# FIELDS OF DEFINITION FOR REPRESENTATIONS OF ASSOCIATIVE ALGEBRAS

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Abstract. We examine situations, where representations of a finite-dimensional $F$-algebra $A$ defined over a separable extension field $K/F$, have a unique minimal field of definition. Here the base field $F$ is assumed to be a $C_1$-field. In particular, $F$ could be a finite field or $k(t)$ or $k((t))$, where $k$ is algebraically closed.

We show that a unique minimal field of definition exists if (a) $K/F$ is an algebraic extension or (b) $A$ is of finite representation type. Moreover, in these situations the minimal field of definition is a finite extension of $F$. This is not the case if $A$ is of infinite representation type or $F$ fails to be $C_1$. As a consequence, we compute the essential dimension of the functor of representations of a finite group, generalizing a theorem of N. Karpenko, J. Pevtsova and the second author.

## 1. Introduction

**Notational conventions.** Throughout this paper $F$ will denote a base field and $A$ a finite-dimensional associative algebra over $F$. If $K/F$ is a field extension (not necessarily algebraic), we will denote the tensor product $K \otimes_F A$ by $A_K$. Let $M$ be an $A_K$-module. Unless otherwise specified, we will always assume that $M$ is finitely generated (or equivalently, finite-dimensional as a $K$-vector space). If $L/K$ is a field extension, we will write $M_L$ for $L \otimes_K M$.

An intermediate field $F \subset K_0 \subset K$ is called a *field of definition* for $M$ if there exists a $K_0$-module $M_0$ such that $M \cong (M_0)_K$. In this case we will also say that $M$ *descends* to $K_0$.

**Minimal fields of definition.** A field of definition $K_0$ of $M$ is said to be *minimal* if whenever $M$ descends to a field $L$ with $F \subset L \subset K$, we have $K_0 \subset L$.

Minimal fields of definition do not always exist. For example, let $F = \mathbb{Q}$ and $A$ be the quaternion algebra

$$A = \mathbb{Q}\{i, j, k\}/(i^2 = j^2 = k^2 = ijk = -1).$$

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Then $A_K$ has a two dimensional module $M$ given by

$$i \mapsto \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \qquad j \mapsto \begin{pmatrix} b & -a \\ -a & -b \end{pmatrix}, \qquad k \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

over any field $K$ of characteristic 0 having two elements $a$ and $b$ such that $a^2 + b^2 = -1$. Examples of such fields include $\mathbb{C}$, $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-5})$. If we take $K$ to be "the generic field" of this type, i.e., the field of fractions of $\mathbb{Q}[a,b]/(a^2+b^2+1)$, then $M$ has no minimal field of definition; see Proposition 6.3(b).

**$C_1$-fields.** Such examples arise because of the existence of noncommutative division rings of finite dimension over $F$. So, it makes sense to develop a theory over fields for which these do not exist. We say that $F$ is a $C_1$-field if any homogeneous polynomial $f_1(x_1, \dots, x_n)$ of degree $d < n$ with coefficients in $F$ has a non-trivial solution in $F^n$. Examples of $C_1$-fields include finite fields, $k(t)$, and $k((t))$, where $k$ is algebraically closed. An algebraic extension of a $C_1$-field is again $C_1$. Over a $C_1$-field every every central division algebra is commutative. For a detailed discussion of this class of fields, including proofs of the above assertions, we refer the reader to [GS, Section 6.2]. Our first main result is as follows.

**Theorem 1.1.** *Let $F$ be a $C_1$-field, $A$ be a finite-dimensional $F$-algebra, $K/F$ be a separable algebraic field extension and $M$ be an an $A_K$-module. Then $M$ has a minimal field of definition $F \subset K_0 \subset K$ such that $[K_0 : F] < \infty$.*

To illustrate Theorem 1.1, let us consider a simple case, where $\mathrm{char}(F) = 0$, $A := FG$ is the group algebra of a finite group $G$, and $M$ is absolutely irreducible $KG$-module. Denote the character of $G$ associated to $M$ by $\chi\colon G \to K$. We claim that in this case the minimal field of definition is $F(\chi)$, the field generated over $F$ by the character values $\chi(g)$, as $g$ ranges over $G$. Indeed, it is clear that $F(\chi)$ has to be contained in any field of definition $F \subset K_0 \subset K$ of $M$. Thus to prove the above assertion, we only need to show that $M$ descends to $F(\chi)$. The minimal degree of a finite field extension $L/F(\chi)$, such that $M$ is defined over $L$ (i.e., there exists an $LG$-module with character $\chi$), is the Schur index $s_M$; cf. [CR, Definition 41.4]. Thus it suffices to show that $s_M = 1$. By [CR, Theorem (70.15)], $s_M$ is the index of the endomorphism algebra $\mathrm{End}_A(M)$ of $M$, which is a central simple algebra over $F(\chi)$. Since $F$ is a $C_1$-field, and $F(\chi)$ is a finite extension of $F$, $F(\chi)$ is also a $C_1$-field. Hence, the index of every central simple algebra over $F(\chi)$ is 1. In particular, $s_M = 1$, and $M$ descends to $F(\chi)$, as claimed.

**Algebras of finite representation type.** A finite-dimensional $F$-algebra $A$ is said to be *of finite representation type* if there are only finitely many indecomposable finitely generated $A$-modules (up to isomorphism).

Our next result shows that for algebras of finite representation type Theorem 1.1 remains valid even if the field extension $K/F$ is not assumed to be algebraic.

**Theorem 1.2.** *Let $F$ be a $C_1$-field, $A$ be a finite-dimensional $F$-algebra of finite representation type, $K/F$ be a field extension, and $M$ be an $A_K$-module. Assume further that $F$ is perfectly closed in $K$. Then $M$ has a minimal field of definition $F \subset K_0 \subset K$ such that $[K_0 : F] < \infty$.*

**Essential dimension.** Given the $A_K$-module $M$, the *essential dimension* $\mathrm{ed}(M)$ of $M$ over $F$ is defined as the minimal value of the transcendence degree $\mathrm{trdeg}(K_0/F)$, where the minimum is taken over all fields of definition $F \subset K_0 \subset K$. The integer $\mathrm{ed}(M)$ may be viewed as a measure of the complexity of $M$. Note that $\mathrm{ed}(M)$ is well-defined, irrespective of whether $M$ has a minimal field of definition or not. We also remark that this number implicitly depends on the base field $F$, which is assumed to be fixed throughout. As a consequence of Theorem 1.2, we will deduce the following.

**Theorem 1.3.** *Let $F$ be a $C_1$-field, $A$ be finite-dimensional $F$-algebra of finite representation type, $K/F$ be a field extension, and $M$ be an $A_K$-module. Then* $\mathrm{ed}(M) = 0$.

Both Theorem 1.2 and 1.3 fail if we do not require $F$ to be a $C_1$-field; see Section 6.

**The essential dimension of the functor of $A$-modules.** We will also be interested in the essential dimension $\mathrm{ed}(\mathrm{Mod}_A)$ of the functor $\mathrm{Mod}_A$ from the category of field extensions of $F$ to the category of sets, which associates to a field $K$, the set of isomorphism classes of $A_K$-modules. By definition,

$$\mathrm{ed}(\mathrm{Mod}_A) := \sup\ \mathrm{ed}(M)\,,$$

where the supremum is taken over all field extensions $K/F$ and all finitely generated $A_K$-modules $M$. The value of $\mathrm{ed}(\mathrm{Mod}_A)$ may be viewed as a measure the complexity of the representation theory of $A$. For generalities on the notion of essential dimension we refer the reader to [BF, Re$_1$, Re$_2$, Me$_1$, Me$_2$]. Essential dimensions of representations of finite groups and finite-dimensional algebras are studied in [KRP] and [BDH, Section 3].

Note that while $\mathrm{ed}(M) < \infty$, for any given $A_K$-module $M$ (see Lemma 2.1), $\mathrm{ed}(\mathrm{Mod}_A)$ may be infinite. In particular, in the case, where $A = FG$ is the group algebra of a finite group $G$ over a field $F$, it is shown in [KRP, Theorem 14.1] that $\mathrm{ed}(\mathrm{Mod}_A) = \infty$, provided that $F$ is a field of characteristic $p > 0$ and $G$ has a subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$. Our final main result is the following amplification of [KRP, Theorem 14.1].

**Theorem 1.4.** *Let $G$ be a finite group and $F$ be a field of characteristic $p$. Then the following conditions are equivalent:*

(1) *The $p$-Sylow subgroup of $G$ is cyclic,*
(2) $\mathrm{ed}(\mathrm{Mod}_{FG}) = 0$,
(3) $\mathrm{ed}(\mathrm{Mod}_{FG}) < \infty$.

## 2. Preliminaries on fields of definition

**Lemma 2.1.** *Let $A$ be a finite-dimensional $F$-algebra, $K/F$ be a field extension and $M$ be an $A_K$-module. Then $M$ descends to an intermediate subfied $F \subset E \subset K$, where $E/F$ is finitely generated.*

*Proof.* Suppose $a_1, \dots, a_r$ generate $A$ as an $F$-algebra. Choose an $F$-vector space basis for $M$. Then the $A$-module structure of $M$ is completely determined by the matrices representing multiplication by $a_1, \dots, a_r$ in this basis. Each of these matrices has $n^2$ entries in $K$, where $n = \dim_F(M)$. Let $E \subset K$ be the field extension of $F$ obtained by adjoining these these $rn^2$ entries to $F$. Then $M$ descends to $E$. □

Next we recall the classical theorem of Noether and Deuring. For a proof, see [CR, (29.7)] or [BP, Lemma 5.1].

**Theorem 2.2.** (Noether-Deuring Theorem) *Let $K/E$ be a field extension, $A$ be a finite-dimensional $E$-algebra, and $M_1$, $M_2$ and $M$ be $A$-modules. If $K \otimes_E M_1$ and $K \otimes_E M_2$ are isomorphic as $A_K$-modules, then $M_1$ and $M_2$ are isomorphic as $A$-modules.* $\square$

**Lemma 2.3.** *Let $F$ be a field, $A$ be a finite-dimensional $F$-algebra, $F \subset E \subset K$ be a field extension, $N$ be $A_E$-module, and $M = N_K$ and $F \subset E_0 \subset E$ be an intermediate field. Then*

*(a) $M$ descends to $E_0$ if and only if $N$ descends to $E_0$.*

*(b) If $F \subset E_{min} \subset K$ is a minimal field of definition for $M$, then $E_{min}$ is a minimal field of definition for $N$.*

*Proof.* (a) If $N$ descends to $E_0$, then clearly so does $M$. Conversely, suppose $M$ descends to $E_0$. That is, there exists a $E_0$-module $N_0$ such that $K \otimes_{E_0} N_0 \simeq M$ as an $A_K$-module. Consider the $A_E$-modules $N_1 := E \otimes_{E_0} N_0$ and $N_2 := N$. Both become isomorphic to $M_K$ over $K$. By Theorem 2.2, $N_1 \simeq N_2$ as $A_E$-modules. In other words, $N$ descends to $E_0$, as desired.

(b) Since $E$ is a field of definition for $M$, we have $E_{min} \subset E$. By part (a), $E_{min}$ is a field of definition for $N$, and part (b) follows. $\square$

We finally come to the main result of this section.

**Proposition 2.4.** *Suppose $F$ is a $C_1$-field, $A$ is a finite-dimensional $F$-algebra, $K/F$ is a field extension, $M$ is a finitely generated $A_K$-module, and $F \subset K_0 \subset K$ is an intermediate field, such that $[K_0 : F] < \infty$.*

*If $M^n$ is defined over $K_0$ for some positive integer $n$, then so is $M$.*

*Proof.* Set $\mathrm{End}^{ss}_{A_K}(M)$ to be the quotient of $\mathrm{End}_{A_K}(M)$ by its Jacobson radical. By our assumption $M^n \simeq K \otimes_{K_0} N$ for some $A_{K_0}$-module $N$. By Fitting's Lemma,

$$\mathrm{End}^{ss}_{A_K}(M^n) \simeq \mathrm{M}_n(D),$$

where $D$ is a finite-dimensional division algebra over some finite field extension $K'$ of $K$. On the other hand,

$$\mathrm{M}_n(D) \simeq \mathrm{End}^{ss}_{A_K}(M^n) \simeq \mathrm{End}^{ss}_{A_K}(K \otimes_{K_0} N) \simeq K \otimes_{K_0} \mathrm{End}^{ss}_{A_{K_0}}(N)\,. \tag{2.5}$$

We conclude that $\mathrm{End}^{ss}_{A_{K_0}}(N)$ is a simple algebra over $K_0$, i.e.,

$$\mathrm{End}^{ss}_{A_{K_0}}(N) \simeq M_m(D_0) \tag{2.6}$$

over $K_0$, for some integer $m \geqslant 0$ and some finite-dimensional central division algebra $D_0$ over a field $K_0'$ such that $K_0'$ is a finite extension of $K_0$. Now recall that we are assuming that $F$ is a $C_1$-field and

$$F \subset K_0 \subset K_0'$$

are finite field extensions. Hence, $K_0'$ is also a $C_1$-field, and thus every finite-dimensional division algebra over $K_0'$ is commutative. In particular, $D_0 = K_0'$, is a field, and

$$\mathrm{M}_n(D) \simeq K \otimes_{K_0} \mathrm{End}^{ss}_{A_{K_0}}(N) \simeq K \otimes_{K_0} \mathrm{M}_m(K_0')\,.$$

Since $\mathrm{M}_n(D)$ is a simple algebra, we conclude that $K \otimes_{K_0} K'_0$ is a field. Moreover, the index of $\mathrm{M}_m(K \otimes_{K_0} K'_0)$ is 1; hence, $D = K'$ is commutative, $K \otimes_{K_0} K'_0 = K'$, and $m = n$.

Now (2.6) tells us that $N \simeq M_0^n$ as a $A_{K_0}$-module, for some indecomposable $A_{K_0}$-module $M_0$. Since $K \otimes_{K_0} N \simeq M^n$, by the Krull-Schmidt theorem $K \otimes_{K_0} M_0 \simeq M$. Thus $M$ descends to $K_0$, as claimed. □

## 3. Proof of Theorem 1.1

We begin with a simple criterion for the existence of a minimal field of definition.

**Lemma 3.1.** *Let $A$ be a finite-dimensional $F$-algebra, and $K/F$ be a field extension, and $M$ be an $A_K$-module, satisfying conditions (a) and (b) below. Then $M$ has a minimal field of definition.*

*(a) Suppose $M$ descends to an intermediate field $F \subset L \subset K$, i.e., $M \simeq K \otimes_L N$ for some $A_L$-module $N$. Then $N$ further descends to a subfield $F \subset E \subset L$, where $[E : F] < \infty$.*

*(b) Suppose $M$ descends to an intermediate field $F \subset E \subset K$ such that $[E : F] < \infty$. That is, $M \simeq K \otimes_E N$ for some $A_E$-module $N$. Then $N$ has a minimal field of definition $E_{min} \subset E$.*

*Proof.* Condition (a) implies that $M$ is defined over some $F \subset E \subset K$ with $[E : F] < \infty$. Let the $A_E$-module $N$ and the field $E_{min} \subset E$ be as in (b).

We claim that $E_{min}$ is independent of the choice of $E$. That is, suppose $F \subset E' \subset K$ is another field of definition of $M$ with $[E' : F] < \infty$, $M := K \otimes_{E'} N'$ for some $A_{E'}$-module $N'$. Let $E'_{min} \subset E'$ be the minimal field of definition of $N'$, so that $N' := E' \otimes_{E'_{min}} N'_{min}$. Then our claim asserts that $E_{min} = E'_{min}$. If we can prove this claim, then clearly $E_{min}$ is the minimal field of definition for $M$. Our proof of the claim will proceed in two steps.

First assume $E \subset E'$. By Lemma 2.3(b), $E'_{min}$ is a minimal field of definition for $N$. By uniqueness of the minimal field of definition for $N$, $E_{min} = E'_{min}$.

Now suppose $F \subset E \subset K$ and $F \subset E' \subset K$ are fields of definition for $M$ such that $[E : F] < \infty$ and $[E' : F] < \infty$. Let $E''$ be the composite of $E$ and $E'$ in $K$ and $E''_{min}$ be the minimal field of definition of $N_{E''} \simeq N'_{E''}$. (Note that $N_{E''}$ and $N'_{E''}$ become isomorphic over $K$; hence, by Theorem 2.2, they are isomorphic over $E''$.) Then, $[E'' : F] < \infty$, and $E, E' \subset E''$. As we just showed, $E_{min} = E''_{min}$ and $E'_{min} = E''_{min}$. Thus $E_{min} = E'_{min}$, as desired. □

We now proceed with the proof of Theorem 1.1.

**Reduction 3.2.** *For the purpose of proving Theorem 1.1, we may assume without loss of generality that*

*(a) $K$ is a finite extension of $F$.*

*(b) $K$ is a Galois extension of $F$.*

*Proof.* (a) follows from Lemma 3.1. Indeed, we are assuming that Theorem 1.1 holds whenever $K$ is a finite extension of $F$. That is, condition (b) of Lemma 3.1 holds. On the other hand, condition (a) of Lemma 3.1 follows from Lemma 2.1.

(b) By part (a), we may assume that $K/F$ is finite. Let $L$ be the normal closure of $K$ over $F$. Then $L/F$ is finite Galois. Lemma 2.3(b) now tells us that if $M_L := L \otimes_K M$ has a minimal field of definition then so does $M$. $\square$

**Lemma 3.3.** *Let $F$ be a $C_1$-field, $A$ be a finite-dimensional $F$-algebra, $K/F$ be a finite Galois extension, and $M$ be an $A_K$-module. The Galois group $G := \mathrm{Gal}(K/F)$ acts on the set of isomorphism classes of $A_K$-modules via*

$$g \colon N \to {}^gN := K \otimes_g N\,.$$

*Let $G_M$ be the stabilizer of $M$ under this action. Then the fixed field $K^{G_M}$ of $G_M$ is the minimal field of definition for $M$.*

*Proof.* Suppose $M$ is defined over $K_0$, where $F \subset K_0 \subset K$. Then clearly ${}^gM \simeq M$ for every $g \in \mathrm{Gal}(K/K_0)$. Hence, $\mathrm{Gal}(K/K_0) \subset G_M$ and consequently, $K^{G_M} \subset K_0$. This shows that $K^{G_M}$ is contained in every field of definition of $M$.

It remans to show that $M$ descends to $K_0 := K^{G_M}$. Write $M = M_1^{d_1} \oplus \cdots \oplus M_r^{d_r}$, where $M_1, \dots, M_r$ are distinct indecomposables. The condition that ${}^gM \simeq M$ for any $g \in G_M$ is equivalent to the following: if $M_j \simeq {}^gM_i$ for some $g \in \mathrm{Gal}(K/K_0)$, then $d_i = d_j$. Grouping $G_M$-conjugate indecomposables together, we see that $M \simeq S_1 \oplus \cdots \oplus S_m$, where each $S_1, \dots, S_m$ is the $G_M$-orbit sum of one of the indecomposable modules $M_i$. (Here the orbit sums $S_1, \dots, S_m$ may not be distinct.) It thus suffices to show that each orbit sum is defined over $K_0$.

Consider a typical $G_M$-orbit sum $S := M_1 \oplus \cdots \oplus M_s$, where we renumber the indecomposable factors of $M$ so that $M_1, \dots, M_s$ are the $G_M$-translates of $M_1$. Let $H$ be the stabilizer of $M_1$ in $G_M$. That is,

$$H := \{h \in G_M \mid {}^hM_1 \simeq M_1\}\,.$$

Let $K_1 := K^H$. Then

$$K \otimes_{K_1} (M_1)_{\downarrow K_1} = \bigoplus_{h\in H} {}^hM_1 = M_1^{|H|}\,.$$

In particular, this tells us that $M_1^{|H|}$ descends to $K_1$. By Proposition 2.4, so does $M_1$. In other words, $M_1 \simeq K \otimes_{K_1} N_1$ for some $K_1$-module $N$. We claim that

$$K \otimes_{K_0} (N_1)_{\downarrow K_0} \simeq S. \tag{3.4}$$

If we can prove this claim, then $S$ descends to $K_0$, and we are done.

To prove the claim, note that on the one hand,

$$K \otimes_{K_0} (M_1)_{\downarrow K_0} = \prod_{g\in G_M} {}^gM_1 = S^{|H|}\,. \tag{3.5}$$

On the other hand, since $M_1 \simeq K \otimes_{K_1} N_1$, we have

$$(M_1)_{\downarrow K_0} \simeq ((M_1)_{\downarrow K_1})_{\downarrow K_0} \simeq (N_1^{|H|})_{\downarrow K_0}\,,$$

and thus

$$K \otimes_{K_0} (M_1)_{\downarrow K_0} = (K \otimes_{K_0} ((N_1)_{\downarrow K_0})^{|H|}) \simeq (K \otimes_{K_0} (N_1)_{\downarrow K_0}))^{|H|}\,. \tag{3.6}$$

Comparing (3.5) and (3.6), we obtain

$$(K \otimes_{K_0} (N_1)_{\downarrow K_0})^{|H|} \simeq S^{|H|}\,.$$

The desired isomorphism (3.4) follows from this by the Krull-Schmidt theorem. □

## 4. Algebras of finite representation type

A finite-dimensional $F$-algebra $A$ is said to be *of finite representation type* if there are only finitely many indecomposable finitely generated $A$-modules (up to isomorphism).

**Theorem 4.1.** *Let $F$ be a $C_1$-field, $A$ be finite-dimensional $F$-algebra of finite representation type, and $K/F$ be a field extension (not necessarily algebraic) such that $F$ is perfectly closed in $K$. (That is, for every subextension $F \subset E \subset K$ with $[E : F] < \infty$, $E$ is separable over $F$.) Suppose $M$ is an indecomposable $A_K$-module. Then*

*(a) $M$ descends to an intermediate subfield $F \subset E \subset K$ such that $[E : F] < \infty$.*

*(b) $M$ is a direct summand of $K \otimes_F N$ for some indecomposable $A_F$-module $N$.*

*Proof.* (a) Consider the $A$-module $M_{\downarrow F}$. Generally speaking this module is not finitely generated over $A$. Nevertheless, since $A$ has finite representation type, thanks to a theorem of Tachikawa [Ta, Corollary 9.5], $M_{\downarrow F}$ can be written as a direct sum of finitely generated indecomposable $A$-modules. Denote one of these modules by $N$. That is,

$$M_{\downarrow F} \simeq N \oplus N' , \tag{4.2}$$

for some $A$-module $N'$ (not necessarily finitely generated).

Let us now take a closer look at $N$. By Fitting's lemma, $E := \mathrm{End}_A^{ss}(N)$ is a finite-dimensional division algebra over $F$. Since $F$ is a $C_1$-field, $E$ is a field extension of $F$. Now set $F' := E \cap K$ and $m = [F' : F]$. Since $F$ is perfectly closed in $K$, $F'$ is finite and separable over $F$. Thus

$$\mathrm{End}_A^{ss}(F' \otimes_F N) \simeq F' \otimes_F \mathrm{End}_A^{ss}(N) \simeq E \times \cdots \times E .$$

This tells us that over $F'$, $N$ decomposes into a direct sum of $m$ indecomposables,

$$F' \otimes_F N = N_1 \oplus \cdots \oplus N_m. \tag{4.3}$$

By the definition of $F'$, $K \otimes_{F'} E$ is a field. Hence, each indecomposable $A_{F'}$-module $N_i$ remains indecomposable over $K$.

Tensoring both sides of (4.2) with $K$, we obtain an isomorphism of $A_K$-modules

$$\begin{aligned} K \otimes M_{\downarrow F} &\simeq (K \otimes_F N) \oplus (K \otimes_F N') \\ &= (\bigoplus_{i=1}^{m} K \otimes_{F'} N_i) \oplus (K \otimes_F N') \\ &= (K \otimes_F N_1) \oplus N'' , \end{aligned}$$

where $N'' := (\bigoplus_{i=2}^{m} K \otimes_{F'} N_i) \oplus (K \otimes_F N')$. Note that

$$K \otimes M_{\downarrow F'} \simeq \bigoplus_B M ,$$

where $B$ is a basis of $K$ as an $F'$-vector space. As we mentioned above, $K \otimes_{F'} N_1$ is an indecomposable $A_K$-module. Since $K \otimes_{F'} N_1$ is finitely generated and is contained in

$\bigoplus_B M$, it lies in the direct sum of finitely many copies of $M$, say, in $M^r := M \oplus \cdots \oplus M$ ($r$ copies). Thus we have maps

$$K \otimes_F N_1 \hookrightarrow M^r \hookrightarrow \bigoplus_B M \twoheadrightarrow K \otimes_F N_1$$

whose composite is the identity, and so $K \otimes_F N_1$ is isomorphic to a direct summand of $M^r$. By the Krull-Schmidt Theorem, $K \otimes_{F'} N_1 \simeq M$. In particular, $M$ descends to $F'$, as claimed.

(b) By (4.3), $N$ is an indecomposable $A$-module, and $N_1$ is a direct summand of $F' \otimes_F N$. Hence, $M \simeq K \otimes_{F'} N_1$ is a direct summand of $K \otimes_F N$, as desired. □

**Corollary 4.4.** *Let $F$ be a $C_1$-field, $A$ be finite-dimensional $F$-algebra of finite representation type, and $K/F$ be a field extension such that $F$ is perfectly closed in $K$. Then $A_K$ is also of finite representation type.*

*Proof.* By our assumption $A$ has finitely many indecomposable modules $N^{(1)}, \dots, N^{(d)}$. By Theorem 4.1(b) every indecomposable $A_K$-module is isomorphic to a direct summand of $K \otimes_F N^{(i)}$ for some $i$. By the Krull-Schmidt Theorem, each $K \otimes_F N^{(i)}$ has finitely many direct summands (up to isomorphism), and the corollary follows. □

## 5. Proof of Theorems 1.2 and 1.3

We will deduce Theorem 1.2 from Lemma 3.1. $M$ satisfies condition (b) of Lemma 3.1 by Theorem 1.1. It thus remains to show that $M$ satisfies condition (a) of Lemma 3.1. For notational simplicity, we may assume that $K = L$ and $M = N$. That is, we want to show that $M$ descends to some intermediate field $F \subset E \subset K$ with $[E : F] < \infty$. Note that in the case, where $M$ is indecomposable, this is precisely the content of Theorem 4.1(a).

In general, write $M = M_1 \oplus \cdots \oplus M_r$ as a direct product of (not necessarily distinct) indecomposables. By Theorem 4.1(a), each $M_i$ descends to an intermediate field $F \subset K_i \subset K$ such that $[K_i : F] < \infty$. Let $E$ be the compositum of $K_1, \dots, K_r$ inside $K$. Then $[E : F] < \infty$, and $M$ descends to $E$. This completes the proof of Theorem 1.2. □

We now proceed with the proof of Theorem 1.3. Denote the perfect closure of $F$ in $K$ by $F^{pf}$. By Theorem 1.2, $M$ descends to an intermediate field $F^{pf} \subset K_0 \subset K$ such that $[K_0 : F^{pf}] < \infty$. Hence, $K_0$ is algebraic over $F$, and consequently, $\mathrm{ed}(M) \leqslant \mathrm{trdeg}_F(K_0) = 0$, as desired. □

## 6. An example

In this section we will show by example that both Theorem 1.2 and 1.3 fail if we do not require $F$ to be a $C_1$-field. Let $F = \mathbb{Q}$ and $A$ be the quaternion algebra

$$A = \mathbb{Q}\{x, y\}/(x^2 = y^2 = -1,\ xy = -yx).$$

and $K/F$ be any field having two elements $a$ and $b$ satisfying $a^2 + b^2 = -1$. Then $A$ has a two dimensional $A_K$-module $M$ given by

$$x \mapsto \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \qquad y \mapsto \begin{pmatrix} b & -a \\ -a & -b \end{pmatrix}. \tag{6.1}$$

**Lemma 6.2.** *The following conditions on an intermediate field $\mathbb{Q} \subset E \subset K$ are equivalent:*

*(a) $\varphi$ descends to $E$,*

*(b) $A$ splits over $E$,*

*(c) there exist elements $a_0$, $b_0 \in E$ such that $a_0^2 + b_0^2 = -1$.*

*Proof.* (a) $\Longrightarrow$ (b). Suppose $M$ descends to an $A_E$-module $N$. Since $A_E := E \otimes_{\mathbb{Q}} A$ is a central simple 4-dimensional algebra over $E$, the homomorphism of algebras given by

$$A_E \to \mathrm{End}_E(N) \simeq \mathrm{M}_2(E)$$

is an isomorphism. In other words, $E$ splits $A$.

(b) $\Longrightarrow$ (a). Conversely, suppose $E$ splits $A$. Then the representation of $A \to \mathrm{End}_K(M)$ factors as follows:

$$A \to E \otimes_{\mathbb{Q}} A \simeq \mathrm{M}_2(E) \to \mathrm{M}_2(K)\,.$$

This shows that $\varphi$ descends to $E$.

The equivalence of (b) and (c) a special case of Hilbert's criterion for the splitting of a quaternion algebra; see the equivalence of conditions (1) and (7) in [Lam, Theorem III.2.7] as well as Remark (B) on [Lam, p. 59]. ☐

**Proposition 6.3.** *Let $a$ and $b$ be independent variables over $\mathbb{Q}$, $E$ be the field of fractions of $\mathbb{Q}[a, b]/(a^2 + b^2 + 1)$, and $M$ be the 2-dimensional $A_E$-module given by* (6.1)*. Then*

*(a)* $\mathrm{ed}(M) = 1$*,*

*(b) $M$ does not have a minimal field of definition.*

*Proof.* (a) The assertion of part (a), follows from [KRP, Example 6.1]. For the sake of completeness, we will give an independent proof.

Suppose $M$ descends to an intermediate subfield $\mathbb{Q} \subset E_0 \subset E$. Since $\mathrm{trdeg}_{\mathbb{Q}}(E) = 1$, $\mathrm{trdeg}_{\mathbb{Q}}(E_0) = 0$ or 1. Our goal is to show that $\mathrm{trdeg}_{\mathbb{Q}}(E_0) \neq 0$. Assume the contrary, i.e., $E_0$ is algebraic over $\mathbb{Q}$.

Note that $E$ is the function field of the conic curve $a^2+b^2+c^2 = 0$ in $\mathbb{P}^2$. Since this curve is absolutely irreducible, $\mathbb{Q}$ is algebraically closed in $E$. Thus the only possibility for $E_0$ is $E_0 = \mathbb{Q}$, On the other hand, $M$ does not descend to $\mathbb{Q}$ by Lemma 6.2, a contradiction.

(b) Suppose $M$ descends to $E_1 \subset E$. Our goal is to show that $M$ descends to a proper subfield $E_3 \subset E_1$. By Lemma 6.2(c) there exist $a_1$ and $b_1$ in $E_1$ such that $a_1^3 + b_1^3 = -1$. If $\mathbb{Q}(a_1, b_1)$ is properly contained in $E_1$, then we are done. Thus we may assume without loss of generality that $E_1 = \mathbb{Q}(a_1, b_1)$. Set $E_3 := \mathbb{Q}(a_3, b_3)$. where $a_3 := a_1^3 - 3a_1b_1^2$ and $b_3 = 3a_1^2b_1 - b_1^3$. We claim that (i) $A$ splits over $E_3$, and (ii) $E_3 \subsetneq E_1$.

In order to establish (i) and (ii), let us consider the following diagram.

$$\begin{tikzcd}
& E_1(i) \arrow[dl, dash] \arrow[d, dash] \\
E_3(i) \arrow[d, dash] & E_1 \arrow[dl, dash] \\
E_3 &
\end{tikzcd}$$

Here as usual, $i$ is a primitive 4th root of 1. It is easy to see that $E_1(i) = \mathbb{Q}(i)(a_1, b_1) = \mathbb{Q}(i)(z)$ is a purely transcendental extension of $\mathbb{Q}(i)$, where $z = a_1 + b_1 i$ and $\dfrac{1}{z} = -a_1 + b_1 i$. Similarly $E_3(i) = \mathbb{Q}(i)(z^3)$, where $z^3 = a_3 + b_3 i$ and $\dfrac{1}{z^3} = -a_3 + b_3 i$. In particular, this shows $a_3^2 + b_3^2 = -1$, thus proving (i). Moreover, since $z$ is transcendental over $\mathbb{Q}(i)$, we have $[E_1(i) : E_3(i)] = [\mathbb{Q}(i)(z) : \mathbb{Q}(i)(z^3)] = 3$ and thus

$$[E_1 : E_3] = \frac{[E_3(i) : E_3] \cdot [E_1(i) : E_3(i)]}{[E_1(i) : E_1]} = \frac{2 \cdot 3}{2} = 3\,.$$

This proved (ii). □

**Remark 6.4.** Write $z^n = a_n + b_n i$ for suitable $a_n, b_n \in E_1$ and set $[E_1 : E_n] = n$. We showed above that $[E_1 : E_3] = 3$ and thus $E_3 \subsetneq E_1$. The same argument yields $[E_1 : E_n] = n$ for any positive integer $n$.

## 7. Proof of Theorem 1.4

We shall actually prove a stronger, more natural theorem, about blocks of finite group algebras. Theorem 1.4 will follow from the fact that $p$-Sylow $p$ of a finite group $G$ are cyclic if and only if every block over a field $F$ of characteristic $p$ has cyclic defect.

**Theorem 7.1.** *Let $B$ be a block of a finite group algebra $FG$, where $F$ is a field of characteristic $p$. Then the following are equivalent:*

(1) *$B$ has cyclic defect,*
(2) $\mathrm{ed}(\mathrm{Mod}_B) = 0$*,*
(3) $\mathrm{ed}(\mathrm{Mod}_B) < \infty$*.*

The implication (1) $\Longrightarrow$ (2) is a direct consequence of Theorem 1.3. The implication (2) $\Longrightarrow$ (3) is obvious.

The remainder of this section will be devoted to proving that (3) $\Longrightarrow$ (1). We shall show that if $B$ has non-cyclic defect, then $\mathrm{ed}(\mathrm{Mod}_B) = \infty$. Let $K$ be an extension field of $F$, let $e$ be the block idempotent of $B$, let $D$ be a defect group of $B$, and let $N = \Phi(D)$, the Frattini subgroup of $D$. If $D$ is not cyclic, $D/N$ is elementary abelian of rank $r \geq 2$, with basis the images of elements $g_1, \dots, g_r \in D$. Since $D$ is a defect group of $B$, any $KD$-module $M$ is a summand of $\mathrm{Res}_{G,D}(e.\,\mathrm{Ind}_{D,G}(M))$.

Now let $n > 0$, and let $K = F(t_{1,1}, \dots, t_{n,r})$ be a function field in $nr$ indeterminates, and let $M_i$ ($1 \leq i \leq n$) be the two dimensional $KD$-module

$$g_j \mapsto \begin{pmatrix} 1 & t_{i,j} \\ 0 & 1 \end{pmatrix}.$$

Then $J^2(KD)$ is in the kernel of $M_i$, so $M_i$ is really a module for $KD/J^2(KD)$, which has a basis $1, (g_1 - 1), \dots, (g_r - 1)$. The last $r$ elements of this list form a basis for $J(KD)/J^2(KD)$, and we form a vector space $V$ with basis $(g_1 - 1), \dots, (g_r - 1)$. The kernel of $M_i$ as a module for $KD/J^2(KD)$ is the codimension one subspace $H_i$ of

$$J(KD)/J^2(KD) \cong V$$

given by

$$H_i := \{\lambda_j(g_j - 1) \mid \sum_j t_{i,j}\lambda_j = 0\}. \tag{7.2}$$

By the Mackey decomposition theorem, the module $M_i' = \mathrm{Res}_{G,D}(e.\mathrm{Ind}_{D,G}(M_i))$ is a direct sum of at least one copy of $M_i$, some conjugates of $M_i$ by elements of $N_G(D)$, and some modules of the form $\mathrm{Ind}_{D\cap {}^gD,D}\mathrm{Res}_{{}^gD,D\cap {}^gD}\,{}^gM$. It follows that the Jordan canonical form of elements of $V$ on $M_i'$ is constant, except on a set $S_i$, which is a finite union of hyperplanes $N_G(D)$-conjugates of $H_i$ and linear subspaces of smaller dimension.

Now let $M := \bigoplus_i M_i$. Our goal is to show that

$$\mathrm{ed}(e.\mathrm{Ind}_{D,G}(M)) \geqslant n(r-1)\,.$$

This will imply that $\mathrm{ed}(\mathrm{Mod}_B) \geq n(r-1)$ for every $n > 0$ and thus $\mathrm{ed}(\mathrm{Mod}_B) = \infty$, as desired.

Note that $e.\mathrm{Ind}_{D,G}(M)$ is a module whose restriction to $D$ is $\bigoplus_i M_i'$. If $e.\mathrm{Ind}_{D,G}(M)$ descends to an intermediate subfield $F \subset K_0 \subset K$, then so does the set $\bigcup_i S_i \subset V$ and its natural image in $\mathbb{P}(V) = \mathbb{P}^{r-1}$, which we will denote by $S$. To complete the proof of Theorem 7.1, it remains to show that if $S$ descends to $K_0$, then

$$\mathrm{trdeg}_F(K_0) \geqslant n(r-1)\,. \tag{7.3}$$

**Lemma 7.4.** *Let $S \subset \mathbb{P}^{r-1}$ be a projective variety defined over a field $K$. Assume that a hyperplane $H$ given by $a_1x_1 + a_2x_2 + \cdots + a_rx_r = 0$ is an irreducible component of $S$ for some $a_1, \ldots, a_r \in K$ (not all zero). Suppose $S$ descends to a subfield $K_0 \subset K$. Then each ratio $a_j/a_l$ is algebraic over $K_0$, as long as $a_l \neq 0$.*

To deduce the inequality (7.3) from Lemma 7.4, recall that in our case $S$ is the union of the hyperplanes $H_1, \ldots, H_n$, a finite number of other hyperplanes (translates of $H_1, \ldots, H_n$ by elements of $N_G(D)$) and lower-dimensional linear subspaces of $\mathbb{P}(V) = \mathbb{P}^{r-1}$. In the basis $(g_1 - 1), \ldots, (g_r - 1)$ of $V$, $H_i$ is given by $t_{i,1}x_1 + t_{i,2}x_2 + \cdots + t_{i,r}x_r = 0$; see (7.2). Thus by Lemma 7.4 the elements $t_{i,j}/t_{i,1}$ are algebraic over $K_0$ for every $i = 1, \ldots, n$ and every $j = 2, \ldots, r$. In other words, if $K_1$ is the algebraic cosure of $K_0$ in $K$, then each $t_{i,j}/t_{i,1} \in K_1$, and thus $\mathrm{trdeg}_F(K_0) = \mathrm{trdeg}_F(K_1) \geqslant n(r-1)$, as desired.

*Proof of Lemma 7.4.* We may assume without loss of generality that $K_0$ is algebraically closed. To reduce to this case, we replace $K_0$ by its algebraic closure $\overline{K_0}$ and $K$ by a compositum of $K$ and $\overline{K_0}$. If we know that each $a_{i,j}$ is algebraic over $\overline{K_0}$ (or equivalently, is contained in $\overline{K_0}$), then $a_{i,j}$ is algebraic over $K_0$.

Now assume that $K_0$ is algebraically closed. Since $S$ is defined over $K_0$, every irreducible component of $S$ is defined over $K_0$. In particular, $H$ is defined over $K_0$. That is, the point $(a_1 : \cdots : a_r)$ of the dual projective space $\check{\mathbb{P}}^{r-1}$ is defined over $K_0$. Equivalently, $a_i/a_j \in K_0$ whenever $a_l \neq 0$. This completes the proof of the claim and thus of Lemma 7.4 and Theorem 7.1. ☐

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