ON A PHENOMENON DISCOVERED BY HEINZ HELLING

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Abstract. The hyperbolic 3-manifolds fibered over the circle with fiber a once punctured torus are considered. Surprisingly, in all of about 200 cases, where the class numbers of their trace fields were calculated, they turned to be equal to one.

This is a report on the work of our friend Heinz Helling during the last several years. Unfortunately, for health problems he is unable to write about it himself. All the main ideas and results presented in this paper belong to him, but some technical details may be different.

In a series of preprints [1], [2], [3], H. Helling with co-authors studied some class of hyperbolic 3-manifolds fibered over the circle, and calculated the trace fields of these manifolds. Ten years later he came to the idea to investigate arithmetic properties of these fields. This lead him to a surprising result: in all the cases, when he managed to calculate the class number (with the help of a computer program), it turned to be equal to one!

1. The construction

Let $T_0 = T \setminus \{\mathrm{pt}\}$ be a once punctured torus. Observe that its fundamental group is a free group F_2 on two generators, say, ξ and η. A fiber bundle over the circle with fiber T_0 is defined by its monodromy, which is an element of the group

$$\mathrm{Out}F_2 = \mathrm{Aut}F_2/\mathrm{Int}F_2 \cong \mathrm{GL}(2,\mathbb{Z}).$$

Denote by $M(A)$ the fiber bundle defined by a matrix $A \in \mathrm{GL}(2,\mathbb{Z})$. The fiber bundles $M(A)$ and $M(A')$ are isomorphic if and only if the matrix A' is conjugate to A or its inverse in $\mathrm{GL}(2,\mathbb{Z})$ (but non-isomorphic fiber bundles can still be homeomorphic manifolds.) The manifold $M(A)$ is hyperbolic (of finite volume) if and only if the matrix A is hyperbolic, i.e. has real eigenvalues distinct from ± 1. It is orientable if and only if $\det A = 1$, i.e. $A \in \mathrm{SL}(2,\mathbb{Z})$.

Since any topological automorphism of the punctured torus T_0 extends to an automorphism of the torus T, the fiber bundle $M(A)$ naturally embeds into a fiber bundle with fiber T. The boundary of this embedding is a circle. Retracting it to a point, we obtain a one point compactification of $M(A)$ with connected punctured neighborhoods of the boundary point. This means that the hyperbolic manifold $M(A)$ has only one cusp.

Let $\alpha \in \mathrm{Aut}F_2$ be a representative of the coset of $\mathrm{Int}F_2$ corresponding to A. Then the fundamental group of $M(A)$ is the semi-direct product $\langle\alpha\rangle \leftthreetimes F_2$, where $\langle\alpha\rangle$ is the cyclic group generated by α. It is known that every automorphism of F_2 takes the commutator (ξ,η) to a conjugate of $(\xi,\eta)^{\pm 1}$. So we may assume that α takes (ξ,η) to $(\xi,\eta)^{\pm 1}$, depending on $\det A$.

From now on, we assume that the manifold $M(A)$ is orientable, i.e. $\det A = 1$, and $\alpha((\xi,\eta)) = (\xi,\eta)$.

If A is hyperbolic with $\det A = 1$, then the fundamental group of $M(A)$ is isomorphic to a torsion-free lattice in $\mathrm{PSL}(2,\mathbb{C})$, which is defined uniquely up to conjugacy. It is known [5] that any such lattice can be isomorphically lifted to $\mathrm{SL}(2,\mathbb{C})$. This means that there is a faithful linear representation

$$\varphi : \langle\alpha\rangle \leftthreetimes F_2 \to \mathrm{SL}(2.\mathbb{C})$$

such that its image is a lattice in $\mathrm{SL}(2,\mathbb{C})$. Denote this lattice with $\Gamma(A)$.

Set

$$X = \varphi(\xi),\ Y = \varphi(\eta),\ L = \varphi(\alpha).$$

Then X, Y and L generate $\Gamma(A)$. Moreover, since the matrices L and (X,Y) commute, they generate a free abelian group of rank 2. It follows that they are unipotent up to a sign. In particular, $\mathrm{tr}(X,Y) = \pm 2$. However, the case $\mathrm{tr}(X,Y) = 2$ is realized, when X and Y have a common eigenvector, which implies that L has the same eigenvector, a contradiction. Hence, $\mathrm{tr}(X,Y) = -2$. Replacing L with $-L$ if needed, one may assume that $\mathrm{tr}L = 2$, i.e. L is unipotent.

2. Calculating the traces

A standard calculation with traces shows that

$$\mathrm{tr}(X,Y) = (\mathrm{tr}X)^2 + (\mathrm{tr}Y)^2 + (\mathrm{tr}XY)^2 - (\mathrm{tr}X)(\mathrm{tr}Y)(\mathrm{tr}XY) - 2$$

for any matrices $X, Y \in \mathrm{SL}(2,\mathbb{C})$. Thus, in our situation we have

$$(\mathrm{tr}X)^2 + (\mathrm{tr}Y)^2 + (\mathrm{tr}XY)^2 = (\mathrm{tr}X)(\mathrm{tr}Y)(\mathrm{tr}XY).$$

In other words, if we set

$$x = \mathrm{tr}X,\ y = \mathrm{tr}Y,\ z = \mathrm{tr}XY.$$

then the point (x,y,z) belongs to the (complex) *Markov surface* defined by the equation

$$x^2 + y^2 + z^2 = xyz.$$

The traces of the matrices

$$\alpha(X) := \varphi(\alpha(\xi)), \quad \alpha(Y) := \varphi(\alpha(\eta)), \quad \alpha(X)\alpha(Y) = \alpha(XY) := \varphi(\alpha(\xi\eta))$$

can be expressed in terms of x, y, z. They do not change when α is multiplied by an inner automorphism of F_2, so they depend only on the matrix A. Moreover, there is a natural action of the group $\mathrm{GL}(2,\mathbb{Z})$ on $\mathbb{C}^3$ leaving the Markov surface invariant such that

$$(\mathrm{tr}\,\alpha(X), \mathrm{tr}\,\alpha(Y), \mathrm{tr}\,\alpha(XY)) = A(x,y,z).$$

It is well known that the group $\mathrm{GL}(2,\mathbb{Z})$ is generated by the matrices

$$R_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$R_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$R_3 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix},$$

acting on the hyperbolic plane H^2 as the reflections in the sides of a triangle with angles $\pi/2, \pi/3, 0$. They lift to the following automorphisms of F_2:

$$\begin{aligned} \rho_1 &: \xi \mapsto \xi,\ \eta \mapsto \eta^{-1}, \\ \rho_2 &: \xi \mapsto \eta,\ \eta \mapsto \xi, \\ \rho_3 &: \xi \mapsto \xi\eta,\ \eta \mapsto \eta^{-1}. \end{aligned}$$

It is easy to see how they change the traces. Thus we obtain:

$$R_1(x,y,z) = (x,y,xy-z),\ R_2(x,y,z) = (y,x,z),\ R_3(x,y,z) = (z,y,x).$$

In our situation, the matrices $\alpha(X)$ and $\alpha(Y)$ are conjugate to X and Y by means of L, so there must be

$$A(x,y,z) = (x,y,z).$$

In other words, (x,y,z) is a fixed point of A lying on the Markov surface.

The field $K = K(A) =: \mathbb{Q}(x,y,z)$ is the trace field of the group $\Gamma(A) \subset \mathrm{SL}(2,\mathbb{C})$. Indeed, it clearly is the trace field of the subgroup $\Delta(A)$ generated by X and Y. It follows that the K-linear span of $\Delta(A)$ is a (generalized) quaternionic algebra over K. But it contains the unipotent matrix $-(X,Y)$, which implies that it is in fact the matrix algebra. This means that X and Y (considered as linear operators) in some basis are written by matrices over K. In the same basis, L is written by a matrix proportional to a matrix over K, because it normalizes $\Delta(A)$. But L is unipotent; hence, it is in fact written by a matrix over K. Thus in some basis the group $\Gamma(A)$ is written by matrices over K, so the traces of all its elements belong to K.

One can also calculate the invariant trace field $k(A)$ for $\Gamma(A)$. By definition, the invariant trace field of a lattice $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$ is the trace field of the group $\mathrm{Ad}(\Gamma)$, and it is the same for commensurable lattices. According to [8, Corollary to Theorem 2],

$$k(A) = \mathbb{Q}(x^2, y^2, xyz) = \mathbb{Q}(x^2, y^2, z^2).$$

3. A series of examples

Consider the series of matrices

$$A_n = \begin{pmatrix} 1 & 1 \\ n & n+1 \end{pmatrix} = R_2(R_1R_3)^n R_2 R_1 R_3 \in \mathrm{SL}(2,\mathbb{Z}).$$

They are hyperbolic for $n \geq 1$. Set

$$A_n(x,y,z) = (x_n, y_n, z_n).$$

We have

$$(x_0, y_0, z_0) = R_1R_3(x,y,z) = (z, y, yz - x),$$

$$(x_n, y_n, z_n) = R_2R_1R_3R_2(x_{n-1},\ y_{n-1},\ z_{n-1}) = (x_{n-1},\ z_{n-1},\ x_{n-1}z_{n-1} - y_{n-1}),$$

whence $x_n = z$ and

$$\begin{pmatrix} z_n \\ y_n \end{pmatrix} = \begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_{n-1} \\ y_{n-1} \end{pmatrix}$$

If (x,y,z) is a fixed point of A_n, there must be $x = z$, so $z_0 = yz - z$ and

$$\begin{pmatrix} z \\ y \end{pmatrix} = \begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} z_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix}^{n+1} \begin{pmatrix} y \\ z \end{pmatrix} \tag{1}$$

One can easily prove by induction that

$$\begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix}^{n+1} = \begin{pmatrix} q_{n+1} & -q_n \\ q_n & -q_{n-1} \end{pmatrix} \text{ for } n \geq 1 \tag{2}$$

where q_n are polynomials in z defined by

$$q_0 = 1, \quad q_1 = z, \quad q_{n+1} = zq_n - q_{n-1} \text{ for } n \geq 1. \tag{3}$$

By taking determinants one obtains the identity

$$q_n^2 - 1 = q_{n-1}q_{n+1} \tag{4}$$

It follows from (3) that

$$q_n(z) = U_n(z/2) = \sum_{i=0}^{[n/2]} (-1)^i \binom{n-i}{i} z^{n-2i}, \tag{5}$$

where U_n is the Chebyshev polynomial of the second kind (of degree n). Note that the polynomial q_n is even (odd) for n even (odd).

From (1) and (2) we obtain

$$y_n = yq_n - zq_{n-1}, \quad z_n = yq_{n+1} - zq_n. \tag{6}$$

We thus see that the fixed points (x, y, z) of A_n on the Markov surface are given by the following equations (having in mind that $x = z$):

$$y^2 + (2-y)z^2 = 0, \tag{7}$$

$$y(q_n(z) - 1) - zq_{n-1}(z) = 0, \tag{8}$$

$$yq_{n+1}(z) - z(q_n(z) + 1) = 0. \tag{9}$$

Since we are interested only in non-real solutions, we may assume that $y, z \neq 0$.

If $q_n(z) = 1$, it follows from (8) that $q_{n-1}(z) = 0$, and then from (3) we get that $q_{n+1}(z) = z$. Substituting this in (9) we obtain $y = 2$, which contradicts (7).

If $q_n(z) = -1$, it follows from (9) that $q_{n+1}(z) = 0$, and then from (3) we get that $q_{n-1}(z) = -z$. Substituting this in (8) we obtain $2y = z^2$, whence by (7) $y = 4$ and therefore $z = \sqrt{8}$. However, all the roots of the polynomial q_{n+1} lie in the interval $(-2, 2)$, so this case does not occur.

Thus, we may assume that $q_n(z) \neq \pm 1$. Under this assumption, equations (8) and (9) are equivalent due to (4). Eliminating y from (7) and (8), one obtains

$$2(q_n(z) - 1)^2 - zq_{n-1}(z)(q_n(z) - 1) + q_{n-1}(z)^2 = 0. \tag{10}$$

It follows from the obvious multiplication law for matrices (2) and from (4) that

$$\begin{aligned} q_{2k} &= (q_k + q_{k-1})(q_k - q_{k-1}), & q_{2k} - 1 &= q_{k-1}(q_{k+1} - q_{k-1}), \\ q_{2k-1} &= q_{k-1}(q_k - q_{k-2}), & q_{2k+1} - 1 &= (q_k + q_{k-1})(q_{k+1} - q_k), \end{aligned}$$

In particular, this shows that if $q_n(z) \neq 1$, then $q_{k-1}(z) \neq 0$ for $n = 2k$ and $q_k(z) + q_{k-1}(z) \neq 0$ for $n = 2k + 1$.

We will use the above formulas to simplify the polynomial F_n from the left hand side in (10) as follows.

For $n = 2k$ we have

$$\begin{aligned} q_{k-1}^{-2} F_n =& 2(q_{k+1} - q_{k-1})^2 - z(q_k - q_{k-2})(q_{k+1} - q_{k-1}) + (q_k - q_{k-2})^2 \\ =& 2(q_{k+1} - q_{k-1})^2 - (q_{k+2} - q_k)(q_k - q_{k-2}) \\ =& (q_{k+1} - q_{k-1})^2 - \begin{vmatrix} q_{k+2} - q_k & q_{k+1} - q_{k-1} \\ q_{k+1} - q_{k-1} & q_k - q_{k-2} \end{vmatrix} \\ =& (q_{k+1} - q_{k-1})^2 - (z^2 - 4) \end{aligned}$$

For $n = 2k + 1$ we have

$$\begin{aligned} (q_k + q_{k-1})^{-2} F_n =& 2(q_{k+1} - q_k)^2 - z(q_k - q_{k-1})(q_{k+1} - q_k) + (q_k - q_{k-1})^2 \\ =& 2(q_{k+1} - q_k)^2 - (q_{k+2} - q_{k+1})(q_k - q_{k-1}) \\ =& (q_{k+1} - q_k)^2 - \begin{vmatrix} q_{k+2} - q_{k+1} & q_{k+1} - q_k \\ q_{k+1} - q_k & q_k - q_{k-1} \end{vmatrix} \\ =& (q_{k+1} - q_k)^2 - (z - 2) \end{aligned}$$

The evaluation of the determinants is obtained in both cases by observing that the underlying 2×2 matrix Q_k obeys $Q_k = \begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix} Q_{k-1}$, so $\det Q_k = \det Q_{k-1}$ and hence $\det Q_k = \det Q_1$ (assuming $q_{-1} = 0$).

Thus, under the assumption $q_n(z) \neq 1$, the solutions to (10) are the roots of the polynomial

$$f_n(z) = \begin{cases} (q_{k+1}(z) - q_{k-1}(z))^2 - (z^2 - 4) & \text{if } n = 2k \\ (q_{k+1}(z) - q_k(z))^2 - (z - 2) & \text{if } n = 2k + 1 \end{cases}$$

These formulas have been given also, without proof, in [3] and [4].

The trace field of $\Gamma(A_n)$ is $\mathbb{Q}(z)$, where z is a suitable root of f_n. The invariant trace field is $\mathbb{Q}(y^2, z^2) = \mathbb{Q}(y, z^2)$. For n even, it follows from (8) that $y \in \mathbb{Q}(z^2)$; in this case the invariant trace field coincides with $\mathbb{Q}(z^2)$.

By means of the computer program Pari[7] (used also by H. Helling), in all the cases checked so far ($n \leq 46$), it was found that the polynomial f_n is irreducible and the trace field $K_n = \mathbb{Q}(z)$ has class number one. Moreover, it is purely imaginary and for n odd has no nontrivial subfield. For n even, it has exactly one nontrivial subfield $k_n \subset K_n$ of index two, which also has class number one. The field k_n has at most one real embedding. It is the invariant trace field of $\Gamma(A_n)$.

For $n = 47$, Pari gave up because it could not cope with the large discriminants showing up, but for several higher values it succeeded, and we never found any class number > 1 for these fields. The irreducibility of the factor f_n has been verified for $n \leq 2000$.

In the table below, we give the polynomial f_n and the discriminant of the trace field K_n for $n \leq 12$ as computed by Pari. Note that in [6] the fields K_n had been calculated for $n = 1$ (p. 138), $n = 2$ (p. 143), and $n = 3$ (p. 143, Ex. 4.6).

The discriminants of the invariant trace fields k_n for $n = 2, 4, 6, 8, 10, 12$ are

$$-7,\ -44,\ 2917,\ 7684,\ -315544,\ -2985968.$$

n	f_n	Discriminant
1	$z^2 - 3z + 3$	-3
2	$z^4 - 5z^2 + 8$	392
3	$z^4 - 2z^3 - z^2 + z + 3$	697
4	$z^6 - 6z^4 + 8z^2 + 4$	-123904
5	$z^6 - 2z^5 - 3z^4 + 6z^3 + 2z^2 - 5z + 3$	-453683
6	$z^8 - 8z^6 + 20z^4 - 17z^2 + 8$	68071112
7	$z^8 - 2z^7 - 5z^6 + 10z^5 + 7z^4 - 14z^3 - 2z^2 + 3z + 3$	628432401
8	$z^{10} - 10z^8 + 35z^6 - 50z^4 + 24z^2 + 4$	-60460908544
9	$z^{10} - 2z^9 - 7z^8 + 14z^7 + 16z^6 - 32z^5 -$ $-13z^4 + 26z^3 + 3z^2 - 7z + 3$	-1537714619747
10	$z^{12} - 12z^{10} + 54z^8 - 112z^6 + 105z^4 - 37z^2 + 8$	79654564209992
11	$z^{12} - 2z^{11} - 9z^{10} + 18z^9 + 29z^8 - 58z^7 - 40z^6 +$ $+40z^6 + 80z^5 + 22z^4 - 44z^3 - 3z^2 + 5z + 3$	5910843534832201
12	$z^{14} - 14z^{12} + 77z^{10} - 210z^8 +$ $+294z^6 - 196z^4 + 48z^2 + 4$	-146079824232841216

4. Conclusion

Heinz Helling calculated the trace fields of the groups $\Gamma(A)$ in about 200 cases, and in all the cases they turned to have class number one. However, he had no time to prove or disprove this for all the groups $\Gamma(A)$, and did not dare to formulate this as a conjecture.

A vague indication to the reason why this phenomenon takes place is the known theorem that the class number of an imaginary quadratic field is equal to the number of cusps for the corresponding Bianchi group (see, for example, [6, Theorem 9.1.1, p. 276]). As was explained above, the group $\Gamma(A)$ has only one cusp. However, it is not arithmetic, unless the invariant trace field is quadratic, and the known proof for Bianchi groups does not go.

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