TOTALLY DECOMPOSABLE QUADRATIC PAIRS

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ABSTRACT. In this paper we show that a split central simple algebra with quadratic pair which decomposes into a tensor product of quaternion algebras with involution and a quaternion algebra with quadratic pair is adjoint to a quadratic Pfister form. This result is new in characteristic two, otherwise it is equivalent to the Pfister Factor Conjecture proven in [3].

Keywords: Central simple algebras, involutions, quadratic pairs, characteristic two, quadratic forms

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1. Introduction

To every symmetric or alternating bilinear form, and hence to every quadratic form when the base field is not of characteristic two, we can associate a central simple algebra with involution. In this way, the theory of quadratic forms is embedded into the larger theory of algebras with involution, and through the use of this correspondence, quadratic form theory has provided inspiration for the study of algebras with involution (see [12]). Pfister forms are a central concept in the modern algebraic theory of quadratic forms. It is therefore natural to look for class of central simple algebras with involution which extends the notion of a Pfister form. This question was first raised in [2]. Involutions adjoint to Pfister forms are tensor products of quaternion algebras with involution. Thus, tensor products of quaternion algebras with involution are a natural candidate.

In characteristic two, quadratic forms and symmetric bilinear forms are not equivalent objects. The relation between bilinear Pfister forms and totally decomposable involutions in characteristic two was studied in [7]. In order to have an object defined on a central simple algebra that corresponds to a quadratic form after splitting, the notion of a quadratic pair was introduced in [12, §5]. In particular, one may use quadratic pairs to give an intrinsic definition of twisted orthogonal groups in a manner that includes fields of characteristic two (see [12, §23.B]).

Algebras with quadratic pair associated to quadratic Pfister forms are tensor products of quaternion algebras with involution and a quaternion algebra with quadratic pair. One may ask whether all such totally decomposable quadratic pairs on a split central simple algebra are adjoint to a quadratic Pfister form. In characteristic different from two, where quadratic pairs are equivalent to orthogonal involutions, this is known to hold by the main result of [3], which says that in this case a totally decomposable orthogonal involution on a split algebra is adjoint to a

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Pfister form. In this article we prove the corresponding result for quadratic pairs over fields of characteristic two.

Our approach is particular to fields of characteristic two. It allows us to capture more information on the resulting quadratic Pfister form (see Corollary 7.2), in a way that is not possible in general over fields of characteristic different from two (see Example 7.4). This approach uses the unusual properties of totally decomposable involutions in characteristic two found in [7]. In particular, we use that the isotropy behaviour of a totally decomposable orthogonal involution can be captured in the isotropy behaviour of an associated bilinear Pfister form (see Proposition 3.4).

The method used in [3] in characteristic different from two is based on a ramification exact sequence for Witt groups of quadratic forms over function fields of conics and the excellence property of these function fields. The latter result is known to extend to the case of arbitrary characteristic, but the former is not yet available in characteristic two in a suitable form. We intend to give a characteristic free proof in a future article. Our solution to the missing case of characteristic two involves several basic properties of quadratic pairs, which we did not find explicitly in the literature. Following [12, §5], we present such statements without assumptions on the characteristic, for ease of future reference.

2. Quadratic forms over fields

In this section we recall the terminology and results we use from quadratic form theory. We refer to [9, Chapters 1 and 2] as a general reference on symmetric bilinear and quadratic forms and for any basic notation and concepts not defined here. For two objects α and β in a certain category, we write $\alpha \simeq \beta$ to indicate that they are isomorphic, i.e. that there exists an isomorphism between them. This applies in particular to algebras with involution or with quadratic pair, but also to quadratic and bilinear forms, where the corresponding isomorphisms are called isometries. Throughout, let F be a field. Let $\mathrm{char}(F)$ denote the characteristic of F and let $F^\times$ denote the multiplicative group of F.

A *bilinear form over F* is a pair (V, b) where V is a finite-dimensional F-vector space and b is a F-bilinear map $b : V \times V \to F$. The *radical of* (V, b) is the set $\mathrm{rad}(V, b) = \{x \in V \mid b(x, y) = 0 \text{ for all } y \in V\}$. We say that (V, b) is *degenerate* if $\mathrm{rad}(V, b) \neq \{0\}$, and *nondegenerate* otherwise. Let $\varphi = (V, b)$ be a bilinear form over F. We say that φ is *symmetric* if $b(x, y) = b(y, x)$ for all $x, y \in V$. We call φ *alternating* if $b(x, x) = 0$ for all $x \in V$. The form φ is said to be *isotropic* if there exists an $x \in V\backslash\{0\}$ such that $b(x, x) = 0$, and *anisotropic* otherwise. We call a subspace $W \subseteq V$ *totally isotropic* (with respect to b) if $b|_{W\times W} = 0$. If φ is nondegenerate and there exists a totally isotropic subspace $W \subseteq V$ such that $\dim_F(W) = \frac{1}{2}\dim_F(V)$, then we call φ *metabolic*.

For $a_1, \dots, a_n \in F^\times$ the symmetric F-bilinear map $b : F^n \times F^n \to F$ given by $(x, y) \mapsto \sum_{i=1}^n a_i x_i y_i$ yields a symmetric bilinear form (F^n, b) over F, which we denote by $\langle a_1, \dots, a_n\rangle$. For a positive integer m, by an *m-fold bilinear Pfister form over F* we mean a nondegenerate symmetric bilinear form over F that is isometric to $\langle 1, a_1\rangle \otimes \dots \otimes \langle 1, a_m\rangle$ for some $a_1, \dots, a_m \in F^\times$. We call $\langle 1\rangle$ the *0-fold bilinear Pfister form.* By [9, (6.3)], a bilinear Pfister form is either anisotropic or metabolic.

By a *quadratic form over F* we mean a pair (V, q) of a finite-dimensional F-vector space V and a map $q : V \to F$ such that, firstly, $q(\lambda x) = \lambda^2 q(x)$ holds for all $x \in V$ and $\lambda \in F$, and secondly, the map $b_q : V \times V \to F$, $(x, y) \longmapsto q(x+y)-q(x)-q(y)$ is

F-bilinear. Then (V, b_q) is a symmetric bilinear form over F, called the *polar form of* (V, q). If b_q is nondegenerate, we say that (V, q) is *nonsingular*, otherwise we say that (V, q) is *singular*. If b_q is the zero map, then we say (V, q) is *totally singular*. By the *quadratic radical of* (V, q) we mean the set $\mathrm{rad}(V, q) = \{x \in \mathrm{rad}(V, b_q) \mid q(x) = 0\}$. We say that (V, q) is *regular* if $\mathrm{rad}(V, q) = \{0\}$. For a symmetric bilinear form (V, b) over F, the map $q : V \to F$ given by $q_b(x) = b(x, x)$ makes (V, q_b) a quadratic form over F. We call (V, q_b) the *quadratic form associated to* (V, b). If $\mathrm{char}(F) = 2$ then this quadratic form is totally singular.

Let $\rho = (V, q)$ and $\rho' = (V', q')$ be quadratic forms over F. By an *isometry of quadratic forms* $\rho \to \rho'$ we mean an isomorphism of F-vector spaces $f : V \to V'$ such that $q(x) = q'(f(x))$ for all $x \in V$. We say ρ is *isotropic* if $q(x) = 0$ for some $x \in V\backslash\{0\}$, and *anisotropic* otherwise. By *a totally isotropic subspace of* ρ we mean an F-subspace W of V such that $q|_W = 0$. We call the maximum of the dimensions of all totally isotropic subspaces of ρ the *Witt index of* ρ, denoted $i_W(\rho)$. Assume ρ is nonsingular. Then $i_W(\rho) \leqslant \frac{1}{2}\dim(\rho)$ (see [9, (7.28)]) and if $i_W(\rho) = \frac{1}{2}\dim(\rho)$ we say that ρ is *hyperbolic*. We denote the anisotropic part of ρ by ρ_{an} (see [9, (8.5)]).

We say that the quadratic forms ρ_1 and ρ_2 over F are *similar* if there exists an element $c \in F^\times$ such that $\rho_1 \simeq c\rho_2$. Recall the concept of a tensor product of a symmetric or alternating bilinear form and a quadratic form (see [9, p.51]). For a quadratic form ρ over F, we say that φ *factors* ρ if there exists a quadratic form ρ' over F such that $\rho \simeq \varphi \otimes \rho'$. Similarly, we say a quadratic form ρ' over F *factors* ρ if there exists a symmetric bilinear form φ over F such that $\rho \simeq \varphi \otimes \rho'$.

For a positive integer m, by an *m-fold (quadratic) Pfister form over F* we mean a quadratic form that is isometric to the tensor product of a 2-dimensional nonsingular quadratic form representing 1 and an $(m-1)$-fold bilinear Pfister form over F. Pfister forms are either anisotropic or hyperbolic (see [9, (9.10)]).

Let ρ be a regular quadratic form over F. If $\dim(\rho) \geqslant 3$ or if ρ is anisotropic and $\dim(\rho) = 2$, then we call the function field of the projective quadric over F given by ρ the *function field of* ρ and denote it by $F(\rho)$. In the remaining cases we set $F(\rho) = F$. This agrees with the definition in [9, §22]. For an anisotropic symmetric bilinear form φ, the quadratic form ρ associated to φ is regular. We call $F(\rho)$ the *function field of* φ and we denote it by $F(\varphi)$. Let K/F be a field extension. Then we write $(V, q)_K = (V \otimes_F K, q_K)$ where q_K is the unique quadratic map such that $q_K(v \otimes k) = k^2 q(v)$ for all $v \in V$ and $k \in K$.

Proposition 2.1. *Let ρ be a nonsingular quadratic form and let φ be an anisotropic bilinear Pfister form over F. If $\rho_{F(\varphi)}$ is hyperbolic then either ρ is hyperbolic or φ factors ρ_{an}.*

Proof. See [10, (5.2)]. □

3. Algebras with involution

We refer to [14] as a general reference on finite-dimensional algebras over fields, and for central simple algebras in particular, and to [12] for involutions. Let A be an (associative) F-algebra. We denote the centre of A by $Z(A)$. For a field extension K/F, the K-algebra $A \otimes_F K$ is denoted by A_K. An element $e \in A$ is called an *idempotent* if $e^2 = e$. An *F-involution on A* is an F-linear map $\sigma : A \to A$ such that $\sigma(xy) = \sigma(y)\sigma(x)$ for all $x, y \in A$ and $\sigma^2 = \mathrm{id}_A$.

Assume now that A is finite-dimensional and simple (i.e. it has no nontrivial two-sided ideals). By Wedderburn's Theorem (see [12, (1.1)]), $A \simeq \mathrm{End}_D(V)$ for a finite-dimensional F-division algebra D and a finite-dimensional right D-vector space V. Furthermore, the centre of A is a field and $\dim_{Z(A)}(A)$ is a square number, whose positive square root is called the *degree of* A and is denoted $\deg(A)$. The degree of D is called the *index of* A and denoted $\mathrm{ind}(A)$. We call A *split* if $\mathrm{ind}(A) = 1$, that is $A \simeq \mathrm{End}_F(V)$ for some finite-dimensional right F-vector space V. If $Z(A) = F$, then we call the F-algebra A *central simple* and we call a field extension K/F such that A_K is split a *splitting field of* A. If A is a central simple F-algebra then we denote $\mathrm{Trd}_A : A \longrightarrow F$ the reduced trace map and $\mathrm{Nrd}_A : A \longrightarrow F$ the reduced norm map, as defined in [12, (1.6)].

By an *F-algebra with involution* we mean a pair (A, σ) of a finite-dimensional central simple F-algebra A and an F-involution σ on A (note that we only consider involutions that are linear with respect to the centre of A, that is involutions of the first kind, here). We use the following notation: $\mathrm{Sym}(A, \sigma) = \{a \in A \mid \sigma(a) = a\}$, $\mathrm{Skew}(A, \sigma) = \{a \in A \mid \sigma(a) = -a\}$ and $\mathrm{Alt}(A, \sigma) = \{a - \sigma(a) \,|\, a \in A\}$. These are F-linear subspaces of A.

Let (A, σ) and (B, τ) be F-algebras with involution. By an *isomorphism of F-algebras with involution* $\Phi : (A, \sigma) \to (B, \tau)$ we mean an F-algebra isomorphism $\Phi : A \to B$ satisfying $\Phi \circ \sigma = \tau \circ \Phi$. On the F-algebra $A \otimes_F B$ we obtain an F-involution $\sigma \otimes \tau$, whereby $(A \otimes_F B, \sigma \otimes \tau)$ is an F-algebra with involution, which we denote by $(A, \sigma) \otimes (B, \tau)$. For a field extension K/F we write $(A, \sigma)_K = (A \otimes_F K, \sigma \otimes \mathrm{id})$.

We call (A, σ) *isotropic* if there exists $a \in A \backslash \{0\}$ such that $\sigma(a)a = 0$, and *anisotropic* otherwise. An idempotent $e \in A$ is called *metabolic with respect to* σ if $\sigma(e)e = 0$ and $\dim_F eA = \frac{1}{2}\dim_F A$. We call (A, σ) *metabolic* if A contains a metabolic idempotent element with respect to σ. For more information on metabolic involutions, see [6].

To every nondegenerate symmetric or alternating bilinear form $\varphi = (V, b)$ over F we can associate an algebra with involution in the following way. Let $A = \mathrm{End}_F(V)$. Then there is a unique involution σ on A such that

$$b(x, f(y)) = b(\sigma(f)(x), y) \qquad \text{for all } x, y \in V \text{ and all } f \in A.$$

We denote this F-involution on A by ad_b. We call (A, ad_b) the *F-algebra with involution adjoint to* φ and we denote it by $\mathrm{Ad}(\varphi)$. For every split F-algebra with involution (A, σ), there exists a nondegenerate symmetric or alternating bilinear form ψ over F such that $(A, \sigma) \simeq \mathrm{Ad}(\psi)$ (see [12, (2.1)]). The following is well-known, but we include a proof for completeness.

Proposition 3.1. *Let φ and ψ be nondegenerate symmetric bilinear forms over F. Then* $\mathrm{Ad}(\varphi \otimes \psi) \simeq \mathrm{Ad}(\varphi) \otimes \mathrm{Ad}(\psi)$.

Proof. Let $\varphi = (V, b)$ and $\psi = (W, b')$. Let $f \in \mathrm{End}_F(V)$ and $g \in \mathrm{End}_F(W)$. Then for all $u, v \in V$, $w, t \in W$ we have

$$\begin{aligned} (b \otimes b')(f \otimes g(u \otimes w), (v \otimes t)) &= (b \otimes b')((f(u) \otimes g(w)), (v \otimes t)) \\ &= b(f(u), v) \cdot b'(g(w), t) \\ &= b(u, \mathrm{ad}_b(f)(v)) \cdot b'(w, \mathrm{ad}_{b'}(g)(t)) \\ &= (b \otimes b')((u \otimes w), (\mathrm{ad}_b(f)(v) \otimes \mathrm{ad}_{b'}(g)(t)))\,. \end{aligned}$$

Therefore, by bilinearity of $b \otimes b'$, we have that $\mathrm{ad}_{b \otimes b'}(f \otimes g) = \mathrm{ad}_b(f) \otimes \mathrm{ad}_b(g)$. Using this, it follows from the linearity of $\mathrm{ad}_{b \otimes b'}$ that the natural isomorphism of

F-algebras $\Phi : \mathrm{End}_F(V) \otimes_F \mathrm{End}_F(W) \to \mathrm{End}_F(V \otimes_F W)$ is an isomorphism of the F-algebras with involution in the statement. □

We distinguish two *types* of F-algebras with involution. An F-algebra with involution is said to be *symplectic* if it becomes adjoint to an alternating bilinear form over some splitting field of the associated F-algebra, and *orthogonal* otherwise. In characteristic different from two, these types are distinguished by the dimensions of the spaces of symmetric and alternating elements. However in characteristic two these dimensions do not depend on the type (see [12, (2.6)]).

An *F-quaternion algebra* is a central simple F-algebra of degree 2. Let Q be an F-quaternion algebra. By [12, (2.21)], the map $Q \to Q$, $x \mapsto \mathrm{Trd}_Q(x) - x$ is the unique symplectic involution on Q; it is called the *canonical involution of Q*. Any F-quaternion algebra has a basis $(1, u, v, w)$ such that

$$u^2 = u + a, v^2 = b \text{ and } w = uv = v - vu\,,$$

for some $a \in F$ with $-4a \neq 1$ and $b \in F^\times$ (see [1, Chap. IX, Thm. 26]); such a basis is called an *F-quaternion basis*. Conversely, for $a \in F$ with $-4a \neq 1$ and $b \in F^\times$ the above relations uniquely determine an F-quaternion algebra (up to F-isomorphism), which we denote by $[a, b)$. By the above, up to isomorphism any F-quaternion algebra is of this form. The following result can be recovered from [8, p.104, Thm. 4], but we include a direct argument.

Lemma 3.2. *Let Q be an F-quaternion algebra and $v \in Q\backslash F$ be such that $v^2 \in F^\times$. There exist an element and $u \in Q$ such that $uv = v(1-u)$ and $u^2 - u = a$ for some $a \in F$ with $-4a \neq 1$. That is, $(1, u, v, uv)$ is an F-quaternion basis of Q.*

Proof. If $\mathrm{char}(F) \neq 2$ then it is well-known that there exists an invertible element $x \in Q$ such that $xv + vx = 0$ and we set $u = x + \frac{1}{2}$. Assume that $\mathrm{char}(F) = 2$. Consider the F-linear map $Q \to Q, x \mapsto xv + vx$. Its kernel and its image are equal to $F[v]$. Hence $uv + vu = v$ for some $u \in Q$, and it follows that $u^2 + u \in F[v] \cap F[u] = F$. □

With an F-quaternion basis $(1, u, v, w)$ of Q, we define F-involutions γ, σ and τ on Q via their action on u and v as follows. We let

$$\begin{aligned}
\gamma:&\quad u \mapsto 1-u, \quad v \mapsto -v\\
\sigma:&\quad u \mapsto 1-u, \quad v \mapsto v\\
\tau:&\quad u \mapsto u, \qquad\;\; v \mapsto -v\,.
\end{aligned}$$

Note that γ is the canonical involution on Q. Further, if $\mathrm{char}(F) = 2$, then $\gamma = \sigma$ and hence σ is symplectic. Otherwise σ is orthogonal. We use the notation

$$[a \cdot\!| \cdot b) = (Q, \gamma), \quad [a \cdot | \, b) = (Q, \sigma), \quad [a \,| \cdot b) = (Q, \tau)\,.$$

Proposition 3.3. *Let (Q, σ) be an F-quaternion algebra with orthogonal involution. Then there exists $a \in F$ with $-4a \neq 1$ and $b \in F^\times$ such that $(Q, \sigma) \simeq [a \,| \cdot b)$.*

Proof. Let γ be the canonical involution of Q. By [12, (2.21)], we have $\sigma = \mathrm{Int}(v) \circ \gamma$ for some invertible element $v \in \mathrm{Skew}(Q, \gamma)\backslash F$. Then $v^2 = -\gamma(v)v \in F^\times$ and we set $b = v^2$. By Lemma 3.2, there exists an element $u \in Q$ such that $uv = v(1-u)$ and $u^2 - u = a$ for some $a \in F$ with $-4a \neq 1$. Hence $(1, u, v, uv)$ is an F-quaternion basis of Q and we have that $\gamma(u) = 1 - u$ and $\gamma(v) = -v$. Hence $\sigma(u) = u$ and $\sigma(v) = -v$, and further $(Q, \sigma) \simeq [a \,| \cdot b)$. □

We call an F-algebra with involution *totally decomposable* if it is isomorphic to a tensor product of F-quaternion algebras with involution. Note that by Proposition 3.1, the F-algebra with involution adjoint to a bilinear Pfister form over F is totally decomposable.

Let (A,σ) be an F-algebra with orthogonal involution. By [12, (7.1)], for any F-algebra with orthogonal involution (A,σ) with $\deg(A)$ even and any $a,b\in\mathrm{Alt}(A,\sigma)$ we have $\mathrm{Nrd}_A(a)F^{\times 2}=\mathrm{Nrd}_A(b)F^{\times 2}$. Therefore, as in [12, §7], we may make the following definition. The *determinant of* (A,σ), denoted $\Delta(A,\sigma)$, is the square class of the reduced norm of an arbitrary alternating unit, that is

$$\Delta(A,\sigma)=\mathrm{Nrd}_A(a)\cdot F^{\times 2}\text{ in }F^\times/F^{\times 2}\qquad\text{for any }a\in\mathrm{Alt}(A,\sigma)\cap A^\times.$$

For the rest of this section, we assume that $\mathrm{char}(F)=2$. Let (A,σ) be a totally decomposable F-algebra with orthogonal involution. That is

$$(A,\sigma)\simeq\bigotimes_{i=1}^{n}(Q_i,\sigma_i)$$

where (Q_i,σ_i) are F-algebras with involution for $i=1,\ldots,n$. Note that we must have that (Q_i,σ_i) is orthogonal for all $i=1,\ldots,n$ by [12, (2.23)]. Let $d_i=\Delta(Q_i,\sigma_i)$. Then the bilinear Pfister form $\pi=\langle\langle d_1,\ldots,d_n\rangle\rangle$ over F does not depend on the choice of the F-quaternion algebras with involution (Q_i,σ_i) in the decomposition of (A,σ) by [7, (7.3)]. We call this bilinear Pfister form *the Pfister invariant of* (A,σ) and denote it by $\mathfrak{Pf}(A,\sigma)$. Note that by [7, (7.3)], for any field extension K/F we have that $\mathfrak{Pf}((A,\sigma)_K)=(\mathfrak{Pf}(A,\sigma))_K$.

Proposition 3.4. *Assume* $\mathrm{char}(F)=2$. *Let* (A,σ) *be a totally decomposable* F-*algebra with orthogonal involution. Then* (A,σ) *is anisotropic (resp. metabolic) if and only if* $\mathfrak{Pf}(A,\sigma)$ *is anisotropic (resp. metabolic).*

Proof. See [7, (7.5)]. □

4. Algebras with quadratic pair

We now recall the definition of and basic results we use on quadratic pairs. Let (A,σ) be an F-algebra with involution. We call an F-linear map $f:\mathrm{Sym}(A,\sigma)\to F$ a *semi-trace on* (A,σ) if it satisfies $f(x+\sigma(x))=\mathrm{Trd}_A(x)$ for all $x\in A$. By [4, (4.3)], if $\mathrm{char}(F)\neq 2$, then $\frac{1}{2}\mathrm{Trd}_A|_{\mathrm{Sym}(A,\sigma)}$ is the unique semi-trace on (A,σ). On the other hand, if $\mathrm{char}(F)=2$, then the existence of a semi-trace on (A,σ) implies that $\mathrm{Trd}_A(\mathrm{Sym}(A,\sigma))=\{0\}$ and hence by [12, (2.6)] that (A,σ) is symplectic.

Given an element $\ell\in A$ with $\ell+\sigma(\ell)=1$, the map $f:\mathrm{Sym}(A,\sigma)\to F$ given by $x\mapsto\mathrm{Trd}_A(\ell x)$ is a semi-trace on (A,σ), and conversely every semi-trace on (A,σ) is of this form by [12, (5.7)] (although the case where $\mathrm{char}(F)\neq 2$ and (A,σ) is symplectic is excluded there, the same proof applies). In this case, we say that the semi-trace f on (A,σ) *is given by* ℓ. For another element $\ell'\in A$ such that $\ell'+\sigma(\ell')=1$, we have that ℓ and ℓ' give the same semi-trace on (A,σ) if and only if $\ell-\ell'\in\mathrm{Alt}(A,\sigma)$ (see [12, (5.7)]).

An F-*algebra with quadratic pair* is a triple (A,σ,f) where (A,σ) is an F-algebra with involution, which is assumed to be orthogonal if $\mathrm{char}(F)\neq 2$ and symplectic if $\mathrm{char}(F)=2$, and where f is a semi-trace on (A,σ). In $\mathrm{char}(F)\neq 2$ the concept of an algebra with quadratic pair is equivalent to the concept of an algebra with orthogonal involution, as then the semi-trace given by $\frac{1}{2}$ is the unique semi-trace on (A,σ).

Given two F-algebras with quadratic pair (A,σ,f) and (B,τ,g), by an *isomorphism of F-algebras with quadratic pair* $\Phi:(A,\sigma,f)\to(B,\tau,g)$ we mean an isomorphism of the underlying F-algebras with involution satisfying $f=g\circ\Phi$.

Let (A,σ,f) be an F-algebra with quadratic pair. We call (A,σ,f) *isotropic* if there exists an element $s\in\mathrm{Sym}(A,\sigma)\backslash\{0\}$ such that $s^2=0$ and $f(s)=0$, and *anisotropic* otherwise. In particular, if (A,σ,f) is isotropic, then A has zero divisors. We call an idempotent $e\in A$ *hyperbolic with respect to σ and f* if $\sigma(e)=1-e$ and $f(eA\cap\mathrm{Sym}(A,\sigma))=\{0\}$. We say that the F-algebra with quadratic pair (A,σ,f) is *hyperbolic* if A contains a hyperbolic idempotent with respect to σ and f.

We describe, following [12, §5], how a nonsingular quadratic form gives rise to an algebra with quadratic pair. Let $\rho=(V,q)$ be a nonsingular quadratic form over F with polar form (V,b_q). By declaring

$$(v_1\otimes w_1)*(v_2\otimes w_2)=b_q(w_1,v_2)\cdot(v_1\otimes w_2)\quad\text{for}\quad v_1,v_2,w_1,w_2\in V$$

a product $*$ is defined on the tensor product $V\otimes_F V$ making it into an F-algebra. By declaring $\sigma(v\otimes w)=w\otimes v$ for $v,w\in V$ we obtain an F-involution σ on the F-algebra $V\otimes_F V$. Then by [12, (5.1)], the F-linear map $\Phi:V\otimes_F V\to\mathrm{End}_F(V)$ determined by

$$\Phi(u\otimes v)(w)=b_q(v,w)u\quad\text{for }u,v,w\in V$$

yields an isomorphism of F-algebras with involution $\mathrm{Ad}(V,b_q)\longrightarrow(V\otimes_F V,\sigma)$. According to [12, (5.11)] there is a unique semi-trace $f_q:\mathrm{Sym}(\mathrm{Ad}(V,b_q))\to F$ such that $f_q(\Phi(v\otimes v))=q(v)$ for $v\in V$, which yields an F-algebra with quadratic pair

$$\mathrm{Ad}(\rho)=(\mathrm{End}_F(V),\mathrm{ad}_{b_q},f_q)\,,$$

called the *adjoint F-algebra with quadratic pair of ρ*. We say that an F-algebra with quadratic pair (A,σ,f) is *adjoint to* ρ if $(A,\sigma,f)\simeq\mathrm{Ad}(\rho)$. By [12, (5.11)], for any split F-algebra with quadratic pair (A,σ,f), there exists a nonsingular quadratic form ρ over F such that $(A,\sigma,f)\simeq\mathrm{Ad}(\rho)$, and for two nonsingular quadratic forms ρ and ρ' over F, we have that $\mathrm{Ad}(\rho)\simeq\mathrm{Ad}(\rho')$ if and only if ρ and ρ' are similar.

Proposition 4.1. *Let ρ be a nonsingular quadratic form over F. Then ρ is isotropic (resp. hyperbolic) if and only if $\mathrm{Ad}(\rho)$ is isotropic (resp. hyperbolic).*

Proof. The statement on isotropy follows from [12, (6.3) and (6.6)]. See [12, (6.13)] for the statement on hyperbolicity. □

For any field extension K/F we will use the notation $(A,\sigma,f)_K$ for the K-algebra with quadratic pair (A_K,σ_K,f_K) where $f_K:\mathrm{Sym}(A_K,\sigma_K)\longrightarrow K$ is the canonical extension of f to a K-linear map.

5. Tensor products of involutions and quadratic pairs

In this section we consider the tensor product of an algebra with involution with an algebra with quadratic pair. This corresponds to notion of the tensor product of a symmetric bilinear form and a quadratic form. We show that this tensor product is associative with the tensor product of algebras with involution. This property underlies our definition of a totally decomposable quadratic pair in the following section.

Proposition 5.1. *Let (A,σ,f) be an F-algebra with quadratic pair and (B,τ) an F-algebra with involution. Then there is a unique semi-trace g on $(B,\tau)\otimes(A,\sigma)$ such that $g(s_1\otimes s_2)=\mathrm{Trd}_B(s_1)\cdot f(s_2)$ for all $s_1\in\mathrm{Sym}(B,\tau)$ and $s_2\in\mathrm{Sym}(A,\sigma)$.*

Moreover, if the semi-trace f on (A,σ) is given by ℓ, then g is the semi-trace on $(B,\tau)\otimes(A,\sigma)$ given by $1\otimes\ell$.

Proof. If $\mathrm{char}(F)\neq 2$ then this result is trivial. Assume that $\mathrm{char}(F)=2$. Note that as $1\otimes\ell+(\tau\otimes\sigma)(1\otimes\ell)=1\otimes 1$, the element $1\otimes\ell$ gives a semi-trace on $(B\otimes_F A,\tau\otimes\sigma)$ by [12, (5.7)]. For all $s_1\in\mathrm{Sym}(B,\tau)$ and $s_2\in\mathrm{Sym}(A,\sigma)$ we have that

$$\mathrm{Trd}_{B\otimes_F A}((1\otimes\ell)(s_1\otimes s_2))=\mathrm{Trd}_B(s_1)\cdot\mathrm{Trd}_A(\ell\cdot s_2)=\mathrm{Trd}_B(s_1)\cdot f(s_2)\,.$$

For the uniqueness statement, see [12, (5.18)]. □

Let (A,σ,f) be an F-algebra with quadratic pair and let (B,τ) be an F-algebra with involution, which is assumed to be orthogonal if $\mathrm{char}(F)\neq 2$. Then by [12, (2.23)], $(B,\tau)\otimes(A,\sigma)$ is orthogonal if $\mathrm{char}(F)\neq 2$ and symplectic if $\mathrm{char}(F)=2$. We denote by $(B,\tau)\otimes(A,\sigma,f)$ the F-algebra with quadratic pair $(B\otimes_F A,\tau\otimes\sigma,g)$, where g is the semi-trace g on $(B,\tau)\otimes(A,\sigma)$ characterised in Proposition 5.1.

Proposition 5.2. *Let φ be a symmetric bilinear form over F and ρ a nonsingular quadratic form over F. Then $\mathrm{Ad}(\varphi\otimes\rho)\simeq\mathrm{Ad}(\varphi)\otimes\mathrm{Ad}(\rho)$.*

Proof. See [12, (5.19)]. □

Proposition 5.3. *Let (B,τ) and (C,γ) be F-algebras with involution that are assumed to be orthogonal if $\mathrm{char}(F)\neq 2$ and let (A,σ,f) be an F-algebra with quadratic pair. Then*

$$((B,\tau)\otimes(C,\gamma))\otimes(A,\sigma,f)\simeq(B,\tau)\otimes((C,\gamma)\otimes(A,\sigma,f))\,.$$

Proof. Let $\Phi:(B\otimes_F C)\otimes_F A\to B\otimes_F(C\otimes_F A)$ be the natural F-algebra isomorphism. Clearly Φ is compatible with the involutions in the statement. By [12, (5.7)], f is given by some $\ell\in A$ with $\ell+\sigma(\ell)=1$. It follows from Proposition 5.1 that the semi-trace associated with $((B,\tau)\otimes(C,\gamma))\otimes(A,\sigma,f)$ is given by $(1\otimes 1)\otimes\ell$ and the semi-trace associated with $(B,\tau)\otimes((C,\gamma)\otimes(A,\sigma,f))$ is given by $1\otimes(1\otimes\ell)$. It then easily follows that Φ is an isomorphism between the F-algebras with quadratic pair. □

By Proposition 5.3, the tensor product of two algebras with involution on the one hand, and the tensor product of an algebra with involution with an algebra with quadratic pair on the other hand, are mutually associative. That is, for F-algebras with involution (A,σ) and (B,τ) and an F-algebra with quadratic pair (C,γ,f), the expression $(A,\sigma)\otimes(B,\tau)\otimes(C,\gamma,f)$ is unambiguous.

Proposition 5.4. *Assume $\mathrm{char}(F)=2$. Let (B,τ) and (C,σ) be F-algebras with symplectic involution. Then there exists a unique semi-trace h on $(B,\tau)\otimes(C,\sigma)$ such that $h(s_1\otimes s_2)=0$ for all $s_1\in\mathrm{Sym}(B,\tau)$ and $s_2\in\mathrm{Sym}(C,\sigma)$. Moreover, for any semi-trace f on (C,σ), h is the semi-trace associated with the F-algebra with quadratic pair $(B,\tau)\otimes(C,\sigma,f)$.*

Proof. For the existence and uniqueness of the semi-trace h, see [12, (5.20)]. Let f be any semi-trace on (C,σ). By [12, (5.7)], f is given by an element $\ell\in C$ with $\ell+\sigma(\ell)=1$. Let g be the semi-trace associated with $(B,\tau)\otimes(C,\sigma,f)$. Then by Proposition 5.1, g is given by $1\otimes\ell$. By [12, (2.6)], as (B,τ) is symplectic we have $\mathrm{Trd}_B(\mathrm{Sym}(B,\tau))=\{0\}$. Hence for all $s_1\in\mathrm{Sym}(B,\tau)$ and $s_2\in\mathrm{Sym}(C,\sigma)$ we have

$$\mathrm{Trd}_{B\otimes_F C}((1\otimes\ell)(s_1\otimes s_2))=\mathrm{Trd}_B(s_1)\cdot\mathrm{Trd}_C(\ell\cdot s_2)=0\,.$$

That is, g satisfies the characterising property of h in the statement. Therefore by the uniqueness of the semi-trace h, we have that $g = h$. $\square$

Given two F-algebras with symplectic involution (B, τ) and (C, σ), we may define a semi-trace h on $(B, \tau) \otimes (C, \sigma)$ in the following manner. If $\mathrm{char}(F) \neq 2$, then $(B, \tau) \otimes (C, \sigma)$ is orthogonal by [12, (2.23)] and we let $h = \frac{1}{2}\mathrm{Trd}_{B \otimes_F C}$. If $\mathrm{char}(F) = 2$, let h be the semi-trace on $(B, \tau) \otimes (C, \sigma)$ characterised in Proposition 5.4. We denote the F-algebra with quadratic pair $(B \otimes_F C, \tau \otimes \sigma, h)$ by $(B, \tau) \boxtimes (C, \sigma)$. If $\mathrm{char}(F) = 2$, then by Proposition 5.4 we have that $(B, \tau) \boxtimes (C, \sigma) \simeq (B, \tau) \otimes (C, \sigma, f)$ for any choice of semi-trace f on (C, σ). In particular, by Proposition 5.3, for an F-algebra with symplectic involution (A, σ), the expression $(A, \sigma) \otimes (B, \tau) \boxtimes (C, \gamma)$ is unambiguous.

Hence given a tensor product of two F-algebras with symplectic involution, there is natural choice of a semi-trace making this product into a quadratic pair. We now consider this quadratic pair in the case where the F-algebras with involution are F-quaternion algebras with their canonical involutions. Let $a \in F$ with $-4a \neq 1$ and $b \in F^\times$ and let $(Q, \gamma) = [a \cdot| \, b)$. Recall that this F-algebra with involution is orthogonal if $\mathrm{char}(F) \neq 2$ and symplectic if $\mathrm{char}(F) = 2$. Let $u \in Q$ be such that $u^2 = u + a$ and $\gamma(u) = 1 - u$ and let f be the semi-trace on (Q, γ) given by u. Then we denote the F-algebra with quadratic pair (Q, γ, f) by $[a \, \| \, b)$.

Proposition 5.5. *Let $a, c \in F$ such that $4a \neq -1 \neq 4c$ and $b, d \in F^\times$. Then*

$$[a \cdot|\cdot b) \boxtimes [c \cdot|\cdot d) \simeq [a + c + 4ac \,|\cdot b) \otimes [c \, \| \, bd)\,.$$

Proof. Let $(B, \sigma, f) = [a \cdot|\cdot b) \boxtimes [c \cdot|\cdot d)$, $(Q_1, \gamma_1) = [a \cdot|\cdot b)$ and $(Q_2, \gamma_2) = [c \cdot|\cdot d)$. Let $i, j \in Q_1$ be such that $i^2 = i + a$, $j^2 = b$ and $ij = j - ji$ and let $u, v \in Q_2$ be such that $u^2 = u + c$, $v^2 = d$ and $uv = v - vu$. In B we have that $\sigma(i \otimes 1) = 1 \otimes 1 - i \otimes 1$, $\sigma(j \otimes 1) = -j \otimes 1$, $\sigma(1 \otimes u) = 1 \otimes 1 - 1 \otimes u$ and $\sigma(1 \otimes v) = -1 \otimes v$.

Let $i' = i \otimes 1 + (1 - 2i) \otimes u$, $j' = j \otimes 1$, $u' = 1 \otimes u$ and $v' = j \otimes v$. Then one easily checks that

$$Q_1' = F \oplus Fi' \oplus Fj' \oplus Fi'j' \quad \text{and} \quad Q_2' = F \oplus Fu' \oplus Fv' \oplus Fu'v'$$

are σ-invariant F-subalgebras of B that commute elementwise with one another. We set $\tau_1 = \sigma|_{Q_1'}$ and $\tau_2 = \sigma|_{Q_2'}$. We have

$$(Q_1', \tau_1) \simeq [a + c + 4ac \,|\cdot b) \text{ and } (Q_2', \tau_2) \simeq [c \cdot| \, bd)\,.$$

Hence $(B, \sigma) \simeq [a + c + 4ac \,|\cdot b) \otimes [c \cdot| \, bd)$. If $\mathrm{char}(F) \neq 2$, then the semi-trace on (B, σ) is uniquely determined, and in this case there is nothing further to show.

Assume $\mathrm{char}(F) = 2$. Then $(B, \sigma, f) \simeq (Q_1, \gamma_1) \otimes (Q_2, \gamma_2, h)$ for any choice of semi-trace h on (Q_2, γ_2) by Proposition 5.4. Let h to be the semi-trace given by u. Then for all $s \in \mathrm{Sym}(Q_1 \otimes_F Q_2, \gamma_1 \otimes \gamma_2) = \mathrm{Sym}(Q_1' \otimes_F Q_2', \tau_1 \otimes \tau_2)$ we have that

$$\mathrm{Trd}_{Q_1 \otimes_F Q_2}((1 \otimes u) \cdot s) = \mathrm{Trd}_{Q_1 \otimes_F Q_2}(u' \cdot s) = \mathrm{Trd}_{Q_1' \otimes_F Q_2'}((1 \otimes u') \cdot s)\,.$$

Hence $(Q_1, \gamma_1) \otimes (Q_2, \gamma_2, h) \simeq (Q_1', \tau_1) \otimes (Q_2', \tau_2, g)$ for the semi-trace g on (Q_2', τ_2) given by u' by Proposition 5.1. That is, $(B, \sigma, f) \simeq [a + c + 4ac \,|\cdot b) \otimes [c \, \| \, bd)$. $\square$

6. Totally decomposable quadratic pairs

We call an F-algebra with quadratic pair *totally decomposable* if it is isomorphic to a tensor product of a totally decomposable F-algebra with involution and an F-quaternion algebra with quadratic pair. It follows from Proposition 5.3 that taking the tensor product of a totally decomposable F-algebra with involution and a totally decomposable F-algebra with quadratic pair gives a totally decomposable algebra with quadratic pair.

Let (A, σ, f) be a totally decomposable F-algebra with quadratic pair. Then there exists a totally decomposable F-algebra with quadratic pair (B, τ) and an F-quaternion algebra with involution (Q, γ, g) such that $(A, \sigma, f) \simeq (B, \tau) \otimes (Q, \gamma, h)$. If $\mathrm{char}(F) \neq 2$ then (B, τ) is necessarily orthogonal. We now show that, even if $\mathrm{char}(F) = 2$, we may always find a decomposition as above where (B, τ) is orthogonal. This will allow us in the next section to use the Pfister invariant of (B, τ) to study the quadratic pair (A, σ, f).

Proposition 6.1. *Let (A, σ, f) be a totally decomposable F-algebra with quadratic pair. Then there exists a totally decomposable F-algebra with orthogonal involution (B, τ) and an F-quaternion algebra with quadratic pair (Q, γ, g) such that $(A, \sigma, f) \simeq (B, \tau) \otimes (Q, \gamma, g)$.*

Proof. The result is trivial if $\mathrm{char}(F) \neq 2$. Assume that $\mathrm{char}(F) = 2$. As (A, σ, f) is totally decomposable, there exist F-quaternion algebras with involution (Q_i, σ_i) for $i = 1, \ldots, n-1$ and an F-quaternion algebra with quadratic pair (Q_n, γ, h) such that

$$(A, \sigma, f) \simeq (Q_1, \sigma_1) \otimes \ldots \otimes (Q_{n-1}, \sigma_{n-1}) \otimes (Q_n, \gamma, h)\,.$$

Suppose σ_i is symplectic. In particular, it is the canonical involution on Q_i, for some $i \in \{1, \ldots, n-1\}$. Then by Proposition 5.4, we have that

$$(Q_i, \sigma_i) \otimes (Q_n, \gamma, h) \simeq (Q_i, \sigma_i) \boxtimes (Q_n, \gamma)\,.$$

Hence, by Proposition 5.5, there exists an F-quaternion algebra with orthogonal involution (Q_i', τ) and an F-quaternion algebra with quadratic pair (Q_n', γ', h') such that

$$(Q_i, \sigma_i) \otimes (Q_n, \gamma, h) \simeq (Q_i', \tau) \otimes (Q_n', \gamma', h')\,.$$

Using this argument repeatedly for all $i = 1, \ldots, n-1$ such that σ_i is symplectic, we modify our expression of (A, σ, f) above to obtain the result. □

For interest, we also record a characteristic two specific counterpart of the previous statement, which produces a symplectic instead of an orthogonal factor.

Proposition 6.2. *Assume that $\mathrm{char}(F) = 2$. Let (A, σ, f) be a totally decomposable F-algebra with quadratic pair with $\deg(A) = 2^n$, where $n \geqslant 2$. Then there exist F-quaternion algebras with canonical involution (Q_i, γ_i) for $i = 1, \ldots, n$ such that $(A, \sigma, f) \simeq (Q_1, \gamma_1) \otimes \ldots \otimes (Q_{n-1}, \gamma_{n-1}) \boxtimes (Q_n, \gamma_n)$.*

Proof. By Proposition 3.3, for every F-quaternion algebra with orthogonal involution (Q, τ), there exists an $a \in F$ and $b \in F^\times$ such that $(Q, \tau) \simeq [a \,|\!\cdot b)$. Similarly, by [4, (5.6)], for every F-quaternion algebra with quadratic pair (Q, γ, f) there exists an $c \in F$ and $d \in F^\times$ such that $(Q, \gamma, f) \simeq [c \,\|\, d)$. The result thus follows using the isomorphism in Proposition 5.5 in a similar way as to how it is used in Proposition 6.1, but in the opposite direction. □

7. Totally decomposable quadratic pairs on a split algebra

We now prove our main result, that over fields of characteristic two a split algebra with totally decomposable quadratic pair is adjoint to a Pfister form. We use the following result, which is an approach unique to fields of characteristic two.

This approach gives more information on the m-fold Pfister form π adjoint to a totally decomposable quadratic pair on an algebra of degree 2^m over a field F of characteristic 2 after extending to splitting field K. Specifically, we show that we can always find an $(m-1)$-fold bilinear Pfister form φ defined over F such that φ_K factors π.

Proposition 7.1. *Assume* $\mathrm{char}(F) = 2$. *Let* (B, τ) *be a totally decomposable orthogonal* F*-algebra with involution,* (Q, γ, h) *be an* F*-quaternion algebra with quadratic pair and* $(A, \sigma, f) = (B, \tau) \otimes (Q, \gamma, h)$. *Then for any field extension* K/F *such that* A_K *is split, there exists a* 1*-fold Pfister form* π *over* K *such that* $(A, \sigma, f)_K \simeq \mathrm{Ad}((\mathfrak{Pf}(B, \tau))_K \otimes \pi)$.

Proof. Let $\varphi = \mathfrak{Pf}(B, \tau)$ and ρ a quadratic form over K with $(A, \sigma, f)_K \simeq \mathrm{Ad}(\rho)$. Note that $\dim(\rho) = 2\dim(\varphi)$.

Assume first that $(B, \tau)_K$ is metabolic. Then by [5, (A.5)] we have that $(A, \sigma, f)_K$ is hyperbolic and by Proposition 4.1 that ρ is hyperbolic. We may thus take π to be the hyperbolic 2-dimensional quadratic form.

Assume now that $(B, \tau)_K$ is not metabolic. Then the bilinear Pfister form φ_K is anisotropic by Proposition 3.4. We consider its function field $L = K(\varphi_K)$. Since φ_L is metabolic, it follows by Proposition 3.4 that $(B, \tau)_L$ is metabolic. Hence, $(A, \sigma, f)_L$ is hyperbolic by [5, (A.5)] and therefore ρ_L is hyperbolic by Proposition 4.1. By Proposition 2.1, there exists a non-trivial nonsingular quadratic form π' over K such that $\rho_{\mathrm{an}} \simeq \varphi_K \otimes \pi'$. We have

$$\dim(\varphi) \cdot \dim(\pi') = \dim(\rho_{\mathrm{an}}) \leqslant \dim(\rho) = 2\dim(\varphi)\,.$$

As $\mathrm{char}(F) = 2$, by [9, (7.32)] $\dim(\pi')$ is even. It follows that $\dim(\pi') = 2$ and $\rho_{\mathrm{an}} \simeq \rho$. In particular, π' is similar to a 1-fold Pfister form π and $\rho \simeq \rho_{\mathrm{an}} \simeq \varphi_K \otimes \pi'$. Hence ρ is similar to $\varphi_K \otimes \pi$. □

Corollary 7.2. *Assume that* $\mathrm{char}(F) = 2$. *Let* $n \in \mathbb{N}$*, let* (A, σ, f) *be a totally decomposable* F*-algebra with quadratic pair with* $\deg(A) = 2^{n+1}$ *and let* K/F *be a field extension such that* A_K *is split. Then there exists an* n*-fold bilinear Pfister form* φ *over* K *and an* $(n+1)$*-fold Pfister form* ρ *over* K *such that* φ_K *factors* ρ *and* $(A, \sigma, f)_K \simeq \mathrm{Ad}(\rho)$.

Proof. For $n = 0$, this is trivial. Otherwise, by Proposition 6.1, there exists a totally decomposable F-algebra with orthogonal involution (B, τ) and an F-quaternion algebra with quadratic pair (Q, γ, h) such that $(A, \sigma, f) \simeq (B, \tau) \otimes (Q, \gamma, h)$. The result then follows from Proposition 7.1 with $\varphi = \mathfrak{Pf}(B, \tau)$. □

Theorem 7.3. *Let* ρ *be a nonsingular quadratic form over* F *with* $\dim(\rho) \geqslant 2$. *Then* $\mathrm{Ad}(\rho)$ *is totally decomposable if and only if* ρ *is similar to a Pfister form.*

Proof. If ρ is a Pfister form over F, then we can write $\rho \simeq \varphi \otimes \pi$ for a bilinear Pfister form φ and a 1-fold quadratic Pfister form π over F. Then by Proposition 5.2 we have that $\mathrm{Ad}(\rho) \simeq \mathrm{Ad}(\varphi) \otimes \mathrm{Ad}(\pi)$. The F-algebra with involution $\mathrm{Ad}(\varphi)$ is totally decomposable by Proposition 3.1. As $\mathrm{Ad}(\pi)$ is an F-quaternion algebra with quadratic pair, it follows that $\mathrm{Ad}(\rho)$ is totally decomposable. Assume conversely that $\mathrm{Ad}(\rho)$ is totally decomposable. If $\mathrm{char}(F) \neq 2$, then any quadratic pair is equivalent to an orthogonal involution and thus the result corresponds to [3, Thm. 1]. If $\mathrm{char}(F) = 2$, then ρ is similar to a Pfister form by Corollary 7.2. □

Let (A, σ, f) be a totally decomposable F-algebra with quadratic pair with $\deg(A) = 2^m$ and let K/F be a field extension such that A_K is split. Let π be the m-fold Pfister form over K such that $(A, \sigma, f)_K \simeq \mathrm{Ad}(\pi)$. In general, it is not possible to find an $(m-1)$-fold quadratic Pfister form ρ over F such that ρ_K factors π. This is illustrated by the following example, which is a variation of an example in [2, (3.9)]. In particular, this example shows that Corollary 7.2 cannot be extended to cover fields of characteristic different from 2, where quadratic and bilinear Pfister forms are equivalent.

Example 7.4. Let $n \in \mathbb{N}$ with $n \geqslant 4$. By [9, (38.4)] and its proof, there exists a field F such that all 3-fold quadratic Pfister forms over F are hyperbolic and there exist F-quaternion algebras $Q_1, \ldots, Q_n$ such that $A = Q_1 \otimes_F \cdots \otimes_F Q_n$ is a division F-algebra. In particular, we have $\deg(A) = \mathrm{ind}(A) = 2^n$. For $i = 1, \ldots, n$, let γ_i be the canonical involution on Q_i if $\mathrm{char}(F) = 2$ and an orthogonal involution on Q_i if $\mathrm{char}(F) \neq 2$. We obtain a totally decomposable F-algebra with quadratic pair

$$(A, \sigma, f) = (Q_1, \gamma_1) \otimes \cdots \otimes (Q_{n-1}, \gamma_{n-1}) \boxtimes (Q_n, \gamma_n)\,.$$

By [11, (3.3) and §2.4] there exists a field extension K/F such that A_K is split and $(A, \sigma, f)_K \simeq \mathrm{Ad}(\rho)$ for some quadratic form ρ over K such that $\mathrm{ind}(A)$ divides $i_W(\rho)$. As for any such ρ we have that $\dim(\rho) = 2^n = \mathrm{ind}(A)$ and $i_W(\rho) \leqslant \frac{1}{2}\dim(\rho)$, it follows that $i_W(\rho) = 0$, that is, ρ is anisotropic. In particular, ρ is not factored by π_K for any $(n-1)$-fold Pfister form π over F, as all such π are hyperbolic.

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