RATIONALITY OF ALGEBRAIC CYCLES OVER FUNCTION FIELD OF SL$_1(A)$-TORSORS

RAPHAËL FINO

Abstract. In this note we prove a result comparing rationality of algebraic cycles over the function field of a SL$_1(A)$-torsor for a central simple algebra $A$ and over the base field.

Keywords: Chow groups, central simple algebras, principal homogeneous spaces.

Contents

1. Introduction 1
2. Preliminaries 2
3. Proof of the result 4
4. Exceptional projective homogeneous varieties 5
References 6

1. Introduction

Let $A$ be a central simple algebra over a field $F$ and let $\text{Nrd} : A^\times \to F^\times$ be the reduced norm homomorphism. We recall that the homomorphism $F^\times \to H^1(F, \text{SL}_1(A))$, associating to $c \in F^\times$ the $\text{SL}_1(A)$-torsor $X_c$ given by the equation $\text{Nrd} = c$, is surjective (with kernel $\text{Nrd}(A^\times)$) – see [7, Proposition 2.7.3] for instance.

The main purpose of this note is to prove the following theorem dealing with rationality of algebraic cycles over function field of SL$_1(A)$-torsors.

Theorem 1.1. Let $A$ be a central simple algebra of prime degree $p$ over a field $F$ and let $X$ be a SL$_1(A)$-torsor. Then

(i) for any equidimensional $F$-variety $Y$, the change of field homomorphism

$$\text{CH}(Y) \to \text{CH}(Y_{F(X)}),$$

where $\text{CH}$ is the integral Chow group, is surjective in codimension $< p + 1$.

(ii) it is also surjective in codimension $p + 1$ for a given $Y$ provided that the variety $X_{F(\zeta)}$ does not have any closed point of prime to $p$ degree for each generic point $\zeta \in Y$.

The method of proof mainly relies on the following statement. This proposition is a version of the result [3, Lemma 88.5] slightly altered to fit our situation (see also the proof of [8, Proposition 2.8]).

Date: 3 June 2015.

2010 Mathematics Subject Classification. 14C25; 20G15.
Proposition 1.2 (Karpenko, Merkurjev). Let $X$ be a smooth variety, and $Y$ an equidimensional variety. Given an integer $m$ such that for any nonnegative integer $i$ and any point $y \in Y$ of codimension $i$ the change of field homomorphism

$$\text{CH}^{m-i}(X) \rightarrow \text{CH}^{m-i}(X_{F(y)})$$

is surjective, the change of field homomorphism

$$\text{CH}^m(Y) \rightarrow \text{CH}^m(Y_{F(X)})$$

is also surjective.

The proof of Theorem 1.1 is given in Section 3. In Section 4, we describe how this theorem can be related to a similar result dealing with rationality of algebraic cycles over function field of projective homogeneous varieties under some groups of exceptional type.

Acknowledgements. This note has been conceived while I was visiting the University of Alberta and I would like to thank the Department of Mathematical and Statistical Sciences for the hospitality. I am very grateful to Nikita Karpenko for suggestions having improved this note and to Philippe Gille for Remark 4.2. This work has been supported by the Université Pierre et Marie Curie - Paris VI.

2. Preliminaries

2.1. Topological filtration and Chow groups. For any smooth variety $X$ over a field $F$ (in this paper, an $F$-variety is a separated scheme of finite type over $F$), one can consider the topological filtration on the Grothendieck ring $K_0(X)$, whose term of codimension $i$ is given by

$$\tau_i(X) = \langle [O_Z] | Z \hookrightarrow X \text{ and codim}(Z) \geq i \rangle,$$

where $[O_Z]$ is the class in $K_0(X)$ of the structure sheaf of a closed subvariety $Z$. We write $\tau_{i/i+1}(X)$ for the successive quotients. We denote by $pr_i$ the canonical surjection

$$\text{CH}^i(X) \rightarrow \tau_{i/i+1}(X),$$

where $\text{CH}$ is the integral Chow group. By the Riemann-Roch Theorem without denominators the $i$-th Chern class induces an homomorphism in the opposite way $c_i: \tau_{i/i+1}(X) \rightarrow \text{CH}^i(X)$ such that the composition $c_i \circ pr$ is the multiplication by $(-1)^{i-1}(i-1)!$.

Note that for any prime $p$, one can also consider the topological filtration $\tau_p$ on the ring $K_0(X)/pK_0(X)$ by replacing $K_0(X)$ by $K_0(X)/pK_0(X)$ in the previous definition. In particular, we get that for any $0 \leq i \leq p$, the map $pr^i_p: \text{Ch}^i(X) \rightarrow \tau_{i/i+1}(X)$, where Ch is the Chow group modulo $p$, is an isomorphism.

Remark 2.1. Assume that $X$ is a $\text{SL}_1(A)$-torsor and let $p$ be a prime. One has $K_0(X) = \mathbb{Z}$ by the result [14, Theorem A] of I. Panin and consequently, for $i \geq 1$, the term $\tau_i(X)$ is equal to zero. Therefore, for any $1 \leq i \leq p$, one has $\text{Ch}^i(X) = 0$. Moreover, by the result [17, Theorem 2.7] of A. Suslin, one has $\text{CH}^i(\text{SL}_p) = 0$ for any $i \geq 1$. Hence, for $A$ of degree $p$ (then there exists a splitting field of $A$ of degree $p$), it follows by transfer argument that $p \cdot \text{CH}^i(X) = 0$ for any $i \geq 1$. Therefore, for $X$ a $\text{SL}_1(A)$-torsor, with $A$
of prime degree \( p \), one has \( \text{CH}^i(X) = 0 \) for any \( 1 \leq i \leq p \). Note that, by Proposition 1.2, this gives Theorem 1.1(i) already.

### 2.2. Brown-Gersten-Quillen spectral sequence.

For any smooth variety \( X \) and any \( i \geq 1 \), the epimorphism \( pr^i \) coincides with the edge homomorphism of the spectral Brown-Gersten-Quillen structure \( E_2^{i-1}(X) \Rightarrow K_0(X) \) (see [16, §7]), that is to say

\[
pr^i : \text{CH}^i(X) \cong E_2^{i-1}(X) \to \cdots \to E_{i+1}^{i-1}(X) = \tau^{i+1}(X).
\]

Assume that \( X \) is a \( \text{SL}_1(A) \)-torus, with \( A \) of prime degree \( p \). Then it follows from Remark 2.1 that \( E_3^{i-1}(X) = 0 \) for \( 3 \leq i \leq p \). Consequently, one has \( A^1(X, K_2) = E_3^{1-2}(X) \).

Moreover, by the result [11, Theorem 3.4] of A. Merkurjev, for any smooth variety \( X \), every prime divisor \( l \) of the order of the differential \( \delta_r \) ending in \( E_r^{p+1-r-1}(X) \) is such that \( l - 1 \) divides \( r - 1 \). Therefore, for any prime \( p \) and \( 2 \leq r \leq p - 1 \), the differential \( \delta_r \) is of prime to \( p \) order. Assume furthermore that \( X \) is a \( \text{SL}_1(A) \)-torus, with \( A \) of prime degree \( p \). Since \( p \cdot \text{CH}^{p+1}(X) = 0 \) (see Remark 2.1), one deduce that, for \( 2 \leq r \leq p - 1 \), the differential \( \delta_r \) is trivial. Consequently, one has \( \text{CH}^{p+1}(X) = E^{p+1-p-1}(X) \).

Therefore, for \( X \) a \( \text{SL}_1(A) \)-torus, with \( A \) of prime degree \( p \), the differential \( \delta_p \) in the BGQ-structure is an homomorphism

\[
\delta : A^1(X, K_2) \to \text{CH}^{p+1}(X).
\]

**Remark 2.2.** Let \( X \) be a principal homogeneous space for a semisimple group \( G \). By [6, Part II, Example 4.3.3 and Corollary 5.4], one has \( E_2^{0,-1}(X) = A^0(X, K_1) = F^\times \) and the composition \( F^\times = K_1(F) \to K_1(X) \to A^0(X, K_1) \) of the pullback of the structural morphism with the inclusions

\[
K_1^{(0/1)}(X) = E_\infty^{0,-1}(X) \subset \cdots \subset E_3^{0,-1}(X) \subset E_2^{0,-1}(X)
\]

given by the BGQ spectral sequence, is the identity. Therefore, for any \( i \geq 2 \), the differential starting from \( E_i^{0,-1}(X) \) is zero, i.e for any \( i \geq 2 \), one has

\[
E_i^{i-1}(X) = \tau^{i+1}(X).
\]

In particular, for \( X \) a \( \text{SL}_1(A) \)-torus, with \( A \) of prime degree \( p \), one has \( E_{p+1}^{p+1-p-1}(X) = 0 \), i.e the differential \( \delta : A^1(X, K_2) \to \text{CH}^{p+1}(X) \) is surjective.

### 2.3. On the group \( A^1(X, K_2) \).

The proof in the next section will use the work of A. Merkurjev on the Rost invariant of simply connected algebraic groups (see [6, Part II]). Let \( X \) be a \( \text{SL}_1(A) \)-torus over \( F \). The group \( A^1(X_{F}(X), K_2) \) is infinite cyclic with generator \( q \) and isomorphic to \( A^1(\text{SL}_n, K_2) \) under restriction (where \( n = \text{deg}(A) \)). Furthermore, the restriction map \( r : A^1(X, K_2) \to A^1(X_{F}(X), K_2) \) is injective with finite cokernel of same order as the element \( R_{\text{SL}_1(A)}(X) \), where

\[
R_{\text{SL}_1(A)} : H^1(F, \text{SL}_1(A)) \to H^3(F, \mathbb{Q}/\mathbb{Z}(2))
\]

is the Rost invariant of \( \text{SL}_1(A) \) (see [6, Theorem 9.10]). Moreover, the homomorphism \( R_{\text{SL}_1(A)} \) is of order \( \exp(A) \) by [6, Theorem 11.5].
If char$(F) = l$ is prime then the modulo $l$ component $H^3(F, \mathbb{Z}/l\mathbb{Z}(2))$ of the Galois cohomology group $H^3(F, \mathbb{Q}/\mathbb{Z}(2))$ is the group $H^3_F(F)$ defined by K. Kato in [10] by means of logarithmic differential forms.

3. PROOF OF THE RESULT

In this section, we prove the result of this note.

Proof of Theorem 1.1. We use notations and materials introduced in the previous section. One can assume that $X$ does not have any rational point over $F$ (or equivalently $X$ does not have any closed point of prime to $p$ degree, by the result [1, Theorem 3.3] of J. Black), if else there is nothing to prove. Note that in this situation, the central simple algebra $A$ is necessarily a division algebra. We recall that conclusion (i) has already been proved (see Remark 2.1). According to Proposition 1.2, it suffices to show that $CH^p(X_{F(\xi)}) = 0$ for each generic point $\xi \in Y$ to get conclusion (ii). Since $X_{F(\xi)}$ does not have any closed point of prime to $p$ degree, it is enough to prove that $CH^p(X) = 0$.

Assume on the contrary that $CH^p(X) \neq 0$. Then $\delta : A^1(X, K_2) \rightarrow CH^p(X)$ is nonzero (since $\delta$ is surjective by Remark 2.2), i.e $E^{1,-2}_{p+1}(X)$ is strictly included in $E^{1,-2}_1(X) = A^1(X, K_2)$. We claim that this implies that, by denoting as $q_X$ the generator of $A^1(X, K_2)$, one has $r(q_X) = q$. Indeed, otherwise one has $r(q_X) = p \cdot q$ by §2.3. Consecutively, by denoting as $c$ the corestriction morphism $A^1(\text{SL}_p, K_2) \rightarrow A^1(X, K_2)$, for any $i \geq 2$, one has $c(E_i^{1,-2}(\text{SL}_p)) = c(A^1(\text{SL}_p, K_2)) = A^1(X, K_2)$ (where the first identity is due to $CH^i(\text{SL}_p) = 0$ for any $i \geq 2$). In particular, one has $E^{1,-2}_1(X) = c(E^{1,-2}_{p+1}(\text{SL}_p)) \subset E^{1,-2}_{p+1}(X)$, which is a contradiction.

Therefore, we have shown that under the assumption $CH^p(X) \neq 0$, the generator $q$ of $A^1(X_{F(\xi)}, K_2)$ is rational. Then it follows that the generator $g$ of $CH^p(X_{F(\xi)})$ is also rational.

However, since $A_{F(X)}$ is still a division algebra (see [17, Corollary 6.5]), by [9, Theorem 7.2 and Theorem 8.2] the cycle $g^{p-1}$ in $CH_0(\text{SL}_1(A_{F(X)}))$ is nonzero and the latter group is cyclic of order $p$ generated by the class of the identity of $\text{SL}_1(A_{F(X)})$. Thus, the degree of the rational cycle $g^{p-1}$ is prime to $p$.

It follows that $X$ has a closed point of prime to $p$ degree, which is a contradiction. The Theorem is proved.

Remark 3.1. Conclusion (i) of Theorem 1.1 holds for central simple algebras of $p$-primary degree (with the same proof). Over a field $F$ of characteristic $\neq p$, one can extend conclusion (ii) of Theorem 1.1 to central simple algebras $A$ of $p$-primary degree and of index $p$ because the kernel of the Rost invariant $R_{\text{SL}_1(A)}$ is trivial by the result [12, Theorem 12.2] of A. Merkurjev and A. Suslin.

Remark 3.2. The end of the above proof shows in particular that for a division algebra $A$ of prime degree $p$ over a field $F$ of arbitrary characteristic, the kernel of the Rost invariant $R_{\text{SL}_1(A)}$ is trivial as well. Indeed, let $\xi \in H^3(F, \text{SL}_1(A))$ and let $X$ be the associated $\text{SL}_1(A)$-torsor. Assume that $R_{\text{SL}_1(A)}(\xi)$ is trivial. It follows then by §2.3 that the generator of $A^1(X_{F(\xi)}, K_2)$ is rational. As we have seen in the above proof, this implies that $X$ has a rational point over $F$, i.e the cocycle $\xi$ is trivial.
Theorem 4.1. Let $E$ be an invariant of type $F$ homogeneous varieties under a group of type $G$ of characteristic different from $p$. The following proof requires the characteristic of the base field to be different from $p$, with $p = 3$ when $G$ is of type $F_4$ and $p = 5$ when $G$ is of type $E_8$, although the original result [4, Theorem 1.1] is valid for arbitrary characteristic.

4. Exceptional projective homogeneous varieties

In this section, we describe how Theorem 1.1 implies a similar version of it for projective homogeneous varieties under a group of type $F_4$ or $E_8$. Namely, we give an alternative proof of Theorem 4.1 below. The following proof requires the characteristic of the base field to be different from $p$, with $p = 3$ when $G$ is of type $F_4$ and $p = 5$ when $G$ is of type $E_8$, although the original result [4, Theorem 1.1] is valid for arbitrary characteristic.

Let $X$ be a nonsplit $\mathbf{SL}_1(A)$-torsor over a field $F$, with $A$ a division algebra of prime degree $p$. There exists a smooth compactification $\tilde{X}$ of $X$ such that the Chow motive $\mathcal{M}(\tilde{X}, \mathbb{Z}/p\mathbb{Z})$ decomposes as a direct sum $R_p \oplus N$, where $R_p$ is the indecomposable Rost motive associated with the symbol $[A \cup (c) \in H^3(F, \mathbb{Z}/p\mathbb{Z}(2))]$, with $c \in F^\times \setminus \text{Nrd}(A^\times)$ giving $X$, see [9, Theorem 1.1]. Note that the projective variety $\tilde{X}$ is a norm variety of $s$.

**Theorem 4.1.** Let $G$ be a linear algebraic group of type $F_4$ or $E_8$ over a field $F$ of characteristic different from $p$, with $p = 3$ when $G$ is of type $F_4$ and $p = 5$ when $G$ is of type $E_8$, and let $X'$ be a projective homogeneous $G$-variety. For any equidimensional variety $Y$, the change of field homomorphism

$$\text{Ch}(Y) \to \text{Ch}(Y_{F(X')})$$

where $\text{Ch}$ is the Chow group modulo $p$, is surjective in codimension $< p + 1$.

It is also surjective in codimension $p+1$ for a given $Y$ provided that $1 \notin \text{deg} \, \text{Ch}_0(X'_{F(\xi)})$ for each generic point $\xi \in Y$.

**Proof.** Since the $F$-variety $X'$ is $A$-trivial in the sense of [8, Definition 2.3], one can assume that $G$ has no splitting field of degree coprime to $p$. Indeed, otherwise $1 \in \text{deg} \, \text{Ch}_0(X')$ by corestriction and this implies that $\text{Ch}(Y) \to \text{Ch}(Y_{F(X')})$ is an isomorphism in any codimension by $A$-triviality, see [8, Lemma 2.9].

Let us now write $G = cG_0$ for a nontrivial cocycle $\xi \in H^1(F, G_0)$, with $G_0$ a split group of the same type as $G$. Then the motive $R_p(G)$ living on the Chow motive (with coefficients in $\mathbb{Z}/p\mathbb{Z}$) of $X'$ given in [15, Theorem 5.17] is the Rost motive of the symbol $R_{G_0,p}(\xi) = [A \cup (c) \in H^3(F, \mathbb{Z}/p\mathbb{Z}(2))]$, where $R_{G_0,p}$ is the the modulo $p$ component of the Rost invariant $R_{G_0}$. A is a division algebra of degree $p$ and $c \in F^\times \setminus \text{Nrd}(A^\times)$ – see [13, §4] and [5, §14] (here the assumption char$(F) \neq p$ is needed).

Let us denote as $X$ the nonsplit $\mathbf{SL}_1(A)$-torsor over $F$ associated with $c$ and as $\tilde{X}$ its smooth compactification. We claim that $X'$ has a closed point of prime to $p$ degree over $F(\tilde{X})$ and vice versa.

Indeed, since $\tilde{X}$ is a norm variety for $[A \cup (c)]$, the motive $R_p(G)$ decomposes as a sum of Tate motives over $F(\tilde{X})$. Therefore, the group $G_{F(\tilde{X})}$ is split by an extension of degree coprime to $p$ and it follows that $X'$ has a closed point of prime to $p$ degree over $F(\tilde{X})$ (this is more generally true for any extension $L/F$ over which $\tilde{X}$ has a closed point of prime to $p$ degree). Moreover, the motive $R_p(G)$ decomposes as a sum of Tate motives.
over $F(X')$ because $G$ is split by $F(X')$. Consequently, $\tilde{X}$ has a closed point of prime to $p$ degree over $F(X')$.

It follows then (note that $\tilde{X}$ is $A$-trivial by [8, Example 5.7]) that the right and the bottom homomorphisms in the commutative square

$$
\begin{array}{ccc}
\text{Ch}(Y) & \longrightarrow & \text{Ch}(Y_{F(X')}), \\
\downarrow & & \downarrow \\
\text{Ch}(Y_{F(\tilde{X})}) & \longrightarrow & \text{Ch}(Y_{F(\tilde{X} \times X')}),
\end{array}
$$

are isomorphisms. Since $F(\tilde{X}) = F(X)$, Theorem 4.1 is now a direct consequence of Theorem 1.1.

The following was pointed out to me by Philippe Gille.

**Remark 4.2.** Let $G_0$ a split group of type $E_8$ over a 5-special field $F$ (i.e $F$ has no proper extension of degree coprime to 5) of characteristic $\neq 5$. The above proof gives rise to a new argument for the triviality of the kernel of the Rost invariant modulo 5

$$H^1(F, G_0) \rightarrow H^3(F, \mathbb{Z}/5\mathbb{Z}(2)).$$

This result is originally due to Vladimir Chernousov (under the assumption $\text{char}(F) \neq 2, 3, 5$, see [2, Theorem]).

Indeed, since $F$ is 5-special, for any nontrivial cocycle $\xi \in H^1(F, G_0)$, the group $\xi G_0$ has no splitting field of degree coprime to 5. Then, as we have seen in the proof, there is a division algebra $A$ of degree 5 such that $R_{G_0, 5}(\xi)$ is equal to a symbol $[A] \cup (c)$ associated with a nonsplit $\text{SL}_1(A)$-torsor $X$. The injectivity of $R_{G_0, 5}$ follows now from Remark 3.2.

**References**


UPMC Sorbonne Universités, Institut de Mathématiques de Jussieu, Paris, FRANCE

Web page: www.math.jussieu.fr/~fino
E-mail address: raphael.fino at imj-prg.fr