RATIONALITY OF ALGEBRAIC CYCLES OVER FUNCTION FIELD OF $\mathbf{SL_1}(A)$-TORSORS

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Abstract. In this note we prove a result comparing rationality of algebraic cycles over the function field of a $\mathbf{SL_1}(A)$-torsor for a central simple algebra A and over the base field.

Keywords: Chow groups, central simple algebras, principal homogeneous spaces.

Contents

1. Introduction

Let A be a central simple algebra over a field F and let $\mathrm{Nrd} : A^\times \to F^\times$ be the reduced norm homomorphism . We recall that the homomorphism $F^\times \to H^1(F, \mathbf{SL_1}(A))$, associating to $c \in F^\times$ the $\mathbf{SL_1}(A)$-torsor X_c given by the equation $\mathrm{Nrd} = c$, is surjective (with kernel $\mathrm{Nrd}(A^\times)$) – see [7, Proposition 2.7.3] for instance.

The main purpose of this note is to prove the following theorem dealing with rationality of algebraic cycles over function field of $\mathbf{SL_1}(A)$-torsors.

Theorem 1.1. *Let A be a central simple algebra of prime degree p over a field F and let X be a $\mathbf{SL_1}(A)$-torsor. Then*

(i) *for any equidimensional F-variety Y, the change of field homomorphism*

$$\mathrm{CH}(Y) \to \mathrm{CH}(Y_{F(X)}),$$

where CH is the integral Chow group, is surjective in codimension $< p + 1$.

(ii) *it is also surjective in codimension $p+1$ for a given Y provided that the variety $X_{F(\zeta)}$ does not have any closed point of prime to p degree for each generic point $\zeta \in Y$.*

The method of proof mainly relies on the following statement. This proposition is a version of the result [3, Lemma 88.5] slightly altered to fit our situation (see also the proof of [8, Proposition 2.8]).

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Proposition 1.2 (Karpenko, Merkurjev)**.** *Let X be a smooth variety, and Y an equidimensional variety. Given an integer m such that for any nonnegative integer i and any point $y \in Y$ of codimension i the change of field homomorphism*

$$\mathrm{CH}^{m-i}(X) \longrightarrow \mathrm{CH}^{m-i}(X_{F(y)})$$

is surjective, the change of field homomorphism

$$\mathrm{CH}^{m}(Y) \longrightarrow \mathrm{CH}^{m}(Y_{F(X)})$$

is also surjective.

The proof of Theorem 1.1 is given in Section 3. In Section 4, we describe how this theorem can be related to a similar result dealing with rationality of algebraic cycles over function field of projective homogeneous varieties under some groups of exceptional type.

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2. Preliminaries

2.1. **Topological filtration and Chow groups.** For any smooth variety X over a field F (in this paper, an F-variety is a separated scheme of finite type over F), one can consider the topological filtration on the Grothendieck ring $K_0(X)$, whose term of codimension i is given by

$$\tau^i(X) = \langle [\mathcal{O}_Z] \mid Z \hookrightarrow X \text{ and } \operatorname{codim}(Z) \geq i \rangle,$$

where $[\mathcal{O}_Z]$ is the class in $K_0(X)$ of the structure sheaf of a closed subvariety Z. We write $\tau^{i/i+1}(X)$ for the successive quotients. We denote by pr^i the canonical surjection

$$\begin{array}{ccc} \mathrm{CH}^i(X) & \twoheadrightarrow & \tau^{i/i+1}(X) \\ [Z] & \longmapsto & [\mathcal{O}_Z] \end{array},$$

where CH is the integral Chow group. By the Riemann-Roch Theorem without denominators the i-th Chern class induces an homomorphism in the opposite way $c_i : \tau^{i/i+1}(X) \rightarrow \mathrm{CH}^i(X)$ such that the composition $c_i \circ pr$ is the multiplication by $(-1)^{i-1}(i-1)!$.

Note that for any prime p, one can also consider the topological filtration τ_p on the ring $K_0(X)/pK_0(X)$ by replacing $K_0(X)$ by $K_0(X)/pK_0(X)$ in the previous definition. In particular, we get that for any $0 \leq i \leq p$, the map $pr^i_p : \mathrm{Ch}^i(X) \twoheadrightarrow \tau_p^{i/i+1}(X)$, where Ch is the Chow group modulo p, is an isomorphism.

Remark 2.1. Assume that X is a $\mathbf{SL_1}(A)$-torsor and let p be a prime. One has $K_0(X) = \mathbb{Z}$ by the result [14, Theorem A] of I. Panin and consequently, for $i \geq 1$, the term $\tau^i(X)$ is equal to zero. Therefore, for any $1 \leq i \leq p$, one has $\mathrm{Ch}^i(X) = 0$. Moreover, by the result [17, Theorem 2.7] of A. Suslin, one has $\mathrm{CH}^i(\mathbf{SL_p}) = 0$ for any $i \geq 1$. Hence, for A of degree p (then there exists a splitting field of A of degree p), it follows by transfert argument that $p \cdot \mathrm{CH}^i(X) = 0$ for any $i \geq 1$. Therefore, for X a $\mathbf{SL_1}(A)$-torsor, with A

of prime degree p, one has $\mathrm{CH}^i(X) = 0$ for any $1 \leq i \leq p$. Note that, by Proposition 1.2, this gives Theorem 1.1(i) already.

2.2. **Brown-Gersten-Quillen spectral sequence.** For any smooth variety X and any $i \geq 1$, the epimorphism pr^i coincides with the edge homomorphism of the spectral Brown-Gersten-Quillen structure $E_2^{i,-i}(X) \Rightarrow K_0(X)$ (see [16, §7]), that is to say

$$pr^i : \mathrm{CH}^i(X) \simeq E_2^{i,-i}(X) \twoheadrightarrow \cdots \twoheadrightarrow E_{i+1}^{i,-i}(X) = \tau^{i/i+1}(X).$$

Assume that X is a $\mathbf{SL_1}(A)$-torsor, with A of prime degree p. Then it follows from Remark 2.1 that $E_{i-1}^{i,-i}(X) = 0$ for $3 \leq i \leq p$. Consequently, one has $A^1(X, K_2) = E_p^{1,-2}(X)$.

Moreover, by the result [11, Theorem 3.4] of A. Merkurjev, for any smooth variety X, every prime divisor l of the order of the differential δ_r ending in $E_r^{p+1,-p-1}(X)$ is such that $l-1$ divides $r-1$. Therefore, for any prime p and $2 \leq r \leq p-1$, the differential δ_r is of prime to p order. Assume furthermore that X is a $\mathbf{SL_1}(A)$-torsor, with A of prime degree p. Since $p \cdot \mathrm{CH}^{p+1}(X) = 0$ (see Remark 2.1), one deduce that, for $2 \leq r \leq p-1$, the differential δ_r is trivial. Consequently, one has $\mathrm{CH}^{p+1}(X) = E_p^{p+1,-p-1}(X)$.

Therefore, for X a $\mathbf{SL_1}(A)$-torsor, with A of prime degree p, the differential δ_p in the BGQ-structure is an homomorphism

$$\delta : A^1(X, K_2) \to \mathrm{CH}^{p+1}(X).$$

Remark 2.2. Let X be a principal homogeneous space for a semisimple group G. By [6, Part II, Example 4.3.3 and Corollary 5.4], one has $E_2^{0,-1}(X) = A^0(X, K_1) = F^\times$ and the composition $F^\times = K_1(F) \to K_1(X) \to A^0(X, K_1)$ of the pullback of the structural morphism with the inclusions

$$K_1^{(0/1)}(X) = E_\infty^{0,-1}(X) \subset \cdots \subset E_3^{0,-1}(X) \subset E_2^{0,-1}(X)$$

given by the BGQ spectral sequence, is the identity. Therefore, for any $i \geq 2$, the differential starting from $E_i^{0,-1}(X)$ is zero, i.e for any $i \geq 2$, one has

$$E_i^{i,-i}(X) = \tau^{i/i+1}(X).$$

In particular, for X a $\mathbf{SL_1}(A)$-torsor, with A of prime degree p, one has $E_{p+1}^{p+1,-p-1}(X) = 0$, i.e the differential $\delta : A^1(X, K_2) \to \mathrm{CH}^{p+1}(X)$ is surjective.

2.3. **On the group $A^1(X, K_2)$.** The proof in the next section will use the work of A. Merkurjev on the *Rost invariant* of simply connected algebraic groups (see [6, Part II]). Let X be a $\mathbf{SL_1}(A)$-torsor over F. The group $A^1(X_{F(X)}, K_2)$ is infinite cyclic with generator q and isomorphic to $A^1(\mathbf{SL_n}, K_2)$ under restriction (where $n = \deg(A)$). Furthermore, the restriction map $r : A^1(X, K_2) \to A^1(X_{F(X)}, K_2)$ is injective with finite cokernel of same order as the element $R_{\mathbf{SL_1}(A)}(X)$, where

$$R_{\mathbf{SL_1}(A)} : H^1(F, \mathbf{SL_1}(A)) \to H^3(F, \mathbb{Q}/\mathbb{Z}(2))$$

is the Rost invariant of $\mathbf{SL_1}(A)$ (see [6, Theorem 9.10]). Moreover, the homomorphism $R_{\mathbf{SL_1}(A)}$ is of order $\exp(A)$ by [6, Theorem 11.5].

If $\mathrm{char}(F) = l$ is prime then the modulo l component $H^3(F, \mathbb{Z}/l\mathbb{Z}(2))$ of the Galois cohomology group $H^3(F, \mathbb{Q}/\mathbb{Z}(2))$ is the group $H^3_l(F)$ defined by K. Kato in [10] by means of logarithmic differential forms.

3. Proof of the result

In this section, we prove the result of this note.

Proof of Theorem 1.1. We use notations and materials introduced in the previous section. One can assume that X does not have any rational point over F (or equivalently X does not have any closed point of prime to p degree, by the result [1, Theorem 3.3] of J. Black), if else there is nothing to prove. Note that in this situation, the central simple algebra A is necessarily a division algebra. We recall that conclusion (i) has already been proved (see Remark 2.1). According to Proposition 1.2, it suffices to show that $\mathrm{CH}^{p+1}(X_{F(\zeta)}) = 0$ for each generic point $\zeta \in Y$ to get conclusion (ii). Since $X_{F(\zeta)}$ does not have any closed point of prime to p degree, it is enough to prove that $\mathrm{CH}^{p+1}(X) = 0$.

Assume on the contrary that $\mathrm{CH}^{p+1}(X) \neq 0$. Then $\delta : A^1(X, K_2) \to \mathrm{CH}^{p+1}(X)$ is nonzero (since δ is surjective by Remark 2.2), i.e $E^{1,-2}_{p+1}(X)$ is strictly included in $E^{1,-2}_p(X) = A^1(X, K_2)$. We claim that this implies that, by denoting as q_X the generator of $A^1(X, K_2)$, one has $r(q_X) = q$. Indeed, otherwise one has $r(q_X) = p \cdot q$ by §2.3. Consecutively, by denoting as c the corestriction morphism $A^1(\mathbf{SL_p}, K_2) \to A^1(X, K_2)$, for any $i \geq 2$, one has $c(E^{1,-2}_i(\mathbf{SL_p})) = c(A^1(\mathbf{SL_p}, K_2)) = A^1(X, K_2)$ (where the first identity is due to $\mathrm{CH}^i(\mathbf{SL_p}) = 0$ for any $i \geq 2$). In particular, one has $E^{1,-2}_p(X) = c(E^{1,-2}_{p+1}(\mathbf{SL_p})) \subset E^{1,-2}_{p+1}(X)$, which is a contradiction.

Therefore, we have shown that under the assumption $\mathrm{CH}^{p+1}(X) \neq 0$, the generator q of $A^1(X_{F(X)}, K_2)$ is rational. Then it follows that the generator g of $\mathrm{CH}^{p+1}(X_{F(X)})$ is also rational.

However, since $A_{F(X)}$ is still a division algebra (see [17, Corollary 6.5]), by [9, Theorem 7.2 and Theorem 8.2] the cycle g^{p-1} in $\mathrm{CH}_0(\mathbf{SL_1}(A_{F(X)}))$ is nonzero and the latter group is cyclic of order p generated by the class of the identity of $\mathbf{SL_1}(A_{F(X)})$. Thus, the degree of the rational cycle g^{p-1} is prime to p.

It follows that X has a closed point of prime to p degree, which is a contradiction.

The Theorem is proved. □

Remark 3.1. Conclusion (i) of Theorem 1.1 holds for central simple algebras of p-primary degree (with the same proof). Over a field F of characteristic $\neq p$, one can extend conclusion (ii) of Theorem 1.1 to central simple algebras A of p-primary degree and of index p because the kernel of the Rost invariant $R_{\mathbf{SL_1}(A)}$ is trivial by the result [12, Theorem 12.2] of A. Merkurjev and A. Suslin.

Remark 3.2. The end of the above proof shows in particular that for a division algebra A of prime degree p over a field F of arbitrary charateristic, the kernel of the Rost invariant $R_{\mathbf{SL_1}(A)}$ is trivial as well. Indeed, let $\xi \in H^1(F, \mathbf{SL_1}(A))$ and let X be the associated $\mathbf{SL_1}(A)$-torsor. Assume that $R_{\mathbf{SL_1}(A)}(\xi)$ is trivial. It follows then by §2.3 that the generator of $A^1(X_{F(X)}, K_2)$ is rational. As we have seen in the above proof, this implies that X has a rational point over F, i.e the cocycle ξ is trivial.

Note also that for a division algebra A of prime degree p over a field F, the Rost invariant $R_{\mathbf{SL_1}(A)}$ coincides, up to sign, with the normalized invariant given by the cup product $[A] \cup (c) \in H^3(F, \mathbb{Z}/p\mathbb{Z}(2))$ for any class $c\,\mathrm{Nrd}(A^\times)$, where $[A]$ is the class of the algebra A in the Brauer group $\mathrm{Br}(F)$, see [6, §11].

4. Exceptional projective homogeneous varieties

In this section, we describe how Theorem 1.1 implies a similar version of it for projective homogeneous varieties under a group of type F_4 or E_8. Namely, we give an alternative proof of Theorem 4.1 below. The following proof requires the characteristic of the base field to be different from p, with $p = 3$ when G is of type F_4 and $p = 5$ when G is of type E_8, although the original result [4, Theorem 1.1] is valid for arbitrary characterisitic.

Let X be a nonsplit $\mathbf{SL_1}(A)$-torsor over a field F, with A a division algebra of prime degree p. There exists a smooth compactification $\tilde{X}$ of X such that the Chow motive $\mathcal{M}(\tilde{X}, \mathbb{Z}/p\mathbb{Z})$ decomposes as a direct sum $\mathcal{R}_p \oplus N$, where $\mathcal{R}_p$ is the indecomposable Rost motive associated with the symbol $[A] \cup (c) \in H^3(F, \mathbb{Z}/p\mathbb{Z}(2))$, with $c \in F^\times \backslash \mathrm{Nrd}(A^\times)$ giving X, see [9, Theorem 1.1]. Note that the projective variety $\tilde{X}$ is a *norm variety* of s.

Theorem 4.1. *Let G be a linear algebraic group of type F_4 or E_8 over a field F of characteristic different from p, with $p = 3$ when G is of type F_4 and $p = 5$ when G is of type E_8, and let X' be a projective homogeneous G-variety. For any equidimensional variety Y, the change of field homomorphism*

$$\mathrm{Ch}(Y) \to \mathrm{Ch}(Y_{F(X')}),$$

where Ch *is the Chow group modulo p, is surjective in codimension $< p + 1$.*

It is also surjective in codimension $p+1$ for a given Y provided that $1 \notin \deg\,\mathrm{Ch}_0(X'_{F(\zeta)})$ for each generic point $\zeta \in Y$.

Proof. Since the F-variety X' is A-trivial in the sense of [8, Definition 2.3], one can assume that G has no splitting field of degree coprime to p. Indeed, otherwise $1 \in \deg\,\mathrm{Ch}_0(X')$ by corestricition and this implies that $\mathrm{Ch}(Y) \to \mathrm{Ch}(Y_{F(X')})$ is an isomorphism in any codimension by A-triviality, see [8, Lemma 2.9].

Let us now write $G = {}_\xi G_0$ for a nontrivial cocycle $\xi \in H^1(F, G_0)$, with G_0 a split group of the same type as G. Then the motive $\mathcal{R}_p(G)$ living on the Chow motive (with coefficients in $\mathbb{Z}/p\mathbb{Z}$) of X' given in [15, Theorem 5.17] is the Rost motive of the symbol $R_{G_0,p}(\xi) = [A] \cup (c) \in H^3(F, \mathbb{Z}/p\mathbb{Z}(2))$, where $R_{G_0,p}$ is the the modulo p component of the Rost invariant R_{G_0}, A is a division algebra of degree p and $c \in F^\times \backslash \mathrm{Nrd}(A^\times)$ – see [13, §4] and [5, §14] (here the assumption $\mathrm{char}(F) \neq p$ is needed).

Let us denote as X the nonsplit $\mathbf{SL_1}(A)$-torsor over F associated with c and as $\tilde{X}$ its smooth compactification. We claim that X' has a closed point of prime to p degree over $F(\tilde{X})$ and vice versa.

Indeed, since $\tilde{X}$ is a norm variety for $[A] \cup (c)$, the motive $\mathcal{R}_p(G)$ decomposes as a sum of Tate motives over $F(\tilde{X})$. Therefore, the group $G_{F(\tilde{X})}$ is split by an extension of degree coprime to p and it follows that X' has a closed point of prime to p degree over $F(\tilde{X})$ (this is more generally true for any extension L/F over which $\tilde{X}$ has a closed point of prime to p degree). Moreover, the motive $\mathcal{R}_p(G)$ decomposes as a sum of Tate motives over $F(X')$ because G is split by $F(X')$. Consequently, $\tilde{X}$ has a closed point of prime to p degree over $F(X')$.

It follows then (note that $\tilde{X}$ is A-trivial by [8, Example 5.7]) that the right and the bottom homomorphisms in the commutative square

$$\begin{tikzcd} \mathrm{Ch}(Y) \arrow[r] \arrow[d] & \mathrm{Ch}(Y_{F(X')}) \arrow[d] \\ \mathrm{Ch}(Y_{F(\tilde{X})}) \arrow[r] & \mathrm{Ch}(Y_{F(\tilde{X}\times X')}) \end{tikzcd}$$

are isomorphisms. Since $F(\tilde{X}) = F(X)$, Theorem 4.1 is now a direct consequence of Theorem 1.1. $\square$

The following was pointed out to me by Philippe Gille.

Remark 4.2. Let G_0 a split group of type E_8 over a 5-special field F (i.e F has no proper extension of degree coprime to 5) of characteristic $\neq 5$. The above proof gives rise to a new argument for the triviality of the kernel of the Rost invariant modulo 5

$$H^1(F, G_0) \to H^3(F, \mathbb{Z}/5\mathbb{Z}(2)).$$

This result is originally due to Vladimir Chernousov (under the assumption char$(F) \neq 2$, 3, 5, see [2, Theorem]).

Indeed, since F is 5-special, for any nontrivial cocycle $\xi \in H^1(F, G_0)$, the group ${}_{\xi}G_0$ has no splitting field of degree coprime to 5. Then, as we have seen in the proof, there is a division algebra A of degree 5 such that $R_{G_0,5}(\xi)$ is equal to a symbol $[A] \cup (c)$ associated with a nonsplit $\mathbf{SL_1}(A)$-torsor X. The injectivity of $R_{G_0,5}$ follows now from Remark 3.2.

References

[1] Black, J. Zero cycles of degree one on principal homogeneous spaces. *Journal of Algebra 335*, 1 (2011), 232-246.

[2] Chernousov, V. Remark on the (mod 5)-invariant of Serre for groups of type E_8. *Math. Notes. 56*, 1 (1994), 730-733.

[3] Elman, R., Karpenko, N., and Merkurjev, A. *The algebraic and geometric theory of quadratic forms*, vol. 56 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2008.

[4] Fino, R. Rationality of cycles over function field of exceptional projective homogeneous varieties. *J. Ramanujan Math. Soc. 29*, 1 (2014), 119-132.

[5] Garibaldi, S. Cohomological invariants: exceptional groups and spin groups. *Memoirs Amer. Math. Soc. 200*, 937 (2009), With an appendix by Detlev W. Hoffmann.

[6] Garibaldi, S., Merkurjev, A., and Serre, J.-P. *Cohomological invariants in Galois cohomology*, vol. 28 of *University Lecture Series*. Amer. Math. Soc., 2003.

[7] Gille, P., and Szamuely, T. *Central simple algebras and Galois cohomology*, vol. 101 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2006.

[8] Karpenko, N., and Merkurjev, A. On standard norm varieties. *Ann. Sci. Ec. Norm. Sup. (4) 46, fascicule 1* (2013), 175-214.

[9] Karpenko, N., and Merkurjev, A. Motivic decomposition of compactifications of certain group varieties. *www.math.uni-bielefeld.de/lag/man* (22 Feb. 2014), 18 pages. (Preprint)

[10] Kato, K. Galois cohomology of complete discrete valuation fields. *Lect. Notes in Math.*, 967 (1982), 215-238.

[11] Merkurjev, A. Adams operations and the Brown-Gersten-Quillen spectral sequence. In *Quadratic forms, linear algebraic groups, and cohomology*, vol. 18 of *Dev.Math.* Springer, New York, 2010, pp. 305-313.

[12] Merkurjev, A., and Suslin, A. K-cohomology of Severi-Brauer varieties and the norm residue homomorphism. *Izv. Akad Nauk SSSR*, 46 (1982), 1011-1046.

[13] Nikolenko, S., Semenov, N., and Zainoulline, K. Motivic decomposition of anisotropic varieties of type F_4 into generalized Rost motives. *J. of K-theory 3*, 1 (2009), 114-124.

[14] Panin, I. Splitting principle and K-theory of simply connected semisimple algebraic groups. *Algebra y Analiz 10*, 1 (1998), 88-131. translation in St. Petersburg Math. J. 10 (1999), n° 1, 69-101.

[15] Petrov, V., Semenov, N., and Zainoulline, K. J-invariant of linear algebraic groups. *Ann. Sci. Ec. Norm. Sup. (4) 41*, 6 (2008), 1023-1053.

[16] Quillen, D. Higher algebraic K-theory. I. (1973). vol. 341 of *Lectures Notes in Math.*, pp. 85-147.

[17] Suslin, A. K-Theory and $\mathcal{K}$-cohomology of certain group varieties. *Advances in Soviet Mathematics 4* (1991), 53-74.

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