AROUND 16-DIMENSIONAL QUADRATIC FORMS IN I_q^3

NIKITA A. KARPENKO

Abstract. We determine the indexes of all orthogonal Grassmannians of a generic 16-dimensional quadratic form in I_q^3. This is applied to show that the 3-Pfister number of the form is ≥ 4. Other consequences are: a new and characteristic-free proof of a recent result by Chernousov–Merkurjev on proper subforms in I_q^2 (originally available in characteristic 0) as well as a new and characteristic-free proof of an old result by Hoffmann-Tignol and Izhboldin-Karpenko on 14-dimensional quadratic forms in I_q^3 (originally available in characteristic $\neq 2$). We also suggest an extension of the method, based on investigation of the topological filtration on the Grothendieck ring of a maximal orthogonal Grassmanian, which applies to quadratic forms of dimension higher than 16.

We work with non-degenerate quadratic forms over arbitrary fields. Recall that a quadratic form, similar to a Pfister form, is called a *general Pfister form*. We refer to [5] for general facts and terminology related to quadratic forms, especially for definition of a (quadratic) Pfister form in arbitrary characteristic. We write $I_q = I_q(F)$ for the Witt group of classes of even-dimensional quadratic forms over a field F. Recall that $I_q(F)$ is a module over the Witt ring $W(F)$ of classes of non-degenerate symmetric bilinear forms. There is a filtration by submodules $I_q = I_q^1 \supset I_q^2 \supset \dots$ defined as follows: for any $d \geq 1$, $I_q^d := I^{d-1}(F) \cdot I_q(F)$, where $I(F) \subset W(F)$ is the fundamental ideal and $I^{d-1}(F)$ is its power.

Let φ be an even-dimensional non-degenerate quadratic form over a field F and let $d \geq 1$ be an integer such that the Witt class $[\varphi] \in I_q(F)$ is in $I_q^d(F)$. Then $[\varphi]$ can be written as a sum of classes of general d-fold Pfister forms. The minimal possible number of the summands is denoted $\mathrm{Pf}_d(\varphi)$ and called the d-Pfister number of φ, cf. [14, §9c].

Given a base field k and a positive even integer m, we are interested to determine

$$\mathrm{Pf}_d(m) := \sup_{\varphi} \mathrm{Pf}_d(\varphi),$$

where φ runs over m-dimensional quadratic forms defined over some field $F \supset k$ and satisfying $[\varphi] \in I_q^d(F)$.

Trivially, $\mathrm{Pf}_1(m) = m/2$ for any m. Also, it is known (and relatively easy to show) that $\mathrm{Pf}_2(m) = (m-2)/2$. In the present paper, we concentrate on the 3-Pfister number $\mathrm{Pf}_3(m)$ which is known to be finite. Finiteness of $\mathrm{Pf}_d(m)$ for $d \geq 4$ is an open question.

Classical results from the theory of quadratic forms provide us with values of $\mathrm{Pf}_3(m)$ for m up to 12: $\mathrm{Pf}_3(m)$ is equal to 0 for $m < 8$, to 1 for $m \in \{8, 10\}$, and to 2 for $m = 12$.

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For k with char $k \neq 2$, the value $\mathrm{Pf}_3(14) = 3$ has been determined independently in [6] and in [7]. More precisely, the lower bound $\mathrm{Pf}_3(14) \geq 3$ has been established in the papers. The upper bound $\mathrm{Pf}_3(14) \leq 3$ is a consequence of the classification of 14-dimensional quadratic forms in I_q^3 obtained by M. Rost in [18], see [6, Proposition 2.3].

The main result of the present paper is Theorem 0.1 providing a lower bound for $\mathrm{Pf}_3(16)$. (It is worthy to mention that no upper bound for $\mathrm{Pf}_3(16)$ is available.) The lower bound for $\mathrm{Pf}_3(14)$ mentioned above, can also be obtained (this time in arbitrary characteristic) by the same method, see Corollary 3.2. Unlike with the previous approaches, this new method applies directly to the generic quadratic form of given dimension in I_q^3 which a priori is expected to have the highest 3-Pfister number.

In view of recent [13, Conjecture 4.5] that the classifying spaces of spinor groups Spin_{16} and Spin_{15} are not retract rational (while the classifying space of Spin_m is retract rational for $m \leq 14$ in characteristic $\neq 2$, see [13, Theorem 4.4]), 16-dimensional forms in I_q^3 and any new piece of information on them become particularly intriguing.

Theorem 0.1. *For any base field k (of arbitrary characteristic) one has* $\mathrm{Pf}_3(16) \geq 4$.

The proof is given in the beginning of §2.

Remark 0.2. A lower bound on $\mathrm{Pf}_3(m)$ for arbitrary m is obtained in [1] via essential dimension of spinor groups (in characteristic $\neq 2$). Although this bound is very impressive for large m, it does not provide any non-trivial information for $m = 16$. This situation is not changed even if the bound on $\mathrm{ed}\,\mathrm{Spin}_{16}$, used in [1], is replaced by the precise value of $\mathrm{ed}\,\mathrm{Spin}_{16}$, obtained later in [3] (in characteristic 0), or if the precise value of the essential dimension of the functor of 16-dimensional quadratic forms in I_q^3, obtained in [3] as well (still in characteristic 0), is inserted into the computations of [1].

Actually, Theorem 0.1 is an immediate consequence of Theorem 2.1 – a stronger result which also has another application: it allows one to recover – now in characteristic-free context – a recent result of [3] on absence of proper even-dimensional subforms of trivial discriminant inside of generic 16-dimensional quadratic forms in I_q^3. Basically, Theorem 2.1 determines the indexes of all orthogonal Grassmannians of a generic 16-dimensional quadratic form in I_q^3, or, equivalently, the maximal possible indexes of orthogonal Grassmannians of arbitrary 16-dimensional quadratic form in I_q^3, see Remark 2.2. Note that the index of the maximal orthogonal Grassmannian of a generic form in I_q^3 of arbitrary dimension m is the torsion index of the spinor group Spin_m and has been computed by B. Totaro in [19]. We show that our method does not apply directly to dimensions higher than 16 but we suggest its enhanced modification which does. This is based on the observation that the topological filtration on the Grothendieck group of the maximal orthogonal Grassmannian (as well as of any flag variety projecting to the maximal orthogonal Grassmannian) of a generic quadratic form in I_q^3 coincides with the gamma filtration, see Theorem 4.3 and Remark 4.5. In general, gamma filtration provides a computable lower bound for a much less accessible topological filtration. Usually, this lower bound is very far from being sharp, see, e.g., [10], where the computations for projective quadrics are performed. One class of projective homogeneous varieties for which this bound is sharp has been previously found in [12, Theorem 3.7] (these were the Severi-Brauer varieties of, in certain sense, *generic* central simple algebras).

The starting point here is matching of gamma and topological filtrations in the case of a generic quadratic form in I^2_q, or, equivalently, of a generic quadratic form of odd dimension, see Proposition 4.2 (the only condition satisfied by the generic form that matters is maximality of the index of the Clifford invariant). This is achieved due to the fact that the orders of cokernels of the change of field homomorphisms to an algebraic closure for the Grothendieck group and the Chow group of the variety, turn out to be equal to each other, which seems to be a lucky coincidence. The remaining step consisting in killing the Clifford invariant generically (thus getting a quadratc form in I^3_q called *generic* here) is standard and similar to [12, Theorem 3.7].

1. Chow groups of maximal orthogonal Grassmannians

In this section, φ is a non-degenerate quadratic of dimension $2n+2$ (for some $n \geq 0$) and of trivial discriminant over a field F. The maximal orthogonal Grassmannian of φ is a smooth projective variety consisting of two isomorphic connected components; let X be one of them. We refer to [5, Chapter XVI] for general information on maximal orthogonal Grassmannians of quadratic forms. Note that X can be identified with the maximal orthogonal Grassmannian of an arbitrary non-degenerate $2n+1$-dimensional subform of φ, [5, Propositions 85.2 and 86.17]. We switch to odd-dimensional forms in §4, but we stay with even-dimensional φ in the present section.

First we assume that the quadratic form φ is hyperbolic. By [5, §86] (see [20] for original proofs), the Chow ring $\mathrm{CH}\,X$ in this case is generated by the special Schubert classes (i.e., classes of the special Schubert varieties) $e_i \in \mathrm{CH}^i X$ with $i = 1, \dots, n$ subject to the relations

$$e_i^2 - 2e_{i-1}e_{i+1} + 2e_{i-2}e_{i+2} + \cdots + 2(-1)^{i-1}e_1e_{2i-1} + (-1)^i e_{2i} = 0, \tag{1.1}$$

where $e_i := 0$ for $i > n$. Let Y be the projective quadric of φ and let $f : Z \to X$ be the projective bundle given by the tautological vector bundle on X. Note that Z is a closed subvariety of $Y \times X$. Let $g : Z \to Y$ be the first projection. For any $i = 1, \dots, n$, the element e_i satisfy (and can be defined by) the formula

$$e_i := f_*g^*(l_{n-i}), \tag{1.2}$$

where $l_{n-i} \in \mathrm{CH}_{n-i}\,Y$ is the class of an $n-i$-dimensional linear subspace lying on Y (given by an $n-i+1$-dimensional totally isotropic subspace of φ). Note that the class l_{n-i} does not depend on the choice of the linear subspace. By [5, Theorem 86.12], the additive group of the Chow ring $\mathrm{CH}\,X$ is free with a basis given by the products $e_I := \prod_{i\in I} e_i$, where I runs over the subsets of the set $\{1, \dots, n\}$. The empty product $e_\emptyset$ is the unity $[X]$ of the ring, while $e_{\{1,\dots,n\}}$ is the class of a rational point.

For arbitrary (i.e., not necessarily hyperbolic) φ, we fix a field extension $\bar F/F$ with hyperbolic $\varphi_{\bar F}$ and write $\bar X$ for $X_{\bar F}$. Let $\overline{\mathrm{CH}}\,X$ be the image of the change of field homomorphism $\mathrm{CH}\,X \to \mathrm{CH}\,\bar X$. This is clearly a subring in $\mathrm{CH}\,\bar X$. Moreover, $2e_i \in \overline{\mathrm{CH}}\,X$ for all i by [5, Proposition 86.13 and Remark 86.14]. Consequently, $2^{|I|}e_I \in \overline{\mathrm{CH}}\,X$ for any I, where $|I|$ is the cardinality.

The even Clifford algebra $C_0(\varphi)$ is a direct product of two copies of a central simple F-algebra of degree 2^n which we, following [5, Remark 13.9], denote by $C^+(\varphi)$. We have

$C^+(\varphi) \simeq C_0(\varphi')$ for any non-degenerate subform $\varphi' \subset \varphi$ of dimension $2n+1$. Besides, the Clifford algebra $C(\varphi)$ is isomorphic to the algebra of 2×2-matrices over $C^+(\varphi)$.

Proposition 1.3. *If $C^+(\varphi)$ is a division algebra, then the subgroup $\overline{\mathrm{CH}}\, X \subset \mathrm{CH}\, \bar{X}$ is generated by the elements $2^{|I|}e_I$, $I \subset \{1, \dots, n\}$. In particular, the subring $\overline{\mathrm{CH}}\, X \subset \mathrm{CH}\, \bar{X}$ is generated by the elements $2e_i$, $i = 1, \dots, n$.*

Proof. Let us take an arbitrary element $x \in \overline{\mathrm{CH}}\, X$ and write it as a linear combination of the basic elements: $x = \sum_I a_I e_I$ with some $a_I \in \mathbb{Z}$. We want to show that for any I, the coefficient a_I is divisible by $2^{|I|}$.

Assume that this is not the case. Multiplying x by an appropriate power of 2, we come to the case with $\min_I(v_2(a_I) - |I|) = -1$, where v_2 is the 2-adic valuation. Let us choose a set I with $v_2(a_I) - |I| = -1$ and minimal $|I|$. Let J be the compliment of I. The product $y := x \cdot 2^{|J|}e_J$ is clearly in $\overline{\mathrm{CH}}\, X$.

Let us consider the degree homomorphism $\deg : \mathrm{CH}\, \bar{X} \to \mathbb{Z}$. By [5, Corollary 86.10], $\deg(e_I \cdot e_J) = 1$. And $\deg(e_{I'} \cdot e_J) = 0 \pmod 2$ for $I' \neq I$ by [5, Lemma 87.6 with Propositions 85.2 and 86.17]. It follows that $v_2(\deg(y)) = n - 1$. Indeed,

$$v_2(\deg(a_I e_I 2^{|J|} e_J)) = v_2(a_I) + |J| = |I| - 1 + |J| = n - 1.$$

At the same time, for any $I' \neq I$, we have $v_2(\deg(a_{I'} e_{I'} 2^{|J|} e_J)) \geq n$ because $v_2(a_{I'}) \geq |I'| - 1 \geq |I| - 1$ and $\deg(e_{I'} \cdot e_J) = 0 \pmod 2$.

On the other hand, since the residue field of any point on X splits the algebra $C^+(\varphi)$, the degree of any element in $\overline{\mathrm{CH}}\, X$ is divisible by $\operatorname{ind} C^+(\varphi) = 2^n$. In particular, $v_2(\deg(y))$ cannot be $n - 1$. □

Corollary 1.4. *For φ and X as in Proposition 1.3, the index $[\mathrm{CH}\, \bar{X} : \overline{\mathrm{CH}}\, X]$ of the subgroup $\overline{\mathrm{CH}}\, X$ in $\mathrm{CH}\, \bar{X}$ is $[\mathrm{CH}\, \bar{X} : \overline{\mathrm{CH}}\, X] = 2^{\sum_{i=0}^{n} i\binom{n}{i}} = 2^{n2^{n-1}}$.* □

Let now S be the Severi-Brauer variety of the central simple F-algebra $C^+(\varphi)$. From now on, we write $\bar{X}$ for $X_{\bar{F}(S)}$ and $\overline{\mathrm{CH}}\, X$ is the image of $\mathrm{CH}\, X$ in $\mathrm{CH}\, \bar{X}$. We also consider an intermediate ring $\overline{\mathrm{CH}}\, X \subset \overline{\mathrm{CH}}\, X_{F(S)} \subset \mathrm{CH}\, \bar{X}$ defined as the image of $\mathrm{CH}\, X_{F(S)}$ in $\mathrm{CH}\, \bar{X}$. The variety $X_{F(S)}$ is a component of the maximal orthogonal Grassmannian of the quadratic form $\varphi_{F(S)}$ whose Clifford invariant is trivial so that $[\varphi_{F(S)}] \in I_q^3(F(S))$, see [5, §16] for fields of characteristic 2 and [5, Chapter VIII] for field of characteristic $\neq 2$.

Proposition 1.5. *The $\overline{\mathrm{CH}}\, X$-algebra $\overline{\mathrm{CH}}\, X_{F(S)}$ is generated by e_1.*

Proof. By [7, Proposition 4.3], the projection $X \times S \to X$ is a projective bundle. Therefore, by the Projective Bundle Theorem for Chow groups (see, e.g., [5, Theorem 57.14]), the $\mathrm{CH}\, X$-algebra $\mathrm{CH}(X \times S)$ is generated by an element of $\mathrm{CH}^1(X \times S)$. The epimorphism of $\mathrm{CH}\, X$-algebras $\mathrm{CH}(X \times S) \twoheadrightarrow \mathrm{CH}\, X_{F(S)}$ given by pull-back with respect to the morphism of schemes $X_{F(S)} \to X \times S$ induced by the generic point of S (for surjectivity of the pull-back see [5, Proposition 57.10]), shows that the $\mathrm{CH}\, X$-algebra $\mathrm{CH}\, X_{F(S)}$ is generated by an element of $\mathrm{CH}^1 X_{F(S)}$. In particular, the $\overline{\mathrm{CH}}\, X$-algebra $\overline{\mathrm{CH}}\, X_{F(S)}$ is generated by an element of $\overline{\mathrm{CH}}^1\, X_{F(S)}$.

The group $\mathrm{CH}^1 \bar{X}$ is generated by e_1. Since $e_1 \in \overline{\mathrm{CH}}\, X_{F(S)}$ by [5, Exercise 88.14(1)], the group $\overline{\mathrm{CH}}^1\, X_{F(S)}$ is also generated by e_1 and we are done.

Here is a solution of the part of [5, Exercise 88.14(1)] that has been used. Since the even Clifford algebra of $\varphi_{F(S)}$ is split, for any field extension $L/F(S)$ the Witt index of φ_L differs from $n-1$. It follows by [5, Proposition 88.8] that an odd multiple of e_1 is in the group $\overline{\mathrm{CH}}\, X_{F(S)}$. Since $2e_1$ is there as well, we get that e_1 is there. □

Corollary 1.6. *If $C^+(\varphi)$ is a division algebra, then the ring $\overline{\mathrm{CH}}\, X_{F(S)}$ is generated by the elements e_1 and $2e_i$, $i = 2, \dots, n$.* □

The quadratic form $\varphi_{F(S)}$ as in Corollary 1.6 will be called a *generic* $2n+2$-dimensional quadratic form in I_q^3. This is a slight abuse of terminology convenient for our purposes. For any given field k, such a form can be constructed over an appropriate field extension of k. An example is given by a *generic* (in a proper sense) form in I_q^3 which is obtained out of a generic form φ in I_q^2 living over a purely transcendental extension F/k of sufficiently large transcendence degree, by passing to the function field of the Severi-Brauer variety of the division algebra $C^+(\varphi)$.

2. Dimension 16

Proof of Theorem 0.1. Let φ be a generic 16-dimensional quadratic form in I_q^3 and let F be its field of definition. Assume that $[\varphi] = [\pi_1] + [\pi_2] + [\pi_3]$ for some general 3-Pfister forms π_1, π_2, π_3 over F. Since non-zero Witt class of any general Pfister form vanishes over a quadratic field extension, there exists a finite field extension L/F of degree dividing 4 such that $[\pi_2]_L = 0 = [\pi_3]_L$. It follows that $[\varphi_L] = [\pi_1]_L$ so that the Witt index $i_W(\varphi_L)$ of φ_L is at least $4 = (\dim\varphi - \dim\pi)/2$. This contradicts Theorem 2.1 below. □

Theorem 2.1. *Let φ/F be a generic 16-dimensional quadratic form in I_q^3. Then the degree $[L:F]$ of any finite field extension L/F with Witt index $i_W(\varphi_L) \geq 3$ is divisible by 8.*

Remark 2.2. Theorem 2.1 completes computation of maximal possible *indexes* of the orthogonal Grassmannians of a 16-dimensional quadratic form in I_q^3. (The answer to the similar question in dimensions ≤ 14 is known; for dimension 14 Theorem 3.1 below can be used.) By *index* $i(X)$ of a variety X we mean the greatest common divisor of the degrees of closed points on X.

Namely, writing i_n for the maximal index of the n-th Grassmannian (the numbering is such that the 1-st Grassmannian is the quadric), Proposition 2.1 says that $8 \mid i_3$. This implies that $i_3 = 8$. For the remaining Grassmannians one has: $i_1 = 2$ (trivial), $i_2 = 4$ (a consequence of [3, Theorem 4.2] as well as of [19, Theorem 0.1]), $i_3 \mid i_4 \mid i_5 = i_6 = i_7 = i_8$, and $i_8 = 8$ (again by [19, Theorem 0.1]).

Proof of Theorem 2.1. Let X be a component of the maximal orthogonal Grassmannian of φ and let Y be the projective quadric of φ. We write $\bar{X}$ for $X_{\bar{F}}$ and $\bar{Y}$ for $Y_{\bar{F}}$, where $\bar{F}$ is an algebraic closure of F.

By Corollary 1.6, the image $\overline{\mathrm{CH}}\, X$ of $\mathrm{CH}\, X$ in $\mathrm{CH}\, \bar{X}$ coincides with the subring generated by $e_1, 2e_2, \dots, 2e_7$.

Assume that there exists a finite field extension L/F with $i_W(\varphi_L) \geq 3$ and with $[L:F]$ not divisible by 8. Let us fix an F-imbedding $L \hookrightarrow \bar{F}$. Then the elements $l_2, l_1, l_0 \in \mathrm{CH}\, \bar{Y}$ are in the image of $\mathrm{CH}\, Y_L$ and therefore, as follows from (1.2), e_5, e_6, e_7 belong to the ring

$\overline{\mathrm{CH}}\, X_L := \mathrm{Im}(\mathrm{CH}\, X_L \to \mathrm{CH}\, \bar{X})$. In particular, the product $e_5e_6e_7$ is in $\overline{\mathrm{CH}}\, X_L$. Applying the norm homomorphism $\mathrm{CH}\, X_L \to \mathrm{CH}\, X$, we get that $[L : F]e_5e_6e_7 \in \overline{\mathrm{CH}}\, X$. Since $8e_5e_6e_7 = (2e_5)(2e_6)(2e_7) \in \overline{\mathrm{CH}}\, X$, it follows that $4e_5e_6e_7 \in \overline{\mathrm{CH}}\, X$. This contradicts Lemma 2.3 (a purely algebraic statement) below. □

Lemma 2.3. *Let R be the ring defined by generators and relations: the generators are $e_1, \dots, e_7$ and relations are as in (1.1) with $n = 7$. Then the subring of R generated by $e_1, 2e_2, \dots, 2e_7$ does not contain $4e_5e_6e_7$.*

Proof. Since relations (1.1) are homogeneous (with $\deg e_i := i$), the ring R is graded. By the shape of (1.1), the 2^7 products of distinct generators additively generate R. Moreover, as we known, these products form a $\mathbb{Z}$-basis of R, cf. [5, Theorem 86.12 and Proposition 86.16].

It follows that the product $(e_2e_4) \cdot 4(e_5e_6e_7)$ is not divisible by 8 in R. To show that $4e_5e_6e_7$ is not a polynomial in $e_1, 2e_2, \dots, 2e_7$, it suffices to show that every monomial M in $e_1, 2e_2, \dots, 2e_7$ of degree $5 + 6 + 7 = 18$, multiplied by e_2e_4, is divisible by 8 in R.

If M contains at least 3 of $2e_2, \dots, 2e_7$ as factors, then already M itself is divisible by 8.

Note that $e_2 = e_1^2$ and $e_4 = 2e_1e_3 - e_1^4$ so that

$$e_2e_4 = 2e_1^3e_3 - e_1^6. \tag{2.4}$$

If M contains precisely 2 of $2e_2, \dots, 2e_7$ as factors, the remaining factors being copies of e_1, it contains $4e_1^4$. Multiplying by e_2e_4 and taking into account formula (2.4) for e_2e_4, we get a sum of two monomials, one of which contains 8 as a factor and the other contains $4e_1^{10}$ as a factor.

However, relations (1.1) modulo 2 show that

$$e_1^2 = e_2,\ e_1^4 = e_2^2 \equiv e_4, \quad \text{and } e_1^8 \equiv e_4^2 \equiv e_8 = 0.$$

In particular, e_1^8 is divisible by 2 in R. Therefore $4e_1^{10}$ is divisible by 8.

Next we consider the case where M contains precisely 1 of $2e_2, \dots, 2e_7$. Then M contains $2e_1^{11}$. Multiplying M by e_2e_4 and taking into account formula (2.4), we get a sum of two monomials, one of which contains $4e_1^{14}$ as a factor and the other contains $2e_1^{17}$ as a factor. Since e_1^{14} is divisible by 2, the first monomial is divisible by 8. Since $e_1^{17} = e_1^8 \cdot e_1^8 \cdot e_1$ is divisible by 4, the second monomial is also divisible by 8.

Finally, when M is a power of e_1, then M is a multiple of e_1^{18}. Multiplying M by e_2e_4, we also get divisibility by 8. □

Using Theorem 2.1, we can recover

Corollary 2.5 ([3])**.** *Generic 16-dimensional quadratic form φ/F in I_q^3 does not contain (2- or) 4- or 6-dimensional subforms of trivial discriminant. Moreover, the same property holds for φ_E with any finite field extension E/F of odd degree.*

Proof. Assume that φ_E does contain a 6-dimensional subform φ' of trivial discriminant. The form φ'_L is hyperbolic for some finite extension L/E of degree dividing 4. We therefore have $i_W(\varphi_L) \geq 3$, contradicting Proposition 2.1.

If φ_E contains 4- (or 2)-dimensional subform of trivial discriminant, we can construct a finite field extension L/E of degree dividing 4 with hyperbolic φ_L. This contradicts not only Proposition 2.1, but also an earlier [19, Theorem 0.1]. □

Let us recall an open question (cf. [3, Theorem 4.2]): does any 16-dimensional quadratic form of trivial discriminant and Clifford invariant contain an 8-dimensional subform of trivial discriminant? The positive answer would provide an upper bound on $\mathrm{Pf}_3(16)$ (at least in characteristic $\neq 2$):

Proposition 2.6. *Let φ be a 16-dimensional quadratic form with trivial discriminant and Clifford invariant over a field F of characteristic $\neq 2$. Assume that φ contains a proper even-dimensional subform φ_1 of trivial discriminant. Then the Witt class of φ is a sum of classes of six general 3-Pfister forms.*

Proof. We have $\varphi = \varphi_1 \perp \varphi_2$ for some φ_2. We may assume that $\dim\varphi_1 \leq \dim\varphi_2$ so that $\dim\varphi_1 \leq 8$. Choose $a \in F^\times$ such that the form $\varphi' := a\varphi_1 \perp \varphi_2$ is isotropic. We have $[\varphi] = [\varphi'] + [\langle\langle a\rangle\rangle] \cdot [\varphi_1] \in I_q(F)$, where $\langle\langle a\rangle\rangle$ is the diagonal form $\langle 1, -a\rangle$. As per [6, Proposition 2.3], $[\varphi']$ is a sum of classes of three general 3-Pfister forms. Since $\dim\varphi_1 \leq 8$, $[\varphi_1]$ is a sum of classes of three general 2-Pfister forms, [5, Lemma 38.1]. □

Unfortunately, the way of proving Corollary 2.5 does not work for 8-dimensional subforms in φ of trivial discriminant. Absence of such subforms would follow from absence of finite field extensions L/F of degree dividing 8 with $i_W(\varphi_L) \geq 4$. However such a field extension does exist. More than that, φ becomes hyperbolic over certain field extension of degree 8.

3. Dimensions 14 and 18

As already mentioned, the following result has been obtained (in characteristic $\neq 2$) independently in [6] and in [7]. The proof presented here is different; it is parallel to the proof of Theorem 2.1.

Theorem 3.1. *Let φ be a generic 14-dimensional quadratic form in I_q^3 over a field F (of arbitrary characteristic). The degree of any finite field extension L/F with $i_W(\varphi_L) \geq 2$ is divisible by 4. Equivalently, for any finite extension E/F of odd degree, the quadratic form φ_L does not contain a 4-dimensional subform of trivial discriminant.*

Proof. Assuming the contrary, we get that $2e_5e_6 \in \overline{\mathrm{CH}}\, X$, where X is a component of the maximal orthogonal Grassmannian of φ. On the other hand, $\overline{\mathrm{CH}}\, X$, being the subring of $\mathrm{CH}\,\bar{X}$ generated by e_1 and $2e_i$ with $i = 2, \ldots, 6$, does not contain $2e_5e_6$. □

Note that any 14-dimensional quadratic form in I_q^3 over a field of characteristic $\neq 2$ does contain a 6-dimensional subform of trivial discriminant as a consequence of M. Rost's classification result.

Corollary 3.2. *For any base field k (of arbitrary characteristic), one has $\mathrm{Pf}_3(14) \geq 3$.*

Proof. Let $\varphi/F \supset k$ be a generic 14-dimensional quadratic form in I_q^3. If the Witt class $[\varphi]$ is the sum of two classes of general Pfister forms, then $i_W(\varphi_L) \geq 3$ for some finite field extension L/F of degree dividing 2. This contradicts Theorem 3.1. □

Now we look at dimension 18 and higher. It has been shown in [3, Theorem 4.2] that for any field k of characteristic 0 and any even $m \geq 18$, there exists a field $F \supset k$ and an m-dimensional quadratic form φ over F with Witt class in $I_q^3(F)$ such that for any finite field extension E/F of odd degree, φ_E does not contain any proper even-dimensional subform of trivial discriminant. We suggest the following conjecture implying this result:

Conjecture 3.3. *For any even $m \geq 18$, the index of the $[m/4]$-th orthogonal Grassmannian of any generic m-dimensional quadratic form φ/F in I_q^3 is equal to $2^{[m/4]}$.*

Conjecture 3.3 actually implies that any generic m-dimensional quadratic form φ/F in I_q^3, for any even $m \geq 18$, has the property of [3, Theorem 4.2]: if this is not the case, i.e., if φ_E does contain a proper even-dimensional subform ψ of trivial discriminant for some finite field extension E/F of odd degree, then, possibly replacing ψ by its complement, we have $d := (\dim \psi)/2 \leq [m/4]$ and ψ becomes hyperbolic over an extension of degree dividing 2^{d-1}. Therefore φ_E acquires Witt index $\geq [m/4]$ over an extension of L of degree dividing $2^{[m/4]-1}$, i.e., the index of the $[m/4]$-th orthogonal Grassmannian of φ divides $2^{[m/4]-1}$.

Unfortunately, the method of proving Theorems 2.1 and 3.1 does not work for Conjecture 3.3. For instance, for $m = 18$ one needs to show that $2^3 e_5 e_6 e_7 e_8 \notin \overline{\mathrm{CH}}\, X$ which is false – see next paragraph. This provides a motivation to develop a modification of the method; we do it in the next section.

Let us show that $2^3 e_5 e_6 e_7 e_8 \in \overline{\mathrm{CH}}\, X$, where X is a component of the maximal orthogonal Grassmannian of a generic 18-dimensional quadratic form in I_q^3. We have $e_2 = e_1^2 \in \overline{\mathrm{CH}}\, X$, $e_4 = e_1(2e_3) - e_2^2 \in \overline{\mathrm{CH}}\, X$, so that $e_8 - 2e_3e_5 = e_1(2e_7) - e_2(2e_6) - e_4^2 \in \overline{\mathrm{CH}}\, X$. Multiplying by $(2e_5)(2e_6)(2e_7) \in \overline{\mathrm{CH}}\, X$, we get that $2^3 e_5 e_6 e_7 e_8 - 2^4 e_3 e_5^2 e_6 e_7 \in \overline{\mathrm{CH}}\, X$. The second summand is in $\overline{\mathrm{CH}}\, X$ as well because $e_5^2 = 2e_4e_6 - 2e_3e_7 + 2e_2e_8$. Therefore the first summand $2^3 e_5 e_6 e_7 e_8$ is in $\overline{\mathrm{CH}}\, X$.

4. Grothendieck ring of maximal orthogonal Grassmannian

For an integer $n \geq 1$, let $\varphi : V \to F$ be a non-degenerate quadratic form of dimension $2n+1$ over a field F (of arbitrary characteristic). The vector space V of definition of φ is a vector space over F of dimension $2n+1$. Let X be the maximal orthogonal Grassmannian of φ, i.e., X is the F-variety of n-dimensional totally isotropic subspaces in V.

We are going to collect information on the Grothendieck group $K(X)$. From Panin's computation of K-theory of projective homogeneous varieties [16], we deduce

Lemma 4.1. *There is a natural (with respect to field extensions of F) group isomorphism of $K(X)$ onto the direct sum of 2^{n-1} copies of $K(F) = \mathbb{Z}$ and 2^{n-1} copies of $K(C_0(\varphi)) = \operatorname{ind} C_0(\varphi) \cdot \mathbb{Z}$, where $C_0(\varphi)$ is the even Clifford algebra of φ.*

Proof. Consider the category of K-correspondences as in [15, §1.8] (where it is called *motivic category*). Objects of this category are pairs (Y, A), where Y is a smooth projective F-variety and A is a separable F-algebra. To simplify notation, one writes Y for the object (Y, F) and A for $(\operatorname{Spec} F, A)$.

As proved in [16], the object in the above category given by the maximal orthogonal Grassmannian X (as well as by any other projective homogeneous variety under the orthogonal group $\mathrm{O}^+(\varphi)$) is isomorphic to a separable algebra A which is a finite direct product of a copies of F and of b copies of $C_0(\varphi)$ for some integers $a, b \geq 0$. As a consequence, we get a natural group isomorphism $K(X) \simeq K(F)^{\oplus a} \oplus K(C_0(\varphi))^{\oplus b}$. The rank of the group $K(X)$ coincides with the rank of $\mathrm{CH}(X)$ which, as we know, is equal to 2^n. Therefore $a + b = 2^n$ and in order to finish the proof of Lemma 4.1 we only need to show that $a = b$ provided that the central simple algebra $C_0(\varphi)$ is not split (i.e., has index > 1).

Let S be the Severi-Brauer variety of $C_0(\varphi)$. The projection $X \times S \to X$ is a projective bundle of rank $2^n - 1$. By Projective Bundle Theorem for K-theory [17, Proposition 4.3], there is a natural isomorphism $K(X \times S) \simeq K(X)^{\oplus 2^n}$. Therefore, $K(X)$ is naturally isomorphic to the direct sum of $2^n a$ copies of $K(F)$ and $2^n b$ copies of $K(C_0(\varphi))$.

On the other hand, S as an object in the category of K-correspondences is isomorphic to the product of 2^{n-1} copies of F and 2^{n-1} copies of $C_0(\varphi)$. (Since the algebra $C_0(\varphi)$ is of exponent 2, its higher tensor powers do not show up.) It follows that the object given by the direct product of varieties $X \times S$, which is the tensor product in the category of K-correspondences of the objects given by X and by S, is isomorphic to the product of $2^{n-1}(a+b)$ copies of F and $2^{n-1}(a+b)$ copies of $C_0(\varphi)$. This gives a natural isomorphism of $K(X \times S)$ with the direct sum of $2^{n-1}(a+b)$ copies of $K(F)$ and $2^{n-1}(a+b)$ copies of $K(C_0(\varphi))$. Comparing the resulting computation for the order of cokernel of the change of field homomorphism for $K(X \times S)$ to an algebraic closure of F, we get that

$$(\operatorname{ind} C_0(\varphi))^{2^n b} = (\operatorname{ind} C_0(\varphi))^{2^{n-1}(a+b)}.$$

Since $\operatorname{ind} C_0(\varphi) \neq 1$, it follows that $2^n b = 2^{n-1}(a+b)$ giving $a = b$. □

Proposition 4.2. *Assume that the index of the even Clifford algebra $C_0(\varphi)$ is maximal:* $\operatorname{ind} C_0(\varphi) = 2^n$. *Then the topological filtration on $K(X)$ coincides with the gamma filtration.*

Proof. Let us consider the filtration of the ring $K(X)$ generated by the K-theoretical Chern classes $c_i(\mathcal{T})$, $i \geq 0$ of the tautological vector bundle $\mathcal{T}$ on X. By definition, the i-th term $\mathcal{F}^i$ of this filtration is additively generated by the products $c_{i_1}(\mathcal{T}) \cdot \dots \cdot c_{i_r}(\mathcal{T})$ with $r \geq 1$ and $i_1 + \dots + i_r \geq i$.

Clearly, for any i, we have a chain of inclusions consisting of 4 terms: $\mathcal{F}^i$ is inside of the i-th term of the gamma filtration, which is inside of the i-term of the topological filtration on $K(X)$, which is inside of the intersection with $K(X)$ of the i-th term of the topological filtration on $K(\bar{X})$. We are going to show that the smallest term in this chain coincides with the largest term meaning that all four terms are the same. This will prove Proposition 4.2.

Let $\bar{F}$ be an algebraic closure of F and $\bar{X} := X_{\bar{F}}$. By Lemma 4.1, the order of the cokernel of the change of field homomorphism $K(X) \to K(\bar{X})$ is equal to $2^{n2^{n-1}}$. On the other hand, the cokernel of the homomorphism of the associated graded rings $GK(X) \to GK(\bar{X})$ with respect to the filtration $\mathcal{F}$ on $K(X)$ and the topological filtration on $K(\bar{X})$, is identified with the cokernel of $\mathrm{CH}\, X \to \mathrm{CH}\, \bar{X}$ whose order also equals $2^{n2^{n-1}}$, see Corollary 1.4. The formula

$$|\operatorname{Ker}(GK(X) \to GK(\bar{X}))| = \frac{|\operatorname{Coker}(GK(X) \to GK(\bar{X}))|}{|\operatorname{Coker}(K(X) \to K(\bar{X}))|}$$

(proved as [11, Proposition 2]) implies that the homomorphism $GK(X) \to GK(\bar{X})$ is injective, i.e., that the filtration $\mathcal{F}$ on $K(X)$ is induced by the topological filtration on $K(\bar{X})$. $\square$

Theorem 4.3. *For φ as in Proposition 4.2, let S be the Severi-Brauer variety of the division algebra $C_0(\varphi)$. Then the topological filtration on $K(X_{F(S)})$ coincides with the gamma filtration.*

Proof. We proceed along the lines of [12, Theorem 3.7]. As we already observed in the proof of Proposition 1.5 and Lemma 4.1, the product $X \times S$ is a projective bundle over X. This implies that $\mathrm{CH}(X \times S)$ and therefore $\mathrm{CH}\, X_{F(S)}$ is generated as $\mathrm{CH}\, X$-algebra by the first Chern class of certain linear vector bundle (well, this Chern class is equal to $\pm e_1$). The epimorphism $\mathrm{CH}\, X_{F(S)} \twoheadrightarrow GK(X_{F(S)})$, where GK is now the associated graded ring of the topological filtration, shows that also $GK(X_{F(S)})$ is generated as $GK(X)$-algebra by the first Chern class of a linear vector bundle. By Proposition 4.2, the ring $GK(X)$ is generated by Chern classes of vector bundles. It follows that also the ring $GK(X_{F(S)})$ is generated by Chern classes of vector bundles. This precisely means that the gamma filtration on $K(X_{F(S)})$ coincides with the topological one. $\square$

Remark 4.4. The proofs of Theorem 4.3 and Proposition 4.2 actually provide a simpler than a priori description of gamma filtration on $K(X_{F(S)})$: it is generated by e_1 and the Chern classes of the tautological vector bundle. More precisely, for any $i \geq 0$, the i-th term of the filtration is additively generated by the products $e_1^{i_0} c_{i_1}(\mathcal{T}) \dots c_{i_r}(\mathcal{T})$ with $r \geq 0$ and $i_0 + i_1 + \dots + i_r \geq i$.

Remark 4.5. Let $\tilde{X}$ be any flag variety of φ projecting to X, i.e., any variety of flags of totally isotropic subspaces in φ, where the flags under consideration include maximal totally isotropic subspaces. In particular, $\tilde{X}$ can be the variety of complete flags. Since the projection $\tilde{X} \to X$ is a flag bundle of a vector bundle on X (namely, of the tautological vector bundle), Theorem 4.3 implies that the topological filtration on $K(\tilde{X}_{F(S)})$ (and on $K(\tilde{X})$) coincides with the gamma filtration.

Using identification of the maximal orthogonal Grassmannian of an odd-dimensional quadratic form with a component of the maximal orthogonal Grassmannian of the corresponding even-dimensional form of trivial discriminant, we get the above statement as well for even-dimensional forms φ of trivial discriminant with division algebra $C^+(\varphi)$.

Finally, using specialization arguments, one can prove coincidence of gamma and topological filtrations for $K(T/P)$, where T is a *generic* principle homogeneous space under Spin_m (with any m: $m = 2n+1$ or $m = 2n+2$) and $P \subset \mathrm{Spin}_m$ is any *special* parabolic subgroup.

Remark 4.6. Instead of completely killing the division algebra $C_0(\varphi)$ by passing to the function field of its Severi-Brauer variety S in the statement of Theorem 4.3, one may partially split $C_0(\varphi)$ by passing to the function field of any of its *generalized* Severi-Brauer variety S'. The conclusion remains the same: gamma filtration on $K(X_{F(S')})$ coincides with the topological filtration. The proof also remains basically the same, cf. [12, Theorem 3.7 and its proof].

It would be very interesting to completely understand the gamma filtration on $K(X_{F(S)})$. Note that for any field extension $L/F(S)$, the change of field homomorphism $K(X_{F(S)}) \to K(X_L)$ is an isomorphism preserving the filtration. So, if this helps, it is enough to perform the computation, say, over an algebraically closed field. Actually, it is even enough to do it for a maximal orthogonal Grassmannian $\bar{X}$ over $\mathbb{C}$. One may consider the additive basis of the group $K(\bar{X})$ given by the Schubert classes; the ring structure is determined by the K-theoretical Littlewood-Richardson formulas obtained in [4]. Alternatively, one may describe the ring $K(\bar{X})$ by generators and relations in the spirit of the discussed description of $\mathrm{CH}\,\bar{X}$, taking for generators the special Schubert classes $e_i \in K(\bar{X})$. (Unfortunately, the relations on e_i in $K(\bar{X})$ look more complicated than (1.1).)

Note that these K-theoretical special Schubert classes still satisfy relations (1.2), where now l_{n-i} stand for the K-theoretical classes of linear subspaces on the quadric. This can be shown using [2, Lemma 2.1].

Of particular interest is to understand the position of the special Schubert classes in the filtration. More specifically, let us consider the class $e_n \in K(X_{F(S)})$ of the special Schubert variety of the lowest dimension (corresponding to the class of a rational point on the quadric).

Conjecture 4.7. *For $n \geq 8$, the special Schubert class $e_n \in K(X_{F(S)})$ does not belong to the term number $n+1-[(n+1)/2]$ of the gamma filtration.*

Conjecture 4.7 implies Conjecture 3.3. Indeed, assume that Conjecture 3.3 fails for some even $m \geq 18$. This means that we can find a field F and a non-degenerate quadratic form φ over F of dimension m and trivial discriminant such that $C^+(\varphi)$ is a division algebra and there exists a finite field extension $L/F(S)$ of degree not divisible by 2^r with $i_W(\varphi_L) \geq r$, where $r := [m/4]$ and S is the Severi-Brauer variety of $C^+(\varphi)$.

Since $i_W(\varphi_L) \geq r$, the projective quadric Y_L of φ_L contains a linear subspace of dimension $r-1$. Its class l_{r-1} in $K(Y_L)$ belongs to the term number $\dim Y - (r-1) = m-r-1$ of the topological filtration on $K(Y_L)$. Applying the norm homomorphism $K(Y_L) \to K(Y_{F(S)})$, preserving the filtration and acting as multiplication by $[L : F(S)]$, we get that the element $2^{r-1}l_{r-1}$ of the group $K(Y_{F(S)})$ is in the $m-r-1$-th term of the topological filtration. Since $2^{r-1}l_{r-1} = l_0 + h^{m-3} + 2h^{m-4} + \cdots + 2^{r-2}h^{m-r-1}$ (see [9, §3.2]), where h is the class of a hyperplane section, we get that l_0 is in the $m-r-1$-th term as well. As in (1.2), we have $e_n = f_*g^*(l_0) \in K(X_{F(S)})$, where X is a component of the maximal orthogonal Grassmannian of φ. The pull-back homomorphism $g^* : K(Y_{F(S)}) \to K(Z_{F(S)})$ preserves the filtration. The push-forward homomorphism $f_* : K(Z_{F(S)}) \to K(X_{F(S)})$ lowers the number of the filtration term by $\dim Z - \dim X = n := (m-2)/2$. It follows that e_n is in the term number $m-r-1-n = n+1-[(n+1)/2]$. Finally, identifying X with the maximal orthogonal Grassmannian of a non-degenerate $2n+1$-dimensional subform of φ, we get a counter-example to Conjecture 4.7.

Remark 4.8. The number $n+1-[(n+1)/2]$ in Conjecture 4.7 is optimal for $n = 8$. Indeed, if φ is an 18-dimensional quadratic form of trivial discriminant and trivial Clifford invariant and Y is its projective quadric, since the Chow groups CH^iY are torsion-free for $i \leq 3$, [8], the class $l_8 \in K(Y)$ is in the 4-th term of the topological filtration, cf. [9, Theorem 3.10]. Since $l_0 = l_8 \cdot h^8$, it follows that l_0 is in the term number $4+8 = 12$.

Therefore e_8 is in the term number $4 = 12 - n$. Conjecture 4.7 claims that e_8 is not in the term number 5.

References

[1] Brosnan, P., Reichstein, Z., and Vistoli, A. Essential dimension, spinor groups, and quadratic forms. *Ann. of Math. (2) 171*, 1 (2010), 533–544.

[2] Buch, A. S., and Ravikumar, V. Pieri rules for the K-theory of cominuscule Grassmannians. *J. Reine Angew. Math. 668* (2012), 109–132.

[3] Chernousov, V., and Merkurjev, A. Essential dimension of spinor and Clifford groups. *Algebra Number Theory 8*, 2 (2014), 457–472.

[4] Clifford, E., Thomas, H., and Yong, A. K-theoretic Schubert calculus for $\mathrm{OG}(n, 2n+1)$ and jeu de taquin for shifted increasing tableaux. *J. Reine Angew. Math. 690* (2014), 51–63.

[5] Elman, R., Karpenko, N., and Merkurjev, A. *The algebraic and geometric theory of quadratic forms*, vol. 56 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2008.

[6] Hoffmann, D. W., and Tignol, J.-P. On 14-dimensional quadratic forms in I^3, 8-dimensional forms in I^2, and the common value property. *Doc. Math. 3* (1998), 189–214 (electronic).

[7] Izhboldin, O. T., and Karpenko, N. A. Some new examples in the theory of quadratic forms. *Math. Z. 234*, 4 (2000), 647–695.

[8] Karpenko, N. Chow groups of quadrics and index reduction formula. *Nova J. Algebra Geom. 3*, 4 (1995), 357–379.

[9] Karpenko, N. A. Algebro-geometric invariants of quadratic forms. *Algebra i Analiz 2*, 1 (1990), 141–162.

[10] Karpenko, N. A. The Grothendieck ring of quadrics, and gamma filtration. In *Rings and modules. Limit theorems of probability theory, No. 3 (Russian)*. Izd. St.-Peterbg. Univ., St. Petersburg, 1993, pp. 39–61, 256.

[11] Karpenko, N. A. On topological filtration for Severi-Brauer varieties. In *K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992)*, vol. 58 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI, 1995, pp. 275–277.

[12] Karpenko, N. A. Codimension 2 cycles on Severi-Brauer varieties. *K-Theory 13*, 4 (1998), 305–330.

[13] Merkurjev, A. S. Invariants of algebraic groups and retract rationality of classifying spaces. Preprint (2015), 20 pages. Available on the web page of the author.

[14] Merkurjev, A. S. Essential dimension: a survey. *Transform. Groups 18*, 2 (2013), 415–481.

[15] Merkurjev, A. S., and Panin, I. A. K-theory of algebraic tori and toric varieties. *K-Theory 12*, 2 (1997), 101–143.

[16] Panin, I. A. On the algebraic K-theory of twisted flag varieties. *K-Theory 8*, 6 (1994), 541–585.

[17] Quillen, D. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*. Springer, Berlin, 1973, pp. 85–147. Lecture Notes in Math., Vol. 341.

[18] Rost, M. On 14-dimensional quadratic forms, their spinors, and the difference of two octonion algebras. Preprint, May 1994, 5 pages. Available on the web page of the author.

[19] Totaro, B. The torsion index of the spin groups. *Duke Math. J. 129*, 2 (2005), 249–290.

[20] Vishik, A. On the Chow groups of quadratic Grassmannians. *Doc. Math. 10* (2005), 111–130 (electronic).

Mathematical & Statistical Sciences, University of Alberta, Edmonton, CANADA
E-mail address: `karpenko` *at* `ualberta.ca`, *web page*: `www.ualberta.ca/~karpenko`