WITT EQUIVALENCE OF FUNCTION FIELDS OF CURVES OVER LOCAL FIELDS

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Abstract. Two fields are Witt equivalent if their Witt rings of symmetric bilinear forms are isomorphic. Witt equivalent fields can be understood to be fields having the same quadratic form theory. The behavior of finite fields, local fields, global fields, as well as function fields of curves defined over archimedean local fields under Witt equivalence is well-understood. Numbers of classes of Witt equivalent fields with finite numbers of square classes are also known in some cases. Witt equivalence of general function fields over global fields was studied in the earlier work [13] by the authors, and applied to study Witt equivalence of function fields of curves over global fields. In this paper we extend these results to local case, i.e. we discuss Witt equivalence of function fields of curves over local fields. As an application, we show that, modulo some additional assumptions, Witt equivalence of two such function fields implies Witt equivalence of underlying local fields.

1. Introduction

One of the fundamental problems in bilinear algebra is to classify fields with respect to Witt equivalence, that is equivalence defined by isomorphism of their Witt rings of symmetric bilinear forms (which also includes fields of characteristic two). The classification problem turns out to be manageable only when restricted to some specific classes of fields and, in fact, is completely resolved only in a few rather special cases. Trivial examples include quadratically closed fields, which are all Witt equivalent, their Witt ring being just $\mathbb{Z}/2\mathbb{Z}$, and real closed fields, which, again, are all Witt equivalent, their Witt ring being $\mathbb{Z}$. It is also not hard to see that finite fields of characteristic different from two are Witt equivalent if and only if they are of the same level, and that there are only two kinds of nonisomorphic Witt rings for this class of fields: $\mathbb{Z}/4\mathbb{Z}$ for fields with the number of elements congruent to $3 \pmod 4$, and the group ring $\mathbb{Z}/2\mathbb{Z}[F^*/F^{*2}]$ for the remaining finite fields F, $\operatorname{char} F \neq 2$.

A slightly more involved, but still to be found in graduate-level textbooks, is the case of local fields, that is complete discrete valued fields with finite residue fields. Recall that archimedean local fields are either $\mathbb{R}$ or $\mathbb{C}$, non-archimedean local fields of characterictic 0 are finite extensions of $\mathbb{Q}_p$ (which include the dyadic case of extensions of $\mathbb{Q}_2$), and non-archimedean local fields of characterictic p are finite extensions of $\mathbb{F}_q((x))$, $q = p^n$, for some $n \in \mathbb{N}$. Each non-archimedean nondyadic local field F is Witt equivalent to either $\mathbb{Q}_3$, or to $\mathbb{Q}_5$, depending on whether the number of elements of its residue field is congruent to $3 \pmod 4$, or to $1 \pmod 4$.

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For a dyadic local field F, that is an extension of $\mathbb{Q}_2$ of degree n, if n is odd, then the Witt equivalence class of F depends only on n, and if n is even, then there are exactly two Witt equivalence classes, one with $\sqrt{-1} \in F$, and one with $\sqrt{-1} \notin F$. These results are to be found, for example, in Lam's classical monograph [30].

Witt equivalence preserves the order of the square class group of the field, and so classification of fields up to Witt equivalence reduces to classification of fields with a given cardinality of the group of square classes. Classification of fields having only finitely many square classes was in the scope of interest of a large part of the quadratic forms community in the late 1970s and early 1980s. Denote by q the number of square classes of a field F, and by $w(q)$ the number of classes of Witt equivalent fields of characteristic $\neq 2$ with q square classes. The exact values of $w(q)$ are known only for $q \leq 32$, were computed in a series of papers by Carson and Marshall [4], Cordes [9], Kula [28], Kula, Szczepanik and Szymiczek [29], Szczepanik [43], and Szymiczek [44], and are summarized in the following table:

q	1	2	4	8	16	32
$w(q)$	1	3	6	17	51	155

The classification of global fields with respect to Witt equivalence proved to be significantly more challenging than the classification of local fields. Recall that a global field is either a number field, i.e. a finite extension of $\mathbb{Q}$, or a function field in one variable over a finite field. Since completions of global fields at their primes are local fields, Witt equivalence of completions of global fields is well-understood. Witt equivalence of global fields was completely resolved by a remarkable local-global principle, whose three different proofs were given by Perlis, Szymiczek, Conner, Litherland [39], and Szymiczek [45], [46], which states that two global fields of characteristic $\neq 2$ are Witt equivalent if and only if their primes can be paired so that corresponding completions are Witt equivalent. Moreover, Baeza and Moresi [2] showed that any two global fields of characteristic 2 are Witt equivalent, and it is not difficult to see that a global field of characteristic 2 is never Witt equivalent to a global field of characteristic different from 2. As a consequence of the local-global principle, it is also possible to provide a complete list of invariants of Witt equivalence for number fields, as shown by Carpenter [3].

For fields K and k, recall that K is a function field over k, if K is a finitely generated field extension of k. If $\operatorname{trdeg}(K : k) = n$ we say K is a function field in n variables over k. The field of constants of K over k is the algebraic closure of k in K; it is a finite extension of k [31, Chapter 10, Proposition 3], so that, in general, we do not require that k is the field of constants of K over k. Two more classes of fields where Witt equivalence is well-understood are function fields in one variable over algebraically closed fields of characteristic $\neq 2$, and function fields in one variable over real closed fields. In the former case, it is implied by an old result by Tsen [47], that two function fields are Witt equivalent if and only if their underlying algebraically closed fields are of the same cardinality, whilst in the latter one, necessary and sufficient conditions for Witt equivalence were given by Koprowski [25] and Grenier-Boley and Hoffmann (with appendix by Scheiderer) [14].

Given the above historical background, it is natural to ask for criteria of Witt equivalence of function fields over global and local fields. This question was first addressed by Koprowski in [24]. Unfortunately, there appears to be a serious error in the proof of the main theorem in [24].

The authors of the present paper considered Witt equivalence of function fields over global fields in their earlier work [13]. In the spirit of the local-global principle by Perlis, Szymiczek, Conner, and Litherland, that established a bijection between valuations of two Witt equivalent global fields of characteristic $\neq 2$, the authors managed to show that Witt equivalence of two function fields over global fields induces in a canonical way a bijection $v \leftrightarrow w$ between Abhyankar valuations v of K having residue field not finite of characteristic 2 and Abhyankar valuations w of L having residue field not finite of characteristic 2 (see [13, Theorem 7.5]). The main tool for setting up this bijection was a method of constructing valuations described in [1], which is based, in turn, on earlier constructions, of a similar sort, described in [18] and [51]. These results were then applied to study Witt equivalence of function fields of curves defined over global fields, that is function fields in one variable over global fields; see [13, Corollary 8.1 and Corollary 8.2].

In the present work the authors extend the results on curves of [13] to function fields of curves defined over local fields. The main result of this article is Theorem 3.5, which states that Witt equivalence of two function fields in one variable over local fields of characteristic $\neq 2$ induces a canonical bijection between certain subsets of Abhyankar valuations of the corresponding fields. Contrary to the intuition that one might have developed based on the necessary and sufficient conditions for Witt equivalence of local and global fields, the case of function fields of curves over local fields (Theorem 3.5) is in no way easier to settle than the case of function fields of curves over global fields ([13, Corollary 8.1]). Theorem 3.5 is then applied to show that, under certain assumptions, Witt equivalence of two function fields of curves over local fields k and ℓ implies Witt equivalence of k and ℓ. This extends [13, Corollary 8.2] to the local case.

The main new results of the paper are found in Section 3. Throughout the entire exposition the authors make extensive use of the notion of hyperfields, which seem to provide a natural and convenient language to study Witt equivalence. In Section 2 we recall basic terminology, establish fundamental connections between hyperfields and valuations, apply the results of [1] to understand the behavior of valuations under Witt equivalence, recall the terminology of function fields and Abhyankar valuations, as well as of nominal transcendence degree. All of this is a summary of the material presented in Sections 2–6 of [13], and we refer to our earlier paper as far as background and most of the details are concerned.

2. Background and preliminaries

The main new tool used to study Witt equivalence here are hyperfields. They are objects like fields, but with addition allowed to be multivalued, and were introduced in 1956 by Krasner [26] and used for the approximation of valued fields. For the decades that followed, structures with multivalued addition have been better known to computer scientists, due to their applications to fuzzy logic, automata, cryptography, coding theory and hypergraphs (see [8], [10] and [53]), as well as, to some extent, to mathematicians with expertise in harmonic analysis (see [32]).

Recently, the hyperstructure theory has witnessed a certain revival in connection with various fields: in a series of papers by Connes and Consani [5], [6], [7], with applications to number theory, incidence geometry, and geometry in characteristic one, in works by Viro [49], [50], with applications to tropical geometry, by Izhakian and Rowen [17] and Izhakian, Knebusch and Rowen [16], with applications to recently introduced algebraic objects such as supertropical algebras, or by Lorscheid [33], [34] to blueprints – these are algebraic objects which aim to provide a firm algebraic foundation to tropical geometry. Jun applied the idea of hyperstructures to generalize the definition of valuations and developed the basic notions of algebraic geometry over hyperrings [19], [20], [21], and, finally, the second author introduced independently the notion of hyperfields (there called multifields) to study quadratic forms in [36] – these ideas were later applied, for example, to develop the theory of orderings of higher level for hyperstructures [11], [12].

Hyperfields provide a convenient and very natural way to describe Witt equivalence, as described in detail in [13]. In what follows we shall review the basic concepts and definitions used later in our paper. We point the reader to Sections 2–6 of [13] as far as background and proofs are concerned.

Just as remarked above, a *hyperfield* is a system $(H, +, \cdot, -, 0, 1)$ where H is a set, $+$ is a multivalued binary operation on H, i.e., a function from $H \times H$ to the set of all subsets of H, $\cdot$ is a binary operation on H, $- : H \to H$ is a function, and $0, 1$ are elements of H such that

- (I) $(H, +, -, 0)$ is a *canonical hypergroup*, terminology as in Mittas [38], i.e. for all $a, b, c \in H$,
  - (1) $c \in a + b \Rightarrow a \in c + (-b)$,
  - (2) $a \in b + 0$ iff $a = b$,
  - (3) $(a + b) + c = a + (b + c)$,
  - (4) $a + b = b + a$;
- (II) $(H, \cdot, 1)$ is a commutative monoid, i.e. for all $a, b, c \in H$,
  - (1) $(ab)c = a(bc)$,
  - (2) $ab = ba$,
  - (3) $a1 = a$;
- (III) $a0 = 0$ for all $a \in H$;
- (IV) $a(b + c) \subseteq ab + ac$;
- (V) $1 \neq 0$ and every non-zero element has a multiplicative inverse.

Hyperfields are made into a category be declaring a morphism from H_1 to H_2, where H_1, H_2 are hyperfields, to be a function $\alpha : H_1 \to H_2$ which satisfies $\alpha(a + b) \subseteq \alpha(a) + \alpha(b)$, $\alpha(ab) = \alpha(a)\alpha(b)$, $\alpha(-a) = -\alpha(a)$, $\alpha(0) = 0$, $\alpha(1) = 1$.

Quotient hyperfields, as defined below, will be of constant use in what follows. For a subgroup T of H^*, where H is a hyperfield, denote by $H/_mT$ the set of equivalence classes with respect to the equivalence relation $\sim$ on H defined by

$$a \sim b \text{ if and only if } as = bt \text{ for some } s, t \in T.$$

The operations on $H/_mT$ are the obvious ones induced by the corresponding operations on H: denoting by $\overline{a}$ the equivalence class of a set $\overline{a}\overline{b} = \overline{ab}$, $-\overline{a} = \overline{-a}$, $0 = \overline{0}$, $1 = \overline{1}$, and

$$\overline{a} \in \overline{b} + \overline{c} \text{ if and only if } as \in bt + cu \text{ for some } s, t, u \in T.$$

$(H/_mT, +, \cdot, -, 0, 1)$ is then a hyperfield that we shall refer to as quotient hyperfield. The group of non-zero elements of $H/_mT$ is H^*/T.

For a hyperfield $H = (H, +, \cdot, -, 0, 1)$ the *prime addition* on H is defined by

$$a +' b = \begin{cases} a+b, & \text{if one of } a, b \text{ is zero ,} \\ a+b \cup \{a,b\}, & \text{if } a \neq 0,\ b \neq 0,\ b \neq -a, \\ H, & \text{if } a \neq 0,\ b \neq 0,\ b = -a. \end{cases}$$

For any hyperfield $H := (H, +, \cdot, -, 0, 1)$, $H' := (H, +', \cdot, -, 0, 1)$ is also a hyperfield [13, Proposition 2.1]. We shall call H' the *prime* of the hyperfield H. One easily verifies that if T is a subgroup of H^* then $H'/_m T = (H/_m T)'$.

Let K be a field. The *quadratic hyperfield* of K, denoted $Q(K)$, is defined to be the prime of the hyperfield $K/_m K^{*2}$. The interest in $Q(K)$ stems from its connection to the algebraic theory of quadratic forms. Denote by $W(K)$ the Witt ring of non-degenerate symmetric bilinear forms over K; see [30], [35] or [52] for the definition in case $\mathrm{char}(K) \neq 2$ and [22], [23] or [37] for the definition in the general case. A (non-degenerate diagonal) *binary form* over K is an ordered pair $\langle \overline{a}, \overline{b} \rangle$, $\overline{a}, \overline{b} \in K^*/K^{*2}$. The *value set* of such a form, denoted by $D_K\langle \overline{a}, \overline{b} \rangle$, is the set of non-zero elements of $\overline{a} + \overline{b}$, i.e., $D_K\langle \overline{a}, \overline{b} \rangle$ is the image under $K^* \to K^*/K^{*2}$ of the subset $D_K\langle a, b \rangle$ of K^* defined by

$$D_K\langle a, b \rangle := \begin{cases} K^*, & \text{if } -ab \in K^{*2}, \\ \{z \in K^* : z = ax^2 + by^2, x, y \in K\}, & \text{otherwise.} \end{cases}$$

Two binary forms $\langle \overline{a}, \overline{b} \rangle$ and $\langle \overline{c}, \overline{d} \rangle$ are considered to be *equivalent,* denoted $\langle \overline{a}, \overline{b} \rangle \approx \langle \overline{c}, \overline{d} \rangle$, if $\overline{c} \in D_K\langle \overline{a}, \overline{b} \rangle$ and $\overline{a}\overline{b} = \overline{c}\overline{d}$.

A hyperfield isomorphism α between two quadratic hyperfields $Q(K)$ and $Q(L)$, where K, L are fields, can be viewed as a group isomorphism $\alpha : K^*/K^{*2} \to L^*/L^{*2}$ such that $\alpha(-\overline{1}) = -\overline{1}$ and

$$\alpha(D_K\langle \overline{a}, \overline{b} \rangle) = D_L\langle \alpha(\overline{a}), \alpha(\overline{b}) \rangle \text{ for all } \overline{a}, \overline{b} \in K^*/K^{*2}.$$

We say two fields K and L are *Witt equivalent,* denoted $K \sim L$, to indicate that $W(K)$ and $W(L)$ are isomorphic as rings. Combining the results in [2], [15], and [35] one gets that $K \sim L$ iff $Q(K)$ and $Q(L)$ are isomorphic as hyperfields (for details, see [13, Proposition 3.2]).

For two hyperfields H_1, H_2 a morphism $\iota : H_1 \to H_2$ induces a morphism $\overline{\iota} : H_1/_m\Delta \to H_2$ where $\Delta := \{x \in H_1^* : \iota(x) = 1\}$. The morphism ι is said to be a *quotient morphism* if $\overline{\iota}$ is an isomorphism, or, equivalently, if ι is surjective, and

$$\iota(c) \in \iota(a) + \iota(b) \text{ if and only if } cs \in at + bu \text{ for some } s, t, u \in \Delta.$$

A morphism $\iota : H_1 \to H_2$ is said to be a *group extension* if ι is injective, every $x \in H_2^* \backslash \iota(H_1^*)$ is *rigid* , meaning $1 + x \subseteq \{1, x\}$, and

$$\iota(1 + y) = 1 + \iota(y), \text{ for all } y \in H_1, y \neq -1.$$

For a field K we adopt the standard notation from valuation theory: if v is a valuation on K, Γ_v denotes the value group, A_v the valuation ring, M_v the maximal ideal, U_v the unit group, K_v the residue field, and $\pi = \pi_v : A_v \to K_v$ the canonical homomorphism, i.e., $\pi(a) = a + M_v$. We say v is *discrete rank one* if $\Gamma_v = \mathbb{Z}$.

We will be interested in the subgroup $T = (1 + M_v)K^{*2}$ of K^*. Consider the canonical group isomorphism $\alpha : U_vK^{*2}/(1 + M_v)K^{*2} \to K_v^*/K_v^{*2}$ induced by

$$x \in U_v \mapsto \pi(x) \in K_v^*.$$

Define $\iota : Q(K_v) \to K/_mT$ by $\iota(0) = 0$ and $\iota(a) = \alpha^{-1}(a)$ for $a \in K_v^*/K_v^{*2}$. If v is non-trivial, then, by a variant of Springer's Theorem [41], [42], the map $Q(K) \to K/_mT$ defined by $\overline{x} \mapsto xT$ is a quotient morphism and ι is a group extension (see [13, Propositions 4.1, 4.2]). The cokernel of the group embedding $\alpha^{-1} : K_v^*/K_v^{*2} \to K^*/T$ is equal to $K^*/U_vK^{*2} \cong \Gamma_v/2\Gamma_v$. For this reason we sometimes say that $K/_mT$ is a *group extension of* $Q(K_v)$ *by the group* $\Gamma_v/2\Gamma_v$. If v is non-trivial and $\mathrm{char}(K_v) \neq 2$, then $K/_mT$ is naturally identified with $Q(\tilde{K}_v)$, where $\tilde{K}_v$ denotes the henselization of (K, v) [13, Proposition 4.3].

If v, v' are valuations on K with $v \preceq v'$, i.e. such that v' is a coarsening of v, meaning $A_v \subseteq A_{v'}$, then $M_{v'} \subseteq M_v$, and, consequently, $(1 + M_{v'})K^{*2} \subseteq (1 + M_v)K^{*2}$. Denote by $\overline{v}$ the valuation on $K_{v'}$ induced by v, that is

$$\overline{v}(\pi_{v'}(a)) = v(a), \text{ for } a \in U_{v'}.$$

The valuations $\overline{v}$ and v have the same residue field. If v and v' are non-trivial and v' is a proper coarsening of v, meaning $A_v \subsetneq A_{v'}$, then $K/_m(1+M_v)K^{*2}$ is a group extension of the hyperfield $K_{v'}/_m(1 + M_{\overline{v}})K_{v'}^{*2}$ in a natural way, and the following diagram of hyperfields and hyperfield morphisms is commutative:

$$\begin{tikzcd}
Q(K) \arrow[r] & K/_m(1+M_{v'})K^{*2} \arrow[r] & K/_m(1+M_v)K^{*2} \\
& Q(K_{v'}) \arrow[u] \arrow[r] & K_{v'}/_m(1+M_{\overline{v}})K_{v'}^{*2} \arrow[u] \\
& & Q(K_v) \arrow[u]
\end{tikzcd}$$

Here, the horizontal arrows are quotient morphisms and the vertical arrows are group extensions.

Let T be a subgroup of K^*. Adopting the well-known terminology from the algebraic theory of quadratic forms, we say that $x \in K^*$ is T-*rigid* if $T + Tx \subseteq T \cup Tx$, and denoting by

$$B(T) := \{x \in K^* : \text{ either } x \text{ or } -x \text{ is not } T\text{-rigid}\}$$

we will refer to the elements of $B(T)$ as to the T-*basic* elements. Note that if $x \in K^*$ is T-rigid and $y = tx$, for some $t \in T$, then y is T-rigid. Consequently, $B(T)$ is a union of cosets of T. The element -1 is not T-rigid, because $0 \in T - T$, so $\pm T \subseteq B(T)$. If, in fact, $\pm T = B(T)$, and either $-1 \in T$ or T is additively closed, we shall say that the subgroup T is *exceptional.* If $H \subseteq K^*$ is a subgroup containing $B(T)$, then there exists a subgroup $\hat{H}$ of K^* such that $H \subseteq \hat{H}$ and $(\hat{H} : H) \leq 2$, and a valuation v of K such that $1 + M_v \subseteq T$ and $U_v \subseteq \hat{H}$. Moreover, $\hat{H}$ can be taken to be simply H, unless T is exceptional [1, Theorem 2.16]. By [35, Theorem 5.18] $B(K^{*2})$ is a subgroup of K^*, and in the case when $T = (1 + M_v)K^{*2}$, for some non-trivial valuation v of K, $B(T) \subseteq U_vK^{*2}$ and

$$B(T) = \{x \in K^* : \overline{x} = \iota(\overline{y}) \text{ for some } y \in B(K_v^{*2})\},$$

where $\iota : Q(K_v) \hookrightarrow K/_mT$ is the morphism described above. $B(T)$ is a group and the group isomorphism $\iota : K_v^*/K_v^{*2} \to U_vK^{*2}/T$ induces a group isomorphism $B(K_v^{*2})/K_v^{*2} \to B(T)/T$. T is exceptional if and only if K_v^{*2} is exceptional. See [13, Proposition 4.6] for details.

We will make frequent use of the following result:

Theorem 2.1 ([13, Theorem 5.3])**.** *Suppose K, L are fields, $\alpha : Q(K) \to Q(L)$ is a hyperfield isomorphism and v is a valuation on K such that Γ_v is finitely generated as an abelian group. Suppose either (i) the basic part of $(1+M_v)K^{*2}$ is U_vK^{*2} and $(1+M_v)K^{*2}$ is unexceptional, or (ii) the basic part of $(1+M_v)K^{*2}$ is $(1+M_v)K^{*2}$ and $(1+M_v)K^{*2}$ has index 2 in U_vK^{*2}. Then there exists a valuation w on L such that the image of $(1+M_v)K^{*2}/K^{*2}$ under α is $(1+M_w)L^{*2}/L^{*2}$ and $(L^* : U_wL^{*2}) \geq (K^* : U_vK^{*2})$. If (i) holds, then the image of U_vK^{*2}/K^{*2} under α is U_wL^{*2}/L^{*2}.*

If K, L are fields and $\alpha : Q(K) \to Q(L)$ is a hyperfield isomorphism such that the image of $(1+M_v)K^{*2}/K^{*2}$ under α is $(1+M_w)L^{*2}/L^{*2}$, then α induces a hyperfield isomorphism $K/_m(1+M_v)K^{*2} \to L/_m(1+M_w)L^{*2}$ such that the obvious diagram

$$\begin{tikzcd} Q(K) \arrow[r] \arrow[d] & Q(L) \arrow[d] \\ K/_m(1+M_v)K^{*2} \arrow[r] & L/_m(1+M_w)L^{*2} \end{tikzcd} \tag{2.1}$$

commutes. If, in addition, the image of U_vK^{*2}/K^{*2} under α is U_wL^{*2}/L^{*2}, then α induces a hyperfield isomorphism $Q(K_v) \to Q(L_w)$ and a group isomorphism $\Gamma_v/2\Gamma_v \to \Gamma_w/2\Gamma_w$ such that the obvious diagrams

$$\begin{tikzcd} K/_m(1+M_v)K^{*2} \arrow[r] & L/_m(1+M_w)L^{*2} \\ Q(K_v) \arrow[r] \arrow[u] & Q(L_w) \arrow[u] \end{tikzcd} \tag{2.2}$$

and

$$\begin{tikzcd} Q(K)^* \arrow[r] \arrow[d] & Q(L)^* \arrow[d] \\ \Gamma_v/2\Gamma_v \arrow[r] & \Gamma_w/2\Gamma_w \end{tikzcd} \tag{2.3}$$

commute. We are assuming here that v, w are non-trivial. See [13, Proposition 5.4] for details.

The *nominal transcendence degree* of K is defined to be

$$\mathrm{ntd}(K) := \begin{cases} \mathrm{trdeg}(K : \mathbb{Q}) & \text{if } \mathrm{char}(K) = 0 \\ \mathrm{trdeg}(K : \mathbb{F}_p) - 1 & \text{if } \mathrm{char}(K) = p \neq 0 \end{cases}.$$

For any abelian group Γ, the *rational rank* of Γ, denoted $\mathrm{rk}_{\mathbb{Q}}(\Gamma)$, is defined to be the dimension of the $\mathbb{Q}$-vector space $\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$. If K is a function field over k and v is a valuation on K, the *Abhyankar inequality* asserts that

$$\mathrm{trdeg}(K : k) \geq \mathrm{rk}_{\mathbb{Q}}(\Gamma_v/\Gamma_{v|k}) + \mathrm{trdeg}(K_v : k_{v|k}),$$

where $v|k$ denotes the restriction of v to k. We will say the valuation v is *Abhyankar* (relative to k) if

$$\mathrm{trdeg}(K : k) = \mathrm{rk}_{\mathbb{Q}}(\Gamma_v/\Gamma_{v|k}) + \mathrm{trdeg}(K_v : k_{v|k}).$$

In this case it is well known that $\Gamma_v/\Gamma_{v|k}$ is finitely generated and K_v is a function field over $k_{v|k}$ [27, Corollary 26].

3. Function fields in one variable over local fields

As remarked in the Introduction, Witt equivalence of local fields is well understood. If k is a local field then

$$|k^*/k^{*2}| = \begin{cases} 1 \text{ if } k = \mathbb{C} \\ 2 \text{ if } k = \mathbb{R} \\ 4 \text{ if } k \text{ is } p\text{-adic}, p \neq 2 \\ 2^{[k:\hat{\mathbb{Q}}_2]+2} \text{ if } k \text{ is dyadic} \\ \infty \text{ if } \operatorname{char}(k) = 2 \end{cases}.$$

If k is p-adic and ℓ is p'-adic, $p, p' \neq 2$, then $k \sim \ell$ iff $-1 \in k^{*2} \Leftrightarrow -1 \in \ell^{*2}$. If k and ℓ are dyadic, then $k \sim \ell$ iff either $[k : \hat{\mathbb{Q}}_2] = [\ell : \hat{\mathbb{Q}}_2]$ is odd or $[k : \hat{\mathbb{Q}}_2] = [\ell : \hat{\mathbb{Q}}_2]$ is even and $-1 \in k^{*2} \Leftrightarrow -1 \in \ell^{*2}$. If $\operatorname{char}(k) = 2$ then $k = \mathbb{F}_{2^s}((x))$ for some $s \geq 1$. It follows that $k^2 = \mathbb{F}_{2^s}((x^2))$ so $[k : k^2] = 2$ and $1, x$ is a basis of k as a vector space over k^2. It follows from this and [2, Proposition 2.10] that if k, ℓ are local fields of characteristic 2 then $k \sim \ell$.

If k is a non-archimedean local field, we denote by v_0 the unique discrete rank one valuation of k.

The following two lemmas appear to be well-known. Although we do not know the precise reference, these results are obviously closely related to results in [40] and [48].

Lemma 3.1. *Suppose w is a non-trivial valuation on $k \in \{\mathbb{R}, \mathbb{C}\}$. Then Γ_w is divisible and k_w is algebraically closed.*

Proof. Case 1: Suppose $k = \mathbb{C}$. Suppose $x \in k^*$. For each integer $n \geq 1$ $\exists$ $y \in k^*$ such that $y^n = x$. Thus $w(x) = w(y^n) = nw(y)$. This proves Γ_w is divisible. Suppose $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ where $a_i \in A_w$, $i = 0, \ldots, n-1$. Clearly $\exists$ $r \in k$, $f(r) = 0$. Since A_w is integrally closed, $r \in A_w$. Thus the image of r in k_w is a root of the image of $f(x)$ in $k_w[x]$. This proves that k_w is algebraically closed.

Case 2: Suppose $k = \mathbb{R}$. Suppose $x \in k^*$. One of x, $-x$ is positive. Also $w(x) = w(-x)$. Thus, replacing x by $-x$ if necessary, we can assume $x > 0$. For each integer $n \geq 1$ $\exists$ $y \in k^*$ such that $y^n = x$. Thus $w(x) = w(y^n) = nw(y)$. This proves Γ_w is divisible. Suppose $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, $a_i \in A_w$, $i = 0, \ldots, n-1$. By the approximation theorem for V-topologies $\exists$ $b_0 \in k$ such that $w(b_0 - a_0) > 0$ and $|b_0 + 1| < 1$ (so $b_0 < 0$). Let $b_i = a_i$ for $i = 1, \ldots, n-1$. Clearly $g(x) := x^n + b_{n-1}x^{n-1} + \cdots + b_0$ has a root $r \in k$. Since A_w is integrally closed, $r \in A_w$. By construction $f(x)$ and $g(x)$ have the same image in $k_w[x]$. Thus the image of r in k_w is a root of the image of $f(x)$ in $k_w[x]$. This proves that k_w is algebraically closed. □

Lemma 3.2. *Suppose w is a non-trivial valuation on a non-archimedean local field k. Then either (1) $w = v_0$, or (2) w, v_0 are independent. In case (2), k_w is algebraically closed and Γ_w is p-divisible for each prime $p \neq \operatorname{char}(k_v)$. If $\operatorname{char}(k) = 0$ then Γ_w is also p-divisible for $p = \operatorname{char}(k_{v_0})$.*

Proof. Denote by u the finest common coarsening of v_0 and w. Since v_0 is discrete rank 1, $u = v_0$ or u is trivial. If $u = v_0$ then $w \preceq v_0$. Since the residue field k_v

is finite, the pushdown of w to k_{v_0} must be the trivial valuation, so $w = v_0$ in this case. If u is trivial then w and v_0 are obviously independent. Suppose now that v_0, w are independent. Let $x \in k^*$. Suppose first that $p \neq \text{char}(k_{v_0})$. By the approximation theorem $\exists\ y \in k^*$ such that $w(y-x) > w(x)$ and $v_0(y-1) > 0$. By Hensel's lemma $\exists\ z \in k^*$, $z^p = y$. Since $w(y-x) > w(x)$, $w(x) = w(y)$. Then $w(x) = w(y) = w(z^p) = pw(z)$. Suppose now that $\text{char}(k) = 0$ and $p \neq \text{char}(k_{v_0})$. In this case pick $y \in k^*$ so that $w(y-x) > w(x)$ and $v_0(y-1) > \frac{ep}{p-1}$ where $e = v_0(p)$. Here, we are identifying $\Gamma_v = \mathbb{Z}$. Then again, $\exists\ z \in k^*$, $z^p = y$, and one can argue as before. This proves the assertion concerning Γ_w. Suppose $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ where $a_i \in A_w$, $i = 0, \dots, n-1$. Let $p = \text{char}(k_{v_0})$. Case 1: n not divisible by p. In this case choose b_i, $i = 0, \dots, n-1$ so that $w(b_i - a_i) > 0$, $i = 0, \dots, n-1$, $v_0(b_0 + 1) > 0$, and $v_0(b_i) > 0$, $i = 1, \dots, n-1$. By Hensel's lemma, $g(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_0$ has a root $r \in k$. Then $r \in A_w$ and the image of r in k_w is a root of the image of $f(x)$ in $k_w[x]$. Case 2: n is divisible by p. In this case choose b_i, $i = 0, \dots, n-1$, so that $w(b_i - a_i) > 0$, $i = 0, \dots, n-1$, $v_0(b_1 + 1) > 0$, and $v_0(b_i) > 0$, $i = 0, \dots, n-1$, $i \neq 1$ and argue as before. □

The following result, which is an extension of [13, Lemma 7.2], appears to be well-known, but, as we are not able to point the reader to a reference, we decided to include a proof.

Lemma 3.3. *Suppose K is a function field in one variable over k, where k is a local field of characteristic $\neq 2$, or any real or algebraically closed field. Then*

(1) There are infinitely many discrete rank one Abhyankar valuations on K trivial on k.

(2) The group K^*/K^{*2} is infinite.

(3) $B(K^{*2}) = K^*$.

Proof. (1) This is clear if K is real or algebraically closed. Otherwise, there exists a subfield $K_0 \subseteq K$ with $\text{ntd}(K_0) = \text{ntd}(K) - 1$. Fix $x \in K$ transcendental over K_0. K is a finite extension of $K_0(x)$. The principal ideal domain $K_0[x]$ has infinitely many irreducibles. Each irreducible f of $K_0[x]$ defines a discrete rank one valuation v_f on $K_0(x)$ with residue field $K_0[x]/(f)$. The valuation v_f extends in some (possibly non-unique) way to a discrete rank one valuation on K whose residue field is some finite extension of $K_0[x]/(f)$.

(2) is true for any field K having infinitely many inequivalent discrete rank one valuations. Let $v_1, \dots, v_n$ be inequivalent discrete rank one valuations on K. Use the approximation theorem to produce $x_i \in K^*$, $i = 1, \dots, n$ so that $v_i(x_j) = \delta_{ij}$ (Kronecker's delta), for $i, j = 1, \dots, n$. Then the 2^n products $x_1^{e_1} \dots x_n^{e_n}$, $e_i \in \{0, 1\}$, belong to distinct square classes. This proves $|K^*/K^{*2}| \geq 2^n$. Since n is can be chosen to be any positive integer, the result follows.

(3) Suppose $x \in K^*$, $x \notin K^{*2}$. Suppose first that k is a p-adic local field. Let v be a discrete rank one Abhyankar valuation on K with $v|k$ trivial. Suppose first that $x \in (1 + M_v)K^{*2}$, say $x = uc^2$, $u \in 1 + M_v$, $c \in K^*$. Since K_v is a finite extension of k it is also a p-adic local field. Consequently, there exists $\pi(z) \in K_v^*$ and $\pi(d), \pi(e) \in K_v^*$ such that $\pi(z) = \pi(d)^2 + \pi(e)^2$, $\pi(z) \notin K_v^{*2}$. Take $y = c^2(d^2 + ue^2) = (cd)^2 + xe^2$. Then $y \notin K^{*2} \cup xK^{*2}$. If such a valuation v does not exist, then there exist inequivalent discrete rank one valuations v, w on K with $x \notin (1 + M_v)K^{*2}$, $x \notin (1 + M_w)K^{*2}$. In this case, use the approximation theorem to choose $a \in K^*$ so that $v(a^2) > v(x)$, $w(a^2) < w(x)$. Define $y = a^2 + x$. Then

$y = x(1+\frac{a^2}{x}) \in x(1+M_v)K^{*2}$, so $y \notin K^{*2}$. Similarly, $y = a^2(1+\frac{x}{a^2}) \in (1+M_w)K^{*2}$, so $y \notin xK^{*2}$.

Next suppose k is algebraically closed. Then K is a C_1-field so every two dimensional form $\langle 1, x\rangle$ is universal, i.e., x is not rigid. Finally, suppose k is real closed. The argument in [14, Corollary 3.8] shows that $u(K) = 2$ and either K is non-real or $K \sim k(t)$. If K is non-real then every two dimensional form $\langle 1, x\rangle$ is universal, i.e., x is not rigid. Suppose $K = k(t)$. We want to show x or $-x$ is not rigid for each $x \in K^*$. We may assume $x \notin K^{*2}$. Modifying x by a square we may assume $x = f(t) \in k[t]$ for some non-constant polynomial $f(t)$. Replacing x by $-x$ if necessary, there exists $p_1, p_2 \in k$ such that $f(p_1) < f(p_2) < 0$. Take $y = r + x$ where $r \in k$, $f(p_1) < -r < f(p_2)$. Then $y \in K^{*2} + xK^{*2}$, $y \notin K^{*2}$ (because y is negative at p_1) and $y \notin xK^{*2}$ (because $\frac{y}{x}$ is negative at p_1), so x is not rigid. □

Our next result can be viewed as some sort of local analog of [13, Theorem 7.3].

Theorem 3.4. *Let K be a function field in one variable over a local field k, $\mathrm{char}(k) \neq 2$. In case k is p-adic, denote by v_0 the canonical valuation on k. Let v be a valuation on K, and let $T = (1 + M_v)K^{*2}$.*

(I) *Abhyankar Cases:*

Case 1 *If v is trivial, then $T = K^{*2}$, $(K^* : T) = \infty$ and $B(T) = K^* = U_vK^{*2}$.*

Case 2 *If $\Gamma_v = \mathbb{Z}$ and $v|k$ is trivial, then*

(a) *if $k = \mathbb{R}$ or $\mathbb{C}$ then $(K^* : U_vK^{*2}) = 2$ and $B(T) = \pm T = U_vK^{*2}$,*

(b) *if k is p-adic, $p \neq 2$, $(K^* : U_vK^{*2}) = 2$, $(U_vK^{*2} : T) = 4$ and $B(T) = \pm T$,*

(c) *if k is dyadic, $(K^* : U_vK^{*2}) = 2$, $(U_vK^{*2} : T) = 2^{[K_v:\hat{\mathbb{Q}}_2]+2}$ and $B(T) = U_vK^{*2}$.*

Case 3 *If k is p-adic and $v|k = v_0$, then*

(a) *if $\Gamma_v = \mathbb{Z} = \Gamma_{v_0}$, $\mathrm{trdeg}(K_v : k_{v_0}) = 1$, then $(K^* : U_vK^{*2}) = 2$, $(U_vK^{*2} : T) = \infty$ and $B(T) = U_vK^{*2}$,*

(b) *If $\Gamma_v = \mathbb{Z} \times \mathbb{Z} = \Gamma_{v_0} \times \mathbb{Z}$, $\mathrm{trdeg}(K_v : k_{v_0}) = 0$, then $(K^* : U_vK^{*2}) = 4$, $(U_vK^{*2} : T) = \begin{cases} 2 \text{ if } p \neq 2 \\ 1 \text{ if } p = 2 \end{cases}$ and $B(T) = \pm T$.*

Case 4 *If $v|k$ has divisible value group and algebraically closed residue field, then*

(a) *if $\Gamma_v = \Gamma_{v|k} \times \mathbb{Z}$, $\mathrm{trdeg}(K_v : k_{v|k}) = 0$ then $(K^* : U_vK^{*2}) = 2$ and $B(T) = T = U_vK^{*2}$,*

(b) *if $\Gamma_v = \Gamma_{v|k}$, $\mathrm{trdeg}(K_v : k_{v|k}) = 1$ then $(K^* : T) = \infty$ and $B(T) = K^* = U_vK^{*2}$.*

(II) *Non-Abhyankar Cases:*

Case 5 *If k is p-adic, $v|k = v_0$, Γ_v/Γ_{v_0} is torsion, and $\mathrm{trdeg}(K_v : k_{v_0}) = 0$, then $\mathbb{Z} \subseteq \Gamma_v \subseteq \mathbb{Q}$, $(K^* : U_vK^{*2}) = 1$ or 2, $(U_vK^{*2} : T) = 1$ or 2 and $B(T) = \pm T$.*

Case 6 *If $v|k$ has divisible value group and algebraically closed residue field, then $\Gamma_v = \Gamma_{v|k}$ is divisible, $K_v = k_{v|k}$ is algebraically closed, and $K^* = U_vK^{*2} = B(T) = T$.*

Proof. We make use of the isomorphism $K^*/U_vK^{*2} \cong \Gamma_v/2\Gamma_v$, the isomorphism $U_vK^{*2}/(1 + M_v)K^{*2} \cong K_v^*/K_v^{*2}$ and the properties of the T-basic part $B(T)$ of K with $T = (1 + M_v)K^{*2}$ described above. The arguments are straightforward and follow, more or less, the same line of reasoning as the proof of [13, Theorem 7.3]. Case 1 uses Lemma 3.3. Case 2 uses the well-known structure of quadratic hyperfields of local fields. Case 3 (a) uses Lemma 3.3. Case 3 (b) uses well-known properties of quadratic hyperfields of finite fields. Case 4 uses Lemmas 3.1 and 3.2, and also Lemma 3.3, in Case 4 (b). Case 5 is more or less trivial. Case 6 uses Lemmas 3.1 and 3.2. □

Let K be a function field in one variable a local field k of characteristic $\neq 2$. Let $\mu_{K,0}$ be the set of valuations v of K such that $(K^* : U_vK^{*2}) = 2$, $2^3 \leq (U_vK^{*2} : (1+M_v)K^{*2}) < \infty$ and $B((1+M_v)K^{*2}) = U_vK^{*2}$. Let $\mu_{K,1}$ be the set of valuations v on K such that $(K^* : U_vK^{*2}) = 2$, $(U_vK^{*2} : (1+M_v)K^{*2}) = \infty$ and $B((1+M_v)K^{*2}) = U_vK^{*2}$. Let $\mu_{K,2}$ be the set of valuations v on K such that $(K^* : U_vK^{*2}) = 4$, $(U_vK^{*2} : (1+M_v)K^{*2}) = 2$ and $B((1+M_v)K^{*2}) = U_vK^{*2}$. Let $\mu_{K,3}$ be the set of valuations v on K such that $(K^* : U_vK^{*2}) = 4$, $(U_vK^{*2} : (1+M_v)K^{*2}) = 2$ and $B((1+M_v)K^{*2}) = (1+M_v)K^{*2}$.

Elements of $\mu_{K,i}$ correspond to various cases of Theorem 3.4: Case 2 (a) if $i = 0$, Case 3 (a) if $i = 2$, and Case 3 (b), (subcase $p \neq 2$) if $i = 2$ or 3. Note that

$$\mu_{K,0} \cup \mu_{K,1} \cup \mu_{K,2} \cup \mu_{K,3}$$

is a subset of the set of all Abhyankar valuations of K over k. Of course, some of the sets $\mu_{K,i}$ are empty. Specifically, $\mu_{K,0} \neq \emptyset$ iff k is dyadic, $\mu_{K,1} \neq \emptyset$ iff k is p-adic, $\mu_{K,2} \cup \mu_{K,3} \neq \emptyset$ iff k is p-adic, $p \neq 2$.

Theorem 3.5. *Suppose K, L are function fields in one variable over local fields of characteristic $\neq 2$ which are Witt equivalent via a hyperfield isomorphism $\alpha : Q(K) \to Q(L)$. Then for each $i \in \{0,1,2,3\}$ there is a uniquely defined bijection between $\mu_{K,i}$ and $\mu_{L,i}$ such that, if $v \leftrightarrow w$ under this bijection, then α maps $(1+M_v)K^{*2}/K^{*2}$ onto $(1+M_w)L^{*2}/L^{*2}$ and U_vK^{*2}/K^{*2} onto U_wL^{*2}/L^{*2} for $i \in \{0,1,2\}$ and such that α maps $(1+M_v)K^{*2}/K^{*2}$ onto $(1+M_w)L^{*2}/L^{*2}$ for $i = 3$.*

Proof. The correspondence $v \leftrightarrow w$ is just the one defined by Theorem 2.1. □

Theorem 3.6. *Let $K \sim L$ be function fields in one variable over local fields k and ℓ respectively, with fields of constants k and ℓ respectively. Then $k \sim \ell$ except possibly when k, ℓ are both dyadic local fields. In the latter case if there exists $v \in \mu_{K,0}$ with $K_v = k$ and $w \in \mu_{L,0}$ with $L_w = \ell$ then $k \sim \ell$.*

We will use the following two technical lemmas for the proof, which are also applicable for the proof of the corresponding result for function fields over global fields.

Lemma 3.7 ([13, Lemma 7.6])**.** *Suppose K is a field,* $\mathrm{char}(K) = 2$*, $\overline{x}, \overline{y} \in K^*/K^{*2}$, $\overline{x}, \overline{y} \neq 1$ and $\overline{y} \in D_K\langle 1, \overline{x}\rangle$. Then $D_K\langle 1, \overline{y}\rangle = D_K\langle 1, \overline{x}\rangle$.*

Lemma 3.8 ([13, Lemma 7.10])**.** *If K is a function field over a field k,* $\mathrm{char}(k) = 2$*, then*

$$[K : K^2] = 2^{\mathrm{trdeg}(K:k)} \cdot [k : k^2].$$

We now proceed to the proof of the theorem:

Proof. Suppose first that k, ℓ have characteristic different from 2. Observe that if k is dyadic then $\mu_{K,0} \neq \emptyset$ so $\mu_{L,0} \neq \emptyset$ and ℓ is also dyadic. Suppose in addition there exists $v \in \mu_{K,0}$ with $K_v = k$ and $w \in \mu_{L,0}$ with $L_w = \ell$. Let $v \leftrightarrow w$ be the bijection between $\mu_{K,0}$ and $\mu_{L,0}$ defined by Theorem 3.5. We know that $K_v \sim L_w$ for any v, w related in this way. Since K_v and L_w are dyadic local fields, this implies $[K_v : \hat{\mathbb{Q}}_2] = [L_w : \hat{\mathbb{Q}}_2]$ for any such v, w. We know also that $k \subseteq K_v$ and $\ell \subseteq L_w$. Choosing $v \leftrightarrow w$ so that $[K_v : \hat{\mathbb{Q}}_2] = [L_w : \hat{\mathbb{Q}}_2]$ is minimal, we see that $K_v = k$ and $L_w = \ell$.

Suppose now that k is p-adic, $p \neq 2$. Then $\mu_{K,2} \cup \mu_{K,3} \neq \emptyset$ and consequently $\mu_{L,2} \cup \mu_{L,3} \neq \emptyset$, so ℓ is p'-adic, for some prime $p' \neq 2$. Then k, ℓ each has level 1 or 2. If k has level 1 then K and consequently also L has level 1. Since ℓ is algebraically closed in L this implies ℓ has level 1. This proves k and ℓ have the same level, so $k \sim \ell$.

Suppose now that $k, \ell \in \{\mathbb{R}, \mathbb{C}\}$. If $k = \mathbb{C}$, $\ell = \mathbb{R}$, then K has level 1, L has level ≥ 2, contradicting $K \sim L$.

Suppose now that k, ℓ both have characteristic 2. Since $[k : k^2] = [\ell : \ell^2] = 2$, Lemma 3.8 implies $[K : K^2] = [L : L^2] = 4$ so, applying [2, Theorem 2.9], we deduce that $K \cong L$. Say $k = \mathbb{F}_{2^s}((x))$, $\ell = \mathbb{F}_{2^{s'}}((x))$. Since the polynomial $f(t) = t^{2^s} + t$ splits in k, it splits in K, and hence also in L. Since ℓ is algebraically closed in L, $f(t)$ splits in ℓ, and hence also in $\mathbb{F}_{2^{s'}}$. This proves $s|s'$. By symmetry, $s = s'$, so $k \cong \ell$.

It remains to show that if $\mathrm{char}(k) \neq 2$ and $\mathrm{char}(\ell) = 2$ then $K \not\sim L$. If $k = \mathbb{C}$ then K has u-invariant 2 so $D_K\langle 1, \overline{x}\rangle = K^*/K^{*2}$ for each $x \in K^*$. On the other hand, $[L : L^2] = 4$ and obviously $L + Lx$ is 2 dimensional over L^2 for any $x \in L^*$, $x \notin L^{*2}$. It follows that $D_L\langle 1, \overline{x}\rangle \neq L^*/L^{*2}$ for any $x \in L^*$, $x \notin L^{*2}$. This implies $K \not\sim L$. Suppose now that $k = \mathbb{R}$. If K is not formally real then K has level 2 and the above argument shows $K \not\sim L$. If K is formally real then obviously $K \not\sim L$.

Suppose now that k is p-adic. Arguing as in [13, Lemma 7.7] we will show that there exists $\overline{x}, \overline{y} \in K^*/K^{*2}$, $\overline{x}, \overline{y} \neq 1$ such that $\overline{y} \in D_K\langle 1, \overline{x}\rangle$, $D_K\langle 1, \overline{y}\rangle \not\subseteq D_K\langle 1, \overline{x}\rangle$. Fix inequivalent discrete rank one Abhyankar valuations v, w on K with $\mathrm{char}(K_v), \mathrm{char}(K_w) \neq 2$. Arguing as in the proof of Lemma 3.3 (3) we see that there exist $x \in K^*$, $y \in K^{*2} + xK^{*2}$ with $y \notin K^{*2} \cup xK^{*2}$. It is thus possible to choose x so that $v(x) = w(x) = 1$ and a_0, b_0 so that $w(a_0) = w(b_0) = 0$ and the image of $c = a_0^2 + b_0^2$ in the residue field of w is not a square. Define $y = a^2 + x$, $z = b^2 + y$ (so $z = a^2 + b^2 + x$) where a, b are such that $v(a) > 0$, $w(a - a_0) > 0$, $w(b - b_0) > 0$. Then $v(y) = v(x) = 1$, so $x, y \notin K^{*2}$ and $w(a^2 + b^2 - c) > 0$ so $z = a^2 + b^2 + x \in c(1 + M_w)$. Thus $x, y, z \in K^*$, $\overline{y} \in D_K\langle 1, \overline{x}\rangle$, $\overline{z} \in D_K\langle 1, \overline{y}\rangle$, $\overline{x} \neq 1$, $\overline{y} \neq 1$. Let $T = (1 + M_w)K^{*2}$. Thus $T + xT = T \cup xT$ and $z \notin T \cup xT$, so $\overline{z} \notin D_K\langle 1, \overline{x}\rangle$. Applying this, along with Lemma 3.7, we see that $K \not\sim L$ holds also in this case. □

References

[1] J.K. Arason, R. Elman, W. Jacob, Rigid elements, valuations, and realization of Witt rings. *J. Algebra* **110** (1987) 449–467.

[2] R. Baeza, R. Moresi, On the Witt-equivalence of fields of characteristic 2. *J. Algebra* **92** (1985), no. 2, 446–453.

[3] J. Carpenter, Finiteness theorems for forms over global fields. *Math. Z.* **209** no. 1 (1992) 153–166.

[4] A.B. Carson, M. Marshall, Decomposition of Witt rings. *Canad. J. Math.* **34** (1982) 1276–1302.

[5] A. Connes, C. Consani, From monoids to hyperstructures: in search of an absolute arithmetic, in: G. van Dijk (ed.) et al., *Casimir force, Casimir operators and Riemann hypothesis.*

Mathematics for innovation in industry and science. Proceedings of the conference, Fukuoka, Japan, November 9–13, 2009, 147-198, de Gruyter, Berlin, 2010.
[6] A. Connes, C. Consani, The hyperring of adele classes, *J. Number Theory* **131** (2011), 159–194.
[7] A. Connes, C. Consani, Universal Thickening of the Field of Real Numbers, in: A. Alaca (ed.) et al., *Advances in the Theory of Numbers. Proceedings of the Thirteenth Conference of the Canadian Number Theory Association*, 11–74, Fields Institute Communications **77**, Springer, New York, 2015.
[8] P. Corsini, V. Leoreanu-Fotea, Applications of hyperstructure theory, *Springer Science & Business Media*, Dordrecht, 2003.
[9] C. Cordes, The Witt group and equivalence of fields with respect to quadratic forms, *J. Algebra* **26** (1973), 400–421.
[10] B. Davvaz, V. Leoreanu-Fotea, Hyperring theory and applications, *International Academic Press*, Palm Harbor, FL, 2007.
[11] P. Gładki, Orderings of higher level in multifields and multirings, *Ann. Math. Silesianae* **24** (2010), 15–25.
[12] P. Gładki, M. Marshall, Orderings and signatures of higher level on multirings and hyperfields, J. K-Theory **10** (2012), 489–518.
[13] P. Gładki, M. Marshall, Witt equivalence of function fields over global fields, *Trans. Amer. Math. Soc.*, to appear
[14] N. Grenier-Boley, D.W. Hoffmann, Isomorphism criteria for Witt rings of real fields. With appendix by Claus Scheiderer. *Forum Math.* **25** (2013) 1–18.
[15] D.K. Harrison, Witt rings. *University of Kentucky Notes*, Lexington, Kentucky (1970).
[16] Z. Izhakian, M. Knebusch, L. Rowen, Layered tropical mathematics, *J. Algebra* **416** (2014), 200–273.
[17] Z. Izhakian, L. Rowen, Supertropical algebra, *Adv. Math.* **225** (2010), 2222-2286.
[18] W. Jacob, On the structure of Pythagorean fields. *J. Algebra* **68** (1981), no. 2, 247–267.
[19] J. Jun, Algebraic geometry over semi-structures and hyper-structures of characteristic one. PhD thesis, Johns Hopkins University, Baltimore, Maryland, USA, May 2015.
[20] J. Jun, Valuations of semirings. arXiv preprint arXiv:1503.01392, 2015.
[21] J. Jun, Hyperstructures of Affine Algebraic Group Schemes. arXiv preprint arXiv:1510.02979, 2015.
[22] J.L. Kleinstein, A. Rosenberg, Succinct and representational Witt rings. *Pacific J. Math.* **86** (1980) 99–137.
[23] M. Knebusch, A. Rosenberg, R. Ware, Structure of Witt rings and quotients of Abelian group rings. *Amer. J. Math.* **94** (1972), 119–155.
[24] P. Koprowski, Local-global principle for Witt equivalence of function fields over global fields. *Colloq. Math.* **91** (2002) 293–302.
[25] P. Koprowski, Witt equivalence of algebraic function fields over real closed fields. *Math. Z.* **242** (2002) 323–345.
[26] M. Krasner, Approximation des corps valués complets de caractéristique $p \neq 0$ par ceux de caractéristique 0. 1957 Colloque d'algèbre supérieure, tenu à Bruxelles du 19 au 22 décembre 1956 pp. 129–206 Centre Belge de Recherches Mathématiques Établissements Ceuterick, Louvain; Librairie Gauthier-Villars, Paris.
[27] F.-V. Kuhlmann, On places of algebraic function fields in arbitrary characteristic. *Advances in Math.* **188** (2004) 399–424.
[28] M. Kula, Fields with non-trivial Kaplanskys radical and finite square class number, *Acta Arith.* **37** (1981), 411–418.
[29] M. Kula, L. Szczepanik, K. Szymiczek, Quadratic forms over formally real fields with eight square classes. *Manuscripta Math.* **29**, (1979) 295–303.
[30] T.-Y. Lam, Introduction to quadratic forms over fields. *Graduate Studies in Mathematics* **67** American Mathematical Society, Providence, RI, 2005.
[31] S. Lang, Algebra. *Addison-Wesley Publishing Co.*, Reading, Mass. 1965
[32] G. Litvinov. Hypergroups and hypergroup algebras. arXiv preprint arXiv:1109.6596, 2011.
[33] O. Lorscheid, The geometry of blueprints: Part I: Algebraic background and scheme theory, *Adv. Math.* **229** (2012), 1804–1846.
[34] O. Lorscheid, Scheme theoretic tropicalization. arXiv preprint arXiv:1508.07949, 2015.

[35] M. Marshall, Abstract Witt rings, *Queen's Papers in Pure and Applied Math.* **57**, Queen's University, Kingston, Ontario (1980).
[36] M. Marshall, Real reduced multirings and multifields, *J. Pure and Appl. Alg.* **205** (2006) 452–468.
[37] J. Milnor, D. Husemoller, Symmetric bilinear forms. *Ergebnisse der Mathematik und ihrer Grenzgebiete*, Band **73**. Springer-Verlag, New York-Heidelberg, 1973.
[38] J. Mittas, Sur une classe d'hypergroupes commutatifs. *C. R. Acad. Sci. Paris Sér. A-B* **269** 1969 A485–A488.
[39] R. Perlis, K. Szymiczek, P.E. Conner, R. Litherland, Matching Witts with global fields. *Contemp. Math.* **155** (1994) 365–378.
[40] F.K. Schmidt, Mehrfach perfekte Körper. *Math. Ann.* **108** (1933), no. 1, 1–25.
[41] T.A. Springer, Quadratic forms over fields with a discrete valuation. I. Equivalence classes of definite forms. *Indag. Math.* **17**, (1955) 352–362.
[42] T.A. Springer, Quadratic forms over fields with a discrete valuation. II. *Indag. Math.* **18** (1956) 238–246.
[43] L. Szczepanik, Fields and quadratic form schemes with the index of radical not exceeding 16, *Ann. Math. Silesianae* **1** (1985), 23–46.
[44] K. Szymiczek, Quadratic forms over fields with finite square class number, *Acta Arith.* **28** (1975), 195–221.
[45] K. Szymiczek, Matching Witts locally and globally. *Math. Slovaca* **41** (1991) 315–330.
[46] K. Szymiczek, Hilbert-symbol equivalence of number fields, *Tatra Mount. Math. Publ.* **11** (1997), 7–16.
[47] Ch. Tsen, Divisionsalgebren über Funktionenkp̌rpern, *Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl.*, **I, No.44, II, No.48** (1933), 335–339.
[48] P. Vámos, Multiply maximally complete fields. *J. London Math. Soc.* (2) **12** (1975/76), no. 1, 103–111.
[49] O. Viro, Hyperfields for tropical geometry i. hyperfields and dequantization. arXiv preprint arXiv:1006.3034, 2010.
[50] O. Viro, On basic concepts of tropical geometry, *Proc. Steklov Inst. Math.* **273** (2011), 252–282.
[51] R. Ware, Valuation rings and rigid elements in fields. *Canad. J. Math.* **33** (1981), no. 6, 1338–1355.
[52] E. Witt, Theorie der quadratischen Formen in beliebigen Körpern. *Journal für die reine und angewandte Mathematik* **176** (1937) 31–44.
[53] P.-H. Zieschang, Theory of association schemes, *Springer Science & Business Media*, Dordrecht, 2006.

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