, of the tautological vector bundle), Theorem 4.3 implies that the topological filtration on $K(\bar{X}_{F(S)})$ (and on $K(\bar{X})$) coincides with the gamma filtration.

Using identification of the maximal orthogonal Grassmannian of an odd-dimensional quadratic form with a component of the maximal orthogonal Grassmannian of the corresponding even-dimensional form of trivial discriminant, we get the above statement as well for even-dimensional forms $\varphi$ of trivial discriminant with division algebra $C^+(\varphi)$.

Finally, using specialization arguments, one can prove coincidence of gamma and topological filtrations for $K(T/P)$, where $T$ is a generic principle homogeneous space under $\text{Spin}_m$ (with any $m$: $m = 2n + 1$ or $m = 2n + 2$) and $P \subset \text{Spin}_m$ is any special parabolic subgroup.

**Remark 4.6.** Instead of completely killing the division algebra $C_0(\varphi)$ by passing to the function field of its Severi-Brauer variety $S$ in the statement of Theorem 4.3, one may partially split $C_0(\varphi)$ by passing to the function field of any of its generalized Severi-Brauer variety $S'$. The conclusion remains the same: gamma filtration on $K(X_{F(S')})$ coincides with the topological filtration. The proof also remains basically the same, cf. [12, Theorem 3.7 and its proof].
It would be very interesting to completely understand the gamma filtration on \( K(X_{F(S)}) \). Note that for any field extension \( L/F(S) \), the change of field homomorphism \( K(X_{F(S)}) \to K(X_L) \) is an isomorphism preserving the filtration. So, if this helps, it is enough to perform the computation, say, over an algebraically closed field. Actually, it is even enough to do it for a maximal orthogonal Grassmannian \( \overline{X} \) over \( \mathbb{C} \). One may consider the basis of the group \( K(\overline{X}) \) given by the Schubert classes; the ring structure is determined by the \( K \)-theoretical Littlewood-Richardson formulas obtained in [4]. Alternatively, one may describe the ring \( K(\overline{X}) \) by generators and relations in the spirit of the discussed description of \( CH X \), taking for generators the special Schubert classes \( e_i \in K(\overline{X}) \). (Unfortunately, the relations on \( e_i \) in \( K(\overline{X}) \) look more complicated than (1.1).)

Note that these \( K \)-theoretical special Schubert classes still satisfy relations (1.2), where now \( l_{n-i} \) stand for the \( K \)-theoretical classes of linear subspaces on the quadric. This can be shown using [2, Lemma 2.1].

Of particular interest is to understand the position of the special Schubert classes in the filtration. More specifically, let us consider the class \( e_n \in K(X_{F(S)}) \) of the special Schubert variety of the lowest dimension (corresponding to the class of a rational point on the quadric).

**Conjecture 4.7.** For \( n \geq 8 \), the special Schubert class \( e_n \in K(X_{F(S)}) \) does not belong to the term number \( n + 1 - [(n + 1)/2] \) of the gamma filtration.

Conjecture 4.7 implies Conjecture 3.3. Indeed, assume that Conjecture 3.3 fails for some even \( m \geq 18 \). This means that we can find a field \( F \) and a non-degenerate quadratic form \( \varphi \) over \( F \) of dimension \( m \) and trivial discriminant such that \( C^+(\varphi) \) is a division algebra and there exists a finite field extension \( L/F(S) \) of degree not divisible by \( 2^r \) with \( i_W(\varphi_L) \geq r \), where \( r := [m/4] \) and \( S \) is the Severi-Brauer variety of \( C^+(\varphi) \).

Since \( i_W(\varphi_L) \geq r \), the projective quadric \( Y_L \) of \( \varphi_L \) contains a linear subspace of dimension \( r - 1 \). Its class \( l_{r-1} \) in \( K(Y_L) \) belongs to the term number \( \dim Y - (r - 1) = m - r - 1 \) of the topological filtration on \( K(Y_L) \). Applying the norm homomorphism \( K(Y_L) \to K(Y_{F(S)}) \), preserving the filtration and acting as multiplication by \( [L: F(S)] \), we get that the element \( 2^{r-1}l_{r-1} \) of the group \( K(Y_{F(S)}) \) is in the \( m - r - 1 \)-th term of the topological filtration. Since \( 2^{r-1}l_{r-1} = l_0 + h^{m-3} + 2h^{m-4} + \cdots + 2^{r-2}h^{m-r-1} \) (see [9, §3.2]), where \( h \) is the class of a hyperplane section, we get that \( l_0 \) is in the \( m - r - 1 \)-th term as well.

As in (1.2), we have \( e_n = f_*g^*(l_0) \in K(X_{F(S)}) \), where \( X \) is a component of the maximal orthogonal Grassmannian of \( \varphi \). The pull-back homomorphism \( g^*: K(Y_{F(S)}) \to K(Z_{F(S)}) \) preserves the filtration. The push-forward homomorphism \( f_*: K(Z_{F(S)}) \to K(X_{F(S)}) \) lowers the number of the filtration term by \( \dim Z - \dim X = n := (m - 2)/2 \). It follows that \( e_n \) is in the term number \( m - r - 1 - n = n + 1 - [(n + 1)/2] \). Finally, identifying \( X \) with the maximal orthogonal Grassmannian of a non-degenerate \( 2n + 1 \)-dimensional subform of \( \varphi \), we get a counter-example to Conjecture 4.7.

**Remark 4.8.** The number \( n + 1 - [(n + 1)/2] \) in Conjecture 4.7 is optimal for \( n = 8 \). Indeed, if \( \varphi \) is an 18-dimensional quadratic form of trivial discriminant and trivial Clifford invariant and \( Y \) is its projective quadric, since the Chow groups \( CH^i Y \) are torsion-free for \( i \leq 3 \), [8], the class \( l_8 \in K(Y) \) is in the 4-th term of the topological filtration, cf. [9, Theorem 3.10]. Since \( l_0 = l_8 \cdot h^8 \), it follows that \( l_0 \) is in the term number \( 4 + 8 = 12 \).
Therefore $e_8$ is in the term number $4 = 12 - n$. Conjecture 4.7 claims that $e_8$ is not in the term number 5.

References