# SYMMETRIC CORRESPONDENCES ON QUADRICS

RAPHAËL FINO

Abstract. We prove a result comparing the rationality of some *elementary* algebraic cycles introduced by Alexander Vishik, defined on orthogonal grassmannians, with the rationality of some algebraic cycles defined on fiber products of the corresponding quadric.

**Keywords:** Chow groups, quadratic forms, orthogonal flag varieties.

Contents



## 1. Introduction

Let $X$ be a smooth projective quadric of dimension $n$ over a field $F$ associated with a non-degenerate $F$-quadratic form $q$ and let $K/F$ be a splitting field extension of $q$. Let us denote $[n/2]$ as $d$. In his paper [4] dedicated to the Kaplansky's conjecture on the $u$-invariant of a field, Alexander Vishik introduced the so-called *Elementary Discrete Invariant*. This invariant encodes information about rationality of algebraic cycles on the $d+1$ orthogonal grassmannians associated with $q$, by means of certain *elementary classes* defined on these grassmannians over $K$: it is the collection of the codimensions of those elementary classes that are already defined at the level of the base field $F$ (such an algebraic cycle is called *rational*).

In the current note, we relate the rationality of the *highest* elementary classes (highest in the sense that, for each grassmannian, the highest elementary class is the one with maximal codimension) with the rationality of certain algebraic cycles defined on some fiber products $X \times X \times \cdots \times X$.

More precisely, for any $I \subset \{0, \ldots, d\}$, we write $\mathcal{F}(I)$ for the partial orthogonal flag variety associated with $q$. So, for any $i \in \{0, \ldots, d\}$, the variety $\mathcal{F}(i)$ is the grassmannian of $i$-dimensional totally isotropic subspaces and we denote it by $G_i$. In particular $G_0$ is the quadric $X$. For $J \subset I$, we write $\pi$ with subindex $I$ with $J$ underlined inside it for the

*Date*: 6 November 2016.

2010 *Mathematics Subject Classification.* 14C25 ; 11E39.

natural projection $\mathcal{F}(I) \to \mathcal{F}(J)$. In particular, for any $i \in \{0, \dots, d\}$, one can consider

$$X \xleftarrow[\pi_{(\underline{0},i)}]{} \mathcal{F}(0,i) \xrightarrow[\pi_{(0,\underline{i})}]{} G_i,$$

and we set

$$Z^i_{n-i} := \pi_{(0,\underline{i})_*} \circ \pi^*_{(\underline{0},i)}(l_0) \in \mathrm{CH}^{n-i}(G_{iK}),$$

where $l_0 \in \mathrm{CH}_0(X_K)$ is the class of a closed point $\mathbf{x} \in X_K$ of degree 1 and CH stands for the Chow ring with $\mathbb{Z}$-coefficients. The cycle $z^i_{n-i} := Z^i_{n-i} \pmod 2 \in \mathrm{Ch}^{n-i}(G_{iK})$ is the highest elementary class for the grassmannian $G_i$ (with Ch the Chow ring with $\mathbb{Z}/2\mathbb{Z}$-coefficients). Note that A. Vishik proved in [4, Proposition 2.5] that the rationality of $z^i_{n-i}$ implies the rationality of $z^j_{n-j}$ for any $j > i$.

We now introduce the other type of algebraic cycles coming into play in the main statement of this note (Theorem 1.1). For any $i \in \{0, \dots, d\}$, let us denote by $\mathrm{sym} : \mathrm{CH}^*(X^{i+1}) \to \mathrm{CH}^*(X^{i+1})$ the homomorphism $\Sigma_{s \in S_{i+1}} s_*$, where $s : X^{i+1} \to X^{i+1}$ is the isomorphism associated with a permutation $s$. We set

$$\rho_i := \mathrm{sym}\left((\times_{j=0}^{i-1} h^j) \times l_0\right) \in \mathrm{CH}^{n+i(i-1)/2}(X_K^{i+1}),$$

where $\times$ is the external product and $h^j$ is the $j$-th power of the hyperplane section class $h \in \mathrm{CH}^1(X)$ (always rational). Note that $\rho_1$ is the *Rost correspondence* $1 \times l_0 + l_0 \times 1$ of $X$ in the sense of [1, §80] (we refer to [1, §62] for an introduction to correspondences). Note that the rationality of $\rho_i$ implies the rationality of $\rho_j$ for any $j > i$. Note also that $\rho_0 = l_0 = Z^0_n$.

The main result of this note is the following statement.

**Theorem 1.1 (Main Theorem).** *Let $i \in \{0, \dots, d\}$. The cycle $z^i_{n-i}$ is rational if and only if the cycle $\rho_i \pmod 2$ is rational.*

This result reduces certain questions about rationality of algebraic cycles on orthogonal grassmannians to the sole level of quadrics. For example, Theorem 1.1 allows one to reformulate Vishik Conjecture [3, Conjecture 3.11] and this reformulation may help prove the conjecture. Note that there are some quadrics for which none of the $z^i_{n-i}$ is rational (hence none of the $\rho_i$ is), this is for example the case for *generic* quadrics (see [3, Statement 3.6]).

The proofs in the present note mainly rely on computations of compositions of correspondences and the use of Chern classes of vector bundles over orthogonal grassmannians.

In section 2, we introduce some materials which will be required to prove Theorem 1.1 in sections 3 and 5. In section 4, we apply Theorem 1.1 (actually, only the direct implication is needed) to relate the first Witt index of the quadric with the rationality of the cycle $z^i_{n-i}$ (Proposition 4.1).

## 2. Materials

In this section, we use notations introduced in the introduction.

2.1. **Rational cycles on powers of quadrics.** We refer to [1, §68] for an introduction to cycles on powers of quadrics. For any $i \in \{1, \dots, d\}$, we set

$$\Delta_i := \operatorname{sym}\left((\times_{j=1}^{i-1} h^j) \times 1 \times l_0\right) + \sum_{k=i}^{d} \operatorname{sym}\left((\times_{j=1}^{i-1} h^j) \times h^k \times l_k\right) \in \mathrm{CH}^{n+i(i-1)/2}(X_K^{i+1}).$$

where $l_k$ is the class in $\mathrm{CH}_k(X_K)$ of a $k$-dimensional totally isotropic subspace of $\mathbb{P}((V_q)_K)$ (with $V_q$ the $F$-vector space associated with $q$). Note that $\Delta_1$ is just the class of the diagonal in $\mathrm{CH}^n(X_K^2)$.

The following observation is crucial for our matter (it will be used in Corollary 3.15 to obtain the first part Theorem 1.1).

**Lemma 2.1.** *For any $i \in \{1, \dots, d\}$, the cycle $\Delta_i \in \mathrm{CH}^{n+i(i-1)/2}(X_K^{i+1})$ is rational.*

*Proof.* We use an induction on $i$. Since $\Delta_1$ is the class of the diagonal in $\mathrm{CH}^n(X_K^2)$, it is rational. Assume $\Delta_{i-1}$ rational and let $\sigma \in S_{i+1}$ be a cyclic permutation ($i \geq 2$). Then the cycle

$$\sum_{j=0}^{i} \sigma_*^j(\Delta_{i-1} \times h^{i-1}) = \operatorname{sym}\left((\times_{j=1}^{i-1} h^j) \times 1 \times l_0\right) + \sum_{k=i-1}^{d} \operatorname{sym}\left((\times_{j=1}^{i-1} h^j) \times h^k \times l_k\right) \tag{2.2}$$

is also rational. Moreover, the cycle

$$\operatorname{sym}\left((\times_{j=1}^{i-1} h^j) \times h^{i-1} \times l_{i-1}\right) \tag{2.3}$$

is always rational. Indeed, since $h^{i-1}$ appears exactly two times in every summand of (2.3), each distinct summand appears exactly two times, therefore it can be rewritten as

$$2 \sum_{s \in A_{i+1}} s_*((\times_{j=1}^{i-1} h^j) \times h^{i-1} \times l_{i-1}),$$

and one has $2l_{i-1} = h^{n-i+1}$. Since $\Delta_i$ is the difference of (2.2) and (2.3), one get the conclusion. ☐

2.2. **Cycles on orthogonal flag varieties.** For $0 \leq i \leq d$ and $n - i - d \leq j \leq n - i$, we set

$$Z_j^i := \pi_{(0,\underline{i})_*} \circ \pi_{(\underline{0},i)}^*(l_{n-i-j}) \in \mathrm{CH}^j(G_{iK})$$

and $z_j^i := Z_j^i \pmod 2 \in \mathrm{Ch}^j(G_{iK})$. The cycles $z_j^i$ are all the elementary classes defining the Elementary Discrete Invariant $EDI(X)$.

For any nonnegative integer $j$ and $i > 0$ such that $j + i \leq d$, we set

$$W_j^i := \pi_{(0,\underline{i})_*} \circ \pi_{(\underline{0},i)}^*(h^{j+i}) \in \mathrm{CH}^j(G_i),$$

$W_j^0 := h^j$ and $w_j^i := W_j^i \pmod 2 \in \mathrm{Ch}^j(G_i)$.

The proofs in the next sections will use the following lemma, which can easily be deduced from [4, Propositon 2.1 and Lemma 2.6] and its proofs. For $0 \leq i \leq d$, let us denote by $T_i$ the tautological vector bundle on $G_i$, i.e. $T_i$ is given by the closed subvariety of the trivial bundle $V\mathbb{1} = V_q \times G_i$ consisting of pairs $(u, U)$ such that $u \in U$. For a vector bundle $E$ over a scheme, we write $c_i(E)$ for the $i$-th Chern class with value in CH.

**Lemma 2.4** (Vishik)**.** *One has*

(i) $\pi_{(0,\underline{i})_*} \circ \pi^*_{(\underline{0},i)}(h^i) = W_0^i = [G(i)]$;

(ii)

$$\pi^*_{(\underline{i-1},i)}(Z_j^{i-1}) = c_1(\mathcal{O}(1)) \cdot \pi^*_{(i-1,\underline{i})}(Z_{j-1}^i) + \pi^*_{(i-1,\underline{i})}(Z_j^i),$$

*where* $\mathcal{O}(1)$ *is the standard sheaf on the projective bundle* $\mathcal{F}(i-1,i) = \mathbb{P}_{G_{i-1}}(T_i^\vee)$, *with* $T_i^\vee$ *the vector bundle dual to* $T_i$;

(iii) *For* $0 \le j < d-i+1$,

$$\pi^*_{(\underline{i-1},i)}(W_j^{i-1}) = c_1(\mathcal{O}(1)) \cdot \pi^*_{(i-1,\underline{i})}(W_{j-1}^i) + \pi^*_{(i-1,\underline{i})}(W_j^i);$$

(iv)

$$\pi^*_{(\underline{i-1},i)}(W_{d-i+1}^{i-1}) = c_1(\mathcal{O}(1)) \cdot \pi^*_{(i-1,\underline{i})}(W_{d-i}^i) + 2\pi^*_{(i-1,\underline{i})}(Z_{d-i+1}^i).$$

The following statement is a direct consequence of the previous lemma.

**Lemma 2.5.** *For any* $1 \le i \le d$, $i \le k \le d$ *and* $0 \le m \le k$, *one has*

$$\pi_{(\underline{i-1},i)_*} \circ \pi_{(i-1,\underline{i})}{}^*(W_{k-i}^i \cdot Z_{n-i-m}^i) = \sum_{j=\max(i-m,0)}^{\min(k-m,i)} W_{k-m-j}^{i-1} \cdot \pi_{(\underline{i-1},i)_*} \circ \pi_{(i-1,\underline{i})}{}^*(Z_{n-2i+j}^i).$$

*Proof.* This is obtained by applying repeatedly Lemma 2.4(ii) after Lemma 2.4(iii), and using the Projection Formula combined with dimensional considerations. □

Finally, the following description of the Chern classes modulo 2 of the tautological vector bundles on orthogonal grassmannians will be needed.

**Lemma 2.6.** *For any* $1 \le i \le d$ *and* $0 \le j \le i$, *one has*

$$c_j(T_{i-1}) \pmod 2 = \pi_{(\underline{i-1},i)_*} \circ \pi_{(i-1,\underline{i})}{}^*(z_{n-2i+j}^i)$$

*in* $\mathrm{Ch}^j(G_{i-1\,K})$.

*Proof.* We use an induction on $j$. Since $Z_{n-2i}^i = \pi_{(0,\underline{i})_*} \circ \pi^*_{(\underline{0},i)}(l_i)$, the cycle $\pi^*_{(i-1,\underline{i})}(Z_{n-2i}^i)$ is the class in $\mathrm{CH}(\mathcal{F}(i-1,i)_K)$ of the closure of $\{(y, < y, x >) | x \in (y^\perp \backslash y) \cap L_i\}$ in $\mathcal{F}(i-1,i)_K$, with $L_i$ a fixed $i$-dimensional totally isotropic subspace of $\mathbb{P}((V_q)_K)$. Consequently, one has $\pi_{(\underline{i-1},i)_*} \circ \pi_{(i-1,\underline{i})}{}^*(Z_{n-2i}^i) = 1$. Let $1 \le j \le i$. By [4, Proposition 2.1], for any $l > d-i+1$, the element $c_l(V\mathbb{1}/T_{i-1})$ is divisible by 2 in $\mathrm{CH}^l(G_{i-1\,K})$. Therefore, by Whitney Sum Formula ([1, Proposition 54.7]), the Chern class $c_j(T_{i-1})$ is congruent modulo 2 to

$$c_j(V\mathbb{1}/T_{i-1}) + \sum_{k=\max(1,j-d+i-1)}^{j-1} c_{j-k}(V\mathbb{1}/T_{i-1}) \cdot c_k(T_{i-1}).$$

By the induction hypothesis and [4, Proposition 2.1], the latter cycle is congruent modulo 2 to

$$c_j(V\mathbb{1}/T_{i-1}) + \sum_{k=\max(1,j-d+i-1)}^{j-1} W_{j-k}^{i-1} \cdot \pi_{(\underline{i-1},i)_*} \circ \pi_{(i-1,\underline{i})}{}^*(Z_{n-2i+k}^i). \tag{2.7}$$

Moreover, by Lemma 2.5, one has

$$(2.8)\quad \pi_{(\underline{i-1},i)_*}\circ\pi_{(i-1,\underline{i})}{}^*(W^i_{d-i}\cdot Z^i_{n-i-d+j}) = \sum_{k=\max(j-d+i,0)}^{j} W^{i-1}_{j-k}\cdot\pi_{(\underline{i-1},i)_*}\circ\pi_{(i-1,\underline{i})}{}^*(Z^i_{n-2i+k}).$$

If $j-d+i-1<0$, combining (2.7) with (2.8), one get that $c_j(T_{i-1})$ is congruent modulo 2 to

$$\pi_{(\underline{i-1},i)_*}\circ\pi_{(i-1,\underline{i})}{}^*(Z^i_{n-2i+j}) + \pi_{(\underline{i-1},i)_*}\circ\pi_{(i-1,\underline{i})}{}^*(W^i_{d-i}\cdot Z^i_{n-i-d+j}).$$

Futhermore, by applying Lemma 2.4(iv) after Lemma 2.4(ii) and using the Projection Formula combined with dimensional considerations, one obtains

$$(2.9)\qquad \pi_{(\underline{i-1},i)_*}\circ\pi_{(i-1,\underline{i})}{}^*(w^i_{d-i}\cdot z^i_{n-i-d+j}) = w^{i-1}_{d-i+1}\cdot\pi_{(\underline{i-1},i)_*}\circ\pi_{(i-1,\underline{i})}{}^*(z^i_{n-i-d+j-1}).$$

Since $j-d+i-1<0$, the cycle $\pi_{(\underline{i-1},i)_*}\circ\pi_{(i-1,\underline{i})}{}^*(Z^i_{n-i-d+j-1})$ is trivial by dimensional reasons.

If $j-d+i-1=0$, combining (2.7) with (2.8), one get that $c_{d-i+1}(T_{i-1})$ is congruent modulo 2 to

$$\pi_{(\underline{i-1},i)_*}\circ\pi_{(i-1,\underline{i})}{}^*(Z^i_{n-i+d+1}) + W^{i-1}_{d-i+1} + \pi_{(\underline{i-1},i)_*}\circ\pi_{(i-1,\underline{i})}{}^*(W^i_{d-i}\cdot Z^i_{n-2i+1})$$

and the two last summands of the previous cycle are equal modulo 2 by identity (2.9) (valid for any $1\le j\le d$) and the base of the induction.

Otherwise – if $j-d+i-1>0$ – one proceeds similarly combining (2.7) with (2.8) and using identity (2.9).

□

## 3. Proof of the first part of Main Theorem

We use notations and materials introduced in sections 1 and 2. For any $i\in\{1,\dots,d\}$, we denote by $\theta_i$ the class of the subvariety

$$\{(y,x_1,\dots,x_{i+1})\mid x_1,\dots x_{i+1}\in y\}\subset G_i\times X^{i+1}$$

in $\mathrm{CH}(G_i\times X^{i+1})$. Viewing the cycle $\theta_i$ as a correspondence $G_i\rightsquigarrow X^{i+1}$, we set

$$\alpha_i := (\theta_i)_*(Z^i_{n-i})+\rho_i\in \mathrm{CH}(X_K^{i+1}),$$

and we view $\alpha_i$ as a correspondence $X_K\rightsquigarrow X_K^i$.

The following computations are the key point in the proof of the first part of Theorem 1.1 (see Corollary 3.15).

**Proposition 3.1.** *For any $i\in\{1,\dots,d\}$, one has*

$$(\alpha_i \ (\mathrm{mod}\ 2))_*(h^k) = \begin{cases} \mathrm{sym}\left((\times_{j=1}^{i-1}h^j)\times 1\right) & \text{if } k=0;\\ 0 & \text{if } 1\le k\le i-1;\\ \mathrm{sym}\left((\times_{j=1}^{i-1}h^j)\times h^k\right) & \text{if } i\le k\le d.\end{cases}$$

*Proof.* The following lemma provides an appropriate formula for $\left((\theta_i)_*(Z^i_{n-i})\right)_*$. We write $p$ with underlined target for projections.

**Lemma 3.2.** *For any $x \in \mathrm{CH}(X_K)$, one has*
$$\left((\theta_i)_*(Z^i_{n-i})\right)_*(x) = p_{G_i\times \underline{X^i}}{}_*\left(\pi_{(0,\underline{i})}{}_*\circ \pi^*_{(\underline{0},i)}(x)\cdot Z^i_{n-i}\cdot \eta_i\right),$$
*where $\eta_i$ is the class in $\mathrm{CH}(G_i\times X^i)$ of the subvariety $\{(y,x_1,\dots,x_i)\mid x_1,\dots,x_i\in y\}\subset G_i\times X^i$.*

*Proof.* By definition, one has
$$(\theta_i)_*(Z^i_{n-i}) = p_{G_i\times \underline{X^{i+1}}}{}_*\left((Z^i_{n-i}\times [X^{i+1}])\cdot \theta_i\right)$$
so
$$\left((\theta_i)_*(Z^i_{n-i})\right)_*(x) = p_{X\times \underline{X^i}}{}_*\left((x\times [X^i])\cdot p_{G_i\times \underline{X^{i+1}}}{}_*\left((Z^i_{n-i}\times [X^{i+1}])\cdot \theta_i\right)\right).$$
Therefore, using the Projection Formula (see [1, Proposition 56.9]) and the following fiber product diagram with respect to projections
$$\begin{array}{ccc} G_i\times X\times X^i & \longrightarrow & X\times X^i\ , \\ \downarrow & & \downarrow \\ G_i\times X^i & \longrightarrow & X^i \end{array}$$
one get that
$$\left((\theta_i)_*(Z^i_{n-i})\right)_*(x) = p_{G_i\times \underline{X^i}}{}_*\circ p_{\underline{G_i}\times X\times \underline{X^i}}{}_*\left((Z^i_{n-i}\times x\times [X^i])\cdot \theta_i\right).$$
Moreover, the cycle $\theta_i$ can be rewritten as
$$\theta_i = p_{\underline{G_i\times X}\times X^i}{}^*([\mathcal{F}(i,0)])\cdot p_{\underline{G_i}\times X\times \underline{X^i}}{}^*(\eta_i),$$
where $[\mathcal{F}(i,0)]$ is the class in $\mathrm{CH}(G_i\times X)$ of the subvariety $\mathcal{F}(i,0)\subset G(i)\times X$. Hence, using the Projection Formula twice, one get that
$$\begin{aligned}\left((\theta_i)_*(Z^i_{n-i})\right)_*(x) &= p_{G_i\times \underline{X^i}}{}_*\left(\left(p_{\underline{G_i}\times X}{}_*\left((Z^i_{n-i}\times x)\cdot [\mathcal{F}(i,0)]\right)\times [X^i]\right)\cdot \eta_i\right)\\ &= p_{G_i\times \underline{X^i}}{}_*\left(p_{\underline{G_i}\times X}{}_*\left(p_{G_i\times \underline{X}}{}^*(x)\cdot [\mathcal{F}(i,0)]\right)\cdot Z^i_{n-i}\cdot \eta_i\right).\end{aligned}$$

Furthermore, by denoting the closed embedding $\mathcal{F}(i,0)\hookrightarrow G_i\times X$ as $in$, one has $in_*(1) = [\mathcal{F}(i,0)]$. It follows again from the Projection Formula that $p_{G_i\times \underline{X}}{}^*(x)\cdot[\mathcal{F}(i,0)] = in_*\circ in^*\left(p_{G_i\times \underline{X}}{}^*(x)\right)$. Consequently, one has
$$p_{\underline{G_i}\times X}{}_*\left(p_{G_i\times \underline{X}}{}^*(x)\cdot [\mathcal{F}(i,0)]\right) = \pi_{(0,\underline{i})}{}_*\circ \pi^*_{(\underline{0},i)}(x)$$
and the lemma is proven. □

For any $x\in \mathrm{CH}^k(X_K)$ with $k\le i-1$, the cycle $\pi_{(0,\underline{i})}{}_*\circ\pi^*_{(\underline{0},i)}(x)$ is trivial by dimensional reasons. Thus, by Lemma 3.2, the cycle $\left((\theta_i)_*(Z^i_{n-i})\right)_*(x)$ is also trivial. Therefore, since $(\rho_i)_*(h^k) = \mathrm{sym}\left((\times_{j=1}^{i-1}h^j)\times 1\right)$ for $k=0$ and is trivial for $0<k\le d$, one get the conclusion of Proposition 3.1 for the cases $k\le i-1$.

Note that for $i\le k\le d$, Lemma 3.2 provides the following identity
$$(\alpha_i \ (\mathrm{mod}\ 2))_*(h^k) = p_{G_i\times \underline{X^i}}{}_*\left(w^i_{k-i}\cdot z^i_{n-i}\cdot \eta_i\right) \text{ in } \mathrm{Ch}(X^i_K), \tag{3.3}$$
where we abuse notation and write $\eta_i$ for $\eta_i$ (mod 2).

We prove the cases $i\le k\le d$ of Proposition 3.1 by backward induction on $i$.

For any $1 \leq i \leq d$, by the very definitions, one has

$$\eta_i = \prod_{j=1}^{i} \left(\mathrm{Id}_{G_i} \times p_{X^i_j}\right)^* ([\mathcal{F}(i,0)]) \text{ in } \mathrm{CH}(G_i \times X^i), \tag{3.4}$$

with $p_{X^i_j}$ the projection from $X^i$ to the $j$-th coordinate, and

$$[\mathcal{F}(i,0)] = \sum_{m=0}^{d} Z^i_{n-i-m} \times h^m + \sum_{m=i}^{d} W^i_{m-i} \times l_m \text{ in } \mathrm{CH}(G_{iK} \times X_K). \tag{3.5}$$

The base of the backward induction $i = d$ (so $i = k = d$) is obtained by combining the identities (3.3), (3.4) and (3.5) for $i = d$ (recall also that $w^i_0 = 1$ for any $0 \leq i \leq d$ by Lemma 2.4(i)) with the fact that, for any integers $0 \leq a_0 \leq a_1 \leq \cdots \leq a_d \leq d$, one has

$$\deg\left(\prod_{j=0}^{d} z^d_{n-d-a_j}\right) = \begin{cases} 1 & \text{if } \{a_0, a_1, \ldots, a_d\} = \{0, 1, \ldots, d\}\,; \\ 0 & \text{otherwise,} \end{cases}$$

where $\deg : \mathrm{Ch}(G_{dK}) \to \mathrm{Ch}(\mathrm{Spec}(K)) = \mathbb{Z}/2\mathbb{Z}$ is the homomorphism associated with the push-forward of the structure morphism, see [1, Lemma 87.6].

The backward induction step will follow from Lemma 3.10, which make use of the following statement.

**Lemma 3.6.** *For any $2 \leq i \leq d$, one has*

$$z^i_{n-i}\cdot\eta_i = \left(\pi_{(i-1,\underline{i})} \times \mathrm{Id}_{X^i}\right)_* \left(\left(\pi_{(i-1,\underline{i})} \times p_{X^i_i}\right)^* ([\mathcal{F}(i,0)]) \cdot \left(\pi_{(\underline{i-1},i)} \times p_{X^i_{\bar{i}}}\right)^* (z^{i-1}_{n-i+1} \cdot \eta_{i-1})\right).$$

*Proof.* It follows from the identity (3.5) and Lemma 2.4(ii), (iii) and (iv) that

$$\begin{aligned}\left(\pi_{(\underline{i-1},i)} \times \mathrm{Id}_X\right)^* ([\mathcal{F}(i-1,0)]) &= (c_1(\mathcal{O}(1)) \times 1) \cdot \left(\pi_{(i-1,\underline{i})} \times \mathrm{Id}_X\right)^* ([\mathcal{F}(i,0)]) \\ &+ \left(\pi_{(i-1,\underline{i})} \times \mathrm{Id}_X\right)^* \left((1 \times h) \cdot \left([\mathcal{F}(i,0)] - Z^i_{n-i-d} \times h^d\right)\right) \\ &+ 2\pi^*_{(i-1,\underline{i})}(Z^i_{d-i+1}) \times l_d\end{aligned}$$

in $\mathrm{CH}(\mathcal{F}(i-1,i)_K \times X_K)$. Considered modulo 2, this gives

$$\left(\pi_{(\underline{i-1},i)} \times \mathrm{Id}_X\right)^* ([\mathcal{F}(i-1,0)]) = (c_1(\mathcal{O}(1)) \times 1 + 1 \times h) \cdot \left(\pi_{(i-1,\underline{i})} \times \mathrm{Id}_X\right)^* ([\mathcal{F}(i,0)]) \tag{3.7}$$

in $\mathrm{Ch}(\mathcal{F}(i-1,i)_K \times X_K)$.

Moreover, by (3.4), one has

$$\left(\pi_{(\underline{i-1},i)} \times p_{X^i_{\bar{i}}}\right)^* (z^{i-1}_{n-i+1} \cdot \eta_{i-1}) = \pi_{(\underline{i-1},i)}{}^*(z^{i-1}_{n-i+1}) \cdot \prod_{j=1}^{i-1} \left(\pi_{(\underline{i-1},i)} \times p_{X^i_j}\right)^* ([\mathcal{F}(i-1,0)]) \tag{3.8}$$

with $p_{X^i_{\bar{i}}}$ the projection from $X^i$ to the $i-1$ first coordinates. By (3.7), the right member of the previous equation can be rewritten as

$$\pi_{(\underline{i-1},i)}{}^*(z^{i-1}_{n-i+1}) \cdot \prod_{j=1}^{i-1} \left(c_1(\mathcal{O}(1)) \times [X^i] + 1 \times p_{X^i_j}{}^*(h)\right) \cdot \prod_{j=1}^{i-1} \left(\pi_{(i-1,\underline{i})} \times p_{X^i_j}\right)^* ([\mathcal{F}(i,0)])\,.$$

Hence, by multiplying the equation (3.8) by $\left(\pi_{(i-1,\underline{i})} \times p_{X^i_i}\right)^* ([\mathcal{F}(i,0)])$ and using Lemma 2.4(ii), one get

$$(3.9)\quad \left(\pi_{(i-1,\underline{i})} \times p_{X^i_i}\right)^* ([\mathcal{F}(i,0)]) \cdot \left(\pi_{(\underline{i-1},i)} \times p_{X^i_i}\right)^* (z^{i-1}_{n-i+1} \cdot \eta_{i-1}) =$$
$$\left(c_1(\mathcal{O}(1)) \times [X^i]\right) \cdot \prod_{j=1}^{i-1} \left(c_1(\mathcal{O}(1)) \times [X^i] + 1 \times {p_{X^i_j}}^*(h)\right) \cdot \left(\pi_{(i-1,\underline{i})} \times \mathrm{Id}_{X^i}\right)^* (z^i_{n-i} \cdot \eta_i).$$

Furthermore, it follows from the Projective Bundle Theorem (see [1, Theorem 53.10]) applied to $\pi_{(i-1,\underline{i})} : \mathcal{F}(i-1,i) \to G_i$ that ${\pi_{(i-1,\underline{i})}}_*(c_1(\mathcal{O}(1))^i) = 1$. Since $\dim \mathcal{F}(i-1,i) - \dim G_i = i$, one obtains the conclusion of the lemma by composing the equation (3.9) by $\left(\pi_{(i-1,\underline{i})} \times \mathrm{Id}_{X^i}\right)_*$ and using the Projection Formula.

□

**Lemma 3.10.** *For any $2 \le i \le d$ and $i \le k \le d$, the cycle in identity (3.3) can be rewritten as*

$$\sum_{m=0}^{k} \sum_{j=\max(i-m,0)}^{\min(k-m,i)} {p_{G_{i-1}\times \underline{X^{i-1}}}}_* \left(w^{i-1}_{k-m-j} \cdot \sigma^j_{i-1} \cdot z^{i-1}_{n-i+1} \cdot \eta_{i-1}\right) \times h^m,$$

*with* $\sigma^j_{i-1} = {\pi_{(\underline{i-1},i)}}_* \circ {\pi_{(i-1,\underline{i})}}^*(Z^i_{n-2i+j}) \pmod 2\ \in \mathrm{Ch}^j(G_{i-1\,K})$.

*Proof.* It follows from Lemma 3.6 and the Projection Formula that the cycle in identity (3.3) can be rewritten as

$${p_{G_{i-1}\times \underline{X^i}}}_* \left(\left(\pi_{(\underline{i-1},i)} \times \mathrm{Id}_{X^i}\right)_* \circ \left(\pi_{(i-1,\underline{i})} \times p_{X^i_i}\right)^* \left(w^i_{k-i} \cdot [\mathcal{F}(i,0)]\right) \cdot z^{i-1}_{n-i+1} \cdot (\eta_{i-1} \times 1)\right).$$

Moreover, by decomposition (3.5) and dimensional considerations, the latter cycle is equal to

$$\sum_{m=0}^{k} {p_{G_{i-1}\times \underline{X^{i-1}}}}_* \left({\pi_{(\underline{i-1},i)}}_* \circ {\pi_{(i-1,\underline{i})}}^*(w^i_{k-i} \cdot z^i_{n-i-m}) \cdot z^{i-1}_{n-i+1} \cdot \eta_{i-1}\right) \times h^m.$$

Lemma 2.5 completes the proof. □

Let $2 \le i \le d$. On the one hand, by identity (3.3) and backward induction hypothesis, for any $i \le k \le d$, one has

$$(3.11)\quad {p_{G_i\times \underline{X^i}}}_* \left(w^i_{k-i} \cdot z^i_{n-i} \cdot \eta_i\right) = \mathrm{sym}\left((\times_{j=1}^{i-1} h^j) \times h^k\right).$$

Therefore, in particular, the coordinate of (3.11) on top right $h^{i-1}$, i.e.,

$${p_{\underline{X^{i-1}}\times X}}_* \left(\left({p_{G_i\times \underline{X^i}}}_* \left(w^i_{k-i} \cdot z^i_{n-i} \cdot \eta_i\right)\right) \cdot [X^{i-1}] \times l_{i-1}\right)$$

is equal to

$$(3.12)\quad \mathrm{sym}\left((\times_{j=1}^{i-2} h^j) \times h^k\right).$$

On the other hand, by Lemma 3.10, this coordinate is also equal to

$$(3.13)\quad \sum_{j=1}^{\min(k-i+1,i)} {p_{G_{i-1}\times \underline{X^{i-1}}}}_* \left(w^{i-1}_{k-i+1-j} \cdot \sigma^j_{i-1} \cdot z^{i-1}_{n-i+1} \cdot \eta_{i-1}\right).$$

Since $W^{i-1}_{k-i+1-j} = c_{k-i+1-j}(V\mathbb{1}/T_{i-1})$ (see [4, Proposition 2.1]) and $\sigma^j_{i-1} = c_j(T_{i-1}) \pmod 2$ (see Lemma 2.6), by Whitney Sum Formula, one has

$$\sum_{j=0}^{\min(k-i+1,i)} w^{i-1}_{k-i+1-j} \cdot \sigma^j_{i-1} = \sum_{j=0}^{\min(k-i+1,i)} c_{k-i+1-j}(V\mathbb{1}/T_{i-1}) \cdot c_j(T_{i-1}) = c_{k-i+1}(V\mathbb{1}) = 0.$$

Consequently, in view of (3.12) and (3.13), one get

$$p_{G_{i-1}\times \underline{X^{i-1}}\,*}\left(w^{i-1}_{k-i+1} \cdot z^{i-1}_{n-i+1} \cdot \eta_{i-1}\right) = \mathrm{sym}\left((\times_{j=1}^{i-2} h^j) \times h^k\right). \tag{3.14}$$

Note that, by identities (3.3) and (3.14), it only remains to prove that

$$p_{G_{i-1}\times \underline{X^{i-1}}\,*}\left(z^{i-1}_{n-i+1} \cdot \eta_{i-1}\right) = \mathrm{sym}\left((\times_{j=1}^{i-1} h^j)\right)$$

to complete the backward induction step. On the one hand, by backward induction hypothesis, the coordinate of (3.11) on top right $h^k$ is

$$\mathrm{sym}\left((\times_{j=1}^{i-1} h^j)\right)$$

and on the other hand, by Lemma 3.10, it is also equal to

$$p_{G_{i-1}\times \underline{X^{i-1}}\,*}\left(z^{i-1}_{n-i+1} \cdot \eta_{i-1}\right).$$

Proposition 3.1 is proven.

□

We are now able to prove the first part of the main result of this note (Theorem 1.1), which we restate below.

**Corollary 3.15.** *Let $i \in \{0, \dots, d\}$. If $z^i_{n-i}$ is rational then $\rho_i \pmod 2$ is also rational.*

*Proof.* Since the conclusion is obvious for $i = 0$, we assume that $i \geq 1$. In view of the ring structure of $\mathrm{CH}(X^{i+1}_K)$ (see [1, §68]), and knowing that the cycle $\alpha_i$ is symmetric, one deduces from Proposition 3.1 that

$$\alpha_i \pmod 2 = \Delta_i \pmod 2 + \beta$$

with $\beta$ a sum of nonessential elements (a nonessential element is an external product of powers of the hyperplane class, it is always rational). Since $\alpha_i = (\theta_i)_*(Z^i_{n-i}) + \rho_i$, the corollary is proved. □

The following statement is a consequence of Proposition 3.1 and its proof.

**Proposition 3.16.** *Let $1 \leq i \leq d-1$, $i+1 \leq k \leq d$ and $1 \leq m \leq i$. For any integers $0 \leq a_1 \leq a_2 \leq \cdots \leq a_i \leq d$, the integer*

$$\deg\left(\left(Z^i_{n-i} \cdot \prod_{l=1}^{i} Z^i_{n-i-a_l}\right) \cdot \left(\sum_{j=0}^{i-m} W^i_{k-m-j} \cdot c_j(T_i)\right)\right)$$

*is congruent to $1 \pmod 2$ if $\{a_1, \dots, a_i\} = \{k\} \cup (\{1, 2, \dots, i\} \backslash \{m\})$ and to $0 \pmod 2$ otherwise.*

*Proof.* Proposition 3.1 and Lemma 3.10 provide two descriptions of the coordinate of the cycle (3.3) on top right $h^m$ (note that one can decompose the cycle $\eta_i \in \mathrm{CH}(G_{iK} \times X_K^i)$ appearing in Lemma 3.10 as a sum of external products by combinng equations (3.4) and (3.5)). The conclusion is obtained by comparing these two descriptions and applying the Whitney Sum Formula to that given by Lemma 3.10. □

**Remark 3.17.** We also retrieve [4, Statement 2.15] by proceeding the same way as in the previous proof but considering the coordinate on top right $h^k$ instead of top right $h^m$. Proposition 3.16 is completed for $i = d$ by [1, Lemma 87.6].

## 4. Elementary Discrete Invariant and first Witt index

In this section, we continue to use notations and materials introduced in the previous sections. Corollary 3.15 implies the following condition on the first Witt index $i_1$ of the quadric $X$.

**Proposition 4.1.** *Assume $X$ anisotropic and let $i \in \{1, \dots, d\}$. If $z_{n-i}^i$ is rational then $i_1 \leq i$.*

*Proof.* In view of Corollary 3.15, the statement follows from the next lemma.

**Lemma 4.2.** *Assume $X$ anisotropic and let $i \in \{1, \dots, d\}$. If $\rho_i \pmod 2$ is rational then $i_1 \leq i$.*

*Proof.* Suppose that $\rho_i \pmod 2$ is rational and that $i_1 > i$. We claim that this implies that $\rho_{i-1} \pmod 2$ is also rational.

Indeed, let us denote by $\pi \in \mathrm{Ch}^{n-i_1+1}(X_K^2)$ the 1-*primordial cycle* (see [1, Definition 73.16] and paragraph right after [1, Theorem 73.26], it is a rational cycle). Even if it means adding a rational cycle to $\pi$, one can assume that $\pi$ decomposes as

$$\pi = 1 \times l_{i_1-1} + l_{i_1-1} \times 1 + \sum_{j=i_1}^{d-i_1+1} a_j \left(h^j \times l_{j+i_1-1} + l_{j+i_1-1} \times h^j\right)$$

for some $a_j \in \mathbb{Z}/2\mathbb{Z}$ (the fact that one can choose to make the previous sum start from $j = i_1$ is due to [1, Proposition 73.27]). Since $i_1 > i$, the computation of the composition of rational correspondences $(\rho_i \pmod 2)_K \circ ((1 \times h^{i_1-i}) \cdot \pi)$ gives the identity

$$(\rho_i \pmod 2)_K \circ \left((1 \times h^{i_1-i}) \cdot \pi\right) = 1 \times (\rho_{i-1} \pmod 2)_K .$$

Therefore, pulling back the latter algebraic cycle with respect to the diagonal morphism

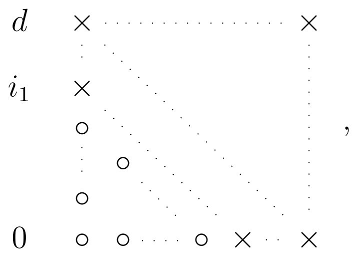
$$\begin{array}{ccc} X^i & \longrightarrow & X^{i+1} \\ (x_1, x_2, x_3, \dots, x_i) & \longmapsto & (x_1, x_1, x_2, x_3 \dots, x_i) \end{array}$$

(for example), one get that $\rho_{i-1} \pmod 2$ is also a rational cycle.

It follows from the claim that $l_0$ is rational, i.e., $X$ is isotropic. □

The proposition is proved. □

**Remark 4.3.** Combining Proposition 4.1 with [4, Propositon 2.5], one get that, representing the *Elementary Discrete Invariant* $EDI(X)$ of an anisotropic quadric $X$ as a $(d+1) \times (d+1)$ coordinate square (see [4, Definition 2.3 and the paragraph right after]), there is no marked integral node below the $i_1$-th diagonal of this square. That is to say, the invariant $EDI(X)$ looks as




where $\circ$ means that the node is unmarked.
We recall that each row is associated with an orthogonal grassmannian, starting with the quadric at the bottom row, and that each integral node corresponds to an *elementary classes* as defined by A. Vishik in [4], with codimension decreasing from left to right (the cycles $z^i_{n-i} \in \mathrm{Ch}^{n-i}(G(i)_K)$ correspond to the first column on the left).

## 5. Proof of the second part of Main Theorem

We use notations and materials introduced in previous sections. In the following statement, the fact that if $\rho_1$ is rational then $Z^1_{n-1}$ is also rational has already been shown by A. Vishik in the proof of [2, Theorem 4.4]. The second part of Main Theorem (Theorem 1.1) is actually valid at the level of integral Chow groups.

**Proposition 5.1.** *Let $i \in \{0, \dots, d\}$. If $\rho_i$ is rational then $Z^i_{n-i}$ is also rational.*

*Proof.* Since the conclusion is obvious for $i = 0$, we assume that $i \geq 1$. Consider the rational cycle

$$\theta'_i := p^*_{\underline{X}\times X\times\underline{\mathcal{F}(0,i)}} \circ \left(\mathrm{Id}_X \times \pi_{(\underline{0},i)}\right)^* (\Delta_1) \cdot p^*_{X\times\underline{X}\times\underline{\mathcal{F}(0,i)}} \circ \left(\mathrm{Id}_X \times \pi_{(0,\underline{i})}\right)^* ([\mathcal{F}(0,i)]_K)$$

in $\mathrm{CH}^{2n-i}(X^2_K \times \mathcal{F}(0,i)_K)$, with $\Delta_1$ the class of the diagonal in $\mathrm{CH}^n(X^2_K)$ (see §2.1). We view it as a correspondence $\theta'_i : X^2_K \rightsquigarrow \mathcal{F}(0,i)_K$. Using decomposition (3.5), one get that for any $(\alpha,\beta) \in \{1, h, \dots, h^{i-1}, l_0\}^2$, with $\alpha \neq \beta$, one has

$$\theta'_{i*}(\alpha\times\beta) = \begin{cases} {\pi_{(0,\underline{i})}}^*(Z^i_{n-i}) & \text{if } \alpha\times\beta = 1\times l_0 \,; \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, since $\rho_i$ is assumed to be rational, one obtains that the cycle

$$(\mathrm{Id}_{X^{i-1}} \times \theta'_i)_* (\rho_i) = \mathrm{sym}\left((\times_{j=1}^{i-1} h^j)\right) \times {\pi_{(0,\underline{i})}}^*(Z^i_{n-i})$$

is rational. By multiplying the previous cycle by $[X^{i-1}] \times {\pi_{(\underline{0},i)}}^*(h^i)$ and then composing by $\left(\mathrm{Id}_{X^{i-1}} \times \pi_{(0,\underline{i})}\right)_*$, one get, using the Projection Formula and Lemma 2.4(i), that the cycle

$$\mathrm{sym}\left((\times_{j=1}^{i-1} h^j)\right) \times Z^i_{n-i}$$

is rational. Therefore, it suffices to prove that, for any $1 \leq k \leq i-1$, the rationality of $\mathrm{sym}\left((\times_{j=1}^{i-k} h^j)\right) \times Z^i_{n-i}$ implies the rationality of $\mathrm{sym}\left((\times_{j=1}^{i-k-1} h^j)\right) \times Z^i_{n-i}$ to conclude. This follows from the next formula (which uses decomposition (3.5))

$$p_{\underline{X^{i-k-1}}\times X\times \underline{G_i}_*}\left([X^{i-k-1}]\times (h^k\times [G_i]\cdot [\mathcal{F}(0,i)]_K)\cdot \mathrm{sym}\left((\times_{j=1}^{i-k}h^j)\right)\times Z^i_{n-i}\right) = \mathrm{sym}\left((\times_{j=1}^{i-k-1}h^j)\right)\times Z^i_{n-i}\ .$$

□

## References


[1] Elman, R., Karpenko, N., and Merkurjev, A. *The algebraic and geometric theory of quadratic forms*, vol. 56 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2008.

[2] Vishik, V. *Symmetric operations* (in russian). In *Trudy Mat. Inst. Steklova 246* (2004), Algebr. Geom. Metody, Svyazi i Prilozh., pp. 92-105. English transl: Proc of the Steklov Institute of Math. 246, 79–92 (2004).

[3] Vishik, V. *Generic points of quadrics and Chow groups*. Manuscripta Math 122, 3 (2007), 365–374.

[4] Vishik, V. *Fields of u-invariant* $2^r+1$. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II,* vol. 270 of *Progr. Math.* Birkhauser Boston Inc., Boston, MA, 2009, pp. 661-685.



Instituto de Matemáticas, Ciudad Universitaria, UNAM, DF 04510, México

*E-mail address*: fino *at* im.unam.mx