Witt rings of quadratically presentable fields

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Abstract

This paper introduces a novel approach to the axiomatic theory of quadratic forms. We work internally in a category of certain partially ordered sets, subject to additional conditions which amount to a strong form of local presentability. We call such partial orders *presentable*. It turns out that the classical notion of the Witt ring of symmetric bilinear forms over a field makes sense in the context of *quadratically presentable fields*, that is fields equipped with a presentable partial order inequationaly compatible with the algebraic operations. As an application, we show that Witt rings of symmetric bilinear forms over fields, of both characteristic 2 and $\neq 2$, are isomorphic to Witt rings of suitably built quadratically presentable fields, which therefore provide a uniform construction of Witt rings for all characteristics.

Keywords: axiomatic theory of quadratic forms, abstract Witt rings, partial orders, hyperfields, special groups.

1 Introduction

The algebraic theory of quadratic forms in its modern form goes back to the seminal work of Witt [20], who introduced the notion of what is now called the Witt ring of a field, and by Pfister [17] and Cassels [2] who identified first significant properties of Witt rings. First attempts to approach the Witt theory from axiomatic point of view go back to the mid 1970's. Objects such as quadratic form

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schemes by Cordes [3, 4], also studied by Carson and Marshall [1], Szczepanik [18, 19] and others, became of interest. Over the years, numerous other axiomatic approaches were considered as well, like the notion of quaternionic maps due to Carson and Marshall [1], abstract Witt rings due to Knebusch, Rosenberg and Ware [11, 12], as well as Marshall [15], strongly represational Witt rings due to Kleinstein and Rosenberg [10], and the theory of special groups by Dickmann and Miraglia [5, 6]. Quadratic form schemes, quaternionic maps, strongly representational Witt rings, and special groups are equivalent descriptions of the same objects. A further formalism used to describe the same phenomena is the theory of rings with multivalued addition, called hyperrings [16].

In this work we propose an approach to the axiomatic theory of quadratic forms by generalising the underlying principles of hyperrings. Our starting point are partially ordered sets subject to additional conditions which amount to a strong form of local presentability when these partial orders are seen as categories. We call such partial orders *presentable*. Roughly speaking, presentable posets reflect the behaviour of *pierced powersets*, that is powersets excluding the empty set ordered by inclusion. The most salient order-theoretic feature of pierced powersets is that they exhibit a generating set of minimal elements, this since a non-empty set is a union of singletons. It is precisely this feature which is captured in the definition of presentable posets. The objective here is to build an axiomatic theory of quadratic forms by describing the behaviour of their value sets.

In Section 2 we formally introduce presentable posets and elaborate on examples including the set of integers greater or equal 1 augmented with a point at infinity and ordered by division, as well as the set of proper ideals of a Noetherian ring reversely ordered by inclusion. We also mention some simple categorical properties of the category of presentable posets.

In Section 3 we introduce presentable algebras, which are monoid objects in the category of presentable posets that satisfy additional properties: *presentable groups*, *presentable rings*, and *presentable fields*, respectively. These algebraic objects are used for the construction of Witt rings peculiar to this setting. A word of caution might be in order here as far as the terminology is concerned: a presentable group is not an internal group, not even a cancellative monoid. We choose to stick to an established vocabulary, but a similar comment can be made about the notion of hypergroup underlying the notion of hyperring [16]. On the other hand, the Witt rings we construct are rings without further ado.

In Section 4 we provide more examples of presentable algebras. In particular, we show how presentable groups, rings and fields arise in a natural way in the study of hypergroups, hyperrings and hyperfields, respectively. This connection provides the main link between our theory and already existing axiomatic theories of quadratic forms.

In Section 5 we define pre-quadratically and quadratically presentable fields, which share certain similarities with groups of square classes of fields, endowed with partial order and addition. We then exhibit a Witt ring structure naturally occuring in quadratically presentable fields. As an application, for every field verifying some mild assumptions one can form a hyperfield by defining on the multiplicative group of its square classes multivalued addition that corresponds to value sets of binary forms. The presentable field induced by this hyperfield is quadratically presentable, and its Witt ring in our sense is isomorphic to its standard Witt ring. What makes our construction of interest is the fact that it uniformely works for fields of both characteristic 2 and $\neq 2$. The construction is technically reminiscent of the one used by Dickmann and Miraglia to build Witt rings of special groups [6].

In Sections 6 and 7 we explain how a pre-quadratically presentable field can be obtained from any presentable field. For that purpose we introduce quotients of presentable fields (with respect to the multiplicative structure) in Section 6, while in Section 7 we use these quotients in order to build pre-quadratically presentable fields from presentable fields. The techniques here heavily rely on the connection between presentable algebras and hyperalgebras.

2 Presentable posets

Recall that a partially-ordered set or *poset* is a set equipped with a reflexive, transitive and anti-symmetric relation. Let A be a poset. An element $a \in A$ is *minimal* if $a' \leqslant a$ implies $a' = a$. Let A be a poset and $\mathcal{S}_A$ be the set of A's minimal elements. We shall write $\mathcal{S}_a \stackrel{\text{def.}}{=} \downarrow a \cap \mathcal{S}_A$ stands for the set of all minimal elements below $a \in A$, and $\mathcal{S}_X \stackrel{\text{def.}}{=} \bigcup_{x \in X} \mathcal{S}_x$ for the set of minimal elements below $X \subseteq A$. The elements of the sets

$$\begin{aligned} \uparrow a &\stackrel{\text{def.}}{=} \{a' \in A | a \leqslant a'\} \\ \downarrow a &\stackrel{\text{def.}}{=} \{a' \in A | a \geqslant a'\} \end{aligned}$$

are called a's *upper bounds* and a's *lower bounds* respectively. Moreover, for a subset $X \subseteq A$, denote

$$\begin{aligned} \uparrow X &\stackrel{\text{def.}}{=} \bigcup_{x \in X} (\uparrow x) \\ \downarrow X &\stackrel{\text{def.}}{=} \bigcup_{x \in X} (\downarrow x) \end{aligned}$$

If it exists, we shall call *supremum* of X the least element of the set $\uparrow X$ and *infimum* of X the greatest element of the set $\downarrow X$. We shall write $\bigsqcup X$ for the supremum of $X \subseteq A$ and $x \sqcup y$ for $\bigsqcup\{x, y\}$.

Definition 2.1. A poset $(A, \leqslant)$ is *presentable* if

i. every non-empty subset $R \subseteq A$ admits a supremum;

ii. $\mathcal{S}_a$ is non-empty and $a = \bigsqcup \mathcal{S}_a$ for each $a \in A$.

We shall call the minimal elements of a presentable poset *supercompact*.

Remark 2.2. Let A be a presentable poset and $x, y \in A$. The following are equivalent

i. $x \leqslant y$;

ii. $\mathcal{S}_x \subseteq \mathcal{S}_y$.

To see this, assume $x \leqslant y$. We have $y = \bigsqcup \mathcal{S}_y$ and supercompacts are not comparable. Conversely, if $\mathcal{S}_x \subseteq \mathcal{S}_y$ then $y = \bigsqcup \mathcal{S}_y$ is an upper bound of $x = \bigsqcup \mathcal{S}_x$.

Proposition 2.3. *Let A be a presentable poset. If $x \leqslant \bigsqcup Y$ then for each $x' \in \mathcal{S}_x$ there is an $y \in Y$ such that $x' \leqslant y$.*

Proof. Let x and Y be as required, and fix $x' \in \mathcal{S}_x$. Clearly $x' \in \mathcal{S}_{\bigsqcup Y}$. Moreover

$$\bigsqcup Y = \bigsqcup \{\bigsqcup \mathcal{S}_y \mid y \in Y\} = \bigsqcup \bigcup_{y \in Y} \mathcal{S}_y$$

and

$$\bigsqcup \bigcup_{y \in Y} \mathcal{S}_y = \bigsqcup \mathcal{S}_{\bigsqcup Y} \Leftrightarrow \bigcup_{y \in Y} \mathcal{S}_y = \mathcal{S}_{\bigsqcup Y},$$

which is the case by Remark 2.2. Thus there exists $y \in Y$ such that $x' \in \mathcal{S}_y$. $\square$

Remark 2.4. Given a poset A one classically defines the *way-below* relation [8] as

$$x \ll y \overset{\text{def}}{\iff} \begin{array}{l} \text{if } y \leqslant \bigsqcup D \text{ then there is } d \in D \text{ such that } x \leqslant d \\ \text{for any directed subset } D \subseteq A \end{array}$$

If A is a presentable poset we have in particular $s \ll x$ for any supercompact $s \in \mathcal{S}_x$, yet in the case of presentable posets the property is stronger. Also, we have $s \ll s$ for any supercompact $s \in \mathcal{S}_x$, so supercompacts are *compact* in the order-theoretical sense.

Example 2.5. The *pierced powerset* $\mathcal{P}^*(X)$of a set X (that is its set of nonempty subsets) is presentable with respect to the ordering by inclusion. The singletons are the supercompacts.

Example 2.6. The set $(\mathbb{Z}_{\geq 1} \cup \{\infty\}, |, n)$, where $n > 0$, of integers greater or equal 1 with a point at infinity added is an accesible poset with respect to the ordering by division. The powers of prime numbers and the number 1 are the supercompacts. Clearly $\mathbb{Z}_{\geq 1}$ here can be replaced with conjugation classes of non-units of any factorization domain (not neccessarily unique) with powers of nonzero irreducibles as supercompacts.

Example 2.7. The set $(\mathcal{I}^*(R), \supseteq, I)$ of all ideals of a Noetherian ring R, where $I \lhd R$, is a presentable poset with respect to the ordering by reverse inclusion. The primary ideals and the trivial ideal R are the supercompacts. Indeed, every element of $\mathcal{I}^*(R)$ is either contained in a maximal ideal (which, in particular, is primary), or is equal to R. Every proper ideal is an intersetion of some primary ideals due to the Noether-Lasker theorem, and, clearly, an intersection of any family of proper ideals is a proper ideal.

Example 2.8. The set $\mathcal{A}^*(F^n)$ of all nonempty affine algebraic sets in F^n, where F is assumed to be algebraically closed, is accesible with respect to the ordering by inclusion. The affine varietes in F^n, i.e. the nonempty irreducible affine algebraic sets, are the supercompacts. This follows immediately from the previous example by the Hilbert's Nullstellensatz. The sup of an arbitrary family of algebraic sets is the Zariski closure of its union.

Recall that a *pointed set* is a pair (A, a) where A is a set and $a \in A$ a distinguished element called *basepoint*. A map $f : A \to B$ is *pointed* if it preserves the basepoint.

Definition 2.9.

1. A map $f : X \to Y$ of presentable posets is *continuous* if it preserves suprema of non-empty sets.

2. A *pointed presentable poset* (A, p) is an presentable poset equipped with a supercompact basepoint $p \in \mathcal{S}_A$.

3. A map $f : X \to Y$ of pointed presentable posets is a *homomorphism* if it is pointed and continuous.

It is easily seen that continous map is monotone. We shall write $\mathbf{aPos}_*$ for the category of pointed presentable posets and their homomorphisms.

Remark 2.10. $\mathbf{aPos}_*$ has a zero-object as well as binary products. Any singleton equipped with the trivial order is a pointed presentable poset, easily seen to be a zero-object. The binary product is the cartesian product equipped with the coordinatewise order.

3 Presentable algebras

Definition 3.1. A *presentable monoid* is a commutative monoid object in the (symmetric monoidal) category $\mathbf{aPos}_*$.

Remark 3.2.

1. Unravelling the definition, a presentable monoid $(M, \leq, 0, +)$ is a pointed presentable poset $(M, \leq, 0)$ with a distinguished supercompact 0 and a suprema-preserving binary addition $+ : M \times M \to M$ such that

   i. $a + (b + c) = (a + b) + c$ for all $a, b, c \in M$;

   ii. $a + 0 = 0 + a = a$ for all $a \in M$;

   iii. $a + b = b + a$ for all $a, b \in M$.

2. The addition is in particular monotone, so

$$(a \leq b) \wedge (c \leq d) \Rightarrow (a + c \leq b + d)$$

   for all $a, b, c, d \in M$.

3. Suppose $a \leq b + c$. We have

$$\begin{aligned} a &\leqslant b + c \\ &= \bigsqcup \mathcal{S}_b + \bigsqcup \mathcal{S}_c \\ &= \bigsqcup \{t + u | t \in \mathcal{S}_b, u \in \mathcal{S}_c\} \end{aligned}$$

   so for any $s \in \mathcal{S}_a$ there are $t \in \mathcal{S}_b$ and $u \in \mathcal{S}_c$ such that

$$s \leqslant t + u$$

Example 3.3. Let $(M, 0, \maltese)$ be a commutative monoid. The pointed presentable poset $(\mathcal{P}^*(M), \subseteq, \{0\})$ is made into a presentable monoid by defining the addition as

$$\begin{array}{rcl} + : \mathcal{P}^*(M) \times \mathcal{P}^*(M) & \longrightarrow & \mathcal{P}^*(M) \\ (A, B) & \mapsto & \{a \maltese b : a \in A, b \in B\} \end{array}$$

This addition preserves suprema, here unions:

$$\left(\bigcup_{i \in I} A_i\right) + \left(\bigcup_{j \in J} B_j\right) = \bigcup_{i \in I, j \in J} (A_i + B_j)$$

Example 3.4. The pointed presentable poset $(\mathbb{Z}_{\geq 1} \cup \{\infty\}, |, 1)$ is made into a presentable monoid by defining the "addition" to be the usual multiplication of integers. This addition preserves suprema, here least common multiples:

$$\operatorname{lcm}(X) \operatorname{lcm}(Y) = \operatorname{lcm}\{xy | x \in X, y \in Y\}$$

Example 3.5. Similarly, the pointed presentable poset $(\mathcal{I}^*(R), \supseteq, R)$, where R is Noetherian, is made into a presentable monoid by defining the binary operation to be the usual multiplication of ideals.

Example 3.6. The presentable set $(\mathcal{A}(k^n), \subseteq, \emptyset)$, where k is a field, is made into a presentable monoid by defining the binary operation to be $\cup$.

Definition 3.7. An *presentable group* G is a presentable monoid together with involutive monotonic endomorphism $- : G \to G$ verifying

$$(s \leq t + u) \Rightarrow (t \leq s + (-u))$$

for all $s, t, u \in \mathcal{S}_G$.

Remark 3.8. Unravelling the definition, the *additive inversion* operation of an presentable group is an involutive endomorphism of presentable sets $- : G \to G$ in $\mathbf{aPos}_*$ such that

$$\begin{tikzcd} G \times G \arrow[r, "- \times -"] \arrow[d, "+"'] & G \times G \arrow[d, "+"] \\ G \arrow[r, "-"'] & G \end{tikzcd}$$

commutes.

Remark 3.9. Assume a presentable group $(G, \leqslant, 0, +, -)$.

1. Notice that the inversion is in particular monotone, so we have quite counter-intuitively

$$(a \leq b) \Rightarrow (-a \leq -b)$$

for all $a, b \in G$.

2. We have $0 \leqslant s+(-s)$, for all $s \in \mathcal{S}_G$, since $s \leqslant 0+s$ implies that $0 \leqslant s+(-s)$. This entails that, in fact

$$0 \leqslant a+(-a)$$

for *any* $a \in G$. Since $\mathcal{S}_a \neq \varnothing$ there is a supercompact $s \in \mathcal{S}_a$ such that $s \leqslant a$, hence

$$\begin{aligned} 0 &\leqslant s-s \\ &\leqslant a-a \end{aligned}$$

3. It is in general not true that $a \leq b+c$ implies $b \leq a-c$ for arbitrary $a, b, c \in G$. Take the presentable group $(\mathcal{P}^*(\mathbb{Z}), \subseteq, \{0\}, +)$, where $\mathbb{Z}$ is endowed with the usual addition. Then

$$\{1,2\} \subseteq \{0,1\}+\{0,2\}=\{0,1,2,3\},$$

but

$$\{0,2\} \nsubseteq \{1,3\}-\{0,2\}=\{1,-1,3\}.$$

Example 3.10. Let $(G, 0, \maltese)$ be an abelian group and denote by $\multimap a$ the opposite element of a with respect to $\maltese$. The presentable monoid $\mathcal{P}^*(G)$ as defined in Example 3.3 is made into a presentable group by defining the subtraction as

$$\begin{array}{rcl} -: \mathcal{P}^*(G) & \longrightarrow & \mathcal{P}^*(G) \\ A & \mapsto & \{\multimap a : a \in A\} \end{array}$$

Example 3.11. The presentable monoid $(\mathbb{Z}_{\geq 1} \cup \{\infty\}, |, 1, \cdot)$ can be made into a presentable group by defining subtraction to be the identity function. The technique still works for presentable monoids $(\mathcal{I}^*(R), \supseteq, R)$, where R is Noetherian, or $(\mathcal{A}(k^n), \subseteq, \emptyset)$, where k is a field.

Definition 3.12. An *presentable ring* R is a presentable group $(R, \leq, 0, +, -)$ equipped with a commutative monoidal structure $(R, \cdot, 1)$, such that $\cdot$ is compatible with $\leq$ and $-$, distributative with respect to $+$, and verifies

$$\mathcal{S}_{ab} \quad = \quad \{st | s \in \mathcal{S}_a, t \in \mathcal{S}_b\}$$

for all $a, b \in R$. a presentable ring R such that $\mathcal{S}_R^* = \mathcal{S}_R \setminus \{0\}$ is a multiplicative group will be also called an *presentable field.*

Example 3.13. Let $(R, 0, \maltese, \bullet, 1)$ be a ring (with the identity 1). The presentable group $(\mathcal{P}^*(R), \subseteq, \{0\}, +)$ defined in Example 3.10 is made into a presentable ring with $\{1\}$ as the identity by defining multiplication as

$$A \cdot B \stackrel{def.}{=} \{a \bullet b : a \in A, b \in B\}$$

Remark 3.14. Assume a presentable ring $(R, \leq, 0, +, -, \cdot, 1)$. We have

1. The element $1 \in R$ is uniquely defined.

2. $1 \neq 0$.

3. $-1 \in \mathcal{S}_R$.

Items 1 and 2 are immediate. For item 3 fix $s \in S(-1)$. Then $s \leq -1$ and, consequently, $-s \leq 1$. But 1 is a supercompact, so $-s = 1$, hence $-1 = s \in \mathcal{S}_R$.

4 Presentable algebras from hyperalgebras

Definition 4.1. A *hypermonoid* is a pointed set $(M, 0, \oplus)$ equipped with a multivalued addition

$$\oplus : M \times M \to \mathcal{P}^*(M)$$

such that

i. $a \oplus 0 = a = 0 \oplus a$ for all $a \in M$;

ii. $a \oplus b = b \oplus a$ for all $a, b \in M$;

iii. $(a \oplus b) \oplus c = \bigcup\{a \oplus x | x \in b \oplus c\} = a \oplus (b \oplus c)$ for all $a, b, c \in M$.

Remark 4.2. Let $(M, 0, \oplus)$ be a hypermonoid. The pointed presentable poset $(\mathcal{P}^*(M), \subseteq, \{0\})$ can be made into a presentable monoid by defining

$$\begin{array}{lcl} + : \mathcal{P}^*(M) \times \mathcal{P}^*(M) & \longrightarrow & \mathcal{P}^*(M) \\ (A, B) & \mapsto & \bigcup\{a \oplus b : a \in A, b \in B\} \end{array}$$

Presentable monoids $\mathcal{P}^*(M)$ are behind the basic intuition underlying this work.

Definition 4.3. A *hypergroup* $(G, 0, \oplus, \ominus)$ is a hypermonoid $(G, 0, \oplus)$ together with a map $\ominus : G \to G$ such that

i. $0 \in a \oplus (\ominus a)$ for all $a \in G$;

ii. $(a \in b \oplus c) \Rightarrow (c \in a \oplus (\ominus b))$ for all $a, b, c \in G$.

Example 4.4. Let $(G, 0, \oplus, \ominus)$ be a hypergroup. The presentable monoid $(\mathcal{P}^*(G), \subseteq, \{0\}, +)$ is made into a presentable group by defining

$$\begin{array}{lcl} - : \mathcal{P}^*(G) & \longrightarrow & \mathcal{P}^*(G) \\ A & \mapsto & \{\ominus a : a \in A\} \end{array}$$

Definition 4.5. A *hyperring* $(R, 0, \oplus, \ominus, 1, \odot)$ is a hypergroup $(R, 0, \oplus, \ominus)$ such that $(R, 1, \odot)$ is a commutative monoid and

i. $0 \odot a = 0$ for all $a \in R$;

ii. $a \odot (b \oplus c) \subset (a \odot b) \oplus (a \odot c)$ for all $a, b, c \in R$;

iii. $0 \neq 1$.

If, in addition, every non-zero element has a multiplicative inverse, then R is called a *hyperfield*.

Example 4.6. Let $(R, 0, \oplus, \ominus, 1, \odot)$ be a hyperring (or a hyperfield). The presentable group $(\mathcal{P}^*(R), \subseteq, \{0\}, +, -)$ is made into a presentable ring (or a presentable field, respectively) by defining the following multiplication:

$$A \cdot B = \{a \odot b : a \in A, b \in B\},$$

for $A, B \in \mathcal{P}^*(R)$, and the identity to be just the singleton $\{1\}$.

Remark 4.7. Let $(\boldsymbol{F}, 0, \oplus, \ominus, 1, \odot)$ be a hyperfield, and let T be a subgroup of the multiplicative group $(\boldsymbol{F}, 1, \odot)$. Denote by $\boldsymbol{F}/_m T$ the set of equivalence classes of the relation $\sim$ defined on $\boldsymbol{F}$ as follows:

$$x \sim y \text{ if and only if } x \odot s = y \odot t, \text{ for some } s, t \in T.$$

Denote by $\bar{x}$ the class of x and introduce the following obvious operations on $\boldsymbol{F}/_m T$ induced from $\boldsymbol{F}$:

$$\bar{x} \in \bar{y} \bar{\oplus} \bar{z} \text{ if and only if } x \odot s \in (y \odot t) \oplus (z \odot u), \text{ for some } s, t, u \in T,$$

$$\bar{x} \bar{\odot} \bar{y} = \overline{x \odot y}, \qquad \bar{\ominus} \bar{x} = \overline{\ominus x}.$$

$(\boldsymbol{F}/_m T, \bar{0}, \bar{\oplus}, \bar{\ominus}, \bar{1}, \bar{\odot})$ is then a hyperfield that we shall call the *quotient hyperfield* of $\boldsymbol{F}$ modulo T (see [16]).

Example 4.8. Let k be a field. Firstly, consider the case when $\operatorname{char} k \neq 2$, $k \neq \mathbb{F}_3, \mathbb{F}_5$. This, in particular, is an example of a hyperfield with $a \oplus b = \{a+b\}$. Let $T = k^{*2}$. It is not difficult to check that the following equivalence holds true, for $x, y, z \in k$:

$$x = s^2 y + t^2 z \text{ for some } s, t \in k \text{ if and only if } \bar{x} \in \bar{y} \bar{\oplus} \bar{z} \text{ in } k/_m k^{*2},$$

so that

$$\bar{y} \bar{\oplus} \bar{z} = D(y, z) \cup \{\bar{0}\},$$

where $D(y, z)$ denotes the value set of the binary quadratic form (y, z). Roughly speaking, $k/_m k^{*2}$ is the same object as the quadratic form scheme of k (terminology as in [13] and [14]) with zero adjoined, and provides a first-order axiomatization of the algebraic theory of quadratic forms over fields.

If $\operatorname{char} k = 2$, or $k = \mathbb{F}_3$, or $k = \mathbb{F}_5$, the equivalence

$$x = s^2 y + t^2 z \text{ for some } s, t \in k \iff \bar{x} \in \bar{y} \bar{\oplus} \bar{z} \text{ in } k/_m k^{*2}$$

fails to hold in general, and it is necessary to modify the definition of addition in $k/_m k^{*2}$, defining $\bar{y} \bar{\oplus} \bar{z}$ "by hand", for $y, z \neq 0$. Namely, we set

$$\bar{y} \bar{\oplus}' \bar{z} = \begin{cases} \bar{y} \bar{\oplus} \bar{z}, & \text{if } \bar{y} = \bar{0} \text{ or } \bar{z} = \bar{0}, \\ \bar{y} \bar{\oplus} \bar{z} \cup \{\bar{y}, \bar{z}\}, & \text{if } \bar{y} \neq \bar{0}, \bar{z} \neq \bar{0}, \bar{y} \neq -\bar{z}, \\ k/_m k^{*2}, & \text{if } \bar{y} \neq \bar{0}, \bar{z} \neq \bar{0}, \bar{y} = -\bar{z}. \end{cases}$$

$k/_m k^{*2}$ equipped with this new addition is again a hyperfield, that we refer to as the *prime hyperfield* of $k/_m k^{*2}$. The construction of the prime addition can be carried out for every hyperfield (see [9, Proposition 2.1]). Note that, for $\operatorname{char} k \neq 2$, $k \neq \mathbb{F}_3, \mathbb{F}_5$, the prime addition $\bar{\oplus}'$ is just $\bar{\oplus}$. The prime hyperfield of $k/_m k^{*2}$ will be also called the *quadratic hyperfield* of k and denoted $Q(k)$.

Example 4.9. Let k be a field with two square classes. Consider, for example, the case when k is formally real. The two sqare classes are represented by $1, -1$, so that k is Euclidean (for example, $k = \mathbb{R}$, or the field of real algebraic numbers, or the field of real constructible numbers etc.), and $Q(k) = \{\bar{0}, \bar{1}, \overline{-1}\}$ with multiplication defined in an obvious way and multivalued addition as follows:

$\bar{\oplus}$	$\bar{0}$	$\bar{1}$	$\overline{-1}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\overline{-1}$
$\bar{1}$	$\bar{1}$	$\bar{1}$	$\{\bar{0}, \bar{1}, \overline{-1}\}$
$\overline{-1}$	$\overline{-1}$	$\{\bar{0}, \bar{1}, \overline{-1}\}$	$\overline{-1}$

The presentable ring $\mathcal{P}^*(Q(k))$ with identity I consists of 7 elements:

$$\begin{array}{llll} \theta = \{\bar{0}\}, & I = \{\bar{1}\}, & \kappa = \{\overline{-1}\}, & \beta = \{\bar{0}, \bar{1}, \overline{-1}\}, \\ \alpha_1 = \{\bar{0}, \bar{1}\}, & \alpha_2 = \{\bar{0}, \overline{-1}\}, & \alpha_3 = \{\bar{1}, \overline{-1}\}, & \end{array}$$

the arithmetic of $\mathcal{P}^*(Q(k))$ is described by the following two tables:

$+$	θ	I	κ	α_1	α_2	α_3	β
θ	θ	I	κ	α_1	α_2	α_3	β
I	I	I	β	I	β	β	β
κ	κ	β	κ	β	κ	β	β
α_1	α_1	I	β	α_1	β	β	β
α_2	α_2	β	κ	β	α_2	β	β
α_3	α_3	β	β	β	β	β	β
β	β	β	β	β	β	β	β

and

	θ	I	κ	α_1	α_2	α_3	β
λ_θ	θ	θ	θ	θ	θ	θ	θ
λ_I	θ	I	κ	α_1	α_2	α_3	β
λ_κ	θ	κ	I	α_2	α_1	α_3	β
λ_{α_1}	θ	α_1	α_2	α_1	α_2	β	β
λ_{α_2}	θ	α_2	α_1	α_2	α_1	β	β
λ_{α_3}	θ	α_3	α_3	β	β	α_3	β
λ_β	θ	β	β	β	β	β	β

and the relation of partial order in $\mathcal{P}^*(Q(k))$ is illustrated by the graph below:

$$\begin{tikzcd} & \beta & \\ \alpha_1 \arrow[ur] & \alpha_2 \arrow[u] & \alpha_3 \arrow[ul] \\ \theta \arrow[u] \arrow[ur] & I \arrow[ul] \arrow[ur] & \kappa \arrow[u] \arrow[ul] \end{tikzcd}$$

where arrows correspond to inclusions and obvious compositions of arrows are omitted for clarity.

5 Witt rings of quadratically presentable fields

Definition 5.1. Let $(R, \leq, 0, +, -, \cdot, 1)$ be a presentable field. We shall call R *pre-quadratically presentable*, if the following conditions hold:

i. $\forall a \in \mathcal{S}_R^*, b \in \mathcal{S}_R [a \leq a + b]$;

ii. $\forall a, b, c \in \mathcal{S}_R [(a \leq 1 - b) \wedge (a \leq 1 - c) \Rightarrow (a \leq 1 - bc)]$;

iii. $a^2 = 1$, for all $a \in \mathcal{S}_R \setminus \{0\}$.

Remark 5.2. Note that in the axiom i. the assumption that $a \in \mathcal{S}_R^*$ is crucial: if $a = 0$ then $a \leq a + b$ is just $0 \leq 0 + b = b$, which means $b = 0$ for all $b \in \mathcal{S}_R$.

Example 5.3. Following Examples 4.8 and 4.9, let k be a field, let $Q(k)$ be its quadratic hyperfield, and $\mathcal{P}^*(Q(k))$ the presentable field that is induced. One easily checks that this, in fact, is a pre-quadratically presentable field with $\leq$ being just $\subseteq$.

Example 5.4. The presentable field $\mathcal{P}^*(R)$ constructed from a ring with identity $(R, 0, \maltese, \bullet, 1)$ in Example 3.13 doest not lead to a pre-quadratically accesible field: it is, in general, not true that $\{a\} \subset \{a\} + \{b\} = \{a + b\}$, for arbitrary $a, b \in R$, so that the axion i. is not satisfied.

Definition 5.5. A *form* ϕ on a pre-quadratically presentable field R is an n-tuple $\langle a_1, \ldots, a_n \rangle$ of elements of ${\mathcal{S}_R}^*$; n is called the *dimension* of ϕ, $\dim \phi$, forms of dimension 1 are called *unary*, and of dimension 2 – *binary*. We define the relation $\cong$ of *isometry* of forms of the same dimension by induction:

i. $\langle a \rangle \cong \langle b \rangle$ iff. $a = b$,

ii. $\langle a_1, a_2 \rangle \cong \langle b_1, b_2 \rangle$ iff. $a_1 a_2 = b_1 b_2$ and $b_1 \leq a_1 + a_2$,

iii. $\langle a_1, \ldots, a_n \rangle \cong \langle b_1, \ldots, b_n \rangle$ iff. there exist $x, y, c_3, \ldots, c_n \in {\mathcal{S}_R}^*$ such that

- $\langle a_1, x \rangle \cong \langle b_1, y \rangle$,
- $\langle a_2, \ldots, a_n \rangle \cong \langle x, c_3, \ldots, c_n \rangle$,
- $\langle b_2, \ldots, b_n \rangle \cong \langle y, c_3 \ldots, c_n \rangle$.

Proposition 5.6. *The relation $\cong$ is an equivalence on the sets of all unary and binary forms of a pre-quadratically presentable field R.*

Proof. The statement is clear for unary forms. For binary forms, reflexivity follows from the axiom i., and for symmetry assume that $\langle a, b \rangle \cong \langle c, d \rangle$, for $a, b, c, d \in {\mathcal{S}_R}^*$. Thus $ab = cd$ and $a \leq c + d$. But then $a = bcd$, so that $bcd \leq c + d$. Thus $b \leq cd(c + d) = c + d$.

It remains to show trasitivity, so assume $\langle a, b \rangle \cong \langle c, d \rangle$ and $\langle c, d \rangle \cong \langle e, f \rangle$, for $a, b, c, d, e, f \in {\mathcal{S}_R}^*$. This means $ab = cd$, $cd = ef$, $a \leq c + d$ and, by symmetry, $e \leq c + d$. Therefore, $c \leq a - d$ and $c \leq e - d$, which gives $-cd \leq 1 - ad$ and $-cd \leq 1 - ed$. By (2) this implies $-cd \leq 1 - ae$. Since $cd = ef$, this is just $-ef \leq 1 - ae$, or, equivalently, $ef \leq ae - 1$. But this is the same as $ae \leq 1 + ef$, so $a \leq e + f$. $\square$

Remark 5.7. Recall that a pre-special group (terminology as is [6, Definition 1.2]) is a group G of exponent 2 together with a distinguished element -1 and a binary operation $\cong$ on $G \times G$ such that, for all $a, b, c, d \in G$:

i. $\cong$ is an equivalence relation,

ii. $(a, b) \cong (b, a)$,

iii. $(a, -a) \cong (1, -1)$,

iv. $[(a, b) \cong (c, d)] \Rightarrow [ab = cd]$,

v. $[(a, b) \cong (c, d)] \Rightarrow [(a, -c) \cong (-b, d)]$,

vi. $[(a,b) \cong (c,d)] \Rightarrow \forall x \in G[(xa,xb) \cong (xc,xd)]$.

Corollary 5.8. *Let $(R, \leq, 0, +, -, \cdot, 1)$ be a pre-quadratically presentable field. Then $(\mathcal{S}_R{}^*, \cong, -1)$ is a pre-special group.*

Proof. The only non-trivial parts to check are that $\langle a, -a\rangle \cong \langle 1, -1\rangle$ and that $\langle a, b\rangle \cong \langle c, d\rangle$ implies $\langle a, -c\rangle \cong \langle -b, d\rangle$, for $a, b, c, d \in \mathcal{S}_R{}^*$. The first statement follows from the fact that $a \leq a + 1$ implies $1 \leq a - a$, and for the second one assume $ab = cd$ and $a \leq c + d$. Thus $d \leq a - c$, so that $\langle d, -b\rangle \cong \langle a, -c\rangle$, and the rest is obvious. ☐

Definition 5.9. A pre-quadratically presentable field $(R, \leq, 0, +, -, \cdot, 1)$ will be called *quadratically presentable*, if the isometry relation is an equivalence on the set of all forms of the same dimension.

Remark 5.10. Let $(G, \cong, -1)$ be a pre-special group. The relation $\cong$ can be extended to the set $\underbrace{G \times \ldots \times G}_{n}$ as follows:

$$(a_1, \ldots, a_n) \cong_n (b_1, \ldots, b_n)$$

provided that there exist $x, y, c_3, \ldots, c_n \in G$ such that

i. $(a_1, x) \cong (b_1, y)$;

ii. $(a_2, \ldots, a_n) \cong_{n-1} (x, c_3, \ldots, c_n)$;

iii. $(b_2, \ldots, b_n) \cong_{n-1} (y, c_3 \ldots, c_n)$.

A *special group* [6, Definition 1.2]) is a pre-special group $(G, \cong, -1)$ such that $\cong_n$ is an equivalence relation for all $n \in \mathbb{N}$.

Corollary 5.11. *Let $(R, \leq, 0, +, -, \cdot, 1)$ be a quadratically presentable field. Then $(\mathcal{S}_R{}^*, \cong, -1)$ is a special group.*

Example 5.12. The pre-quadratically presentable field $\mathcal{P}^*(Q(k))$, for a field k, is quadratically presentable. That $\cong$ is an equivalence relation on the set of all forms of the same dimension follows from the well-known inductive description of the isometry relation of quadratic forms.

Definition 5.13. Let R be a pre-quadratically presentable field, let $\phi = \langle a_1, \ldots, a_n\rangle$, $\psi = \langle b_1, \ldots, b_m\rangle$ be two forms. The *orthogonal sum* $\phi \oplus \psi$ is defined as the form

$$\langle a_1, \ldots, a_n, b_1, \ldots, b_m\rangle,$$

and the *tensor product* $\phi \otimes \psi$ as

$$\langle a_1b_1, \ldots, a_1b_m, a_2b_1, \ldots, a_2b_m, \ldots, a_nb_1, \ldots, a_nb_m\rangle.$$

We will write $k \times \phi$ for the form $\underbrace{\phi \oplus \ldots \oplus \phi}_{k\ times}$.

Proposition 5.14. *1. Let R be a pre-quadratically presentable field. The direct sum and the tensor product of isometric forms are isometric.*

2. (Witt cancellation) Let R be a quadratically presentable field. If $\phi_1 \oplus \psi \cong \phi_2 \oplus \psi$, then $\phi \cong \psi$.

Proof. One proceeds by induction on the dimension of forms using basic properties of pre-quadratically presentable fields expresses in Corollary 5.8 and the transitivity of $\cong$ on forms of dimension higher than 2 for Witt cancellation. The arguments are essentially the same as in the proof of corresponding results for pre-special and special groups and will be omitted here. See [6, Proposition 1.6] for details. $\square$

Definition 5.15. Let R be a quadratically presentable field. Two forms ϕ and ψ will be called Witt equivalent, denoted $\phi \sim \psi$, if, for some integers $m, n \geq 0$:

$$\phi \oplus m \times \langle 1, -1 \rangle \equiv \psi \oplus n \times \langle 1, -1 \rangle.$$

Remark 5.16. It is easily verified that $\sim$ is an equivalence relation on forms over R, compatible with (and, clearly, coarser than) the isometry. One also easily checks that Witt equivalence is a congruence with respect to orthogonal sum and tensor product of forms. Denote by $W(R)$ the set of equivalence classes of forms over R under Witt equivalence, and by $\overline{\phi}$ the equivalence class of ϕ. With the operations

$$\overline{\phi} + \overline{\psi} = \overline{\phi \oplus \psi}, \qquad \overline{\phi} \cdot \overline{\psi} = \overline{\phi \otimes \psi}$$

$W(R)$ is a commutative ring, having as zero the class $\overline{\langle 1, -1 \rangle}$, and $\overline{\langle 1 \rangle}$ as multiplicative identity. The proof depends is a cruicial way on Witt cancellation, but essentially mimics the proof of the fact that similarity classes of forms over fields form a ring under orthogonal sum and tensor product. The details are left to the reader.

Definition 5.17. Let R be a quadratically presentable field. $W(R)$ with binary operations as defined above is called the *Witt ring of R*.

As one might expect, the main example of a Witt ring of a quadratically presentable field, is the Witt ring of the quadratically presentable field induced by the quadratic hyperfield of a field:

Theorem 5.18. *For a field k, $W(\mathcal{P}^*(Q(k)))$ is just the usual Witt ring $W(k)$ of non-degenerate symmetric bilinear forms of k.*

Proof. The map $W(k) \to W(\mathcal{P}^*(Q(k)))$ that sends the Witt equivalence class of the form $(a_1, \ldots, a_n)$, $a_1, \ldots, a_n \in k/{k^*}^2$, to the Witt equivalence class of the form $\langle \{a_1\}, \ldots, \{a_n\} \rangle$ is easily checked to be a well-defined isomorphism of rings. $\square$

Remark 5.19. Notice that Theorem 5.18 provides a uniform construction of the Witt ring for all charateristics as well as for $\mathbb{F}_3$ and $\mathbb{F}_5$.

6 Quotients in presentable fields

In order to investigate Witt rings of presentable fields, one needs to know how to pass from presentable fields to quadratically presentable fields. We are "almost" able to do that, and will show how one can build a pre-quadratically presentable field from arbitrary presentable field – it is, however, not clear when the resulting presentable field is quadratically presentable. The main tool to be used are quotients of presentable fields. Before we proceed to general quotients, we focus on a rather special case of quotients "modulo" multiplicative subsets of supercompacts. These are, in fact, the only quotients that we need in the sequel, which explains why we choose to present our exposition in this particular manner.

Theorem 6.1. *Let $(R, \leq, 0, +, -, \cdot, 1)$ be a presentable field. Let $T \subseteq \mathcal{S}_R^*$ be a multiplicative set i.e. for all $s, t \in T$, $st \in T$. Define the relation $\sim$ on $\mathcal{S}_R$ by*

$$a \sim b \text{ if and only if } \exists s, t \in T\, as = bt.$$

This is an equivalence relation, whose equivalence classes will be denoted by $\overline{a}$, $a \in \mathcal{S}_R$. Let

$$\overline{a} \odot \overline{b} = \overline{ab}, \qquad \ominus\overline{a} = \overline{-a},$$

and let

$$\overline{a} \in \overline{b} \oplus \overline{c} \text{ if and only if } \exists s, t, u \in T[as \leq bt + cu].$$

Then $(\mathcal{S}_R/\sim, \overline{0}, \oplus, \ominus, \overline{1}, \odot)$ is a hyperfield.

Proof. The relation $\sim$ is clearly reflexive and symmetric, and for transitivity assume $as = bt$ and $bu = cv$, for some $a, b, c \in \mathcal{S}_R$, $s, t, u, v \in T$. Then $asu = btu$ and $btu = cvt$ with $su, tu, vt \in T$ thanks to the multiplicativity of T.

Next, the operation $\odot$ is clearly well-defined, and to see that so is $\oplus$, assume $\overline{b} = \overline{b'}$ and $\overline{c} = \overline{c'}$, say, $vb = v'b'$ and $wc = w'c'$, for some $v, v', w, w' \in T$. Then

$$\begin{aligned}
\overline{a} \in \overline{b} + \overline{c} &\Leftrightarrow \exists s, t, u \in T[as \leq bt + cu] \\
&\Rightarrow \exists s, t, u[asvw \leq bvwt + cwvu] \\
&\Leftrightarrow \exists s, t, u[asvw \leq b'v'wt + c'w'vu] \\
&\Leftrightarrow \overline{a} \in \overline{b'} \oplus \overline{c'}.
\end{aligned}$$

In order to show that $\mathcal{S}_R/\sim$ with operations defined as above is, indeed, a hyperring, we note that both the commutativity of $\oplus$ and the fact that $(\mathcal{S}_R/\sim, \overline{1}, \odot)$ forms a commutative group are obvious, that $\overline{0} \in \overline{a} \ominus \overline{a}$, for all $\overline{a} \in \mathcal{S}_R/\sim$, follows immediately from $0 \leq a - a$ for all $a \in \mathcal{S}_R$, that $\overline{0} \odot \overline{a} = \overline{0}$ is clear in view of $0 \cdot 1 = 0$, and that $\overline{0} \neq \overline{1}$ is apparent, as $1 \cdot t = 0$, for some $t \in T$, leads to $0 = 1$. It remains to show the neutrality of $\overline{0}$, associativity of $\oplus$, cancellation and distributativity of $\oplus$ and $\cdot$.

Assume $\overline{b} \in \overline{a} \oplus \overline{0}$, so $bs \leq at + 0 = at$, for some $s, t \in T$. But then $bst^{-}1 \leq a$, and, since a is a supercompact, this yields $bst^{-1} = a$ and, consequently, $\overline{b} = \overline{a}$.

Assume $\overline{d} \in \overline{a} \oplus (\overline{b} \oplus \overline{c})$, so that $\overline{d} \in \overline{a} \oplus \overline{e}$ with $\overline{e} \in \overline{b} \oplus \overline{c}$. Hence $ds \leq at + eu$ and $es' \leq bt' + cu'$, for some $s, t, u, s', t', u' \in T$. Thus $dss' \leq ats' + eus'$ and $eus' \leq but' + cuu'$, so that $dss' \leq ats' + (but' + cuu') = (ats' + but') + cuu'$. It follows that there exist supercompacts $d', f, c' \in \mathcal{S}_R^*$ with $d' \leq dss'$, $f \leq ats' + but'$ and $c' \leq cuu'$ with $d' \leq f + c'$. Using the same argument as in the proof of neutrality of $\overline{0}$, we easily check that $d' = d$ and $c' = c$. Therefore $d \leq f + c$ and $f \leq ats' + but'$. This yields $\overline{d} \in \overline{f} \oplus \overline{c}$ with $\overline{f} \in \overline{a} \oplus \overline{b}$, so that $\overline{d} \in (\overline{a} \oplus \overline{b}) \oplus \overline{c}$.

Assume $\overline{a} \in \overline{b} \oplus \overline{c}$, so that $at \leq bs + cu$, for some $s, t, u \in T$. Then there are supercompacts $a' \leq at$, $b' \leq bs$ and $c' \leq cu$ such that $a' \leq b' + c'$. Using the same trick as before we conclude $a = a'$, $b = b'$, $c = c'$, so that, in fact, $a \leq b + c$, and thus $b \leq a - c$, which implies $\overline{b} \in \overline{a} \ominus \overline{c}$.

Finally, if $\overline{d} \in \overline{a} \odot (\overline{b} \oplus \overline{c})$, then $\overline{d} = \overline{ae}$ with $\overline{e} \in \overline{b} \oplus \overline{c}$, and thus $es \leq bt + cu$, for some $s, t, u \in T$. But then $aes \leq abt + acu$, so $\overline{ae} \in \overline{ab} \oplus \overline{ac}$. □

Remark 6.2. We note here that the above result remains true for any presentable ring $(R, \leq, 0, +, -, \cdot, 1)$ and a subgroup $T \subseteq \mathcal{S}_R^*$ of the multiplicative monoid $\mathcal{S}_R^*$. That is, we only need to be able to invert the elements of T for the argument to go through.

Definition 6.3. The *quotient* of $(R, \leq, 0, +, -, \cdot, 1)$ *modulo the multiplicative set* T is the presentable field $(\mathcal{P}^*(\mathcal{S}_\mathcal{R}/\sim), \subseteq, \{\overline{0}\})$ with the hyperfield $(\mathcal{S}_R/\sim, \overline{0}, \oplus, \ominus, \overline{1}, \odot)$ defined in Theorem 6.1 and will be denoted by $R/_m T$.

Theorem 6.1, as remarked before, is a special case of the following, more general result:

Theorem 6.4. *Let* $(R, \leq, 0, +, -, \cdot, 1)$ *be a presentable field. Let* $\sim$ *be a nontrivial congruence on the set* $\mathcal{S}_R^*$ *of supercompacts of* R*, i.e. an equivalence relation such that* $0 \nsim 1$*, and, for all* $a, a', b, b' \in \mathcal{S}_R^*$*, if* $a \sim a'$*, and* $b \sim b'$ *then*

$$ab \sim a'b', \qquad a + b \sim a' + b', \qquad -a \sim -a'.$$

Denote by $\overline{a}$ *the equivalence class of* $a \in \mathcal{S}_R$*. Let*

$$\overline{a} \odot \overline{b} = \overline{ab}, \qquad \ominus\overline{a} = \overline{-a},$$

and let

$$\overline{a} \in \overline{b} \oplus \overline{c} \text{ if and only if } \exists a' \in \overline{a}, b' \in \overline{b}, c' \in \overline{c}[a' \leq b' + c'].$$

Then $(\mathcal{S}_R/\sim, \overline{0}, \oplus, \ominus, \overline{1}, \odot)$ *is a hyperfield.*

The proof mimics the one of Theorem 6.1. That $\overline{0} \neq \overline{1}$ follows from the fact that $0 \nsim 1$.

7 From presentable fields to pre-quadratically presentable fields

In this section it remains to explain how a pre-quadratically presentable field can be obtained from arbitrary presentable field.

Theorem 7.1. *Let* $(R, \leq, 0, +, -, \cdot, 1)$ *be a presentable field and define the following operations on the set* $\mathcal{S}_R$ *of supercompacts of* R*:*

$$a \odot b = a \cdot b, \qquad \ominus a = -a$$

and

$$a \in b \oplus c \text{ if and only if } a \leq b + c.$$

Then $(\mathcal{S}_R, 0, \oplus, \ominus, 1, \odot)$ *is a hyperfield. Further, define the* prime addition *on* $\mathcal{S}_R$ *as follows:*

$$a \oplus' b = \begin{cases} a \oplus b, & \text{if } a = 0 \text{ or } b = 0, \\ a \oplus b \cup \{a, b\}, & \text{if } a \neq 0, b \neq 0, a \neq -b, \\ \mathcal{S}_R, & \text{if } a \neq 0, b \neq 0, a = -b. \end{cases}$$

Then $(\mathcal{S}_R, 0, \oplus', \ominus, 1, \odot)$ *is again a hyperfield, called the* prime hyperfield *of* $(R, \leq, 0, +, -, \cdot, 1)$*, and the induced presentable field* $(\mathcal{P}^*(\mathcal{S}_R), \subseteq, \{0\}, +', -, \cdot, \{1\})$*, that will be called the* prime presentable field*, satisfies the condition:*

$$\forall \{a\} \in \mathcal{S}^*_{\mathcal{P}^*(\mathcal{S}_R)}, \forall \{b\} \in \mathcal{S}^*_{\mathcal{P}}(\mathcal{S}_R)[\{a\} \subseteq \{a\} +' \{b\}].$$

Proof. That $(\mathcal{S}_R, 0, \oplus, \ominus, 1, \odot)$ is a hyperfield is apparent. That $(\mathcal{S}_R, 0, \oplus', \ominus, 1, \odot)$ is also a hyperfield is [9, Proposition 2.1]. The extra condition satisfied by the presentable field $\mathcal{P}^*(\mathcal{S}_R)$ is then obvious. □

Theorem 7.2. *Let* $(R, \leq, 0, +, -, \cdot, 1)$ *be a presentable field such that*

$$\forall a \in \mathcal{S}^*_R, \forall b \in \mathcal{S}_R[a \leq a + b].$$

Let

$$T = \{s \in \mathcal{S}^*_R \mid s \leq a^2 \text{ for some } a \in R\}.$$

Then T *is a multiplicative set and the quotient* $R/_m T$ *of* R *modulo* T *is a pre-quadratically presentable ring.*

Proof. That T is a multiplicative set is apparent, for if $s \leq a^2$ and $t \leq b^2$, for some $s, t \in \mathcal{S}^*_R$, $a, b \in R$, then $st \leq a^2 b^2 = (ab)^2$ and $st \neq 0$, since $\mathcal{S}^*_R$ is a group. The condition

$$\forall a \in \mathcal{S}^*_R, \forall b \in \mathcal{S}_R[a \leq a + b]$$

carries over to $R/_m T$, non-zero supercompacts of $R/_m T$ form a group, since in the process of taking a quotient modulo multiplicative set we end up with a presentable field, and, finally, squares of all non-zero supercompacts of $R/_m T$ are equal to identity, as they are just classes of squares of non-zero supercompacts in R, which are, by definition, equivalent to 1.

It remains to show that for all supercompacts $\{\overline{a}\}$, $\{\overline{b}\}$ and $\{\overline{c}\}$ in $R/_m T$, if $\{\overline{a}\} \subseteq \{\overline{1}\} - \{\overline{b}\}$ and $\{\overline{a}\} \subseteq \{\overline{1}\} - \{\overline{c}\}$, then $\{\overline{a}\} \subseteq \{\overline{1}\} - \{\overline{bc}\}$. Fix three supercompacts as above and assume the antedecent. This is equivalent to $\overline{a} \in \overline{1} \ominus \overline{b}$ and $\overline{a} \in \overline{1} \ominus \overline{c}$ in the hyperfield $\mathcal{S}_R/\sim$, which, in turn, is equivalent to

$$sa \leq t - ub \text{ and } s'a \leq t' - u'c,$$

for some non-zero supercompacts $s, s', t, t', u, u' \in R$ such that $s \leq x^2$, $s' \leq {x'}^2$, $t \leq y^2$, $t' \leq {y'}^2$, $u \leq z^2$, $u' \leq {z'}^2$, for some $x, x', y, y', z, z' \in R$. Since $\mathcal{S}_R^*$ is a group, the elements $sa, s'a, ub, u'c$ are also supercompacts, which allows switching terms between both sides of the above inequalities, and gives

$$ub \leq t - sa \text{ and } u'c \leq t' - s'a',$$

and, in turn

$$ub \leq y^2 - x^2 a \text{ and } u'c \leq {y'}^2 - {x'}^2 a.$$

Hence

$$\begin{array}{rcl} uu'bc & \leq & (y^2 - x^2 a)({y'}^2 - {x'}^2 a) \\ & = & y^2{y'}^2 - y^2{x'}^2 a - x^2{y'}^2 a + x^2{x'}^2 a^2 \\ & \leq & y^2{y'}^2 - y^2{x'}^2 a - x^2{y'}^2 a + x^2{x'}^2 a^2 + 2xx'yy'a - 2xx'yy'a \\ & = & (yy' + axx')^2 - a(x'y + xy')^2 \end{array}$$

Let v and w be supercompacts with $uu'bc \leq v - w$ and $v \leq (yy' + axx')^2$ and $w \leq a(x'y + xy')^2$. If both v and w are equal to zero, then one of b or c is zero, so $\overline{a} \in \overline{1} \ominus \overline{bc}$ is just $\overline{a} \in \overline{1} \ominus \overline{b}$ or $\overline{a} \in \overline{1} \ominus \overline{c}$. If $v = 0$ and $w \neq 0$, then $w = w'w''$ for $w', w'' \in \mathcal{S}_R^*$ with $w' \leq a$ and $w'' \leq (x'y + xy')^2$. Thus $w' = a$, since a is a supercompact itself, and hence a minimal element, and $w'' \in T$, so that $uu'bc \leq -aw''$, $w'' \in T$. But $-aw''$ is again a supercompact, as $\mathcal{S}_R^*$ is a group, so $uu'bc = -aw''$. But $-aw'' \leq -aw'' + 1$, so $uu'bc = -aw'' \leq 1 - aw''$, yielding $\overline{a} \in \overline{1} \ominus \overline{bc}$. Similarly, if $v \neq 0$ and $w = 0$, then $\overline{bc} = \overline{1} \in \overline{1} - \overline{a}$.

This leaves us with the case $v \neq 0$ and $w \neq 0$. Then $v \in T$ and $w = w'w''$, for some $w', w'' \in \mathcal{S}_R^*$ with $w' \leq a$ and $w'' \leq (x'y + xy')^2$. But then $w' = a$, and $w'' \in T$. So, at the end we obtain

$$uu'bc \leq v - aw'',$$

with $uu', v, w'' \in T$, or, equivalently

$$aw'' \leq v - uu'bc,$$

which is the same as $\{\overline{a}\} \subseteq \{\overline{1}\} - \{\overline{bc}\}$. $\square$

Remark 7.3. It is, in general, not clear when the pre-quadratically presentable field that results is quadatically presentable.

Example 7.4. Let k be a field, let $(\mathcal{P}^*(k), \subseteq, \{0\})$ be the induced presentable field. It follows from the construction that applying Theorems 7.1 and 7.2 we obtain the pre-quadratically presentable field $\mathcal{P}^*(\mathcal{S}_{\mathcal{P}^*(k)})/_m T$, where $T = \{\{s\} \in \mathcal{S}_{\mathcal{P}^*(\mathcal{S}_{\mathcal{P}^*(k)})} \mid \{s\} \subseteq \{a\}^2 \text{ for some } \{a\} \in \mathcal{P}^*(\mathcal{S}_{\mathcal{P}^*(k)})\}$, which is isomorphic to $\mathcal{P}^*(Q(k))$, and hence quadratically presentable. In particular, $W(\mathcal{P}^*(\mathcal{S}_{\mathcal{P}^*(k)})) \cong W(\mathcal{P}^*(Q(k))) \cong W(k)$.

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