The chain equivalence of totally decomposable orthogonal involutions in characteristic two

A.-H. Nokhodkar

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Abstract

It is shown that two totally decomposable algebras with involution of orthogonal type over a field of characteristic two are isomorphic if and only if they are chain equivalent.

1 Introduction

The chain equivalence theorem for bilinear Pfister forms describes the isometry class of n-fold Pfister forms in terms of the isometry class of 2-fold Pfister forms (see [6, (3.2)] and [1, (A.1)]). There exist some related results in the literature for certain classes of central simple algebras over a field. In [11], the chain equivalence theorem for biquaternion algebras over a field of characteristic not two was proved (see [2] for the corresponding result in characteristic two). Also, the chain equivalence theorem for tensor products of quaternion algebras over a field of arbitrary characteristic was recently obtained in [3].

Let F be a field of characteristic 2. An algebra with involution (A, σ) over F is called *totally decomposable* if it decomposes as tensor products of quaternion F-algebras with involution. In [4], a bilinear Pfister form $\mathfrak{Pf}(A, \sigma)$, called the *Pfister invariant*, was associated to every totally decomposable algebra with orthogonal involution (A, σ) over F. In [9, (6.5)], it was shown that the Pfister invariant can be used to classify totally decomposable algebras with orthogonal involution over F. Regarding this result, an analogue chain equivalence for these algebras was defined in [9, (6.7)]. A relevant problem then is whether the isomorphism of such algebras with involution implies that they are chain equivalent (see [9, (6.8)]). In this work we present a solution to this problem.

2 Preliminaries

In this paper, F is a field of characteristic 2.
Let V be a finite dimensional vector space over F. A bilinear form $\mathfrak{b} : V \times V \to F$ is called *anisotropic* if $\mathfrak{b}(v, v) \neq 0$ for every nonzero vector $v \in V$. The form $\mathfrak{b}$ is called *metabolic* if V has a subspace W with $\dim W = \frac{1}{2} \dim V$ and $\mathfrak{b}|_{W \times W} = 0$. For $\lambda_1, \cdots, \lambda_n \in F^\times$, the form $\langle\langle \lambda_1, \cdots, \lambda_n \rangle\rangle := \bigotimes_{i=1}^n \langle 1, \lambda_i \rangle$ is called a *bilinear Pfister form*, where $\langle 1, \lambda_i \rangle$ is the diagonal form $\mathfrak{b}((x_1, x_2), (y_1, y_2)) = x_1 y_1 + \lambda_i x_2 y_2$. By [5, (6.3)], a bilinear Pfister form is either metabolic or anisotropic. We say that $\mathfrak{b} = \langle\langle \alpha_1, \cdots, \alpha_n \rangle\rangle$ and $\mathfrak{b}' = \langle\langle \beta_1, \cdots, \beta_n \rangle\rangle$ are *simply P-equivalent*,

if either $n = 1$ and $\alpha_1 F^{\times 2} = \beta_1 F^{\times 2}$ or $n \geqslant 2$ and there exist $1 \leqslant i < j \leqslant n$ such that $\langle\langle \alpha_i, \alpha_j \rangle\rangle \simeq \langle\langle \beta_i, \beta_j \rangle\rangle$ and $\alpha_k = \beta_k$ for all other k. We say that $\mathfrak{b}$ and $\mathfrak{b}'$ are *chain P-equivalent*, if there exist bilinear Pfister forms $\mathfrak{b}_0, \cdots, \mathfrak{b}_m$ such that $\mathfrak{b}_0 = \mathfrak{b}$, $\mathfrak{b}_m = \mathfrak{b}'$ and every $\mathfrak{b}_i$ is simply P-equivalent to $\mathfrak{b}_{i-1}$.

A *quaternion algebra* over F is a central simple F-algebra of degree 2. Every quaternion algebra Q has a *quaternion basis*, i.e., a basis $\{1, u, v, w\}$ satisfying $u^2 + u \in F$, $v^2 \in F^\times$ and $uv = w = vu + v$. It is easily seen that every element $v \in Q \setminus F$ with $v^2 \in F^\times$ extends to a quaternion basis $\{1, u, v, uv\}$ of Q. A tensor product of two quaternion algebras is called a *biquaternion algebra*.

An *involution* on a central simple F-algebra A is an antiautomorphism of A of period 2. Involutions which restrict to the identity on F are said to be of *the first kind*. An involution of the first kind is either *symplectic* or *orthogonal* (see [7, (2.5)]). The *discriminant* of an orthogonal involution σ is denoted by $\operatorname{disc}\sigma$ (see [7, (7.2)]). If K/F is a field extension, the scalar extension of (A, σ) to K is denoted by $(A, \sigma)_K$. We also use the notation $\operatorname{Alt}(A, \sigma) = \{a - \sigma(a) \mid a \in A\}$.

Let (A, σ) be a totally decomposable algebra of degree 2^n with orthogonal involution over F. In [9], it was shown that there exists a unique, up to isomorphism, subalgebra $S \subseteq F + \operatorname{Alt}(A, \sigma)$ such that (i) $x^2 \in F$ for $x \in S$; (ii) $\dim_F S = \deg_F A = 2^n$; (iii) S is self-centralizing; (iv) S is generated as an F-algebra by n elements. Also, S has a *set of alternating generators*, i.e., a set $\{u_1, \cdots, u_n\}$ consisting of units such that $S \simeq F[u_1, \cdots, u_n]$ and $u_{i_1} \cdots u_{i_l} \in \operatorname{Alt}(A, \sigma)$ for every $1 \leq l \leq n$ and $1 \leq i_1 < \cdots < i_l \leq n$. We denote the isomorphism class of the subalgebra S by $\Phi(A, \sigma)$. Note that $\Phi(A, \sigma)$ is commutative by [9, (3.2 (i))]. Also, if $\deg_F A \leqslant 4$, then $\Phi(A, \sigma)$ is unique as a set. In fact if A is a quaternion algebra, then $\Phi(A, \sigma) = F + \operatorname{Alt}(A, \sigma)$ by dimension count. If A is a biquaternion algebra, then $\Phi(A, \sigma) = F + \operatorname{Alt}(A, \sigma)^+$, where $\operatorname{Alt}(A, \sigma)^+$ is the set of square-central elements in $\operatorname{Alt}(A, \sigma)$ (see [10, (4.4)] and [10, (3.9)]).

Let $(A, \sigma) = \bigotimes_{i=1}^n (Q_i, \sigma_i)$ be a decomposition of (A, σ) and choose $\alpha_i \in F^\times$ such that $\operatorname{disc}\sigma_i = \alpha_i F^{\times 2}$, $i = 1, \cdots, n$. As in [4], we call the form $\langle\langle \alpha_1, \cdots, \alpha_n \rangle\rangle$ the *Pfister invariant* of (A, σ) and we denote it by $\mathfrak{Pf}(A, \sigma)$. Note that by [4, (7.2)], the Pfister invariant is independent of the decomposition of (A, σ).

With the above notations, we have the following results.

Theorem 2.1. ([9, (5.7)]) *For a totally decomposable algebra of degree 2^n with orthogonal involution (A, σ) over F, the following conditions are equivalent: (i) $(A, \sigma) \simeq (M_{2^n}(F), t)$, where t is the transpose involution. (ii) $\mathfrak{Pf}(A, \sigma) \simeq \langle\langle 1, \cdots, 1 \rangle\rangle$. (iii) $x^2 \in F^2$ for every $x \in \Phi(A, \sigma)$.*

Theorem 2.2. ([9, (6.5)]) *Let (A, σ) and (A', σ') be two totally decomposable algebras with orthogonal involution over F. If $A \simeq A'$ and $\mathfrak{Pf}(A, \sigma) \simeq \mathfrak{Pf}(A', \sigma')$, then $(A, \sigma) \simeq (A', \sigma')$.*

3 The chain lemma

Our first result, which strengthens [9, (5.6)], gives a natural description of the Pfister invariant.

Lemma 3.1. *Let (A, σ) be a totally decomposable algebra of degree 2^n with orthogonal involution over F. If $\mathfrak{Pf}(A, \sigma) \simeq \langle\langle \alpha_1, \cdots, \alpha_n \rangle\rangle$ for some $\alpha_1, \cdots, \alpha_n \in F^\times$, then there exists a decomposition $(A, \sigma) \simeq \bigotimes_{i=1}^n (Q_i, \sigma_i)$ into quaternion F-algebras with involution such that $\operatorname{disc}\sigma_i = \alpha_i F^{\times 2}$, $i = 1, \cdots, n$.*

Proof. By [9, (5.5)] and [9, (5.6)] there exists a set of alternating generators $\{u_1,\cdots,u_n\}$ of $\Phi(A,\sigma)$ such that $u_i^2=\alpha_i$, $i=1,\cdots,n$. If $\alpha_i\in F^2$ for every i, the result follows from (2.1). Thus (by re-indexing if necessary) we may assume that $\alpha_n\notin F^2$. It is enough to prove that there exists a decomposition $(A,\sigma)\simeq\bigotimes_{i=1}^n(Q_i,\sigma_i)$ such that $u_i\in\mathrm{Alt}(Q_i,\sigma_i)$, $i=1,\cdots,n$. We use induction on n. The case $n=1$ is evident, so suppose that $n>1$. Let $B=C_A(u_n)$ be the centralizer of u_n in A and set $K=F[u_n]=F(\sqrt{\alpha}_n)$. By [9, (6.3)] and [9, (6.4)], $(B,\sigma|_B)$ is a totally decomposable algebra with orthogonal involution over K and $\{u_1,\cdots,u_{n-1}\}$ is a set of alternating generators of $\Phi(B,\sigma|_B)$. By induction hypothesis there exists a decomposition

$$(B,\sigma|_B)\simeq_K(Q'_1,\sigma'_1)\otimes_K\cdots\otimes_K(Q'_{n-1},\sigma'_{n-1}),$$

into quaternion K-algebras with involution such that $u_i\in\mathrm{Alt}(Q'_i,\sigma'_i)$ for $i=1,\cdots,n-1$. By dimension count we have $\Phi(Q'_i,\sigma'_i)=K+Ku_i$. Since $K^2\subseteq F$ and $u_i^2\in F$, we get $x^2\in F$ for every $x\in\Phi(Q'_i,\sigma'_i)$. By [9, (6.1)] there exists a quaternion F-algebra $Q_i\subseteq Q'_i$ such that $(Q'_i,\sigma'_i)\simeq_K(Q_i,\sigma|_{Q_i})\otimes(K,\mathrm{id})$ and $u_i\in\mathrm{Alt}(Q_i,\sigma|_{Q_i})$, $i=1,\cdots,n-1$. Set $Q_n=C_A(Q_1\otimes\cdots\otimes Q_{n-1})$. Then Q_n is a quaternion F-algebra and $(A,\sigma)\simeq(Q_1,\sigma|_{Q_1})\otimes\cdots\otimes(Q_n,\sigma|_{Q_n})$. Since $u_n\in K=Z(B)\subseteq C_A(Q_1\otimes\cdots\otimes Q_{n-1})=Q_n$, we obtain $u_n\in Q_n$. Finally [8, (3.5)] implies that $u_n\in\mathrm{Alt}(Q_n,\sigma|_{Q_n})$. This completes the proof. □

Lemma 3.2. *Let K/F be a field extension satisfying $K^2\subseteq F$. Let Q and Q' be quaternion algebras over F and let $v'\in Q'\setminus F$ with $v'^2\in F^\times$. If there exists an isomorphism of K-algebras $f:Q'_K\simeq Q_K$ such that $f(v'\otimes1)\in Q\otimes F$, then there exists $\eta\in K$ such that $f(Q'\otimes F)\subseteq Q\otimes F[\eta]$. In addition, if $\{1,u,v,uv\}$ and $\{1,u',v',u'v'\}$ are respective quaternion bases of Q and Q' and $f(v'\otimes1)=v\otimes1$, then $f(u'\otimes1)=1\otimes\lambda+u\otimes1+v\otimes\eta$ for some $\lambda\in F$.*

Proof. The first statement follows from the second, since $f(u'\otimes1)$ and $f(v'\otimes1)$ generate $f(Q'\otimes F)$ as an F-algebra. To prove the second statement write $f(u'\otimes1)=1\otimes\eta_1+u\otimes\eta_2+v\otimes\eta_3+uv\otimes\eta_4$ for some $\eta_1,\cdots,\eta_4\in K$. Since

$$\begin{aligned}v\otimes1&=f(v'\otimes1)=f((u'v'+v'u')\otimes1)\\&=f(u'\otimes1)(v\otimes1)+(v\otimes1)f(u'\otimes1)=v\otimes\eta_2+v^2\otimes\eta_4,\end{aligned}$$

we get $\eta_4=0$ and $\eta_2=1$, i.e., $f(u'\otimes1)=1\otimes\eta_1+u\otimes1+v\otimes\eta_3$. Hence

$$\begin{aligned}f((u'^2+u')\otimes1)&=f(u'\otimes1)^2+f(u'\otimes1)\\&=1\otimes\eta_1^2+u^2\otimes1+v^2\otimes\eta_3^2+(uv+vu)\otimes\eta_3\\&\quad+1\otimes\eta_1+u\otimes1+v\otimes\eta_3\\&=1\otimes\eta_1^2+(u^2+u)\otimes1+v^2\otimes\eta_3^2+1\otimes\eta_1.\end{aligned}$$

As $f((u'^2+u')\otimes1)\in F$ and $K^2\subseteq F$, the above relations imply that $\eta_1\in F$, proving the result. □

The following definition was given in [9, (6.7)].

Definition 3.3. Let $(A,\sigma)=\bigotimes_{i=1}^n(Q_i,\sigma_i)$ and $(A',\sigma')=\bigotimes_{i=1}^n(Q'_i,\sigma'_i)$ be two totally decomposable algebras with orthogonal involution over F. We say that (A,σ) and (A',σ') are *simply equivalent* if either $n=1$ and $(Q_1,\sigma_1)\simeq(Q'_1,\sigma'_1)$

or $n \geqslant 2$ and there exist $1 \leqslant i < j \leqslant n$ such that $(Q_i, \sigma_i) \otimes (Q_j, \sigma_j) \simeq (Q'_i, \sigma'_i) \otimes (Q'_j, \sigma'_j)$ and $(Q_k, \sigma_k) \simeq (Q'_k, \sigma'_k)$ for $k \neq i, j$. We say that (A, σ) and (A', σ') are *chain equivalent* if there exist totally decomposable algebras with involution $(A_0, \tau_0), \cdots, (A_m, \tau_m)$ such that $(A, \sigma) = (A_0, \tau_0)$, $(A', \sigma') = (A_m, \tau_m)$ and for every $i = 0, \cdots, m-1$, (A_i, τ_i) and (A_{i+1}, τ_{i+1}) are simply equivalent. We write $(A, \sigma) \approx (A', \sigma')$ if (A, σ) and (A', σ') are chain equivalent.

Since the symmetric group is generated by transpositions, for every isometry ρ of $\{1, \cdots, n\}$ we have

$$\bigotimes_{i=1}^{n}(Q_i, \sigma_i) \approx \bigotimes_{i=1}^{n}(Q_{\rho(i)}, \sigma_{\rho(i)}).$$

Lemma 3.4. *Let $(A, \sigma) = \bigotimes_{i=1}^{3}(Q_i, \sigma_i)$ and $(A', \sigma') = \bigotimes_{i=1}^{3}(Q'_i, \sigma'_i)$ be totally decomposable algebras with orthogonal involution over F. Let $\alpha_i \in F^\times$ (resp. $\alpha'_i \in F^\times$) be a representative of the class $\operatorname{disc}\sigma_i$ (resp. $\operatorname{disc}\sigma'_i$), $i = 1, 2, 3$. Suppose that $A \simeq A'$, $\langle\langle \alpha_1, \alpha_2 \rangle\rangle \simeq \langle\langle \alpha'_1, \alpha'_2 \rangle\rangle$ and $\alpha_3 = \alpha'_3$. If $\langle\langle \alpha_1, \alpha_2 \rangle\rangle$ is metabolic, then (A, σ) and (A', σ') are chain equivalent.*

Proof. By [1, (A.5)] there exists $\beta \in F$ such that $\langle\langle \alpha_1, \alpha_2 \rangle\rangle \simeq \langle\langle 1, \beta \rangle\rangle$. Thus, according to (3.1) and (2.1), one can write

$$(Q_1, \sigma_1) \otimes (Q_2, \sigma_2) \simeq (M_2(F), t) \otimes (Q_0, \sigma_0),$$
$$(Q'_1, \sigma'_1) \otimes (Q'_2, \sigma'_2) \simeq (M_2(F), t) \otimes (Q'_0, \sigma'_0),$$

where (Q_0, σ_0) and (Q'_0, σ'_0) are quaternion algebras with orthogonal involution over F and $\operatorname{disc}\sigma_0 = \operatorname{disc}\sigma'_0 = \beta F^{\times 2}$. It follows that $A \simeq M_2(F) \otimes Q_0 \otimes Q_3$ and $A' \simeq M_2(F) \otimes Q'_0 \otimes Q'_3$. Since $A \simeq A'$, we get $Q_0 \otimes Q_3 \simeq Q'_0 \otimes Q'_3$. We also have

$$\mathfrak{Pf}(Q_0 \otimes Q_3, \sigma_0 \otimes \sigma_3) \simeq \langle\langle \beta, \alpha_3 \rangle\rangle \simeq \mathfrak{Pf}(Q'_0 \otimes Q'_3, \sigma'_0 \otimes \sigma'_3),$$

which implies that $(Q_0, \sigma_0) \otimes (Q_3, \sigma_3) \simeq (Q'_0, , \sigma'_0) \otimes (Q'_3, \sigma'_3)$ by (2.2). Thus,

$$\begin{aligned}(A, \sigma) = \bigotimes_{i=1}^{3}(Q_i, \sigma_i) &\approx (M_2(F), t) \otimes (Q_0, \sigma_0) \otimes (Q_3, \sigma_3) \\ &\approx (M_2(F), t) \otimes (Q'_0, \sigma'_0) \otimes (Q'_3, \sigma'_3) \approx \bigotimes_{i=1}^{3}(Q'_i, \sigma'_i) = (A', \sigma'). \end{aligned}$$ □

Lemma 3.5. ([9, (6.3)]) *Let (A, σ) be a totally decomposable algebra with orthogonal involution over F. For every $v \in \Phi(A, \sigma)$ with $v^2 \in F^\times \setminus F^{\times 2}$, there exists a σ-invariant quaternion F-algebra $Q \subseteq A$ such that $v \in \Phi(Q, \sigma|_Q)$.*

Proof. By [9, (6.3)], there exists a σ-invariant quaternion F-algebra $Q \subseteq A$ containing v. Write $v = \lambda + w$ for some $\lambda \in F$ and $w \in \operatorname{Alt}(A, \sigma)$. Then $w \in \operatorname{Alt}(Q, \sigma|_Q)$ by [8, (3.5)], hence $v = \lambda + w \in F + \operatorname{Alt}(Q, \sigma|_Q) = \Phi(Q, \sigma|_Q)$. □

Lemma 3.6. *Let $(A, \sigma) = \bigotimes_{i=1}^{3}(Q_i, \sigma_i)$ and $(A', \sigma') = \bigotimes_{i=1}^{3}(Q'_i, \sigma'_i)$ be two totally decomposable algebras with orthogonal involution over F. If $\mathfrak{Pf}(A, \sigma)$ and $\mathfrak{Pf}(A', \sigma')$ are simply P-equivalent and $A \simeq A'$, then (A, σ) and (A', σ') are chain equivalent.*

Proof. Choose invertible elements $v_i \in \operatorname{Alt}(Q_i, \sigma_i)$ and $v'_i \in \operatorname{Alt}(Q'_i, \sigma'_i)$, $i = 1, 2, 3$. Set $\alpha_i = v_i^2 \in F^\times$ and $\alpha'_i = v_i'^2 \in F^\times$. Then $\mathfrak{Pf}(A, \sigma) \simeq \langle\langle \alpha_1, \alpha_2, \alpha_3 \rangle\rangle$ and $\mathfrak{Pf}(A', \sigma') \simeq \langle\langle \alpha'_1, \alpha'_2, \alpha'_3 \rangle\rangle$. By re-indexing if necessary, we may assume that $\alpha_3 = \alpha'_3$ and $\langle\langle \alpha_1, \alpha_2 \rangle\rangle \simeq \langle\langle \alpha'_1, \alpha'_2 \rangle\rangle$. In view of (3.4), it suffices to consider the case where $\langle\langle \alpha_1, \alpha_2 \rangle\rangle$ is anisotropic. Set $K = F[v_1, v_2]$ and $K' = F[v'_1, v'_2]$. Then $K \simeq K' \simeq F(\sqrt{\alpha_1}, \sqrt{\alpha_2})$, because $\langle\langle \alpha_1, \alpha_2 \rangle\rangle$ is anisotropic. Consider $C_A(K) \simeq_K Q_3 \otimes_F K$ and $C_{A'}(K') \simeq_{K'} Q'_3 \otimes_F K'$. As $K \simeq K'$, one may consider $Q'_3 \otimes_F K'$ as a quaternion algebra over K, which is isomorphic to $Q_3 \otimes_F K$. Since $\operatorname{disc}(\sigma_3 \otimes \mathrm{id}) = \operatorname{disc}(\sigma'_3 \otimes \mathrm{id}) = \alpha_3 K^{\times 2}$, by [7, (7.4)] there exists an isomorphism of K-algebras with involution

$$f : (Q'_3 \otimes K', \sigma'_3 \otimes \mathrm{id}) \to (Q_3 \otimes K, \sigma_3 \otimes \mathrm{id}). \tag{1}$$

Dimension count shows that $\operatorname{Alt}(Q_3 \otimes K, \sigma_3 \otimes \mathrm{id}) = v_3 \otimes K$ and $\operatorname{Alt}(Q'_3 \otimes K', \sigma'_3 \otimes \mathrm{id}) = v'_3 \otimes K'$, hence $f(v'_3 \otimes 1) = v_3 \otimes \beta$ for some $\beta \in K$. The relations $v_3^2 = v_3'^2 = \alpha_3$ then imply that $\beta = 1$, i.e., $f(v'_3 \otimes 1) = v_3 \otimes 1 \in Q_3 \otimes F$. By (3.2) there exists $\eta \in K$ such that $f(Q'_3 \otimes F) \subseteq Q_3 \otimes F[\eta]$.

If $\eta \in F$, then $Q'_3 \simeq Q_3$. Hence $(Q'_3, \sigma'_3) \simeq (Q_3, \sigma_3)$ by [7, (7.4)]. The isomorphism $A \simeq A'$ then implies that $Q_1 \otimes Q_2 \simeq Q'_1 \otimes Q'_2$. Thus, $(Q_1, \sigma_1) \otimes (Q_2, \sigma_2) \simeq (Q'_1, \sigma'_1) \otimes (Q'_2, \sigma'_2)$ by (2.2), i.e., (A, σ) and (A', σ') are simply equivalent. Suppose that $\eta \notin F$, hence $\eta^2 \in F^\times \setminus F^{\times 2}$. As $\eta \in K \simeq \Phi(Q_1 \otimes Q_2, \sigma_1 \otimes \sigma_2)$, by (3.5) there exists a σ-invariant quaternion algebra $Q_4 \subseteq Q_1 \otimes Q_2$ such that $\eta \in \Phi(Q_4, \sigma|_{Q_4})$. Let Q_5 be the centralizer of Q_4 in $Q_1 \otimes Q_2$. Then

$$(Q_1, \sigma_1) \otimes (Q_2, \sigma_2) \simeq (Q_4, \sigma|_{Q_4}) \otimes (Q_5, \sigma|_{Q_5}), \tag{2}$$

which implies that

$$\mathfrak{Pf}(Q_4 \otimes Q_5, \sigma|_{Q_4} \otimes \sigma|_{Q_5}) \simeq \langle\langle \alpha_1, \alpha_2 \rangle\rangle, \tag{3}$$

by [4, (7.2)]. Since $f(Q'_3 \otimes F) \subseteq Q_3 \otimes F[\eta]$ and $\eta \in Q_4$, we get $f(Q'_3 \otimes F) \subseteq Q_3 \otimes Q_4$. Let Q_6 be the centralizer of $f(Q'_3 \otimes F)$ in $Q_3 \otimes Q_4$. Then

$$(Q_3, \sigma_3) \otimes (Q_4, \sigma|_{Q_4}) \simeq (Q_6, \sigma|_{Q_6}) \otimes f(Q'_3 \otimes F, \sigma'_3 \otimes \mathrm{id}). \tag{4}$$

By (2) and (4) we have

$$\begin{aligned} (A, \sigma) &= (Q_1, \sigma_1) \otimes (Q_2, \sigma_2) \otimes (Q_3, \sigma_3) \\ &\approx (Q_4, \sigma|_{Q_4}) \otimes (Q_5, \sigma|_{Q_5}) \otimes (Q_3, \sigma_3) \\ &\approx (Q_5, \sigma|_{Q_5}) \otimes (Q_3, \sigma_3) \otimes (Q_4, \sigma|_{Q_4}) \\ &\approx (Q_5, \sigma|_{Q_5}) \otimes (Q_6, \sigma|_{Q_6}) \otimes (Q'_3, \sigma'_3). \end{aligned} \tag{5}$$

We claim that $\operatorname{disc} \sigma|_{Q_6} = \operatorname{disc} \sigma|_{Q_4}$. If this is true, then

$$\mathfrak{Pf}(Q_5 \otimes Q_6, \sigma|_{Q_5} \otimes \sigma|_{Q_6}) \simeq \mathfrak{Pf}(Q_5 \otimes Q_4, \sigma|_{Q_5} \otimes \sigma|_{Q_4}).$$

Thus, using (3) we obtain

$$\mathfrak{Pf}(Q_5 \otimes Q_6, \sigma|_{Q_5} \otimes \sigma|_{Q_6}) \simeq \langle\langle \alpha_1, \alpha_2 \rangle\rangle \simeq \mathfrak{Pf}(Q'_1 \otimes Q'_2, \sigma'_1 \otimes \sigma'_2). \tag{6}$$

The chain equivalence (5) together with $A \simeq A'$ yields $A' \simeq Q_5 \otimes Q_6 \otimes Q'_3$, hence $Q_5 \otimes Q_6 \simeq Q'_1 \otimes Q'_2$. By (6) and (2.2) we have $(Q_5, \sigma|_{Q_5}) \otimes (Q_6, \sigma|_{Q_6}) \simeq (Q'_1, \sigma'_1) \otimes (Q'_2, \sigma'_2)$. This, together with (5) yields the desired chain equivalence:

$$\begin{aligned} (A, \sigma) &\approx (Q_5, \sigma|_{Q_5}) \otimes (Q_6, \sigma|_{Q_6}) \otimes (Q'_3, \sigma'_3) \\ &\approx (Q'_1, \sigma'_1) \otimes (Q'_2, \sigma'_2) \otimes (Q'_3, \sigma'_3) = (A', \sigma'). \end{aligned}$$

We now proceed to prove the claim. Let $v_6 \in \mathrm{Alt}(Q_6, \sigma|_{Q_6}) \subseteq Q_3 \otimes Q_4$ and $v_4 \in \mathrm{Alt}(Q_4, \sigma|_{Q_4})$ be two units. It is enough to show that $v_6 = \mu(1 \otimes v_4)$ for some $\mu \in F$. The element

$$v_6 \in \mathrm{Alt}(Q_6, \sigma|_{Q_6}) \subseteq \mathrm{Alt}(Q_3 \otimes Q_4, \sigma_3 \otimes \sigma|_{Q_4})$$

is square-central. Hence $v_6 \in \Phi(Q_3 \otimes Q_4, \sigma_3 \otimes \sigma|_{Q_4}) = F[v_3 \otimes 1, 1 \otimes v_4]$ by [10, (4.4)], i.e., there exist $a, b, c, d \in F$ such that

$$v_6 = a(v_3 \otimes 1) + b(1 \otimes v_4) + c(v_3 \otimes v_4) + d.$$

Since $\sigma_3 \otimes \sigma|_{Q_4}$ is orthogonal, by [7, (2.6)] we have $1 \notin \mathrm{Alt}(Q_3 \otimes Q_4, \sigma_3 \otimes \sigma|_{Q_4})$, hence $d = 0$. On the other hand by extending $\{v_3\}$ and $\{v_3'\}$ to quaternion bases $\{u_3, v_3, u_3v_3\}$ of Q_3 and $\{u_3', v_3', u_3'v_3'\}$ of Q_3' and using (3.2) for the map f in (1), we get

$$f(u_3' \otimes 1) = 1 \otimes \lambda + u_3 \otimes 1 + v_3 \otimes \eta \in Q_3 \otimes F[\eta] \subseteq Q_3 \otimes Q_4,$$

for some $\lambda \in F$. Thus,

$$\begin{aligned} v_6 f(u_3' \otimes 1) + f(u_3' \otimes 1) v_6 &= a(v_3u_3 + u_3v_3) \otimes 1 + bv_3 \otimes (v_4\eta + \eta v_4) \\ &\quad + c(v_3u_3 + u_3v_3) \otimes v_4 + c\alpha_3 \otimes (v_4\eta + \eta v_4) \\ &= av_3 \otimes 1 + (bv_3 + c\alpha_3) \otimes (v_4\eta + \eta v_4) + cv_3 \otimes v_4. \quad (7) \end{aligned}$$

As $v_4, \eta \in \Phi(Q_4, \sigma|_{Q_4})$, we have $\eta v_4 = v_4\eta$. Also, $v_6 \in Q_6$ commutes with $f(u_3' \otimes 1) \in f(Q_3' \otimes F)$, i.e., $v_6 f(u_3' \otimes 1) + f(u_3' \otimes 1)v_6 = 0$. Therefore, (7) leads to $a = c = 0$, hence $v_6 = b(1 \otimes v_4)$, proving the claim. ☐

Remark 3.7. Let $K = F(\sqrt{\alpha})$ be a quadratic field extension. Consider a bilinear Pfister form $\mathfrak{b}$ over F and let $\mathfrak{b}_K$ be the scalar extension of $\mathfrak{b}$ to K. If $\mathfrak{b}_K$ is metabolic then $\mathfrak{b} \otimes \langle\langle\alpha\rangle\rangle$ is also metabolic. In fact if $\mathfrak{b}$ is itself metabolic, the conclusion is evident. Otherwise, $\mathfrak{b}$ is anisotropic and the result follows form [5, (34.29 (2))] and [5, (34.7)]. Note that this implies that if $\mathfrak{b}_K \simeq \mathfrak{b}'_K$ for some bilinear Pfister form $\mathfrak{b}'$ over F, then $\mathfrak{b} \otimes \langle\langle\alpha\rangle\rangle \simeq \mathfrak{b}' \otimes \langle\langle\alpha\rangle\rangle$.

The following result gives a solution to [9, (6.8)].

Theorem 3.8. *Let $(A, \sigma) \simeq \bigotimes_{i=1}^n (Q_i, \sigma_i)$ and $(A', \sigma') \simeq \bigotimes_{i=1}^n (Q_i', \sigma_i')$ be two totally decomposable algebras with orthogonal involution over F. Then $(A, \sigma) \simeq (A', \sigma')$ if and only if (A, σ) and (A', σ') are chain equivalent.*

Proof. The "if" part is evident. To prove the converse, let $\deg_F A = 2^n$. The case $n \leqslant 2$ is trivial, so suppose that $n \geqslant 3$. By [4, (7.2)] we have $\mathfrak{Pf}(A, \sigma) \simeq \mathfrak{Pf}(A', \sigma')$. Hence by [1, (A. 1)], there exist bilinear Pfister forms $\mathfrak{b}_0, \cdots, \mathfrak{b}_m$ such that $\mathfrak{b}_0 = \mathfrak{Pf}(A, \sigma)$, $\mathfrak{b}_m = \mathfrak{Pf}(A', \sigma')$ and for $i = 0, \cdots, m-1$, $\mathfrak{b}_i$ is simply P-equivalent to $\mathfrak{b}_{i+1}$. Set $(A_0, \tau_0) = (A, \sigma)$ and $(A_m, \tau_m) = (A', \sigma')$. By (3.1) every $\mathfrak{b}_i$, $i = 1, \cdots, m-1$, can be realised as the Pfister invariant of a totally decomposable algebra with orthogonal involution (A_i, τ_i) over F with $A_i \simeq A$. We show that for $i = 0, \cdots, m-1$, (A_i, τ_i) and (A_{i+1}, τ_{i+1}) are chain equivalent. By induction, it suffices to consider the case where $m = 1$ (i.e., we may assume that $\mathfrak{Pf}(A, \sigma)$ and $\mathfrak{Pf}(A', \sigma')$ are simply P-equivalent). If $n = 3$ the result follows from (3.6). So suppose that $n \geqslant 4$.

For $i = 1, \cdots, n$, choose invertible elements $v_i \in \mathrm{Alt}(Q_i, \sigma_i)$ and $v'_i \in \mathrm{Alt}(Q'_i, \sigma'_i)$ and set $\alpha_i = v_i^2 \in F^\times$ and $\alpha'_i = v_i'^2 \in F^\times$. Then $\mathfrak{Pf}(A, \sigma) \simeq \langle\langle \alpha_1, \cdots, \alpha_n \rangle\rangle$ and $\mathfrak{Pf}(A', \sigma') \simeq \langle\langle \alpha'_1, \cdots, \alpha'_n \rangle\rangle$. By re-indexing if necessary, we may assume that

$$\langle\langle \alpha_1, \alpha_2 \rangle\rangle \simeq \langle\langle \alpha'_1, \alpha'_2 \rangle\rangle \quad \text{and} \quad \alpha_i = \alpha'_i \quad \text{for} \quad i = 3, \cdots, n. \tag{8}$$

Suppose first that $\alpha_n \in F^{\times 2}$. Then $(Q_n, \sigma_n) \simeq (Q'_n, \sigma'_n) \simeq (M_2(F), t)$ by (2.1). Set $C = \bigotimes_{i=1}^{n-1} Q_i$ and $C' = \bigotimes_{i=1}^{n-1} Q'_i$, so that $C \simeq C_A(Q_n) \simeq C_{A'}(Q'_n) \simeq C'$. Using (8) we have

$$\mathfrak{Pf}(C, \sigma|_C) \simeq \mathfrak{Pf}(C', \sigma'|_{C'}) \simeq \langle\langle \alpha_1, \cdots, \alpha_{n-1} \rangle\rangle,$$

hence $(C, \sigma|_C) \simeq (C', \sigma'|_{C'})$ by (2.2). By induction hypothesis, $(C, \sigma|_C)$ and $(C', \sigma'|_{C'})$ are chain equivalent. Thus, $(A, \sigma) \approx (C, \sigma|_C) \otimes (M_2(F), t)$ and (A', σ') $\approx (C', \sigma'|_{C'}) \otimes (M_2(F), t)$ are also chain equivalent.

Suppose now that $\alpha_n \notin F^{\times 2}$. Set $B = C_A(v_n)$, $B' = C_{A'}(v'_n)$, $K = F[v_n]$ and $K' = F[v'_n] \simeq F(\sqrt{\alpha_n}) \simeq K$. Then

$$(B, \sigma|_B) \simeq_K \bigotimes_{i=1}^{n-1} (Q_i, \sigma_i)_K \quad \text{and} \quad (B', \sigma'|_{B'}) \simeq_K \bigotimes_{i=1}^{n-1} (Q'_i, \sigma'_i)_{K'},$$

are totally decomposable algebras with orthogonal involution over K and K' respectively. Since $A \simeq_F A'$, we have $B \simeq_K B'$. Also, using (8) we get

$$\mathfrak{Pf}(B, \sigma|_B) \simeq \langle\langle \alpha_1, \cdots, \alpha_{n-1} \rangle\rangle_K \simeq \langle\langle \alpha'_1, \cdots, \alpha'_{n-1} \rangle\rangle_K \simeq \mathfrak{Pf}(B', \sigma'|_{B'}).$$

Thus, $(B, \sigma|_B) \simeq_K (B', \sigma'|_{B'})$ by (2.2). By induction hypothesis we get $(B, \sigma|_B)$ $\approx (B', \sigma'|_{B'})$. Again, using induction, it suffices to consider the case where $(B, \sigma|_B)$ and $(B', \sigma'|_{B'})$ are simply equivalent (note that every totally decomposable algebra with involution (B'', σ'') over K with $(B'', \sigma'') \simeq (B, \sigma|_B)$ has a decomposition of the form $\bigotimes_{i=1}^{n-1} (Q''_i, \sigma''_i)_K$, where every (Q''_i, σ''_i) is a quaternion algebra with orthogonal involution over F). By re-indexing, we may assume that $(Q_{n-2}, \sigma_{n-2})_K \otimes (Q_{n-1}, \sigma_{n-1})_K \simeq_K (Q'_{n-2}, \sigma'_{n-2})_K \otimes (Q'_{n-1}, \sigma'_{n-1})_K$ and

$$(Q_i, \sigma_i)_K \simeq_K (Q'_i, \sigma'_i)_{K'}, \quad \text{for} \quad i = 1, \cdots, n-3, \tag{9}$$

In particular, $\bigotimes_{i=2}^{n-1} (Q_i, \sigma_i)_K \approx \bigotimes_{i=2}^{n-1} (Q'_i, \sigma'_i)_{K'}$, which implies that

$$\langle\langle \alpha_2, \cdots, \alpha_{n-1} \rangle\rangle_K \simeq \langle\langle \alpha'_2, \cdots, \alpha'_{n-1} \rangle\rangle_K. \tag{10}$$

Since $n - 3 \geqslant 1$, (9) gives an isomorphism of K-algebras with involution

$$f : (Q_1, \sigma_1) \otimes (K, \mathrm{id}) \simeq (Q'_1, \sigma'_1) \otimes (K', \mathrm{id}).$$

The element $1 \otimes v_n$ lies in the center of $Q_1 \otimes K$. Thus, there exist $a, b \in F$ such that $f(1 \otimes v_n) = 1 \otimes (a + bv'_n)$. Squaring both sides implies that $\alpha_n = a^2 + b^2 \alpha_n$. The assumption $\alpha_n \notin F^2$ then yields $a = 0$ and $b = 1$, i.e., $f(1 \otimes v_n) = 1 \otimes v'_n$. As $(K', \mathrm{id}) \subseteq (Q'_n, \sigma'_n)$, the isomorphism f induces a monomorphism of F-algebras with involution

$$g : (Q_1, \sigma_1) \otimes (K, \mathrm{id}) \hookrightarrow (Q'_1, \sigma'_1) \otimes (Q'_n, \sigma'_n),$$

with $g(1 \otimes v_n) = 1 \otimes v'_n$. Let Q'_0 be the centralizer of $g(Q_1 \otimes F)$ in $Q'_1 \otimes Q'_n$. Then

$$(Q'_1, \sigma'_1) \otimes (Q'_n, \sigma'_n) \simeq_F (Q'_0, \sigma'|_{Q'_0}) \otimes (Q_1, \sigma_1). \tag{11}$$

Since the element $1 \otimes v_n \in Q_1 \otimes K$ commutes with $Q_1 \otimes F$, we get $1 \otimes v'_n = g(1 \otimes v_n) \in Q'_0$, which implies that $1 \otimes v'_n \in \mathrm{Alt}(Q'_0, \sigma'|_{Q'_0})$ by [8, (3.5)]. Thus,

$$\operatorname{disc} \sigma'|_{Q'_0} = \alpha_n F^{\times 2} \in F^\times / F^{\times 2}. \tag{12}$$

Using (11) we have

$$(A', \sigma') \approx (Q'_2, \sigma'_2) \otimes \cdots \otimes (Q'_{n-1}, \sigma'_{n-1}) \otimes (Q'_0, \sigma'|_{Q'_0}) \otimes (Q_1, \sigma_1). \tag{13}$$

This, together with $A \simeq A'$ implies that

$$Q_2 \otimes \cdots \otimes Q_n \simeq Q'_2 \otimes \cdots \otimes Q'_{n-1} \otimes Q'_0. \tag{14}$$

The isometry (10) and (3.7) show that

$$\langle\langle \alpha_2, \cdots, \alpha_{n-1}, \alpha_n \rangle\rangle \simeq \langle\langle \alpha'_2, \cdots, \alpha'_{n-1}, \alpha_n \rangle\rangle,$$

hence, thanks to (12), the Pfister invariants of $(D, \tau) := (Q_2, \sigma_2) \otimes \cdots \otimes (Q_n, \sigma_n)$ and $(D', \tau') := (Q'_2, \sigma'_2) \otimes \cdots \otimes (Q'_{n-1}, \sigma'_{n-1}) \otimes (Q'_0, \sigma'|_{Q'_0})$ are isometric. Using (14) and (2.2) we obtain $(D, \tau) \simeq (D', \tau')$. By induction hypothesis, (D, τ) and (D', τ') are chain equivalent. Thus, by (13) we have

$$\begin{aligned}(A', \sigma') &\approx (Q'_2, \sigma'_2) \otimes \cdots \otimes (Q'_{n-1}, \sigma'_{n-1}) \otimes (Q'_0, \sigma'|_{Q'_0}) \otimes (Q_1, \sigma_1) \\ &= (D', \tau') \otimes (Q_1, \sigma_1) \approx (D, \tau) \otimes (Q_1, \sigma_1) \\ &= (Q_2, \sigma_2) \otimes \cdots \otimes (Q_n, \sigma_n) \otimes (Q_1, \sigma_1) \approx (A, \sigma). \qquad \square\end{aligned}$$

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A.-H. Nokhodkar, `a.nokhodkar@kashanu.ac.ir`
Department of Pure Mathematics, Faculty of Science, University of Kashan, P. O. Box 87317-53153, Kashan, Iran.